Lecture Notes on History of Mathematics

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Preface

These lecture notes are written for the history part of the course MATH3208 (Algebra and History of Mathematics at Sultan Qaboos University). This course is designed for giving mathematical background to future mathematics teachers in Oman. These lecture notes contain a brief general history of mathematics covering from the ancient civilisations to the beginning of the 20th century. These contents are based on several textbooks [1][2][3][5][6].

One of evidence of pre-historic mathematical activity is a carved wolf's bone with 55 notches indicating the number of some objects, created around 30,000 B.C. found in the Czech Republic. Mathematics is a mind activity therefore, it relies on the culture and thus on the civilisation; namely every civilisation has its own mathematics. This implies that to learn the history of mathematics is to learn the history of mind activity of mankind. If we look at the history, we will see that the motives to develop mathematics were not always to make mathematics for practical use. It seems that the real motive of mathematics is based on a curiosity of intelligence. In other words, as far as the curiosity exsits, mathematics exists. I believe this is the main reason why mathematics has survived for six thousand years.

These lecture notes consist of the following chapters. The first and second chapters describe the history of mathematics in the Mesopotamian and Egyptian civilisations. Then in Chapter 3, Greek mathematics with a rigorous style of proof is described. Chapter 4 and 5 describe mathematics in China and India respectively. In Chapter 6 the Islamic mathematics is described. Chapter 7 describes the Middle Age in Europe. Chapter 8 describes Renaissance and theory of equations. Chapter 9 describes the calculus. Chapter 10 describes non-Euclidean geometry and modern geometry. Finally Chapter 11 describes Probability and Statistics. The appendices contain a brief history of Japanese mathematics and counting rods.
I hope the students who will become school teachers, find this course fruitfull

for them.

Contents

Chapter 1

Babylonian Mathematics

Around 3500 B.C. the first civilisation appeared in the Tigris and Euphrates river valleys, known as the civilisation of Mesopotamia. Many distinct civilisations ruled this area including a kingdom based at the city of Babylon.

1.1 Cuneiform

These people in the Mesopotamian valley wrote documents by stylus on clay tablets in cuneiform, meaning "wedge-shaped" in Greek. Thousands of tablets have been excavated during past two centuries. These tablets contain many mathematical problems and solutions thus we can see what kind of mathematics were used and developed at the time. In old Babylonian Period (1800-1600 B.C.) Babylonians developed algebra with their numeration system with base 60. They had only two symbols: a unit symbol, and a tens

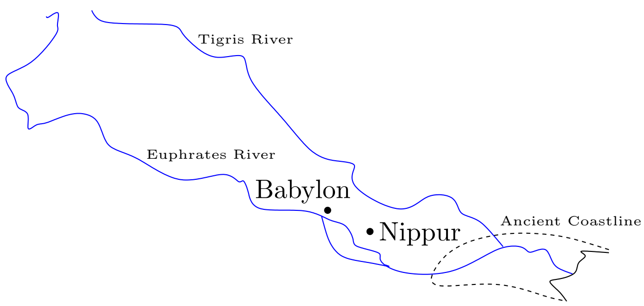

Figure 1.1: Ancient civilisations between two rivers.

symbol (see Figure 1.2). They had to invent the positional notation.

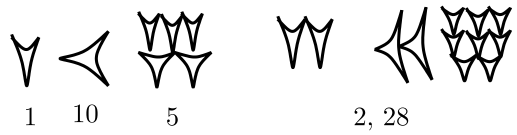

Figure 1.2: Cuneiforms

Number (10)	Number (60)
60	1, 0
64	1, 4
3600	1, 0, 0
3604	1, 0, 4
1/60	0; 1

1.2 Geometric formulae

They developed mathematics such as:

(1) a formula of a solution to a quadratic equation,

(2) they considered systems of equations in two unknowns,

(3) a good approximation of square root of 2 as a fraction.

(4) the Pythagorean theorem at least a thousand years earlier than Pythagoras.

Their algebra was essentially rhetorical.
(1) They solve the follwing problem.

Problem 1.2.1. The length and width of canal are together 6;30 GAR; the area of the canal is 7;30 SAR. What are the length and width?

The solution of this problem is the following.

(a) Take the half of the sum of the length and width which is 3;15.

(b) Square 3;15 to get 10;33, 45.

(c) Subtract the product of length and width, 7;30, from 10;33, 45 to get 3;3, 45.

(d) Take its square root, which is 1;45.

(e) Add it to the half of the sum of the length and width, to get 5 GAR, the length, and subtract it from the sum, to get 1;30 GAR, the length.

With today's notation, we describe the problem as follows.

$$\begin{aligned} x + y &= b \\ xy &= c \end{aligned}$$

Find x and y This is equivalent to solving

$$x^2 - bx + c = 0$$

(a): $\frac{b}{2}$, (b) and (c): $\left(\frac{b}{2}\right)^2 - c$, (d): $\sqrt{\frac{b^2}{4} - c}$, and (e): $\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$.
The last result is

$$\frac{b + \sqrt{b^2 - 4c}}{2}$$

which is the familiar quadratic formula.
(3) They obtained the approximated value of $\sqrt{2}$:

$$\sqrt{2} \approx 1.4141296...$$

A possible method to obtain the approximation of $\sqrt{2}$ is the following. Let a be an approximation and guess the value, say $a = 1$. Then

$$a = 1 < \sqrt{2} \tag{1.1}$$

The fraction $\frac{2}{a}$ is also an approximation of $\sqrt{2}$ and we have

$$1 = a < \sqrt{2} < \frac{2}{a} = 2 \tag{1.2}$$

Take the average of the lower and upper bounds to obtain 1.5. We repeat the process with $a = 1.5$;

$$\frac{2}{1.5} < \sqrt{2} < 1.5. \tag{1.3}$$

Then the average will be $\frac{17}{12}$ Next step will give

$$\sqrt{2} \approx 1.41421569. \tag{1.4}$$

It is obvious that there is no need for this degree of accuracy of $\sqrt{2}$. The motive to find such approximation was possibly just a curiosity of intelligence.

(4) For the Pythagorean theorem, it is thought they may have realised the relation between sides of a right triangle from the diagram shown in Figure 1.3 [1]. In the diagram the area of the square formed by the hypotenuse of a shaded right triangle is the white square in the left. This is equal to the sum of the areas of two squares formed by the opposite and the base. This is because the area of the white square in the left is obtained by subtracting the area of four copies of the right triangle from the biggest square. This is exactly the sum of areas of two white squares in the right.

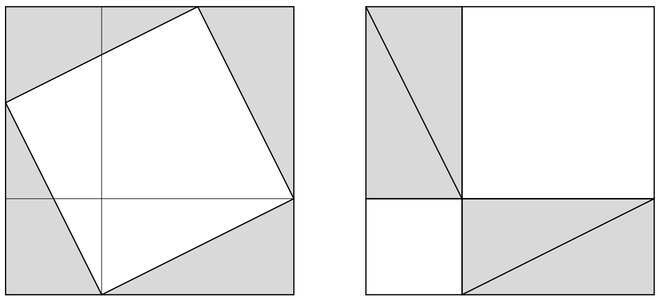

Figure 1.3

They obtained the formula of the area A of a circle with diameter d:

$$A = \frac{\ell d}{4} \tag{1.5}$$

where ℓ is the circumference of the circle [2].

Chapter 1 Exercises

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1. Write 2015 in cuneiform.

2. Using the formula (1.5), find the area of the disc with the diameter 3 when the $\pi \approx 3.16$.

3. Using the method to approximate $\sqrt{2}$, find an approximation of $\sqrt{3}$.
4. Discuss about both advantages and disadvantages when the clay tablets are used to record documents.

Chapter 2

Egyptian Mathematics

The ancient Egyptian civilisation was developed at the Nile delta. The Greek historian Herodotus (5th century B.C.) called Egypt ”the Gift of the Nile.” The Nile Delta forms the Lower Egypt, and the region between the delta and the First Cataract or waterfalls at Aswan forms the Upper Egypt. The Upper and Lower Egypt were united around 3100 B.C. by the first Pharaoh of Egypt, Menes.

2.1 Nile Delta

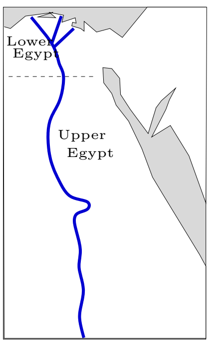

Figure 2.1: The upper Egypt and the lower Egypt.

Egyptians wrote on papyrus made from reeds from the Nile River or leather or cloth made from cotton or linen or stone. The first three materials have advantages such as they are cheap, easy to write, easy to correct, they are flexible and so easy to store. The disadvantage is they easily decay. Therefore, the documents written on stones are survived. Stones are common materials to build temples at the time and they wrote a story on walls of temples. The letters they used are called hieroglyphics (”sacred writings” in Greek).

The hieroglyphic numbering system is a *tally system*; that is, there were symbols for the unit, 10, 100 and all 10^n for all $n \geq 1$ (see Figure 2.2).

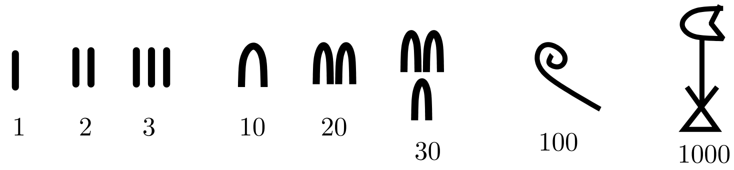

Figure 2.2: The hieroglyphic numbers.

The writing hieroglyph was difficult for non-artists so another form was developed, known as *hieratic*.
In 1858 Henry Rhind purchased a rolled papyrus, today it is called the Rhind Papyrus which contains mathematical contexts. The Rhind Papyrus was written by A'h-mosé and he claimed that the writing is based on an older manuscript. The original text was thought as it was written in between 1849 and 1801 B.C.

2.2 Multiplication

Here we use pseudo hieratic numeration instead using hieratic letters. For example, 13 is written as 10, 3. 27 is written as 20, 7. The ancient Egyptians carried out the multiplication 13×27 as follows.

1	$20, 7$
2	$50, 4$
4	$100, 8$
8	$200, 10, 6$

In the first row, write 1 followed by one of these numbers, here it is 27, written as $20, 7$ in pseudo hieratic expression. In the second row, write the double of all numbers; that is, 2 $50, 2$. Then continue this process for several times so that we can obtain the number 13 as the sum of the some numbers in the first column; that is,

$$13 = 1 + 4 + 8.$$

Add the corresponding numbers in the second column; that is,

$$27 + 108 + 216 = 351.$$

We can interpret this calculation in terms of today's notations:

$$13 \times 27 = (1 + 4 + 8) \times 27 = 27 + 108 + 216 = 351 \qquad (2.1)$$

2.3 Geometry

Some geometric problems are explained in the Rhind Papyrus. The slope of a pyramid is the ratio of the "run" and "rise":

$$\frac{\text{run}}{\text{rise}}$$

which is the reciprocal of our familiar definition of slope.

Problem 2.3.1. If a pyramid is 100, 50 cubits high, and the side of its base is 400 cubits, what is its seked?

Solution. Half of the base is 200. divide 200 by 100, 50 to get 1, $\frac{1}{3}$ of a cubit, which is its seked.

They could calculate the area of isosceles triangle with the base b and the height h. A'h-mosé justifies the formula by dividing the triangle into two right triangles and combine them into a rectangle. Thus the area is $\frac{1}{2}bh$.

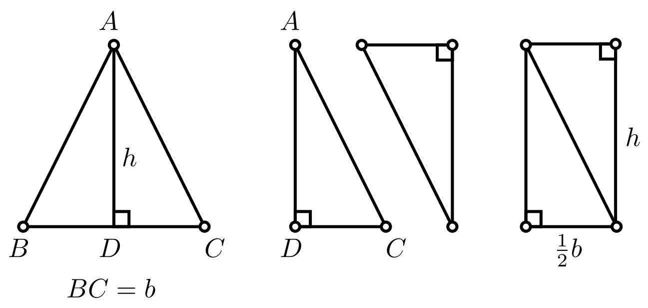

Figure 2.3

The area of isosceles trapezoid is also obtained in the similar way.

The area of disc with the diameter 9 is obtained by the following procedure. Subtract 1 from the diameter to obtain 8. Then square it to obtain 64. He claims this is the area of the disc. From this example, we can estimate the circumference ratio as $3\frac{1}{6}$. It is not known whether or not Egyptians recognised $3\frac{1}{6}$ as an approximation of π.

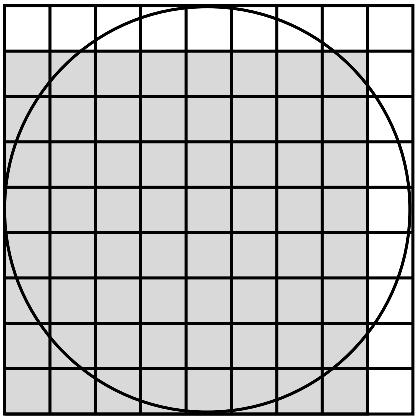

Figure 2.4: The circle with the diameter 9

They obtained the volume of a frustum with the base side a, the top side b and the height h.

$$V = \frac{1}{3}h(a^2 + ab + b^2) \tag{2.2}$$

It is thought that the formula is obtained from the decomposition of the frustum as depicted in Figure 2.5.

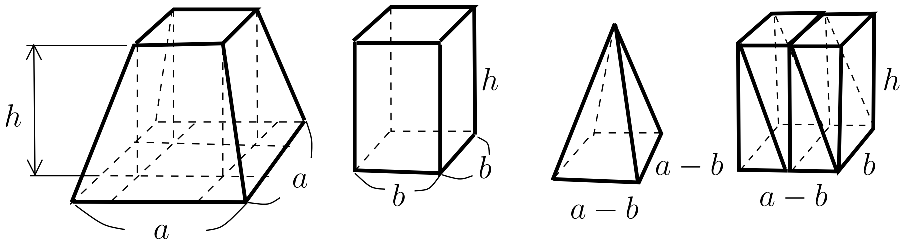

Figure 2.5: The decomposition of the frustum.

$$\begin{aligned} V &= b^2h + \frac{1}{3}(a-b)^2h + (a-b)bh \\ &= \frac{1}{3}h(3b^2 + (a-b)^2 + 3ab - 3b^2) \\ &= \frac{1}{3}h(3b^2 + a^2 - 2ab + b^2 + 3ab - 3b^2) \\ &= \frac{1}{3}h(a^2 + ab + b^2) \end{aligned} \tag{2.3}$$

Egyptians used a formula for quadrilateral with a, b, c and d as lengths of consecutive sides.

$$K = \frac{1}{4}(a+c)(b+d). \tag{2.4}$$

However, this formula is obviously incorrect. This incorrect formula had appeared 3000 years earlier in ancient Babylonia.

Chapter 2 Exercises

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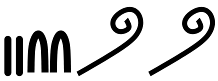
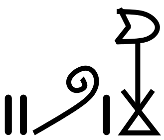
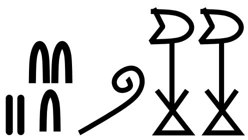
1. Write the number 2015 in hieroglyph.

2. What numbers are represented below?

   (A)  **(B)**  **(C)** 

3. Discuss advantages and disadvantages of hieroglyphic numeration.

4. If a pyramid is 100, 40 cubits high, and the side of its base is 180 cubits, what is its seked?

Chapter 3

Greek Mathematics

The civilisation in river and valleys declined and new culture started along the Mediterranean Sea. The civilisation was from roughly 800 B.C. to 800 C.E.

3.1 Thales and Pythagoras

The Greeks made mathematics into a systematic subject equipped with logic and abstract concepts. This distinguishes Greek mathematics from Babylonians' or Egyptians'.

Thales (circa 625–547 B.C.) is said to be the person who introduced logical proofs based on deductive reasoning. It is said that he predicted solar eclipse in 585 B.C. and applied angle-side-angle criterion of triangle congruence to measure the distance to a ship at sea. Also he discovered that the base angles of an isosceles triangle are equal and that vertical angles are equal and proved that the diameter of a circle divides the circle into two equal parts.

Pythagoras (circa 580–500 B.C.) founded a religious sect committed to the study of philosophy, mathematics and natural science. The sect believed in the idea: the positive numbers form the basic organising principle of the universe. For example, they believed that musical harmonies depend on numerical ratios.
They probably represented numbers by dots or pebbles so that they visually grasped properties of numbers. For example, Figure 3.1 depicts "square numbers" and "triangular numbers". Observing these numbers, they found

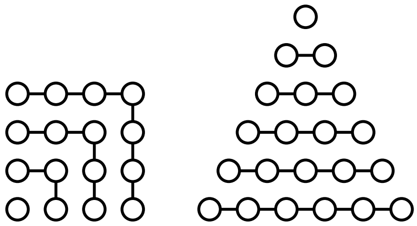


Figure 3.1: Square numbers and triangular numbers.

the fact that the sum of odd numbers is the square of the number of terms; for example,

$$1+3=2^2, \qquad 1+3+5+7=4^2, \qquad 1+3+5+7+9=5^2.$$

The Pythagoras were interested in the triple of numbers (a, b, c) such that $a^2+b^2=c^2$; called a Pythagorean triple.
Although the Pythagorean theorem had been known in other cultures long before Pythagoreans, their study on the Pythagorean theorem led the discovery of the irrational. They discovered the number $\sqrt{2}$ is irrational. This discovery, however caused a crisis which led them to recast their algebra in geometric form.
The Greeks found the existence of irrational numbers however, it is not known when and who made it. For example, they could prove that the number $\sqrt{2}$ is irrational.
Their proof in terms of today's notaion could be the following.

Proof. For a given square $ABCD$, suppose that $AC : AB = m : n$ where m and n are coprime integers.

$$\begin{aligned} AC^2 : AB^2 &= m^2 : n^2 \\ AB^2 + BC^2 : AB^2 &= m^2 : n^2 \\ 2AB^2 : AB^2 &= m^2 : n^2 \\ 2 : 1 &= m^2 : n^2 \\ \therefore \quad 2n^2 &= m^2 \end{aligned}$$

Therefore, m is even. Thus n^2 is even and thus n is even. This contradicts that m and n are coprime. □

Comparing with Babylonian's approximation of $\sqrt{2}$, it is clear that the difference between Babylonian and Greek mathematics.

3.2 Paradoxes of Zeno

Pythagoreans believed in that numbers constitute the entire heaven. This doctrine however faced many problems such as the discovery of the irrational number. Also, some philosophers such as Eleatics attacked their belief by showing inconsistencies of their belief. Zeno of Eleatics (fl. ca. 450 B.C.) started from his opponents' premises to deduce to an absurdity. Four paradoxes of Zeno are well known.

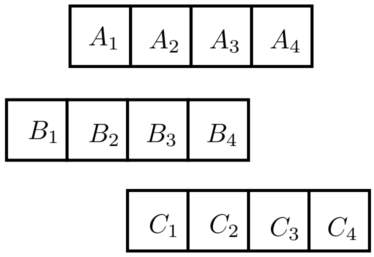
(1) The Dichotomy,

(2) the Achilles,

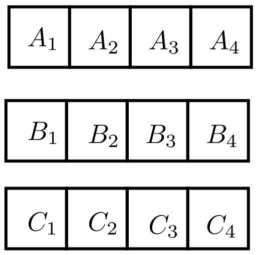
(3) the Arrow and

(4) the Stade (Studium).

(1) To travel from a point A to another point B_0 one must reach the midpoint B_1 on the way. To reach B_1, one must reach the midpoint B_2 and so on. Therefore, one has to pass infinitely many points thus it takes infinite time but it is impossible so one cannot travel from A to B_0.
(2) Achilles is in a footrace with the tortoise. Achilles gives the tortoise a head start; say, put its starting point in 100 meters ahead of Achilles. They start at the same time. When Achilles reach the starting point of the tortoise, the tortoise has moved away from the point. then Achilles reach the second point of the tortoise, the tortoise has move away from the second point and so on. Achilles has to reach infinite number of points, thus he can never overtake the tortoise.
(3) A flying arrow occupies an equal space at any moment, thus the arrow is motionless.
(4) Let A_1, A_2, A_3 and A_4 be bodies in the line of equal size that are stationary; let B_1, B_2, B_3 and B_4 be bodies in the line, of the same size as the A's, and let C_1, C_2, C_3 and C_4 be bodies in the line of the same size as the A's and B's. The bodies B's are moving to the right parallel to the A's and C's are moving to the left parallel to the A's so that each B passes each A and each C passes each A in an instant of time. Let us assume that at a given time, the bodies have the following positions.



Then after a single instant time interval, the positions will be as follows:



It is clear that C_1 has passed two of B's in an instant time interval. Therefore, the instance is not the minimum unit of time interval as we can create smaller time interval.

3.3 The Elements

Euclid of Alexandria is known as the author of the *Elements of Geometry*, which was organised into 13 books. Euclid unified a collection of isolated discoveries into a single deductive system based on a set of initial postulates, definitions and axioms. Little is known about the life of Euclid. He founded a school of mathematics in Alexandria.
Euclid's Elements consists of 13 books and they contain 465 propositions. The contents are:

1. Triangles, parallels, and area.

2. Geometric algebra

3. Circles

4. Inscribed and circumscribed objects

5. Abstract proportions
6. Similarity
7. Fundamentals of number theory
8. Continued proportions in number theory
9. Number theory
10. Incommensurable (rational vs. irrational) numbers
11. Solid geometry
12. Measurement
13. Regular solids

In Elements, there are 23 definitions. Some of them are:

Definition

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The ends of a line are points.
4. A *straight line* is a line which lies evenly with the ponts on itself.
11. An *obtuse angle* is an angle greater than a right angle.
12. An *acute angle* is an angle less than a right angle.

Postulates

(1) To draw a straight line from any point to any point.

(2) To produce a finite straight line continuously in a straight line.

(3) To describe a circle with any centre and distance.

(4) That all right angles are equal to one another.

(5) That, if a straight line intersecting two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The fifth postulate is longer and more complicated than other postulates. It is natural question to ask whether or not the fifth postulate could be proved by using other postulates. Since then people had made great effort to prove the fifth postulate before non-Euclidean geometry was discovered in the 19th century.

Common Notions

(1) Things which equal the same thing also equal one another.

(2) If equals are added to equals, then the whole are equal.

(3) If equals are subtracted from equals, then the remainders are equal.

(4) Things which coincide with one another equal one another.

(5) Then whole is greater than the part.

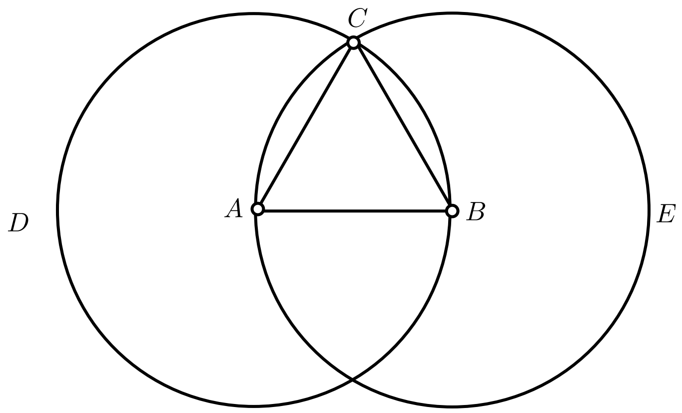
Proposition 3.3.1 (I-1)**.** *To construct an equilateral triangle on a given finite straight line.*

Let AB be the given finite straight line.

It is required to construct an equilateral trinagle on the straight line AB.

Describe the circle BCD with center A and radius AB. Again describer the circle ACE with center B and radius BA. Join the straight lines CA and CB from the point C at which the circles cut one another to the points A

and B.



Now, since the point A is the center of the circle CDB, therefore each of the straight lines AC and BC equals AB. And things which equal the same thing also equal one another, therefore AC also equals BC. Therefore the three straight lines AC, AB and BC equal one another. Therefore the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB.

Q.E.F. (*quod erat faciendum* "which was to be done")

(Also Q.E.D. *quod erat demonstrandum* "which was to be deomonstrated."

This proof seems clear and logically rigorous. However, there are many logical gaps in it.

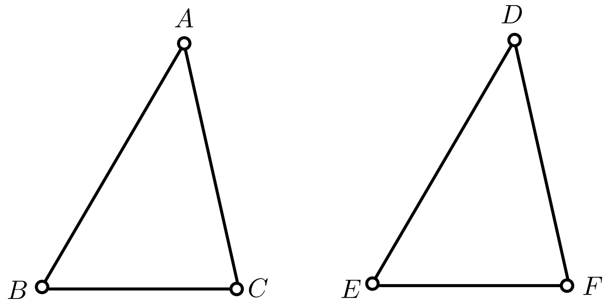
Q1. Find the gaps in the proof of Proposition 3.3.1 as many as possible.

Proposition 3.3.2 (I-4)**.** *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

Let ABC and DEF be two triangles having the two sides DE and DF respectively, namely AB equals to DE and AC equal to DF, and the angle BAC equal to the angle EDF.

I say that the base BC also equals the base EF, the triangle ABC equals the triangle DEF, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle ABC

equals the angle DEF, and the angle ACB equals the angle DFE.

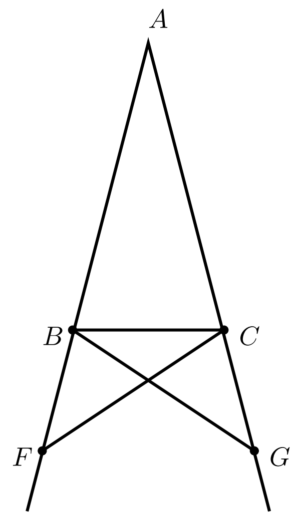


If the triangle ABC is superposed on the triangle DEF, and if the point A is placed on the point D and the straight line AB on DE, then the point B also coincides with E, because AB equals DE. Again, AB coinciding with DE, the straight line AC also coincides with DF, because the angle BAC equals the angle EDF. Hence the point C also coincides with the point F, because AC again equals DF. But B also coincides with E, hence the base BC coincides with the base EF and equals it. Thus the whole triangle ABC coincides with the whole triangle DEF and equals it. And the remaining angles also coincide with the remaining angles and equal them, the angle ABC equals the angle DEF, and the angle ACB equals the angle DFE. Therefore if two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Q.E.D.

Proposition 3.3.3 (I-5)**.** *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight lines BD and CE be produced further in a straight line with AB and AC. I say that the angle ABC equals the angle ACB, and the angle CBD equals the angle BCE. Take an arbitrary point F on BD. Cut off AG from AE the greater equal to AF the less, and join the straight lines

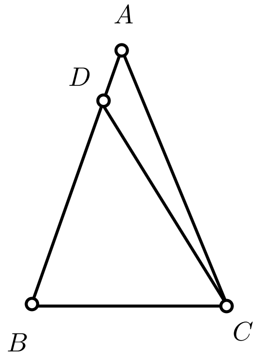
FC and GB.



Since AF equals AG, and AB equals AC, therefore the two sides FA and AC equal the two sides GA and AB, respectively, and they contain a common angle, the angle FAG. Therefore the base FC equals the base GB, the triangle AFC equals the triangle AGB, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle ACF equals the angle ABG, and the angle AFC equals the angle AGB. Since the whole AF equals the whole AG, and in these AB equals AC, therefore the remainder BF equals the remainder CG. But FC was also proved equal to GB, therefore the two sides BF and FC equal the two sides CG and GB respectively, and the angle BFC equals the angle CGB, while the base BC is common to them. Therefore the triangle BFC also equals the triangle CGB, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle FBC equals the angle GCB, and the angle BCF equals the angle CBG. Accordingly, since the whole angle ABG was proved equal to the angle ACF, and in these the angle CBG equals the angle BCF, the remaining angle ABC equals the remaining angle ACB, and they are at the base of the triangle ABC. But the angle FBC was also proved equal to the angle GCB, and they are under the base. Therefore in isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another. Q.E.D.

Proposition 3.3.4 (I-6)**.** *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

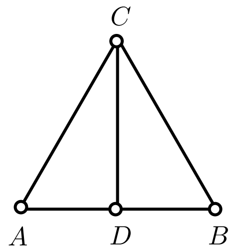
Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that the side AB also equals the side AC.



If AB does not equal AC, then one of them is greater.
Let AB be greater. Cut off DB from AB the greater equal to AC the less, and join DC.
Since DB equals AC, and BC is common, therefore the two sides DB and BC equal the two sides AC and CB respectively, and the angle DBC equals the angle ACB. Therefore the base DC equals the base AB, and the triangle DBC equals the triangle ACB, the less equals the greater, which is absurd.
Therefore AB is not unequal to AC, it therefore equals it. Q.E.D.

Theorem 3.3.1. *To bisect a given finite straight line.*

Let AB be the given finite straight line.

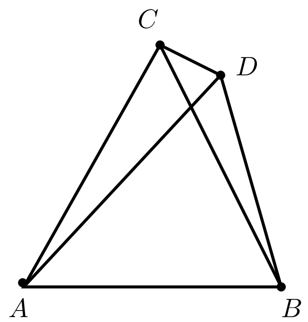


It is required to bisect the finite straight line AB.
Construct the equilateral triangle ABC on it. and bisect the angle ACB by the straight line CD.
I say that the straight line AB is bisected at the point D.
Since CA equals CB, and CD is common, therefore the two sides CA and CD equal the two sides CB and CD respectively, and the angle ACD equals the angle BCD, therefore the base AD equals the base BD.

Therefore the given straight line AB is bisected at D.

Proposition 3.3.5 (I-7)**.** *Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.*

If possible, given two straight lines AC and CB constructed on the straight line AB and meeting at the point C, let two other straight lines AD and DB be constructed on the same straight line AB, on the same side of it, meeting in another point D and equal to the former two respectively, namely each equal to that from the same end, so that AC equals AD which has the same end A, and CB equals DB which has the same end B.



Join CD. Since AC equals AD, therefore the angle ACD equals the angle ADC. Therefore the angle ADC is greater than the angle DCB. Therefore the angle CDB is much greater than the angle DCB. Again, since CB equals DB, therefore the angle CDB also equals the angle DCB. But it was also proved much greater than it, which is impossible. Therefore given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end. Q.E.D.

Proposition 3.3.6 (I-8)**.** *If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.*

Proposition 3.3.7. *In any triangle, if one of the sides is produced, then the exterior angle is greater than ither of the interior and opposite angles.*

Let ABC be a triangle, and let one side of it BC be produced to D.
I say that the exterior angle ACD is greater than either of the interior and opposite angles CBA and BAC.
References
http://aleph0.clarku.edu/~djoyce/elements/bookI/propI4.html

3.4 Primes and Perfect Numbers

Let A and B be numbers with $A < B$. If $kA = B$ for some number k, then A is said to *measure* B. If B has no measure but the unit, then B is said to be *prime*.
Two quantities A and B are *commensurable* if there is a quantity C such that C measures both A and B. Otherwise, A and B are *incommensurable*.

Theorem 3.4.1 (Book IX of the Elements)**.** *There are infinitely many primes.*

Proof. This is proved by a contradiction. Suppose there are only finite number of primes. Given any finite collection of primes $p_1, p_2, \ldots, p_n$, we consider the number

$$p = p_1 p_2 \cdots p_n + 1.$$

This p is not divisible by any prime $p_1, p_2, \ldots, p_n$. There are two cases:
Case 1. p is prime and $p > p_1, p_2, \ldots, p_n$. This is a contradiction.
Case 2. p has a prime divisor which is not in $\{p_1, p_2, \ldots, p_n\}$. This is a contradiction. Therefore, the number of primes is infinite. □

A *perfect number* is a number that is the sum of its divisors.

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 28 &= 1 + 2 + 4 + 7 + 14 \end{aligned}$$

Theorem 3.4.2. *If* $2^n - 1$ *is prime, then* $2^{n-1}(n^n - 1)$ *is perfect.*

Is there an odd perfect number?

3.5 The Euclidean Algorithm

This algorithm first appeared in Book VII of the Elements. Thus it is called the *Euclidean algorithm* but it is said that the algorithm and related results were probably known earlier.
The Euclidean algorithm is used to find the greatest common divisor of two positive integers a, b. The first step is to construct the pair (a_1, b_1), where

$$\begin{aligned} a_1 &= \max(a,b) - \min(a,b), \\ b_1 &= \min(a,b). \end{aligned}$$

Then repeat the process recursively:

$$\begin{aligned} a_{i+1} &= \max(a_i,b_i) - \min(a_i,b_i), \\ b_{i+1} &= \min(a_i,b_i). \end{aligned}$$

The algorithm terminates when $a_{i+1} = b_{i+1}$ and this value is the greatest common divisor of a and b.

3.6 Diophantus

Diophantus of Alexandria is not known exactly which century he was active. It seems to be between 150 B.C. and 350 A.D. He wrote the text *Arithmetica*, which consists of 13 books. This is only Greek algebra known to the present day.
An epigram commemorates Diophantus's life such as:

Problem 3.6.1. For one-sixth of his life, Diophantus was a boy. A twelfth part later, he grew a beard, and married after a seventh. Five years later had a son. The son lived only half the father's life, and Diophantus died four years later.

Arithmetica dealt with indeterminate problems that are today called "Diophatine Equations".

Problem 3.6.2 (8th Problem of Book II)**.** To divide a given number into two suares. Given square number 16.

Solution. Let x^2 be one of squares. The other is $16 - x^2$ which must be equal to a square such as $(2x - 4)^2$.

$$\begin{aligned} 16 - x^2 &= 4x^2 - 16x + 16 \\ 5x^2 - 16x &= 0 \end{aligned}$$

Therefore, one of desired numbers is $x = 16/5$. The other is $16 - \frac{16^2}{25} = \frac{144}{25} = (\frac{12}{5})^2$.

Problem 3.6.3 (15th Problem of Book VI)**.** Given two numbers and a square. If one of the numbers times the square minus the other number is a square, another square larger than the first square can be found that satisfies the same property.

This problem deals with the equation called "Pell's equation".

$$Ax^2 - B = y^2$$

3.7 Archimedes

Archimedes (287-212 B.C.) was born in Syracuse. In his youth, he studied in Alexandria. He showed that the circular constant π is about $3.14\ldots$ obtained the volume of the sphere (ball) and studied conic curves. He obtained the area between a line crossing a parabola and the parabola. He solved the problem of balance. He made the static mechanics a science. His work included many areas in science such as physics, mathematics, hydrostatics. When Roman legions led by Marcellus besieged Syracuse Archimedes was called by King Hiero II to protect Syracuse. He provided catapults to hurl stones against Roman ships, huge crane to lift up Roman ships to turn over, and a giant mirror set Roman ships on fire. Roman soldiers were frightened by the power of these war machines. When Syracuse was fallen, although Marcellus strictly ordered not to kill Archimedes, Archimedes was killed by Roman soldiers.

3.7.1 π of Archimedes

He wrote: Every circumference is greater than 3 times of the diagmeter and the exceeding amount is between $\frac{1}{7}$ and $\frac{10}{71}$; that is, he showed:

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$$

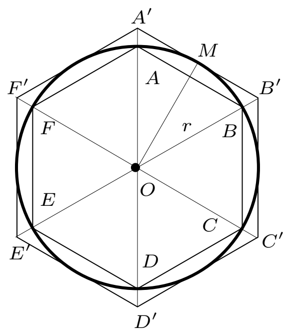


Figure 3.2: The hexisagon is inscribed in the circle with radius r.

He obtained these inequalities by the following method. The regular hexagon is inscribed in the circle with radius r (see Figure3.2).
Let L be the circumference of the circle iwth the radius r and let d be the diameter. Then we have:

$$\pi = \frac{L}{d} > \frac{6r}{2r} = 3$$

Therefore, π is larger than 3. From the diagram in Figure 3.2

$$\begin{aligned} OM &= r \\ \angle OMA' &= \angle OMB' = 90^\circ \\ \therefore \quad \angle A'OM &= 30^\circ \\ A'M &= r\tan 30^\circ = \frac{r}{\sqrt{3}} \\ \therefore \quad A'B' &= \frac{2r}{\sqrt{3}} \end{aligned}$$

The perimeter of the external hexiagon is $6 \times \frac{2r}{\sqrt{3}} = 4r\sqrt{3}$. It is obvious that

$$L < 4r\sqrt{3}$$

Therefore,

$$\pi = \frac{L}{d} < \frac{4r\sqrt{3}}{2r} = 2\sqrt{3}$$

He made the central angle half so that 12-gons are obtained so that he obtained better approximation of π.

Repeating this procedure four times after the hexiagon, he obtained 96-gon and then he obtained the inequalities:

$$\frac{223}{71} < \pi < \frac{22}{7}$$

3.8 The law of the lever

He dealt with the principles of the lever in the beginning of the treatise "Planes in Equilibrium". He began the argument from seven postulates:

1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance.

2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline twards that weight to which the addtion was made.

3. Similarly, if anything be taken away from one of the weights, they ar not in equilibrium but incline towards the weight from which nothing was taken.

4. When equal and similar plane figures coincide if appied to one another, their centres of gravity similarly coincide.

5. In figures which are unequal but similar the centre of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at cettain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.

7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure.

Proposition 3.8.1. *Weights which balance at equal distnace are equal.*

Proposition 3.8.2. *Unequal weights at equal distances will not balance but will incline towards the greater weight.*

Proposition 3.8.3. *Unequal weights will balance at unequal distnces, the greater weight being at the lesser distance.*

Proof. Let A and B be two unequal weights, say $A > B$, balancing at C at distance AC and BC respectively.

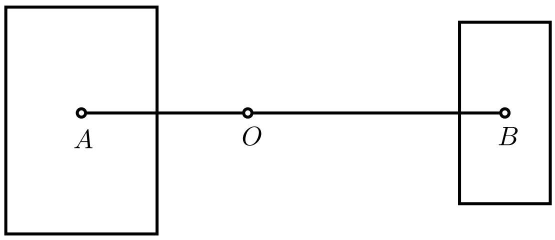


Figure 3.3: Propostion3.8.3.

Then $AC < BC$. If not, take away from A the weight $A-B$. The remainders will then incline towards B [Post. 3]. But this is impossible, for

(1) if $AC = CB$, then the equal remainders will balance, or

(2) if $AC > CB$, then they will incline towards A at the greater distance [Post. 1].

Hence $AC < CB$.
Conversely, if the weights balance, and $AC < CB$, then $A > B$. □

Proposition 3.8.4. *Two magnitudes, whether commensurable or incommensurable, balance at distances reciprocally proportional to the magnitudes.*

Proposition 3.8.4 in today's terms is the following.
Suppose the magnitudes A and B to be commensurable or incommensurable, and the points A and B to be their centres of gravity. Let DE be a straight line so divided at C that

$$A : B = DC : CE$$

Then if A is placed at E and B at D, then C is the centre of gravity of the two taken together.

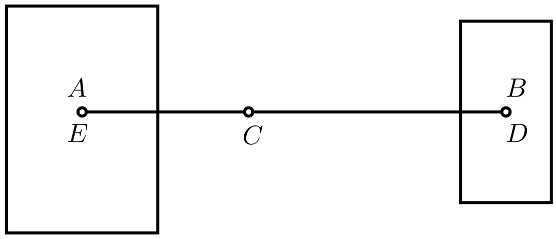


Figure 3.4: Propostion3.8.4.

This is well known as the law of lever.

3.8.1 Parabola of Archimedes

Archimedes obtained areas and volumes of many geometric objects and solids. He used the method of exhaustion. Here we present one example of parabola to show an outline of his method.

Proposition 3.8.5. *Let L be a parabola and let P and Q be two points on it. Let O be a point on L such that the tangent line at O is parallel to the chord PQ. Then the area of segment of the parabola POQ is:*

$$POQ = \frac{4}{3}\triangle POQ \tag{3.1}$$

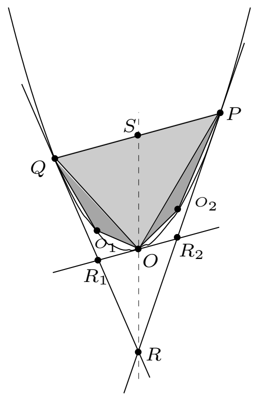


Figure 3.5: The method of exhaustion.

It is known that $SO = OR$. This implies that

$$\triangle RPQ = 2\triangle OPQ \tag{3.2}$$

Since $PQ \parallel R_1R_2$, $R_1R_2 = \frac{1}{2}PQ$. Therefore,

$$\triangle RR_1R_2 = \frac{1}{2}\triangle OPQ \tag{3.3}$$

$$\begin{aligned}
\triangle RPQ - \triangle RR_1R_2 &= \frac{3}{2}\triangle OPQ \\
&= \triangle R_1OQ + \triangle R_2OP + \triangle OPQ \\
\therefore \quad \frac{3}{2}\triangle OPQ &= \triangle R_1OQ + \triangle R_2OP + \triangle OPQ \\
\frac{1}{2}\triangle OPQ &= \triangle R_1OQ + \triangle R_2OP
\end{aligned}$$

Applying (3.2) to the pair $(\triangle R_1OQ, \triangle O_1OQ)$ and $(\triangle R_2OQ, \triangle O_2OQ)$ we obtain the follwoing relations:

$$\begin{aligned}
\frac{1}{2}\triangle OPQ &= \triangle R_1OQ + \triangle R_2OP \\
&= 2\triangle O_1OQ + 2\triangle O_2OP \\
\therefore \quad \frac{1}{4}\triangle OPQ &= \triangle O_1OQ + \triangle O_2OP
\end{aligned}$$

In order to obtain the area of the segment of the parabola, OPQ, we repeat to create the inscribed triangles on the edges of the inscribed polygon. Putting $A_0 = \triangle OPQ$, the sum to the area of the parabola segment is:

$$A_0 + \frac{1}{4}A_0 + \left(\frac{1}{4}\right)^2 A_0 + \cdots + \left(\frac{1}{4}\right)^n A_0 + \cdots$$

In the time of Archimedes, the concept of infinity was not established. He proved:

$$A_0 + \frac{1}{4}A_0 + \left(\frac{1}{4}\right)^2 A_0 + \cdots + \left(\frac{1}{4}\right)^n A_0 + \frac{1}{3}\left(\frac{1}{4}\right)^n A_0 = \frac{4}{3}\triangle OPQ \tag{3.4}$$

Let A be the area of the segment of the parabola. Using (3.4), Archimedes deduced contradictions from both $A < \frac{4}{3}\triangle OPQ$ and $A > \frac{4}{3}\triangle OPQ$. Therefore, $A = \frac{4}{3}\triangle OPQ$.

3.8.2 Floating bodies of Archimedes

He discovered that the relation between weight loss of an object in water and the water displaced by the object.
Let W be the weight of an object mixed with gold and silver. Let the weight of gold be w_1 and of silver be w_2.

$$W = w_1 + w_2$$

Let the weight loss of the object be F. Let F_1 be the weight loss of a gold object with the weight W. Similarly, let F_2 be the weight loss for silver with the weight W. Then we have

$$\frac{w_1}{w_2} = \frac{F - F_2}{F_1 - F}.$$

Proposition 3.8.6. *Let A be the area of a circle with radius r. Then A is equal to the area of the right triangle with height r and with the base the circumference:*

$$A = \frac{1}{2} r \times (\textit{the circumference})$$

3.9 Apollonius's Conics

Apollonius of Perge (circa 262- 190 B.C.) wrote *Conics*. He studied the conic sections such as ellipse, parabola and hyperbola. With today's notation, a cone C is defined by

$$x^2 + y^2 = z^2$$

in $\mathbf{R}^3$. The vertex V of C is the origin. Let E be a plane in $\mathbf{R}^3$.
If $V \in E$, then the conic sections are called *singular conic sections*. There are three cases.

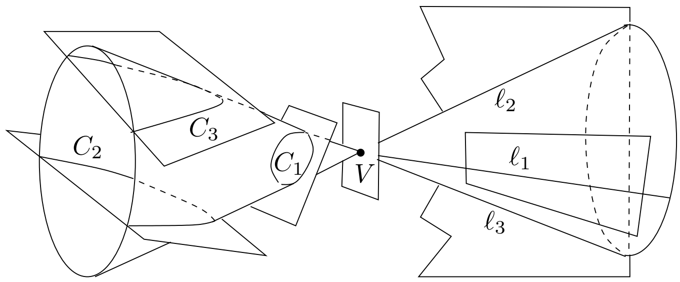
1. $C \cap E$ is one point (V in diagram).
2. $C \cap E$ is a straight line (ℓ_1 in diagram).
3. $C \cap E$ is a pair of two straight lines (ℓ_2 and ℓ_3 in diagram).

If $V \notin E$, then the conic sections are called *non–singular conic sections*.

1. $C \cap E$ is an ellipse (C_1 in diagram).

2. $C \cap E$ is a parabola (C_2 in diagram).

3. $C \cap E$ is a hyperbola (C_3 in diagram).



3.10 Ptolemy

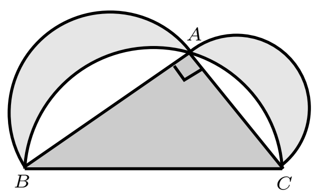
Claudius Ptolemy (100-170 C.E.) was a writer in Alexandria known as a mathematician, astronomer, geographer, astrologer, and poet of a single epigram in the Greek Anthology. He wrote three important scientific treatises such as the astronomical treatise now known as the *Almagest* (*The Great Treatise*), the *Geography* and the astrological treatise *Four Books*. Islamic and European scholars were influenced by these treatises.

Decline of Greek Mathematics

From the 4th to the 3rd century B.C. Greek Mathematics decline dramatically. This is due to the change of the sociopolitical climate. In this period Roman Republic expanded their territory to southward and reached Magna Graecia. By the second century B.C. old kingdoms were swept out from east Mediterranean world. Mathematical creativity requires intellectual curiosity rather than practical needs. In the Greek civilisation the intellectual curiosity supported philosophy and mathematics. On the other hand, it is clear that Roman imperial government did not give much priority to mathematical research. Nevertheless, Greek tradition still continued in Egypt as Alexandrian Museum and Library existed until the 5th century.

Chapter 3 Exercises

1. What is the significance of Greek mathematics?
2. Discuss about the influence of Greek mathematics to mathematics after Greek?
3. Discuss about the significant difference between Babylonian and Greek mathematics.
4. How do you explain "Achilles", one of Paradoxes of Zeno, with today's knowledge?
5. Discuss why Euclid's Elements had been used as a standard text for learning mathematics.
6. Discuss why Greek mathematics declined dramatically from the 3rd to the 4th century.
7. Demonstrate the Archimedes's method to approximate π with the regular 24-gon.
8. A right angle triangle $\triangle ABC$ is given. The semicircles with the diameters AB, AC and BC bound two crescents (see the diagram below). Prove that the area of $\triangle ABC$ is the sum of areas of the two crescents.



Chapter 4

Chinese mathematics

The earliest excavated records of the Chinese civilisation are dated to about 1600 B.C. The civilisation was established along the Yellow river.

There are many mathematical concepts which are Chinese origin such as the negative numbers, the binomial theorem, and matrix algebra etc.

4.1 Chinese Numeration

Unlike Bbylonians, Chinese did not use the sexagesimal fractions, instead they used the decimal fractions. The Chinese numeral expression uses certain units of numbers; ten (十), hundred (百), thousand (千), ten thousand (万). For example, 2017 is expressed by two 千 十 seven. Here we use alphabet T for ten, H for hundred S for thousand, and M for ten thousand. Therefore, 345701 is expressed as

$$34\ M\ 5\ S\ 7\ H\ 1.$$

They used eighteen symbols expressing numbers from 1 to 9 in two different forms (see Figure 4.2). These two forms were used to distinguish the digits clearly.

一	二	三	四	五	六	七	八	九	十
one	two	three	four	five	six	seven	eight	nine	ten

十一	十二	十三	二十	三十	百	二百
eleven	twelve	thirteen	twenty	thirty	hundred	two hundred

一億二千七百三十万
one hundred twenty-seven million three hundred thousand

Figure 4.1: Chinese numeration.

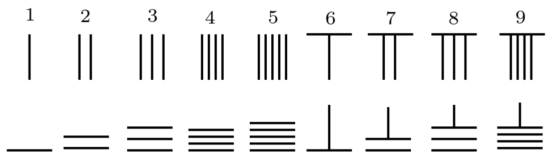


Figure 4.2: These eighteen symbols were used alternately from right to left.

The ancient Chinese used the tabular calculation. The following demonstrates the multiplication 83×27. Write the number 83 in the first row and

		8	3
	2	7	

		8	3
1	6		
	5	6	
	2	7	

			3
2	1	6	
		2	7

			3
		6	
		2	1
2	1	6	
		2	7

2	2	4	1
		2	7

Figure 4.3: $83 \times 27 = 2241$

the second number in the bottom row such that 7 is placed below the first digit of 83. Multiply 8 by 2 to obtain 16 and place it on the second row and multiply 8 and 7 to obtain 56. Place it under 16 such that 5 is placed under 6.
Delete 8 from the first row and shift 27 one unit to the right. Multiply 3 by 27 to repeat the same procedure.
The division is done as follows. Divide 27015 by 51.
Set two numbers 27015 and 51 in the last two rows such that the divisor 51 is in the last row and last digit is placed in the first column and place 27015 at second last row and put to the left. Compare the first digits 27 and 51.

Obviously, the division cannot be done. Thus shift 51 one unit to the right. Now compare 270 and 51. Subtract 51 from 270 to obtain 219 and put 1 above 9. Now, the last two rows contain 21715 and 51. Repeat the procedure above.

2	7	0	1	5
5	1			

2	7	0	1	5
	5	1		

		1		
2	1	9	1	5
	5	1		

		2		
1	6	8	1	5
	5	1		

		5		
	1	3	1	5
	5	1		

		5		
	1	3	1	5
		5	1	

		5	2	
		2	9	5
		5	1	

		5	2	
		2	9	5
			5	1

		5	2	5
			4	4
			5	1

Figure 4.4: $22015 = 525 \times 51 + 44$

4.2 Root Extractions

Chinese knew the methods for finding n-th root of a number $K > 0$.
The idea is to use the expansion of

$$(a+b)^n = K. \tag{4.1}$$

If b is samll, then $a^n \approx K$.
A variation of the method is called *Horner's method*. As early as the eleven century, the *Pascal Triangle* appeared in Chinese text and even in the tenth century it appeared in Indian text too.
By the thirteenth century, an algorithm for finding the n-th roots of a number was known to Chinese mathematicians. The work of *Yang Hui* (fl. 13 CE) contains the method.

Example 4.2.1. Consider the problem to find the fourth root of $K = (a+b)^4$. Applying the Yang Hui's method:

Here we transpose the original Chinese table so that we can use the modern notations. (Compare with the tables in Appendix B).

HF	HL	SL	LF	S	SS
$\underline{1}$	a	a^2	a^3	K	a
	a	$2a^2$	$3a^3$	$-\underline{a^4}$	b
	$2a$	$3a^2$	$\underline{4a^3}$	$K - a^4$	
	a	$3a^2$	$6a^2b + 4ab^2 + b^3$	$-4a^3b - 6a^2b^2 - 4a^2b^3 - b^4$	
	$3a$	$\underline{6a^2}$	$4a^3$	$K - a^4$	
			$+6a^2b + 4a^2b^2 + b^3$	$-4a^3b - 6a^2b^2 - 4a^2b^3 - b^4$	
	a	$4ab + b^2$		$= K - (a + b)^4$	
	$\underline{4a}$	$6a^2 + 4ab + b^2$			
	b				
	$4a + b$				

In the table, the binary coefficients of $(a+b)^4$ appear as underlined numbers, a^4, $4a^3$, $6a^2$, $4a$ and 1.

Example 4.2.2. Use Yang Hui's method to find the 4-th root of 1048576.

The following is a set of procedures for finding 4-th root of 1048576. They use the following table. HF, HL, SL, S and SS are notations (corresponding to some Chinese characters) to distinguish the columns. We use these letters for both the columns and the values.
Put the number 1048576 in the column S. Put 1 in the column HF.

HF	HL	SL	LF	S	SS
1				1048576	

We guess that the number 1048576 is between 30^4 and 40^4:

$$30^4 < 1048576 < 40^4$$

This means that in the formula (4.1) $a = 30$. Then put 30 in the column SS. The first procedure starts from putting $HF \times SS = 30$ in the column HL. Put $HL \times SS = 900$ in the column SL, and put $SL \times SS = 27000$ in the column LF. Subtract $LF \times SS$ from S:

HF	HL	SL	LF	S	SS
1	30	900	27000	1048576	30
				-810000	
				238576	

Here, $810000 = a^4 = 30^4$.
We put $HF \times SS = 30$ in the column HL and add to the previous value 30 to obtain 60. Put $60 \times SS = 1800$ in the column SL and add to obtain 2700. Add $2700 \times SS = 81000$ to LF to obtain 108000 in the column LF:

HF	HL	SL	LF	S	SS
1	30	900	27000	1048576	30
	30	1800	81000	-810000	
	60	2700	108000	238576	

We continue this until the one clumn is made short of the previous step.

HF	HL	SL	LF	S	SS
1	30	900	27000	1048576	30
	30	1800	81000	-810000	
	60	2700	108000	238576	
	30	2700			
	90	5400			
	30				
	120				

Next find new SS so that it maximizes the number $108000 \times SS + 5400 \times (SS)^2 < 238576$. By trial and error, we find 2. Put 2 in the column SS uder 30. This means that in formula (4.1) $b = 2$.
Then apply the same procedure.

HF	HL	SL	LF	S	SS
1	30	900	27000	1048576	30
	30	1800	81000	-810000	2
	60	2700	108000	238576	
	30	2700	11288	-238576	
	90	5400	119288	0	
	30	244			
	120	5644			
	2				
	122				

Finally, we get 0 in the column S.

$$\therefore \quad \sqrt[4]{1048576} = 32.$$

4.3 Chinese Remainder Theorem

The theorem is stated in today's terms as follows:

Theorem 4.3.1 (The Chinese Remainder Theorem)**.** *Let* $m_1, m_2, \ldots, m_n$ *be pairwise relatively prime positive numbers and* $a_1, a_2, \ldots, a_n$ *arbitrary integers. Then the system*

$$\begin{aligned} x &\equiv a_1 \ (\mathrm{mod} m_1) \\ x &\equiv a_2 \ (\mathrm{mod} m_2) \\ &\vdots \\ x &\equiv a_n \ (\mathrm{mod} m_n) \end{aligned}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The simultaneous solution is given by:

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n,$$

where $M_k = m_1 m_2 \cdots m_n / m_k = m/m_k$ and y_k is an inverse of M_k modulo m_k.
This theorem was first published in *Suzi's Mathematical Classic* in the 3rd to 5th centuries by a Chinese mathematician Sun Tzu. It was given as a problem such as:

Problem 4.3.2. There are things. The number of the things is not known. The total number of things. If divide the number by 3, the remainder is 2, if divide the number by 5, then the remainder is 3 and if divide the number by 2, then the remainder is 2. What is the number?

Answer 23.
Solution. Put 140 as the number divided by 3 with the remainder 2, put 63 as the number divided by 5 with the remainder 3 and put 30 as the number divided by 7 with the remainder 2. Add these three numbers to obtain 233. Subtract 210 from 233 to obtain the answer.
We can interpret this problem and its solution as follows.
Problem. Solve:

$$\begin{aligned} x &\equiv 2 \ (\mathrm{mod}\ 3) \\ x &\equiv 3 \ (\mathrm{mod}\ 5) \\ x &\equiv 2 \ (\mathrm{mod}\ 7) \end{aligned}$$

Solution. Let $M = 3 \cdot 5 \cdot 7 = 105$. Then $M_1 = 35$, $M_2 = 21$ and $M_3 = 15$. We need to find an inverse of $M_1 = 35$ modulo 3 that is 2 and so $y_1 = 2$. An inverse of $M_2 = 21$ modulo 5 is 1 and so $y_2 = 1$. For $M_3 = 15$, an inverse $y_3 = 1$.
Therefore, the solution is:

$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 = 233 \equiv 23 \pmod{105}.$$

4.4 Nine Chapters

Around 150 B.C., *Nine Chapters* was written. The author is unknown. This work includes 246 problems on surveying, agriculture, partnerships, engineering, taxation, calculation, the solution of equations, and the properties of right triangles. The significant part of the work is that it deals with the simultaneous linear equations, using positive and negative numbers.
In the first volume of Nine Chapters, there are descriptions about aclculation of fractions. It explains

1. Simplification.

$$\frac{12}{18} = \frac{2}{3}$$

2. Addition of fractions.

$$\begin{aligned} \frac{1}{3} + \frac{2}{5} &= \frac{5}{15} + \frac{6}{15} \\ &= \frac{5+6}{15} = \frac{11}{15} \end{aligned}$$

3. Subtraction of fractions.

$$\begin{aligned} \frac{2}{3} - \frac{2}{5} &= \frac{10}{15} - \frac{6}{15} \\ &= \frac{10-6}{15} = \frac{4}{15} \end{aligned}$$

4. Division of fractions. They multiply the denominators to obtain the common divisor. The numerators are multiplied by the corresponding denominators.

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{14}{21} \div \frac{15}{21} \\ &= 14 \div 15 = \frac{14}{15}.\end{aligned}$$

5. Multiplication of fractions.

$$\begin{aligned}\frac{1}{3} \times \frac{2}{5} &= \frac{1 \times 2}{3 \times 5} \\ &= \frac{2}{15}.\end{aligned}$$

The book dealt with the matrix algebra.

Problem 4.4.1 ([6], p. 214, **Problem 8.3**). Three sheaves of a good harvest, 2 sheaves of a mediocre harvest and 1 sheaf of a bad harvest yield a profit of 39 tous. Two sheaves of a good harvest, 3 sheaves of a mediocre harvest, and 1 sheaf of a bad harvest yield a profit of 34 tous. One sheaf of a good harvest, 2 sheaves of a mediocre harvest, and 3 sheaves of a bad harvest yield a profit of 26 tous. How much is the profit from each sheaf of good, mediocre and bad harvest?

Chinese language is written from top to bottom and from right to left. So, they express this problem as the following table.

1	2	3	good
2	3	2	mediocre
3	1	1	bad
26	7 34	39	

Solution. Here, we convert the table into our familiar style.

good	mediocre	bad	
3	2	1	39
2	3	1	34
1	2	3	26

Multiply 3 (the leftmost number of the topmost row) to the second and the third rows.

3	2	1	39
6	9	3	102
3	6	9	78

Subtract the first row from the second and the third row to obtain empty in the second and the third row in the first column. We obtain the following.

3	2	1	39
	5	1	24
	4	8	39

Multiply 5 (the leftmost number of the second row) to the third row.

3	2	1	39
	5	1	24
	20	40	195

Subtract the second row from the third row four times.

3	2	1	39
	5	1	24
		36	99

Divide 99 by 36 to obtain the value of the bad harvest. Then use this value to find other two values.
In this method, negative numbers may appear but Chinese did not have any difficulty to handle them.

4.5 Sea Island Mathematical Manual

Liu Hui approximated the area of a circle with radius one unit and obtained the approximation 3.141024.
Around 263 A.D. Liu Hui's *Sea Island Mathematical Manual* It contains problems such as:

Problem 4.5.1 ([6] p. 207)**.** Two poles are erected, 5 bu high and 1000 bu apart and in line with the top of a sea island. Viewed from the ground level 123 bu behind the front pole, the top of a sea island coincides with the top of the pole. Viewed from the ground level 127 bu behind the rear pole, the top of the sea island coincides with the top of the pole. Find the height of the island and the distance from the poles.

This problem was solved with a diagrammatic method.

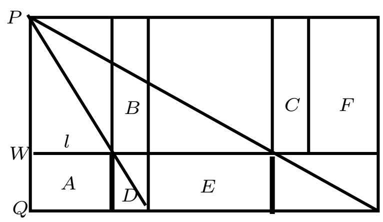


Figure 4.5

In the diagram, A B, C D, E and F denote rectangles as well as their areas. Let the height and width of A be h and l respectively. Let the height and width of B be h' and l' respectively. Then

$$\frac{h'}{l} = \frac{h}{l'}$$
$$h'l' = hl$$

This means $A = B$.
Suppose $B = C$ and thus $A = B = C$. On the other hand, the both sides of the diagonal, there are two pairs of equal triangles. Therefore,

$$\begin{aligned} A + D + E &= C + F \\ D + E &= F \end{aligned}$$

Here $D + E$ is known as 5×1000 and F is the product of PW and the difference $127 - 123 = 4$. Therefore,

$$PW = \frac{5 \times 1000}{4}$$

The height of the island $PQ = PW + 5 = 5 \times 251 = 1255$. The distance l fromm the first pole is obtained as follows.

$$\begin{aligned} PW : 5 &= l : 123 \\ 5l &= 123 \times \frac{5 \times 1000}{4} \\ l &= 123 \times \frac{1000}{4} = 30750 \end{aligned}$$

The distance from the second pole is $30750 + 1000 = 31750$.

4.6 Zu Chongzhi

Zu Chongzhi (429 A.D.-500 A.D.) was a mathematician, astronomer and an engineer. He found an approximation of π as

$$3.1415926 < \pi < 3.1415927.$$

He found the fractions $\frac{355}{113}$ or $\frac{22}{7}$ as approximations of π. He made a significant contribution to mathematics and he wrote a book but his treatise was lost.

Chapter 4 Exercises

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1. Use the tabular calculation to demonstrate
   (a) 74×83 (b) $22016 \div 532$

2. Verify that the procedure explained in Example 4.2.2 gives the binomial coefficients of $(30 + 2)^4$.

3. Use Yang Hui's method to find:
   (a) $\sqrt[4]{50625}$ (b) $\sqrt[5]{6436343}$

4. Two poles are erected, 4 m high and 300 m apart and in line with the top of a hill. Viewed from the ground level 12 m behind the front pole, the top of a hill coincides with the top of the pole. Viewed from the ground level 10 m behind the rear pole, the top of the hill coincides with the top of the pole. Find the height of the hill and the distance from the poles.

Chapter 5

Indian Mathematics

Indian mathematics were significantly different from Greek mathematics. In Greek mathematics, treated arithmetic relates to a geometry. While Indians treated arithmetic with purely numerical method. This leads them to use zero without difficulties. India was divided into many independent states. This is one of main reason why it is difficult to trace the ancient Indian mathematics. In this chapter we introduce few of the medieval Indian mathematicians.

5.1 Aryabhata

Aryabhata (476-550) (or Aryabhata I) was born in "Kusumapura", the capital of the Gupta empire, near Pataliputra. At the time, there were two major centre of learning; "Kusumapura" and "Ujjain". "Kusumapura" is the centre of communications network. Because of this, Aryabhata and his school could reach mathematical and astronomical advances. In 499 Aryabhata wrote *Aryabhatia*, a collection of mathematical problems and procedures. The mathematical section of *Aryabhatia* covers:

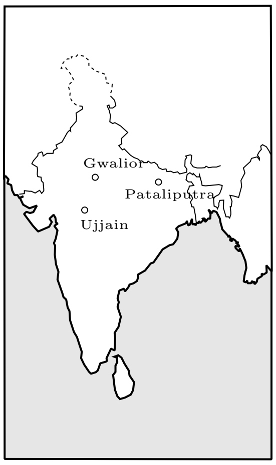

Figure 5.1: Medieval India

1. Arithmetic,
2. Algebra,

3. Plane geometry,
4. Spherical trigonometry,
5. Continued fractions,
6. Qudratic equations,
7. Sums of power series, and
8. Table of sines.

He is also famous for calculating π as $\pi \approx 3.1416$.

5.2 Brahmagupta

Brahmagupta (598-670) wrote *Brahmasphutasiddhanta* (*The Opening of Universe*) in 628. It consists of 10 chapters with additional 15 commentary chapters. He wrote the second treatise *Khandakhadyaka* in 665. He became the head of the astoronmical observatory at Ujjian. In *Brahmasphutasiddhanta* he wrote about calculation on zero. He recognized some facts; in today's notation:

1. $a + 0 = a$ for any number a and
2. $a \times 0 = 0$ for any number a.

He gave arithmetical rules of positive numbers and negative numbers.

1. $a - 0 = a$ for $a < 0$.
2. $a - 0 = a$ for $a > 0$.
3. $0 - 0 = 0$.
4. $0 - a > 0$ for $a < 0$.
5. $0 - a < 0$ for $a > 0$.
6. The product of zero multiplied by a negative number or a positive number is zero.
7. The product of zero multiplied by zero is zero.

8. $a \times b > 0$ and $a/b > 0$ for $a > 0$ and $b > 0$.
9. $a \times b > 0$ and $a/b > 0$ for $a < 0$ and $b < 0$.
10. $a \times b < 0$ and $a/b < 0$ for $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$.

He also claimed $\frac{0}{0} = 0$ and $\frac{a}{0} = 0$ but these are incorrect.

5.3 Quadratic Equations

Medieval Indians solved quadratic equations such as the following problem.

Problem 5.3.1. One hundred is loaned for one month, and the interest received is loaned for six months. The total of the interest and the interest on the interest is 16. Find the interest on the principal.

Solution by the today's method.
Let r be the rate of the interest. We obtain the

$$100r + 100r \times 6r = 16 \tag{5.1}$$

Use the quadratic formula we have

$$\begin{aligned} r &= \frac{-50 \pm \sqrt{2500 + 9600}}{600} \\ &= \frac{-50 \pm \sqrt{12100}}{600} \\ &= \frac{-50 \pm 110}{600} \\ &= \frac{1}{10} \end{aligned} \tag{5.2}$$

Thus the answer is $100 \times \frac{1}{10} = 10$.
Brahmagupta gave the solution as follows ([6] p. 221): "Add the interest on the principal and the interest on this interest, and multiply this by the time and by the principal: the result is 9600. Add this to the square of half the principal, 2500, making 12100. Find its square root, 110, then subtract half the principal and divide by the time, obtaining 10, which is the interest on the principal."

5.4 The Rule of Three

The rule of three is like follows: The "fruit" was multiplied by the "desire" and divided by the "measure", which would give the "fruit of the desire". We can make the diagram to explain the Rule of Three:

$$\frac{\text{The desire} \times \text{The fruit}}{\text{The measure}} = \text{The fruit of the desire}$$

We apply this rule to solve the following problem.

Problem 5.4.1. If one and one quarter pala of sandalwood cost ten and a half para, how much does nine and a quarter pala of sandalwood cost?

The measure is $1\frac{1}{4}$ pala, the fruit is $10\frac{1}{2}$, the desire is $9\frac{1}{4}$ and the fruit of the desire is the unknown solution x.

$$\frac{\frac{21}{2} \times \frac{37}{4}}{\frac{5}{4}} = 77\frac{7}{10}.$$

Bhaskara Atscharja (1114-1185) discussed the problem:

Problem 5.4.2. One fifth of a troop of monkeys less three, squared, went into a cave. One monkey was left outside. How many were there?

He found the answers 50 and 5 and 5 is rejected. Today, we solve this problem by solving the quadratic equation.

$$\begin{aligned} \left(\frac{x}{5} - 3\right)^2 + 1 &= x \\ \frac{x^2}{25}\frac{11}{5}x + 10 &= 0 \\ x^2 - 55x + 250 &= 0 \\ (x-50)(x-5) &= 0 \end{aligned} \quad (5.3)$$

Thus the solutions are 50 and 5 but 5 does not meet the condition. Therefore, the solution is 50.

Bhaskara correctly noted on the operation with zero.

Medieval Indian mathematicians showed their genius to solve indeterminate systems of linear equations.

Bramagputa dealt with the equation:

$$61x^2 + 1 = y^2 \tag{5.4}$$

and he found the smallest whole number solution ([6], p. 224):

$$x = 22613980, \quad y = 1766319049$$

Problem 5.4.3. Given a, b, find whole numbers x, y so $ax^2b = y^2$.

Bhaskara gave a method to find all solutions when one pair of solutions is given.

5.5 Bhaskara

Bhaskara Atschrja (1114-1185), also known as Bhaskara II made two important mathematical treatise, *Lilavati* (The Beautiful) and *Vija Ganita* (Seed Counting).
In the treatise *Vija Ganita*, Bhaskara gave an explanation of negarive and positive numbers and also rules of signed numbers. He wrote that the square root of a positive number may have positive or negative signs.
Zero had been known but some confusion existed but Bhaskara correctly delt with zero. He stated if we devide a non-zero number a by 0, then it is impossible to obtain the result b. If it exists, then $a = 0 \times b$ which is not possible because the product with zero is always zero.
Vija Ganita includes solutions to:

1. quadratic equations,
2. degenerate cubic and biquadratic equations,
3. multilinear equations,
4. non-linear equations in several variables.

5.6 Combinatorics

Combinatrics is a study to determine the number of possible combinations of a finite set of objects. The earliest statement is in the sixth century B.C.

An Indian physician noted in a medical text that there are 63 possible combinations of six tastes.
Varahamihira (05-587) worked and taught in Ujjain discussed the problem:

Problem 5.6.1. Find the number of perfumes that could be made by choosing four substances out of sixteen.

He gave the answer 1820.
Mahavira (800-870) gave a correct formula to find the number of combinations of m objects out of n.

5.7 Indian Geometry

The Medieval Indian mathematicians knew the following rules:

R-1 The area of a triangle is half the base times the perpendicular. Half this area, times the height, is the volume of hte pyramid.

Let A be the area of a triangle with the base b and the height h:

$$A = \frac{1}{2}bh$$

R-2 Half the circumference times half the diameter is the area of the circle. This area meultiplied by its square root is the exact volume of the sphere.

Let d be the diameter of the circle with radius r. Let A be the area of the circle:

$$A = \left(\frac{1}{2}d\right)^2 \pi = \pi r^2$$

R-3 The area of any plane figure is the product of two of its sides. The chord of one-sixth the circumference of a circle equals the radius.

R-4 Add four to one hundred, multiply by eight and add sixty-two thousand. The result is the approximate value of the circumference of a circle with diameter of twenty thousand.

R-5 The product of half the sides and opposite sides is the rough area of a triangle of quadrilateral. Half the sum of the sides set down four times and each lessened by the sides being multiplied together–the square-root of the product is the exact area.

Chapter 5 Exercises

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1. What are possible causes of the discovery of *zero* by Indian mathematics?
2. Check if the rules from R-1 to R-3 in Section 5.7 are correct.
3. Use the rule R-4 to calculate the Brahmagupta's approximation of π.

Chapter 6

Islamic States

In the seventh century a new civilisation came out of Arabia. Under the inspiration of the prophet Muhammad, an Islamic state was established in Arabian peninsula. In 632 Muhammad died in Medina. Within few years after his death, Damascus and Jerusalem and much of the Mesopotamian Valley fell to the state. By 641 Alexandria was captured. This city had been known as the mathematical centre of the world. By 766 the caliph al-Mansur founded his new capital in Bagdad, which shortly became new centre of mathematics. The caliph Harun al-Rashid, who ruled from 786-809, established a library in Bagdad. The collection of manuscripts were brought from ancient academies in Athens and Alexandria including classic Greek mathematical and scientific texts. The translation program for the manuscripts into Arabic began.

6.1 The House of Wisdom

A caliph Al-Mamum (809–833) established at Baghdad a “House of Wisdom” comparably with the museum at Alexandria. By the end of the 9th century, many of works of Euclid, Archimedes, Apollonius, Diophantus, Ptolemy and other Greek mathematics had been translated into Arabic.

6.2 Al-khwarizmi

A mathematician and astronomer Al Khwarizmi, his ancestors came from Khwarism the region south of the Aral Sea, was an early member of “House

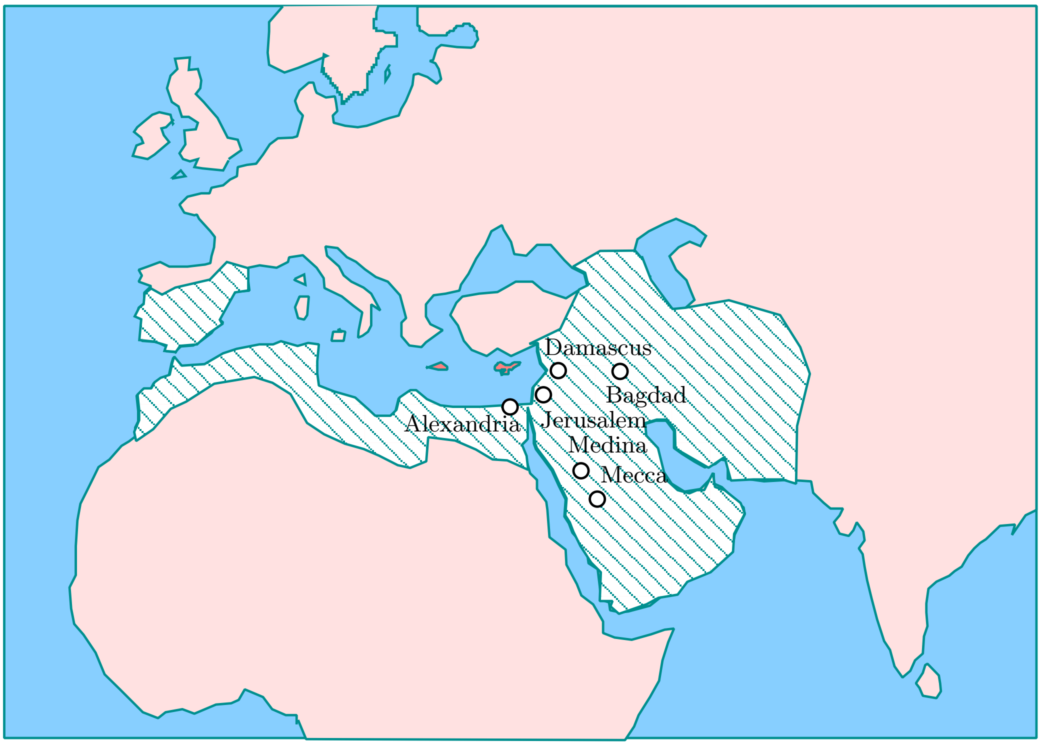


Figure 6.1: The Islamic State (about the 8th century).

of Wisdom". He wrote texts on Hindu mathematics. He introduced nine characters to describe numbers in place value system. Then he described addtion, sbtraction, multiplication, division, determining square roots, and gave examples.
He wrote two books of arithmetic and algebra.

- "De numen idorem" (concerning the Hindu Art of Recknoing)

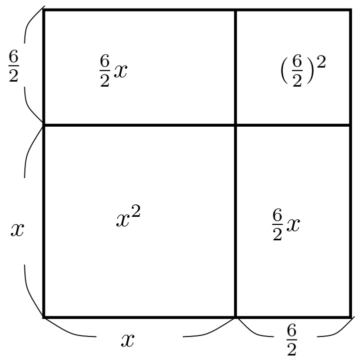
- "Al-jabr wa'l muqiabalah" ("restaration and balancing" the 9th century)

The name of the second book became the word "*algebra*". He studied equations and their solutions. He gave a process to solve the equations and their applications. The word "*algorithm*" came from his name because of such his work.

He solved the quadratic equation

$$x^2 + 6x = 16 \tag{6.1}$$

in terms of today's notations as follows:
Consider the square with the length of each side $x + \frac{6}{2}$.



The area of the square is given by

$$\left(x + \frac{6}{2}\right)^2 = x^2 + 6x + 9. \tag{6.2}$$

If x satisfies the given equation, then

$$x^2 + 6x + 9 = 25, \tag{6.3}$$

and thus

$$\begin{aligned} x + \frac{6}{2} &= 5 \\ x &= 2. \end{aligned}$$

This method does not give the negative solution since x is the length of a side of the square. The following is one of al-Khārizmī's problem.

Problem 6.2.1. Subtract three roots from a square, then multiply the remainder by itself, and the square is restored.

6.3 Ibn Sinan

Arabic geometers were interested in conic sections as they wanted to make an accurate sundial (see Figure 6.2). Several treatises about conic sections were written by Ibrahim ibn Sinan (908-946). He wrote "On the Drawing of the Three Conic Sections" which contains the procedure of construction of the parabola, hypabola and ellipse.

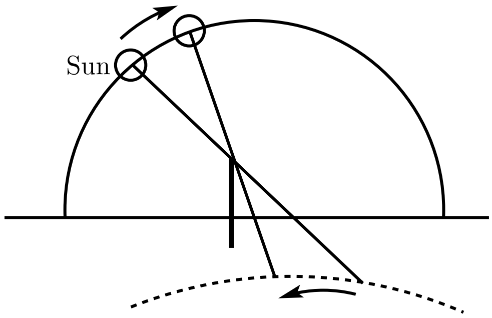


Figure 6.2: Conic section and sundial.

6.4 Omar Khayyam

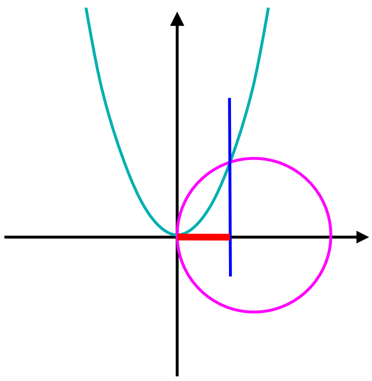


Figure 6.3: Khayyam's solution of the equation $x^3 + a^2x = b$.

The Percian mathematician, astronomer, and poet Omar Khayyam (1050-1123) expressed roots of cubic equations with intersection of conic sections but he could not find a formula for the roots. He solved the cubic equation

$$x^3 + a^2x = b \tag{6.4}$$

as follows. He constructed the parabola $x^2 = ay$ a circle with diameter b/a^2, and a vertical line through the intersection point. The root of the equation is given by the length between the origin and the intersection of the vertical line and the horizontal axis.

6.5 The fifth postulate

Several attempts to prove the fifth postulate were done by Islamic mathematicians. The most sophisticated attempt was done by Nasir Eddin al Tusi (1201-1274). His proof depended on an assumption equivalent to Euclid's fifth postulate. Nasir Eddin's assumption was that *if a line u was perpendicular to a line w at A, and another line v crossed w obliquely at B, then lines perpendicular to u and intersecting v are less than AB on the side on which v, w make acute angles, and the lines are greater on the other side.*

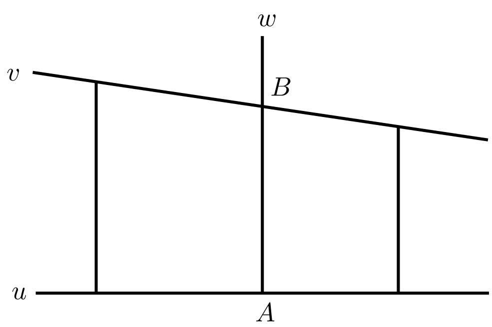


Figure 6.4: Eddin's assumption.

6.6 Trigonometry

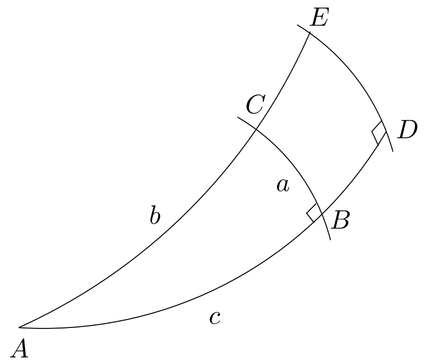
An Indian Siddhanta (Hindu astronomy) was brought to Bagdad late in the eighth century and translated in Arabic. Thus the trigonometric knowledge came from Hindu. Islamic scholars also studied Ptolemy's trigonometry. Trigonometry was written as a part of astronomical works. Mathematicians were interested in using trigonometry to solve spherical triangles because Muslims needed to know the direction of Mecca as well as to know the time for prayers.

Ptolemy used only one trigonometric function the chord while Hindu modified that into sine. In the ninth century cosine, tangent, cotangent secant, cosecant were used in Islamic work.

6.6.1 Spherical trigonometry

An astronomer in Bagdad Muhammad Abul-Wafa al-Buzjani (940-997) had the following result.

Theorem 6.6.1. *If ABC and ADE are two spherical triangles with right angles at B, D, respectively, and a common acute angle at A, then* $\sin BC : \sin CA = \sin DE : \sin EA$.



Also he gave a proof for the following theorem:

Theorem 6.6.2. *In any spherical triangle ABC,*

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Chapter 6 Exercises

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1. Discuss what was the historic role of Islamic mathematics.
2. Investigate the contributions from Islamic mathetmaticians to today's mathemtics.
3. Show that the Eddin's assumptoin is equivalent to the Euclid's fifth postulate.

Chapter 7

The Middle Ages

The period after the decline of civilisation of the Greeks and Romans called *Middle Ages*. During this time, mathematics were developed by Muslims, Indians and Chinese. During this period, Hindu-Arabic Numerals were introduced to Europe. By the year 1000 negative numbers reached Europe.

7.1 Western World

In 395 the Roman empire was divided into the Western and Eastern halves. The Western Roman empire existed between the 3rd and the 5th century but it was eroded by abuses of power, civil wars, barabarian migrations and invasions and economic depression. In 476 finally the empire was fallen. The Eastern Roman empire (Byzantine Empire) continued until 1453. This empire is later called the Byzantine empire and the capital city was Constantinople (Istanbul).

The *Migration Period* (Barbarian Invasions) was a period of enormous scale of Germanic migrations with or without wars in Europe. The orign of Germanic people are coasts of Baltic Sea and South part of Scandinavian Peninsula.

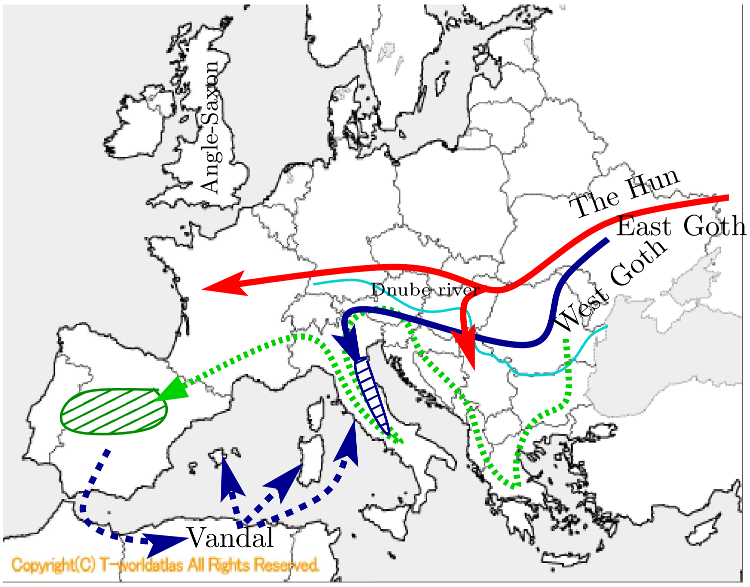


Figure 7.1: Period of Germanic Migration

They gradually migrated into Roman teritories. As they developed the agriculuture, they faced the shortage of land and also In the late of the 4th century, the Huns invaded into Europe. Germanic people moved into Roman teritories. They established many kingdoms inside West Roman empire but most of them perished from 200 to 300 AD. Only Franks continued and extended the teritory. The kingdom was associated with Roman cathoric church.

Roman Catholic Church was developed as the Franks extended their teritories. The most influential churches were the church in Constantinoble and the church in Roma. The bishop of Roman Catholic Church was called a *pope* and the church obtained an authority.

Charlemagne (Charles the great; 742-814) unified large part of West Europe and established the great kingdom. He promoted industries, revived classic learning and strengthened the central government. These policy made the kingdom stronger than the East Roman empire. In 800 Charlemagne was given the crown by the pope Leo III and thus the West Roman empire was revived.
After Charles the great, the kingdome was split into three countries: East Frank, West Frank and Italy. These three kingdoms are today called Ger-

many, France and Italy. In the East Frank, Otto I was given the crown by the pope. This is the biginning of the *Holy Roman Empire*.

The Angles and the Saxons landed Britania and established seven kingdoms (449-1066). In 829 Egbert unified the kingdoms.

During the period of migration, in Europe, feudal societies were organized. It was between the 5th and the 10th centuries. However, the level of culture in Europe was very low comparing with the East. Very limited people could read and write. There was little motivation to develop mathematics as feudal societies were self-sufficient. The trade was not active after Muslem conqured the Maditerreanean sea routes. In the monastries, the subject survived as some of four basic subjects, arithmetic, geometry, music and astronomy. Also available texts in Latin were limited.

7.2 Crusades

The Eastern Roman empire (Byzantine Empire) continued until 1453. This empire is later called the Byzantine Empire and the capital city was Constantinople (Istanbul).
In 1095 Bizantine emperor Alexous I in Constantinople asked Pope Urban II in Italy for sending troops to support the empire against Turkish threat. The Pope called Catholic soldires to join the military campaign. This was the first Crusade (1096-1099) to recapture the Holy land (Jerusalem) which was under the Muslim control. The first crusaide successfully recaptured the holy land (Jerusalem) but soon occupied by Muslims.
There were seven of military campaigns were carried out during 200 years. Although these military campaigns was originally organised to recapture the Holy Land (Jerusalem), as times went, the purpose and destination changed so that they wanted to obtain their land and polytical powers. For instance, the fourth Crusade (1202-204) invaded even the Chritian's cities, Zara (1202), the rival city of Venice and Constantinople (1204), the capital of Byzantine empire. After the capital was fallen, the Bysantine Empire was divided into number of samll states. The Cursaiders founded Latin Empire.
After the fourth Crusade the destination became Egypt rather than Jerusalem. The last campaingn was done in 1270.
These campaigns however, developed trades in the Mediterenian Sea and introduce a sophisticated Eastern culture to Europe. As a result, Italian

cities Genoa and Venice arose.

7.3 The Century of Translation

Considering the cultual development, there were some disadvantages in Europe. One of them was that the higher learning did not exist in Europe. They had to establish the higher education. On the other hand, Islamic world had advantages:

1. Holly Koran required that every male be able to read the Holly Koran, meaning the literacy was widely spread.

2. The empire was tolerate the differences and encouraged their subjects to do the same.

In Spain, Judaism, Islam and Christianity were mixed. This situation helped to introduce the intellectual heritage to Europe.
Not many European could read Arabic but all educated European could read Latin. Therefore, the translation into Latin had an important role to recover the ancient learning.
Adelard of Bath (fl. 1116-1142) was a translator who translated the astronomical tables of al-Khārizmī(1126) and Euclid (1142).
Gerald Cremona (1114-1187) translated Euclid, Ptolemy's Almagest (about 1155), correction of Almagest by Jabir and Algebra of al-Khārizmīinto Latin.

7.4 Leonaldo of Pisa

Leonardo Pisano Bigollo (about 1180-1250) was later, in the 19th century, known as Fibonacci (son of Banaccio). He wrote *Book of Abacus* (1202, 2nd edition 1227).

Problem 7.4.1. A lion ate a sheep in 4 hours, a leopard in 5 hours, a bear in 6: If one sheep is thrown to them in how many hours will it be devoured?

Solution. Suppose that they ate sheep for 60 hours. A lion can eat 15 sheep, leopard can 12 and a bear 10. Total 37 sheep.

$$\frac{60}{37} = 1\frac{23}{37} \text{ hours.}$$

Today called *Fibonacci sequence* came from the following problem included in *Book of abacus*.

Problem 7.4.2. How many pairs of rabbits can be bred from one pair in a year, if the pairs produce a pair every month and begun to breed in the second month.

Solution. In the first month, the first pair breeds one pair, so there are 2 pairs. In the second month, the first pair breeds another pair, so there are 3 pairs. Of these two pairs will be fertile in the third month, so in the third month there will be 5 pairs. Continueing this fashion, we will otain 377 at the end.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	5	8	13	21	34	55	89	144	233	377

The formula explained here is called **Binet's Formula** as it was said that Jacques Phillippe Marie Binet (1786-1856) first expressed Fibonacci numbers in closed form. However, Abraham de Moivre (1667-1754) already discovered the same formula.
The Fibonacci sequence is recrusively defined by

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_{n+2} &= a_{n+1} + a_n \end{aligned} \tag{7.1}$$

The characteristic equation is

$$x^2 - x - 1 = 0$$

Let α and β be the roots of the equation. Then $\alpha + \beta = 1$ and $\alpha\beta = -1$.

$$\begin{aligned} a_{n+2} - (\alpha+\beta)a_{n+1} + \alpha\beta a_n &= 0 \\ a_{n+2} - \alpha a_{n+1} &= \beta(a_{n+1} - \alpha a_n) \\ &= \beta^2(a_n - \alpha a_{n-1}) \\ &= \beta^n(a_2 - \alpha a_1) = \beta^{n+1} \\ a_{n+2} - \beta a_{n+1} &= \alpha(a_{n+1} - \beta a_n) \\ &= \alpha^2(a_n - \beta a_{n-1}) \\ &= \alpha^n(a_1 - \beta a_1) = \alpha^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} a_{n+2} - \alpha a_{n+1} &= \beta^{n+1} \\ a_{n+2} - \beta a_{n+1} &= \alpha^{n+1} \end{aligned}$$

From these equations we obtain the following relation.

$$a_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$\therefore \quad a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{7.2}$$

Black Death (14C. Pandemic)

The plague reached Sicily in October 1347 and rapidly spread all over the island. It reached Italy in January 1348, the outbreak in Pisa. The disease struck European countries such as France, Spain, Portugal and England by June 1348. Then spread into Germany and Scandinavia from 1348 to 1350. It was introduced in Norway in 1349.

From 1346 to 1370, the plague pandemic In Europe occurred. As the illness were previously unknown, nobody could save the ill. As a result, the number of death tall reached about 20 to 30 million which was the 1/3 to 2/3 of the European population.

The population of farmers decreased. The short of farmers changed the style of farming, from the cultivation to the grazing.

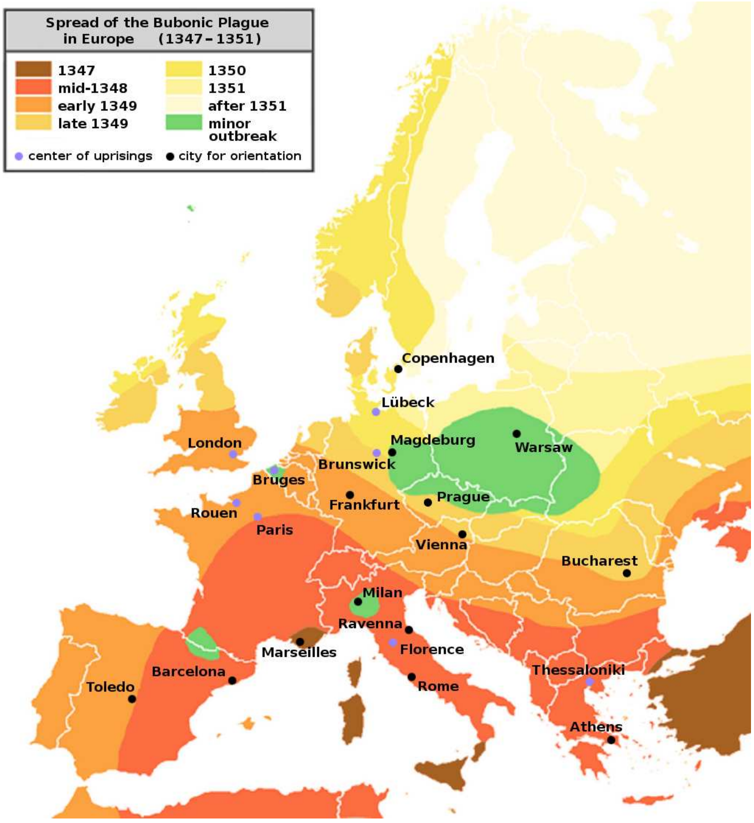


Figure 7.2: 14C. Pandemic

Hundred Years' War (1337-1453)

The series of conflicts during about hundred years (1337-1453) between the House of Plantagenet, rulers of the Kingdom of England, against the House of Valois, rulers of the Kingdom of France, for control of Kingdom of France. This series of conflicts were not conflicts between Kingdom of Enagland and the Kingdom of France but between feudal loads in France belonging to two Kingdoms. This series of wars gave the downfall of the feudal loads.

Chapter 7 Exercises

1. Discuss how the 14th century pandemic affect people's life style and their minds.

2. Use mathematical induction to prove Binet's formula (7.2).

3. Let a_n $(n = 0, 1, 2, \ldots)$ be the Fibonacci sequence defined by $a_0 = 0$, $a_1 = 1$, $a_{n+2} = a_{n+1} + a_n$ $(n \geq 0)$. Find

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Chapter 8

From Diophantus to Fermat

Through the period of crusaides, transportations between the eastern worlds and Europe were developed and trades made some Italian cities wealthy. Europeans started to learn the Greek classic through the Latin translations of Arabic texts.

8.1 The Renaissance

In the fourteenth century the Renaissance began in Italy and spread northward over the next few centuries. Renaissance scholars studied Greek geometry texts. They studied Euclid, whose *Elements*. In the sixteenth century vernacular versions of *Elements* began to appear such as Italian, German, French and Spanish versions. The significant version was the English translation which included all 13 original books of the *Elements* as well as three additional books traditionally ascribed to Euclid.

By the seventeenth century astronomers had found that Ptolemy's predictions of planetary positions or lunar eclipse were greatly in error. European explorers needed improved navigational techniques; that is, correct astronomical tables. They also found that Ptolemy's geography was in error. Julian Calendar also had serious inadequacies.

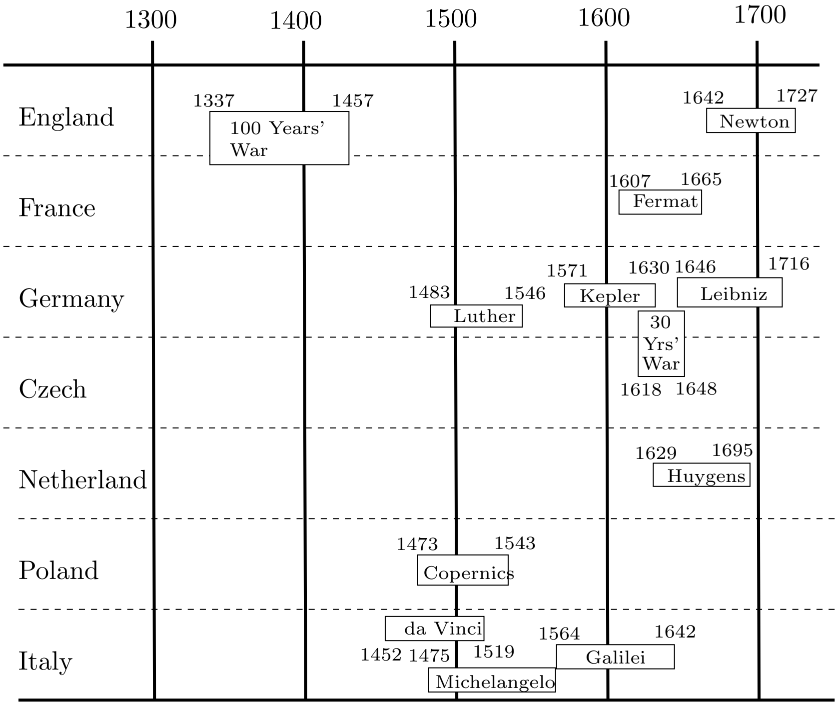


Figure 8.1: The renaissance and the revolution of science.

8.2 Theory of Equations

Early in the 16th century, the Italian mathematicians Scipinoe del Ferro, Niccolò Tartaglia and Gerolamo Cardano solved the general cubic equation in terms of the constants appearing in the equation.
The genral cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{8.1}$$

was considered.
It can be reduced by introducing $y = x + \frac{1}{3}a$, to the simple form

$$y^3 + py + q = 0 \tag{8.2}$$

where $p = b - \dfrac{a^2}{3}$ and $q = \dfrac{2a^3}{27} - \dfrac{ab}{3} + c$.
If we consider positive coefficients and positive x, then there are three types,

$$
\begin{aligned}
x^3 + px &= q && (8.3)\\
x^3 &= px + q && (8.4)\\
x^3 + q &= px && (8.5)
\end{aligned}
$$

Scipione del Ferro solved the type (8.3).
In his book “Ars Magna sive de regulis algebraicis” (1545) Cardano explained Ferro’s method with the following example.

$$x^3 + 6x = 20 \tag{8.6}$$

as follows. Putting $x = u - v$,

$$
\begin{aligned}
x^3 + 6x &= (u-v)^3 + 6(u-v)\\
&= (u^3 - v^3) + 3uv(u-v) + 6(u-v)\\
&= (u^3 - v^3) - 3(uv-2)(u-v) = 20
\end{aligned}
$$

Suppose u and v satisfy the following conditions.

$$
\begin{aligned}
u^3 - v^3 - 20 &= 0\\
3(uv-2)(u-v) &= 0 && (8.7)
\end{aligned}
$$

Then $x = u - v$ satisfies the equation (8.6). As $u - v \neq 0$, $uv = 2$ and thus $u^3v^3 = 8$. The difference and the product of u^3 and v^3 are given so that u^3 and $-v^3$ are rotts of the quadratic equation:

$$t^2 - (u^3 - v^3)t - u^3v^3 = 0$$

Thus

$$
\begin{aligned}
u^3 &= \sqrt{108} + 10\\
v^3 &= \sqrt{108} - 10
\end{aligned}
\tag{8.8}
$$

thus

$$x = \sqrt[3]{\sqrt{108} + 10} + \sqrt[3]{\sqrt{108} - 10}. \tag{8.9}$$

In 1539 Tartaglia told Cardano his method to solve the cubic equation of type (8.3). Then Cardano found the method to solve equations of types (8.4) and (8.5).

The method for type (8.4) is the following.

$$\begin{aligned} x^3 - px &= q \\ x &= u + v \\ x^3 - px &= u^3 + v^3 + 3uv(u+v) - p(u+v) = q \\ &= u^3 + v^3 + (3uv - p)(u+v) = q \end{aligned}$$

$$\begin{aligned} u^3 + v^3 &= q \\ 3uv - p &= 0 \end{aligned} \tag{8.10}$$

Thus

$$\begin{aligned} u^3 &= \frac{1}{2}q + w \\ v^3 &= \frac{1}{2}q - w \end{aligned}$$

where $w = \sqrt{(\frac{1}{2}q)^2 - (\frac{1}{3}p)^3}$, and hence

$$x = u + v = \sqrt[3]{\frac{1}{2}q + w} + \sqrt[3]{\frac{1}{2}q - w} \tag{8.11}$$

Lodovico Ferrari found an exact solution to equations of the fourth degree. Rafael Bombelli wrote three books intitled l'Algebra (1572). He recognized the imaginary number i and introduced a notation for it.

René Descartes (1596-1650) introduced algebraic notation in his explanation of the principles of his discovery "analytic geometry". He established essential notation in algebra such as

$$a + b, a - b, ab, \frac{a}{b}, \sqrt{a}.$$

8.3 Generalised Algebra

In the sixteenth century Europeans started to develop algebra. François Viète(1540-1603) wrote *In artemanalyticei lsagoge* in 1691. His contribution was to generalise algebra using alphabetical letters. He used vowel letters, A, E, I, O, U, Y for unknown and consonants $B, G, D, \ldots$ for known values such as constants. He also used plus symbol '+', which was introduced by Widmann. His letters are however, linked with geometric meanings. In fact, a square of unknown cannot be added to side but can be added to plane. Therefore, he had to add special units for constant letters (see Table 8.1 from [6] p. 327).

Modern	Variable	Homogeneous quantity
x	A side	B side
x^2	A square	B plane
x^3	A cube	B solid
x^4	A square-square	B plano-plane
x^5	A square-cube	B plano-cube
x^6	A cube-cube	B cube-cube

Table 8.1: Viète's Notation

Example 8.3.1. A cube and some sides equal a number.

Solution. Let the unknown be A and let the known sides be B and G. Then it is written as A cube + AB plane equal G solid.
Renè Descartes (1596-1650) finally established the modern style mathematical notation system that close to today's style.

8.4 Fermat's Little Theorem

Diophantus influenced on modern number theory. In 1621, a Latin version of Diophantus was published and Pierre de Fermat (1601-1665) read it and was led to his theorems.
He observed the sequence of numbers of the form $2^n = 1$:

Exponent n	1	2	3	4	5	6	7	8	9	10
Root $2^n - 1$	1	3	7	15	31	63	127	255	511	1023

He noticed the following.

1. If the exponent is composite, then so is the root.

2. If n is prime, then $2n \mid 2^n - 2$.
3. If $2^n - 1$ is composite, then the only factors are primes $2n + 1$.

Then he conjectured:

Conjecture 8.4.1. Given prime p, $p \mid r^n - 1$ for some n; moreover, $n \mid p - 1$. Furthermore, $p \mid r^{kn} - 1$ for all k.

Fermat gave no proof.

8.5 Fermat's Last Theorem

Fermat's most famous claim appeared in 1670 when Fermat's son, Samuel published his father's copy of Diophantus, complete with his notes. In the note, Fermat had a "wonderful proof".

Conjecture 8.5.1 (Fermat's Last Conjecture)**.** The equation $x^n + y^n = z^n$ has no non-trivial integer solutions for $n > 2$.

Note that Fermat's Last Conjecture was finally proved and thus became a theorem in 1994 by Wildes; 320 years after the conjecture was announced.

8.6 Rational Right-Angled Triangles

Fermat proved the following.

Proposition 8.6.1. *The area of right triangle whose sides are rational numbers could not be a square number.*

Proof. (cf. [6] p. 358) If the sides of a right triangle are $2xy$, $x^2 - y^2$, and $x^2 + y^2$, and the area of this triangle is a square, then x and y must be squares, say $x = u^2$ and $y = v^2$, and moreover, $u^4 - v^4$ is a square.
$u^2 + v^2$ and $u^2 - v^2$ are also squares, say

$$\begin{aligned} u^2 + v^2 &= p^2 \\ u^2 - v^2 &= q^2 \end{aligned}$$

Then $p^2 = 2v^2 + q^2$ and $v^2 + q^2 = u^2$.

$$\begin{aligned} 2v^2 &= p^2 - q^2 \\ &= (p+q)(p-q) \end{aligned} \tag{8.12}$$

$$v^2 = 2\frac{p+q}{2}\frac{p-q}{2} \tag{8.13}$$

One of $\frac{p+q}{2}$ and $\frac{p-q}{2}$ is even and the other is odd. As the left hand side is a square, one of them is in the form 2 times a square and the other is a square. There are integers m and n such that $v^2 = 2m^2n^2$, where n is odd and $2m$ is even.

$$\begin{aligned} 2n^2 &= \frac{p+q}{2} \\ m^2 &= \frac{p-q}{2} \end{aligned}$$

or

$$\begin{aligned} 2n^2 &= \frac{p-q}{2} \\ m^2 &= \frac{p+q}{2} \end{aligned}$$

Either case, we obtain

$$p = 2n^2 + m^2 \tag{8.14}$$

On the other hand,

$$\begin{aligned} u^2 &= \frac{p^2+q^2}{2} \\ &= \frac{(2n^2+m^2)+(2n^2-m^2)}{2} \\ &= 4n^4m^4 = x < x^2 < x^2 + y^2 \end{aligned}$$

This means that there is a right triangle which has rational sides and the square area. The area is obviously smaller than the area of the original triangle. We can construct smaller triangle satisfying the conditions of the theorem. However, this is impossible to find a infinite sequence of the constructions. Hence, there is no right triangle whose sides are whole numbers and whose area is a square. □

8.7 Modern Algebra

Italian origin mathematician Joseph Louis Lagrange (1736-1813) reached the idea that in order to solve an equation, one should express intermediate quantities as rational functions of roots of the equation. Thus he recognized the importance of study of the behaviour of rational functions.

Norwegian mathematician Niels Abel (1802-1829) proved that an equation with degree $m \geq 5$ is not solvable by radicals. French mathematician Evariste Galois (1811-1832) discovered the concept of groups as systems of permutations and combinations of roots of polynomials. He determined when a given equation is solvable in radicals. Groups became one of biggest achievement of the 19th-century mathematics.

Carl Friedrich Gauss (1777-1855) proved that every polynomial equation can be factored into linear factors over the complex numbers. By the time of Gauss, mathematicians had shifted their attention to studying the structure of abstract mathematical systems such as groups and quaternions rather than solving polynomial equations.

The British mathematician and astronomer William Rowan Hamilton discovered quaternions. The German mathematician Hermann Grassman investigated vectors and American physicist J. W. Gibbs recognized the utility for physicists. George Boole wrote an algebraic treatment of basic logic in The Laws of Thought (1854). Since then modern algebra called abstract algebra has been developed.

8.8 Works of Kepler

The model of Claudius Ptolemy (100-170 CE) dominated European astoronomy. The model shows the center of the universe is the earth and the sun orbits around the earth. The wondering stars (planets) are put on the epicylce whose center moves along circular orbit around the earth.

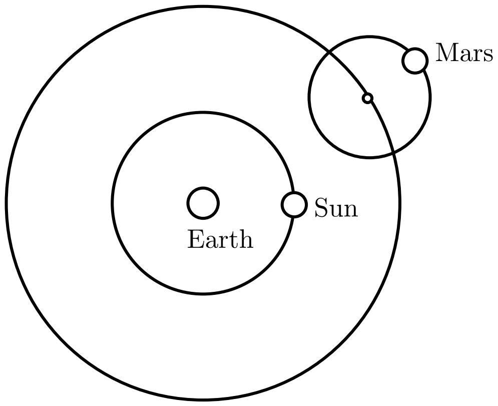


Figure 8.2: Ptolemy's model.

Nicolaus Copernics (1473-1543) reached a conclusion that the earth-centred approach was no longer worked. He found Greek philosophers once proposed a sun-centred (helico centric) system. Copernicus's book De Revolutimibus Orbium Coelelstium (On the Revolutions of the Heavenly Spheres) describe the universe as a series of nested spheres containing planets. His book was published in 1543 after his death. His model however, still needs epicylces (see Figure 8.3).

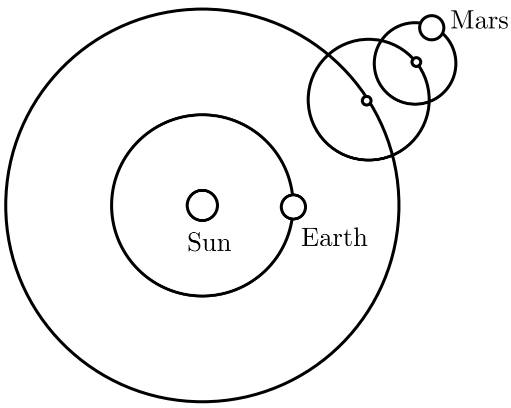


Figure 8.3: Copernics's model.

Tycho Brahe (1546-1601) devoted his life time to making observations to produce better astronomical tables. He convinced king Frederick II of Demark to allow him to have an observatory and newly consturcted instruments also hiring assistants. According to his careful observations he reached the idea that the universe should be filled with some material rather than nested spheres. His model sets the earth at the centre (see Figure 8.4).

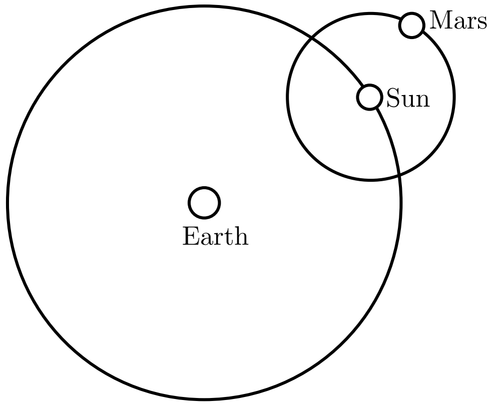


Figure 8.4: Tycho Brahe's model.

Johannes Kepler (1571-1630) worked with Tycho Brahe for the final two years of Tycho Brahe's life in Prague. During Kepler's era, Aristotelian physics was believed. According to Arisotle, action in the universe must follow the perfect form that is a circular form.
After examining Tycho's enormous data and worked for decades, he finally found that the orbits of planets are not circuler but elliptical orbits and the sun sits at one of foci of the ellipical orbit (see Figure 8.5).

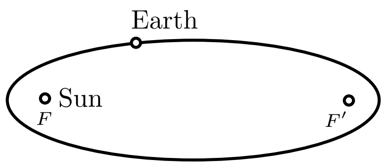


Figure 8.5: Kepler's model.

Kepler discorvered the following three laws:

1. The orbit of a planet is an ellipse with the Sun at one of two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

The first two laws were published in 1609 and the third law was published in 1619.

8.9 Gallileo Galilei

Gallileo Galilei (1564-1642), Italian astoronomer, physicist and mathematician improved the telescope. Using the improved telescope to observe the sky and found that the universe is different from the description claimed by Church. He discovered the followings:

1. The isochronism of pendulum.
2. A freely falling body is uniformly accelerated (by 1604).
3. The law of inertia.

He believed heliocentric solar system based on his observation. However, this belief conflicted with the Church authority and he was charged and convicted in 1633. He was house arrested until his death in 1642. He wrote *Diaglog Concerning Two New Sciences* (1638).

Chapter 9

Calculus

Greek mathematicians such as Archimedes already reached the idea of integral by the method of exhaustion. However, the method used by Archimedes depended on the geometric properties of the geometric object and thus the method found for a particular geometric object is difficult to apply for other geometric shapes. Finding the universal method of integral was a problem. On the other hand, finding tangent line to a given curve was also a common problem among mathematicians. These problems were finally solved in 17th century when calculus was established.

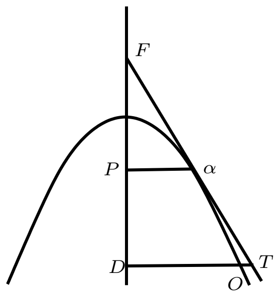
9.1 John Wallis

John Wallis (1616-1703) is considered as one of founders of analytic geometry. In *Treatise on Conic Sections* he treated the conic sections. He proved the following proposition.

Proposition 9.1.1. *Given a point α on a parabola P on diameter PA with ordinate $R\alpha$, if PA intersects the extension of diameter at F so that $AF = AP$, then $F\alpha$ touches the parabola.*

We follow the outline of his proof.

Proof. Take any point D on PA. Draw ordinate DO to T on $F\alpha$. Either DT is equal to DO, in which P coincides D, or DT is greater than DO.



Let $P\alpha = p$, $PA = d$, hence $PF = 2d$; let $PD = a$, and so $DA = d \pm a$, $DF = 2d \pm a$. By the property of the parbola,

$$\begin{aligned} PA : DA &= (P\alpha)^2 : (DO)^2 \\ d : d \pm a &= p^2 : (DO)^2 \end{aligned}$$

We obtain:

$$(DO)^2 = \frac{d \pm a}{d} p^2$$

By similar triangles,

$$\begin{aligned} PF : DF &= P\alpha : DT \\ 2d : d \pm a &= p : DT \end{aligned}$$

We obtain:

$$DT = \frac{2d \pm a}{2d} p$$

Thus $(DT)^2 = \frac{4d^2 \pm 4d\alpha + \alpha 2}{4d^2} p^2$.

$$(DT)^2 - (DO)^2 = \frac{\alpha^2}{4d^2} p^2 > 0$$

Hence $DT > DO$ or if D, P coincide, then $a = 0$ and $DT = DO$. □

He published *Arithmetic of the Infinite, or New Mathod of Inquiry into the Quadrature of Curves* in 1656. This publication influenced Newton.

9.2 Isaac Barrow

Isaac Barrow (1630-1677) proved a version of the fundamental theorem of calculus. His expression is in a classic style.

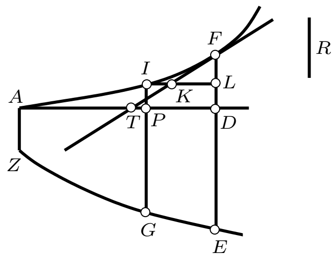


Figure 9.1

Proposition 9.2.1 (Fundamental Theorem of Calculus, Part 1.)**.** *Let ZGE be any curve whose ordinates are increasing and whose axis is AD. Let AIF be a curve, where if any straight line EDF is down perpendicular to AD, then the recutangle on DF and a given length R is equal the area $ADEZ$. Also let $DE : DF = R : DT$. Then TF is tangent to AIF.*

Note 9.2.1. *In Proposition 9.2.1, the length R is used. At the time of Barrow, R is used to match the dimensions between quantitles; that is, the length DF and the area of $ADEZ$ are not comparable but $DF \cdot R$ makes them comparable.*

Proof. The area of $ADEZ$,

$$\begin{aligned} \text{area}(ADEZ) &= R \cdot FD \\ \text{area}(APGZ) &= R \cdot LD \\ \text{area}(PDEG) &= R \cdot (FD - LD) = R \cdot FL \end{aligned}$$

Since area$(PDEG) < PD \cdot DE$ and $PG < DE$,

$$\begin{aligned} R \cdot FL &< PD \cdot DE \\ \frac{R}{DE} &< \frac{PD}{FL} \end{aligned} \tag{9.1}$$

By the assumption,

$$\frac{DF}{DT} = \frac{FL}{LK} = \frac{DE}{R}.$$

$$\therefore \quad LK = \frac{R \cdot FL}{DE} = \text{area}(PDEG) < PD = IL$$

Hence TF does not intersect AIF on the side toward A. On the other hand, if I is the other sides of F, then $LK > LI$ and does not intersect on the side away from A. Hence TF tangents to the curve. □

In modern terms he stated:

Proposition 9.2.2. *If $\int_0^x f(t) = F(x)$, then $F'(x) = f(x)$.*

He also stated the following proposition in a geometric style.

Proposition 9.2.3. *Let AMB be a curve with axis AD; let BD be drawn perpendicular to AD. Let KZL be a curve such that for any point M on AB, with MT the tangent and MFZ is parallel to DB, and R a given length, we have $TF : FM = R : FZ$. Then the area DLK will equal the rectangle on R, DB (see Figure 9.2).*

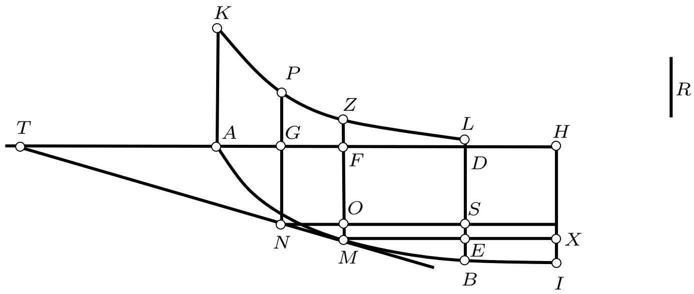


Figure 9.2: The Fundamental Theorem of Calculus Part II.

In the modern form the statement is:

Proposition 9.2.4 (Fundamental Theorem of Calculus, Part 2.). *If $F'(x) = f(x)$ (and $F(0) = 0$), then*

$$\int_0^x f(t)dt = F(t).$$

Note 9.2.2. *In the time of Barrow there was no notation $\int$ and $'$, also functions are not used in the statements and their proofs.*

9.3 Isaac Newton

Isaac Newton (1642-1727) made four major contributions to mathematics: the binomial theorem; calculus; the theory of gravitation: and the discovery of the nature of light and colours. Newton's first discoveries early months of 1665. In 1665, he began to think of the rate of change or fluxion of continuously varying quantities, or fluids, lengths, areas, volumes, distances and temperatures. He expressed a function in terms of infinite series. He linked the infinite series and rates of changes.

9.3.1 The Binomial Theorem

Newton considered the areas under the curves:

$$(1-x^2)^{\frac{0}{2}},\quad (1-x^2)^{\frac{2}{2}},\quad (1-x^2)^{\frac{4}{2}},\quad (1-x^2)^{\frac{6}{2}}\cdots \tag{9.2}$$

The areas under the curves form a sequence

$$x,\quad x-\frac{1}{3}x^3,\quad x-\frac{2}{3}x^3+\frac{1}{5}x^5,\quad x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7,\cdots \tag{9.3}$$

He thought the area under the curve $(1-x^2)^{\frac{1}{2}}$ could be found by an interpolation.
In the table below shows the terms of the areas under the curves $(1-x^2)^{\frac{q}{2}}$.
Powers of x in the first column

q	0	1	2	3	4	5	6
x	1		1		1		1
x^3	$-\frac{0}{3}$		$-\frac{1}{3}$		$-\frac{2}{3}$		$-\frac{3}{3}$
x^5					$\frac{1}{5}$		$\frac{3}{5}$
x^7							$-\frac{1}{7}$

The second row has common denominator and the numerators form a sequence $-0,-1,-2,-3,\ldots$. Newton interpolated between $q=0$ and $q=1$ to obtain the area under the curve $(1-x^2)^{\frac{1}{2}}$ as

$$x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{16}x^7}{5}-\frac{\frac{5}{128}x^9}{9}-\cdots \tag{9.4}$$

Then he found that $(1-x^2)^{1/2}$ could be found as

$$(1-x^2)^{1/2}=1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\cdots \tag{9.5}$$

9.3.2 Finding Fluxions

Newton viewed x and y as functions of time t or flowing quantities called *fluents*. The rates of change of x and y are denoted by p and q respectively. They are called *fluxions* and he used the notation: $p=\dot{x}$ and $q=\dot{y}$.

He introduced the infinitesimal small number o (omicron) and defined the slope of the tangent line at (x, y) as the ratio of

$$\frac{oq}{op} = \frac{q}{p} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} \tag{9.6}$$

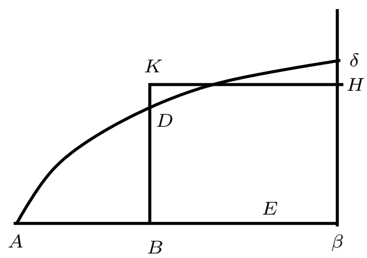
Example 9.3.1. Given $y = x^2$, find an equation of the tangent line at the point (a, b) on the curve.

Solution. For the infinitesimal samll time o, (a, b) moves to $(a', b') = (a + op, b + oq)$. Thus the slope of the tangent line is $\frac{oq}{op} = \frac{q}{p}$

$$\begin{aligned} b + oq &= (a + op)^2 \\ &= a^2 + 2a(op) + (op)^2 \\ &= b + 2a(op) + (op)^2 \\ oq &= 2a(op) + (op)^2 \\ q &= 2a(p) + op^2 \\ \therefore \frac{q}{p} &= 2a + op \end{aligned}$$

As o is the infinitesimally samll, the slope will be $2a$.
Newton published *Analysis by Equations of an Infinite Number of Terms* in 1669. He proved the following proposition.

Proposition 9.3.1 (Definite Integral of a Rational Power)**.** *If* $ax^{\frac{m}{n}} = y$, *then the* area$(ABD) = \frac{an}{m+n} x^{\frac{m+n}{n}}$.



Let $ABD = z$, $B\beta = o$, $BK = v$. Thus the area $B\beta HK = ov$. Therefore, $A\beta = x + o$, area $A\delta\beta = z + ov$, He consider the specific function $\frac{2}{3}x^{\frac{3}{2}} = z$, and so $\frac{4}{9}x^3 = z^2$.

$$\begin{aligned}
\frac{2}{3}x^{\frac{3}{2}} &= z \\
\frac{4}{9}x^3 &= z^2 \\
\frac{4}{9}(x+o)^3 &= (z+ov)^2 \\
\frac{4}{9}(x^3+3x^2o+3xo^2+o^3) &= z^2+2ovz+o^2v^2 \\
\frac{4}{9}(3x^2o+3xo^2+o^3) &= 2ovz+o^2v^2 \\
\frac{4}{9}(3x^2+3xo+o^2) &= 2vz+ov^2
\end{aligned}$$

Now suppose $B\beta$ is infinitesimally small. Then o is nothing and $v = y$, thus $\frac{4}{9}3x^2 = 2zy$.

9.3.3 The Principia

In 1687 Newton published *Mathematical Principles of Natural Philosophy*, known as *Principia*.
It contains the generalised Galileos ideas on motion, called *Newton's laws of motion*:

(1) A object at rest will stay at rest, while an object in motion will stay in motion unless acted upon by an external force.

(2) The change in motion of an object is proportional to the force exerted on it.

(3) For every action, there is an equal and opposite reaction.

The second low is today called the *Newton's Second Law*

$$F = ma,$$

where a is the acceleration.

9.3.4 Optics

Newton published the paper *Philosophical Transactions* (1672) explains the nature of light. He announced

(1) The white light is a combination of rays of varying colour.

(2) Each colour has its own characteristic index of refraction.

In 1704 the *Optics* was published.

9.3.5 Newton's Method

In the *Method of Fluxions* and in *De analysi*, Newton's method for the the approximate solution of equations can be found.

9.4 Leibniz

Gottifried Wilhelm Leibniz (1646-1716) He was born in Lepzig and took his doctorate at the University of Alfdorf. He entered the diplomatic service and so traveled widely. When he visted Paris, he met Huygens who asked Leibniz to find the series:

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$$

Leibniz made each term the sum of two fractions:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

The sum is 2. After he read Pascal's arithmetical triangle, Leibniz invented the harmonic triangle.
Each term of the harmonic triangle is the difference of two terms dirctly above it and to the right. This means that each term is the sum of the all terms in the row below the term. For example,

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

Leibniz claimed the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

1	1	1	1	1	1	$\cdots$
1	2	3	4	5	$\cdots$	
1	3	6	10	$\cdots$		
1	4	10	20	$\cdots$		
1	5	15	35	$\cdots$		

Table 9.1: Arithmetic triangle

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\cdots$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}$	$\cdots$	
$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{60}$	$\cdots$		
$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{60}$	$\cdots$			
$\frac{1}{5}$	$\frac{1}{30}$	$\cdots$				

Table 9.2: Harmonic triangle

is $\frac{1}{0}$.
Leibniz noted the following pattern in the harmonic triangle; for the series

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=\frac{1}{1},$$

Divide every denominator by the denominator of the first fraction to obtain the following.

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\cdots=\frac{2}{1},$$

9.4.1 Leibniz's Calculus

Leibniz is known as one of founder of Calculus. By 1676, Leibniz had arrived at the same conclusion that Newton had arrived several years earlier. One of significant contributions of him to calculus is that he invented good notations for calculus such as differentials dx and dy. In 1684 Leibniz published "A New Method for Maxima and Minima, and Also for Tangents, which Is Not Obstructed by Irrational Quantities". He gave the formulae

$$\begin{aligned} dxy &= xdy + ydx, \\ d\left(\frac{x}{y}\right) &= \frac{ydx - xdy}{y^2} \\ dx^n &= nx^{n-1}dx \end{aligned}$$

Two years later, Leibniz published and he mensioned about the inverse relationship between differentiation and integration.

9.5 Mathematical Analysis

Newton and Leibniz established the calculus but it does not mean that they established calculus on functions. There were some ambiguity on the infinitesimal samll number o (omicron).

Functions and Limits

When Newton and Leibniz established calculus, they worked on plane curves and the concept of function was not clearly defined. Leonhard Euler (1707-1783) introduced the concept of mathematical functions. Bernard Bolzano (1781-1848) introduced the modern definition of continuity in 1816.
Augustin Louis Cauchy (1789-1857) put calculus on the regorous foundation using infinitesimals. He also introduced the concept of the Cauchy sequence.
Karl Theodor Wilhelm Weierstrass (1815-1897) developed (ε, δ)-definition of limit:
A function $f(x)$ is continuous at $x = a$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon.$$

Chapter 10

Non-Euclidean Geometry

For more than 2000 years, mathematicians had made great efffort to prove the fifth postulate of the Element of Euclid. However, all attempts had failed before the 19th century. Then some people started to suspect that the fifth postulate was independent from others and also, it could be replaced. Three mathematicians Carl Friedrich Gauss (1777-1855), John Bolyai (1802-1860) and Nicolai Lobachevsky (1793-1856) independently succeeded to build non-Euclidean geometries. In this section we learn how the non-Euclidean geometries were found.

10.1 Perspective

Artists (painters) in the 15th century started to study how the three dimensional object is projected in the plane (canvas). Renaissance artists used the following principle for drawing perspective picture:

1. A straight line in persepective remains straight.

2. Parallel lines either remian parallel or converge to a single point (their vanishing point).

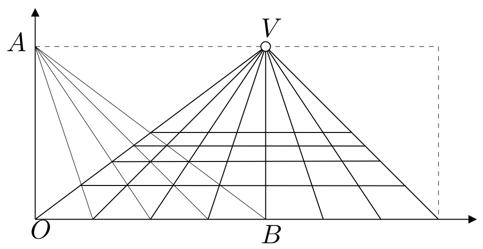
The picture plane is a section of the projection from the observers eye (station point) to the scene to be pictured.
Battista Alberti (1404-1472) wrote the first text on the perspective. He described a construction method of perspective drawing. Suppose we are drawing a checker board on the floor and the picture plane is vertical to the

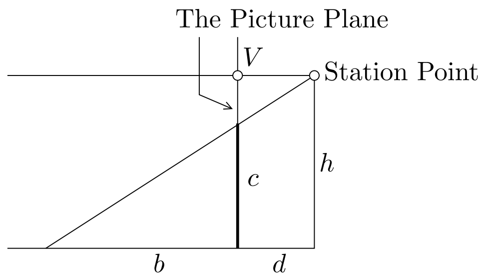
floor. Also suppose that the horizontal lines are parallel to the base of the picture plane. The intersection point between the picture plane and vertical line from the station point is called the *vanishing point*. The vertical lines on the checker board to the base of the picture plane will meet at the vanishing point. The horizontal lines on the checker board can be drawn as following. Suppose the distance to a horizontal line from the base of the picture plane is b.

Let the left vertical side of the picture plane be the y-axis and let the base of the picture plane be the x-axis. Then the left bottom corner is the origin O. Take point $A(0, h)$ and $B(b, 0)$. From the right diagram, we have:

$$\begin{aligned} c : b &= h : (d + b) \\ c &= \frac{hb}{d + b} \end{aligned} \tag{10.1}$$

The line $OV : y = \frac{h}{d}x$ intersects with the line AB at a point. the y-coordinate of the intersection point is $y = \frac{hb}{d+b} = c$. This means we can draw the horizontal line as a parallel line to the base line passing through the intersection point.

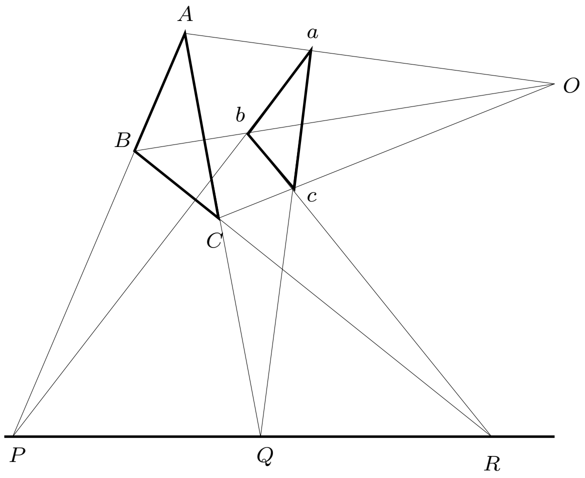




10.2 Projective Geometry

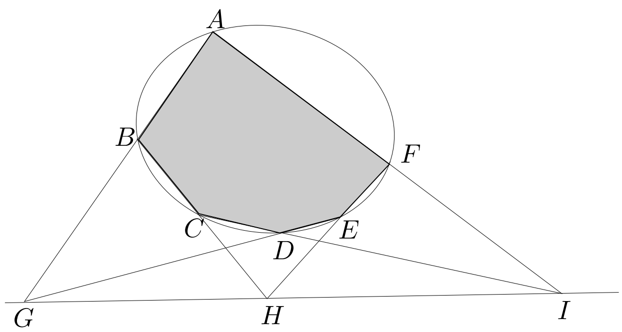
French engineer and architect Girard Desargues (1591-1661) invented modern projective geometry in his work titled “Rough draft for an essay on the results of taking plane sections of a cone” (1639). Desargues’ most famous theorem is the following.

Theorem 10.2.1 (Desargues)**.** *Let $\triangle ABC$ and $\triangle abc$ be two trinagles. Suppose that lines Aa, Bb and Cc meet at a point O. Then crossing points of lines AB and ab, AC and ac and BC and bc lie on the same line.*



This theorem and the Pascal's theorem given below are called fundamental theorems in projective geometry.

Theorem 10.2.2. *If a hexagon is inscribed in a conic, then the opposite sides intersect in three collinear points.*



10.3 Analytic Geometry

In the seventeenth century Rene Descartes (1596-1650) applied algebra to geometry and created analytic geometry.
Piere de Fermat (1601-1665) is also recognized as a creator of analytic geometry. He published the paper "Ad Locos Planos et Slidos Isagoge" (1636). He developed a method for determining maxima, minima and tangents to curved lines.

10.4 Non-Euclidean Geometry

Postulate 5 in the Elements is known as Euclid's parallel postulate. This has become one of most famous and controversial statement in mathematical history. The reason of this is that the statement is beyond the reach of possible observation, this is not brief, simple and self-evident.

From the moment the Elements appeared and continuing into the nineteenth century mathematicians have tried to prove the postulate 5 with first four postulates. This effort led to the discovery of non–Euclidean geometries, in which four postulates hold except the fifth one and all Euclidean theorems based on these first four postulates are true.

In the first third of the nineteenth century, three mathematicians, Carl Friedrich Gauss (1777–1855), John Bolyai (1802–1860) and Nicolai Lbachevsky (1793–1856) independently succeeded to build non–Euclidean geometries. Euclidean geometry is no longer "the" geometry of space.

Postulate 5 can be expressed as follows:

The fifth postulate

Let BC be a line and let A be a point not on the line. Let AD be a perpendicular from A to BC and let AE be a line from A. If the angle $\angle DAE$ is less than $\pi/2$, then AE meets BC on the same side of the angle.

Lobachevsky replaced the Postulate 5 with its negation:

Lobachevsky's postulate

there exist a line AE such that $\angle DAE < \pi/2$ and AE does not meet BC.

He defined that AE and BC are parallel if they do not meet. On the other hand, a line AF meets BC. Thus there is an angle α such that if $\angle DAH < \alpha$, the line AH meets BC. The angle α is called the *angle of parallelism.* The other side of AD there is another parallel line AK with the angle of parallelism α.

The discovery of existence of new geometry other than Euclidean geometry led the idea that geometry itself can be a mathematical object.

Bernhard Riemann (1826–1866) gave a lecture at Göttingen in 1854 entitled "On the Hypotheses which lie at the Foundation of Geometry" about

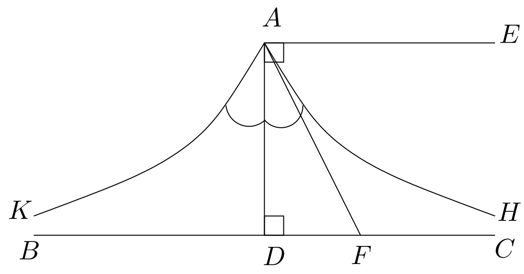


Figure 10.1

"points" in an "n–dimensional space" are n–tuples of numbers. He dealt with the concept, today it is called "n-dimensional manifold". Then he talked about metric on an n-dimensional manifold given by

$$ds^2 = \sum_{i=1}^{n} \sum_{i=1}^{n} g_{ij} dx_i dx_j,$$

where $g_{ij} = g_{ji}$ and all of the g_{ij} are continuous functions on the manifold. In the Euclidean space it can be simplified to:

$$ds^2 = \sum_{i=1}^{n} dx_i^2.$$

He named the space with this simple metric "flat". He define curvature for "nonflat" manifolds and showed it depends only on the coefficients g_{ij}.

10.5 Erlangen Program

Felix Klein (1849–1925) proposed a way of classifying geometries in terms of groups of transformations at the University of Erlangen in 1872. This concept has become known as Klein's Erlangen Program. According to Klein a geometry is the study of properties of a set S that remain invariant under some transformations of some transformation group Γ. The geometry is denoted by $G(S, \Gamma)$.

Chapter 11

Probability and Statistics

The study of probability began in 16th century when gamblers wanted to know the chance to win the gamble. In 18th century Bernoulli and De Moivre gave the mathematical foundation. On the other hand, Statistics began as a study of recording the crop yields or the population of a country. Babylonians recorded their crop yields on clay tablets. Ancient Egyptian already investigated the number of people in the country to call certain number of labours for building a pyramid.

11.1 Probability

The systematic study on probability began in 16th century Italy with Cardano. He gambled with rolling three dices. His book "Book on Gambling" was published about century after his death. He stated the multiplicative rule: the probability of getting a six on each of three successive rolls with one dice is

$$\left(\frac{1}{6}\right)^3$$

He also recognised the law of large numbers. In France by Blaise Pascal (1623-1662) and Pierre de Fermat (1607-1665)

Jacob Bernoulli (1665-1705) "Ars Conjectandi" (1713). He considered the repeated experiments; today called "Bernoulli trials": For a particular outcome either success with probability p or failure with probability q, he gave the probability of getting at least m successes in n trials as

$$\sum_{k=m}^{n} \binom{n}{k} p^k q^{n-k}. \tag{11.1}$$

Binomial distribution

If a random variable follows the binomial distribution $\mathrm{Bin}(n, p)$, then

$$P(X = x) = \sum_{i=x}^{n} \binom{n}{x} p^x q^{n-x}.$$

Abraham de Moivre published "The Doctrine of Chances" (1718). He introduced the concept of normal distribution.

De Moivre-Laplace's Theorem

If the random variable X follows the binomial distribution $\mathrm{Bin}(n, p)$, then

$$Y = \frac{X - np}{\sqrt{np(1-p)}}$$

follows the normal distribution.

Example 11.1.1. Toss a coin 10000 times. Find the probability of getting heads 5100 times or more: $P(X \geq 5100)$.

Solution.
We apply De Moivre-Laplace's Theorem to obtain the random variable Y:

$$Y = \frac{X - 10000 \cdot \frac{1}{2}}{\sqrt{10000 \cdot \frac{1}{2}\frac{1}{2}}} = \frac{X - 5000}{50}$$

Thus

$$P(X \geq 5100) = P(Y \geq 2) \approx 2.28\%$$

Legendre dealt with a fitting problem in his treatise (1805) and he gave a solution. Suppose that the observation data follow an equation $y = f(x)$

where $f(x)$ involves parameters $\alpha, \beta, \ldots$, and one has data points $(x_k, y_k,$ $k = 1, 2, \ldots, n$. Then find the parameters $\alpha, \beta, \ldots$, so that

$$E(\alpha, \beta, \ldots) = \sum_{k=1}^{n} (f(x_k - y_k)^2.$$

is minimised.
In 1816 Carl Friederich Gauss (1777-1855) published a paper on observation errors. He found the likelihood of an error of size x to be

$$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2},$$

where h was called *measure of precision*. Today, the term $\frac{1}{\sqrt{2}h}$ is called the standard deviation.
The Russian mathematician Chebyshëv (1846) showed that the probability that the the propotion of scuccesses will differ from the expected proportion by less than any $\varepsilon > 0$ tends to 1 as the number of trials increases. In 1867 he proved "Chebyshëv's inequality".

11.2 Statistics

A systematic study of statistics began in the 17th century in England. Gottfried Leibniz (1646-1716) and Edmond Halley (1656-1742) studied about mortality statistics.
The development of the theory of probability enabled to establish the theoretical background for statistics.
Karl Pearson (1856-1936) wrote a paper (1893) introduced "standard deviation" to denote the natural unit of probability and in 1901 he introduced "chi-square test" of significance: If

$$P(a \leq X_k \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}t^2} dt,$$

where each of these probabilities is independent of all the others, then the probability $P(X_1^2 + \cdots + X_n^2 \leq c)$ lies between 0 and c by the chi-square density with n degrees of freedom:

$$P(X_1^2 + \cdots + X_n^2 \leq c) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} \int_0^c x^{(n/2)-1} e^{-x/2} dx$$

If $X_1, \ldots, X_n$ are independent random variables with expected positive values $\mu_1, \ldots, \mu_n$, the random variable

$$\chi^2 = \frac{(X_1 - \mu_1)^2}{\mu_1} + \cdots + \frac{(X_n - \mu_n)^2}{\mu_n}$$

has the chi-square distribution with $n - 1$ degrees of freedom.

Appendices

Appendix A

Japanese Mathematics

This is a short survey of a history of Japanese mathematics up to the 19th century. Japan has been keeping the civilization more than thousand years. At the early time of the country's history, the influence of civilization mainly came from China through Korean peninsula. Mathematics was imported from China and well developed during the Edo-era (1603-1868).
The first time mathematics arrived Japan in the seventh century with masters of making calendar from Korean peninsula. The position, master of mathematics, was made as the administration job in the government. This position was a hereditary title and thus mathematics did not settle in Japan. The multiplication table (times table), which is familier to today's Japanese as 'Kuku', was transfered at this period.
Second time mathematics arrived Japan was in 'Muromachi–era' (14th -16th century). This time mathematics settled down properly in Japan. What came from China at this period were abacus, arithmetrics and also 'sangi' (counting rods; see Appendix B).
Such mathematics was used for commerce, trading, topography, managing water supply and building castles. The way of calculation using an abacus was wide spread among people. Especially, the multiplication table 'Kuku' enabled people to compute divisions easily.

A.1 Development of Japanese Mathematics

In 1627 Yoshida Mitsuyoshi published a remarkable book *Jinkooki*, which instructed not only how to use abacuses but also it explained names of num-

bers, exchanging silver and gold, evaluating interests, trading, transport fair, civil engineering and topography. It also included many problems, whose solutions were not given. Readers tried to solve those questions also they add new questions to own publications. This was continued repeatedly. Finally, this kind of pubilication contained hundreds of questions and they became difficult to solve.
There was an advanced way of solving complicated problems, which was invented in the 12th century in China. This was called 'Tengen–jutsu' in Japanese; Today's theory of linear equations systems. It came to Japan as a book, however it did not contain any instruction for using 'sangi'.
In 1671, Sawaguchi Kazuyuki published *Kokon Sampo-ki*, which described Chinese algebra with 15 problems. From this point, Japanese mathematics jumped from arithmetics using abacuses to mathematics using calculators (sangi).

A.2 Seki Takakazu (Kowa)

Not much is known about the life of Seki Takakazu (Kowa) (1635?-1708). It is said that he was born in between 1635-1643. He established the base of development of Japanese mathematics (Wasan).
He studied the method of solving equations of Chinese mathemtics (*Ten-gen jutsu*). In 1674, he published *Hatsubi-Sanpo*, in which he modified the Chinese method to established *Ten-zan jutsu*. His method could solve multi-variable linear equation systems. He also solved the 15 problems given in Sawaguchi's *Kokon Sampo-ki*.
Around 1681, he calculated π and obtained the approximation up to 16th decimals:

$$\pi \approx 3.141526503589793238\underline{6}$$

(the numbers before the underlined are accurate.) He also discovered the determinant of matrix.

Figure A.1: Seki Takakazu (Japanese Post Stamp (1992)).

Yoriyuki Arima collected Seki's methods and published *Shuki Sanpo* (1769). His pupile Kenko Takebe (1664-1739) obtained better approximation of π as:

$$\pi \approx 3.14152650358979323864338327950288419716\underline{8}$$

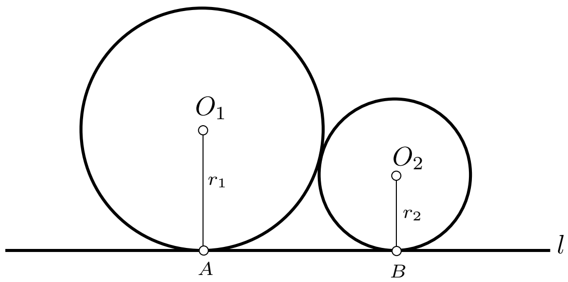
(the numbers before the underlined are accurate.)

A.3 Sangaku

During Edo era (1603-1867) Tokugawa clan governed Japan. It was a feudal period and there were strict social classes; Samurais, Farmers, Craftsmen and Merchants. But mathematics was studied by people from all social classes. And they produced theorems in Euclidean geometry. It was common when they solved a problem, then they made wooden tablets with beautifully colored drawings. The tablet was called a *sangaku* which means a mathematics tablet in Japanese. They dedicated sangakus to a shrine or temple to thank god for the discovery of a theorem. Those sangakus werer hung under the roof so that many people could recognize the achivement. Today, most of sangaku were lost, but about 820 sangakus have survived.

Example A.3.1. The following problem was found in a sangaku (1882). Two circle $O_1(r_1)$ and $O_2(r_2)$ are externally tangent each other. Suppose that there is a common tangent line l to three circles. Prove the follwoing relation.

$$(AB)^2 = 4r_1r_2$$



A.4 After 1868

When Meiji restration (1867-1877) started, the samurai's government was replaced by new government (Meiji restration) and the capital was moved from Kyoto to Edo and the name of the city was changed to *Tokyo* (1869). Meiji governmnet introduced modern education system and started to teach western style mathematics in schools. As the educated graduates from the modern schools increased, the traditional mathematics (wasan) declined. It seemed that people easily accepted and learned the western mathematics. This is because the foundation of the mathematical knowledge among people was already provided by the traditional mathematics.

Appendix B

Counting Rods

In the 17th century Japanese developed their own mathematics. These Japanese mathematicians used the Counting Table and Counting Rods. Counting rods were invented in China and introduced to Japan in the 8th century.

Here, we demonstrate the operations with counting rods and the table. We will see the basic operations and solving equations. Originally, it is written in mathematical texts that there were two types of expressions of a number with the counting rods. Using these two types avoids a confusion between neighbouring numbers (see Figure B.1).

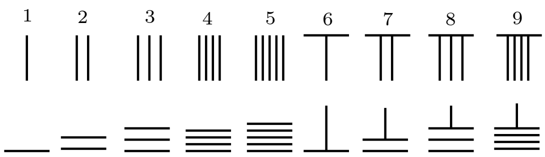


Figure B.1: Two types of expressions of numbers.

It is also known with the counting table, we can use only one type of expression, therefore, we use one type of expression and will demonstrate how the calculation goes.

There are two types of rods one is in red and the othre is in black. The red rods represent the positive numbers, and the black rods represent the negative numbers. Here we use white rods as red rods in our diagrams (see Figure B.2).

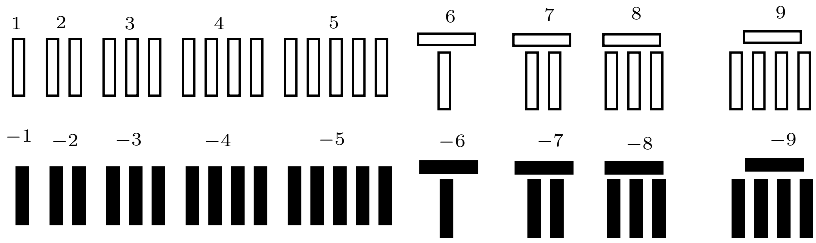
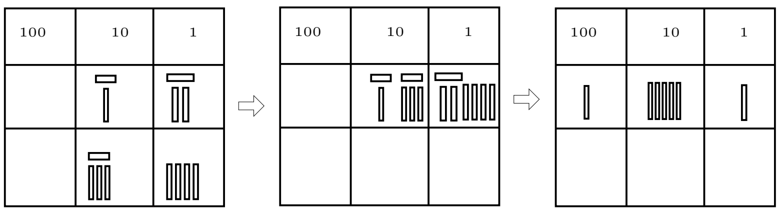


Figure B.2: Counting rods in colors. Here white rods mean ‘red rods’ representing positive numbers and black rods represent negative numbers.

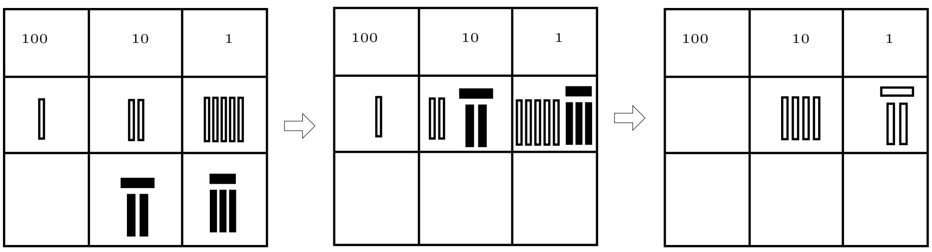
B.1 Addition and Subtraction

In the following tables $67 + 84 = 151$ is done:



$67 + 84 = 151$

In the following tables $125 - 78 = 47$ is done:



$125 - 78 = 47$

B.2 Solving equation

The equation $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ is expressed in the table. The following table expresses the equation

$$x^3 + 7x^2 + 841x - 80885 = 0 \tag{B.1}$$

First we add the pointers (circle) to the first row (10) and the first right column (a_0). We guess the answer x is in the order 30. Put 3 in the top of the 10-column (see the right diagram of Figure B.3). Then shift the numbers in the rows of a_1 , a_2 and a_4 to the left 1 cell, 2 cells and 3 cells respectively.

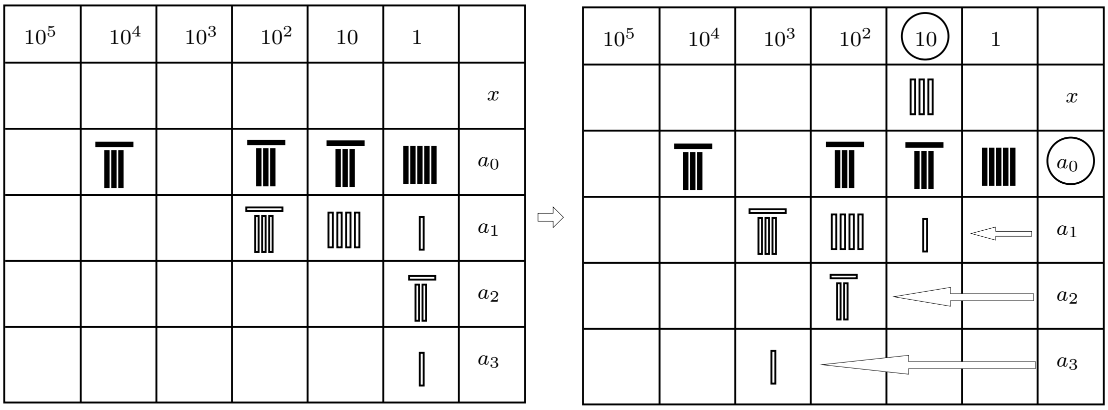


Figure B.3: Put the pointers to the top of the column of 10 and right of the a_0-row. Shift the a_1, $_2$ and a_3-rows.

Then apply the following operation:

1. Multiply 3 to the a_3-row and add the result to the a_2-row.

2. Multiply 3 to the a_2-row and add the result to the a_1-row.

3. Finally, multiply 3 to a_1 and add the result to the a_0-row.

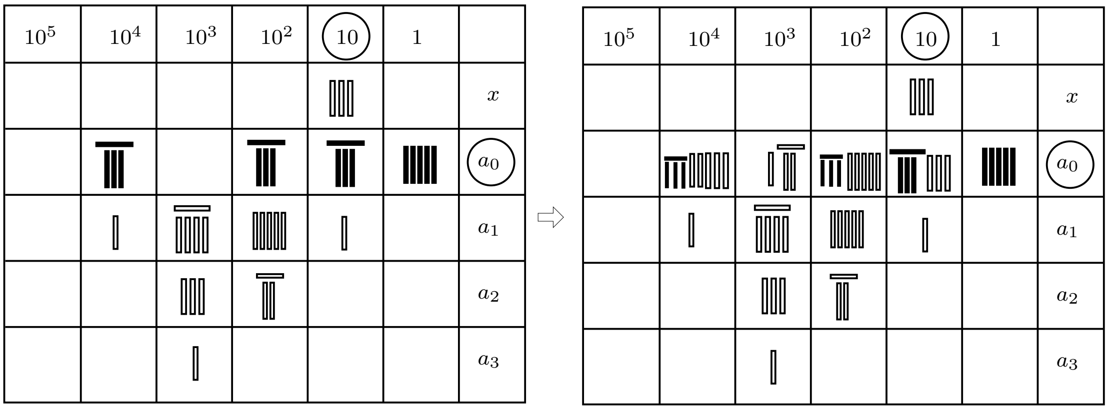


Figure B.4: In the left diagram $3(3(3a_3 + a_2) + a_1)$ is done. In the right diagram, $3(3(3a_3 + a_2) + a_1) + a_0$ is done.

The a_0-row is not zero. Then we shift the pointer on the right column from a_0 to a_1 (see Figure B.5). Multiply 3 to the a_3-row and add the result to the a_2-row. Multiply 3 to the a_2-row and add the result to the a_1-row.

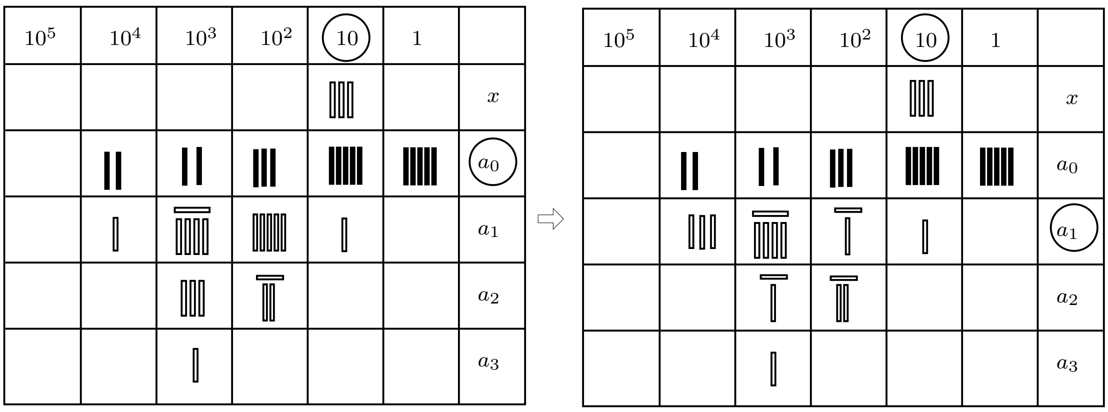


Figure B.5: The number in a_1-row is $3(3a_3 + a_2) + a_1$.

Shift the pointer on the right column from a_1 to a_2. Multiply 3 to the a_3-row and add the result to the a_2-row (see the left diagram of Figure B.6). Then shift the a_1, a_2 and a_3-rows to the right. Move the pointer from a_3 to a_0 (see the right diagram of Figure B.6).

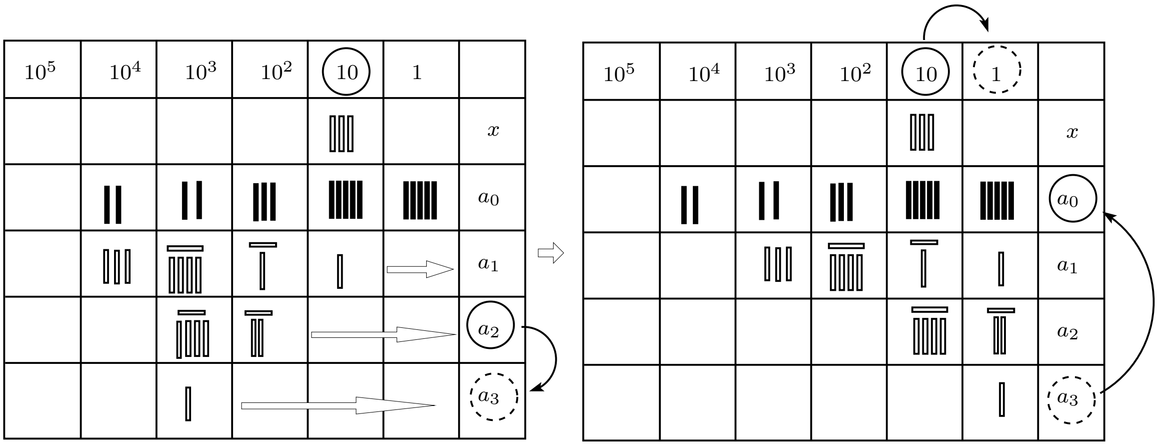


Figure B.6: The number in a_2-row is $3a_3 + a_2$.

Apply the same operation above. This time a_0-row becomes zero.

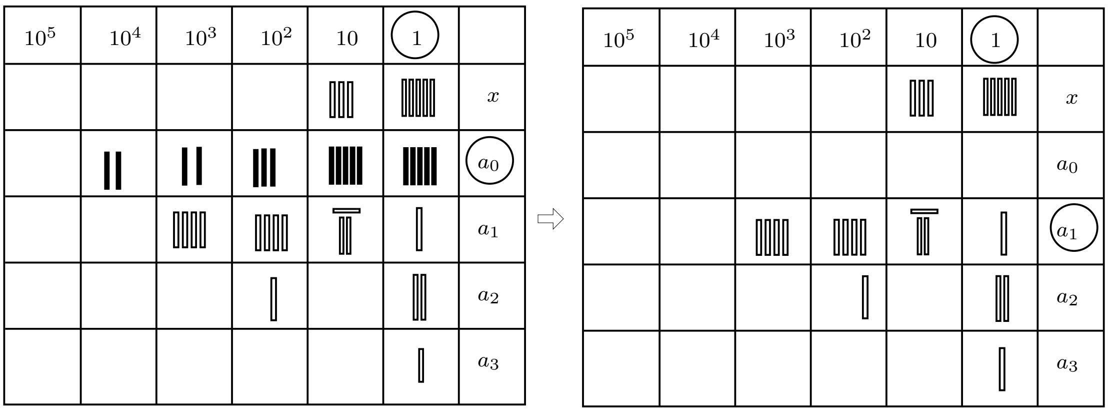


Figure B.7: The number in a_0-row is $5a_1 + a_0 = 0$.

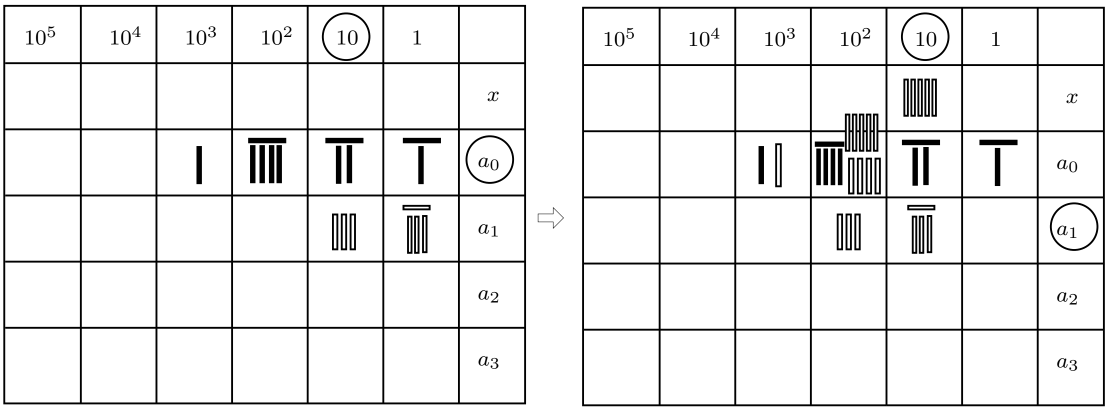
The solution is 35.

B.3 Division

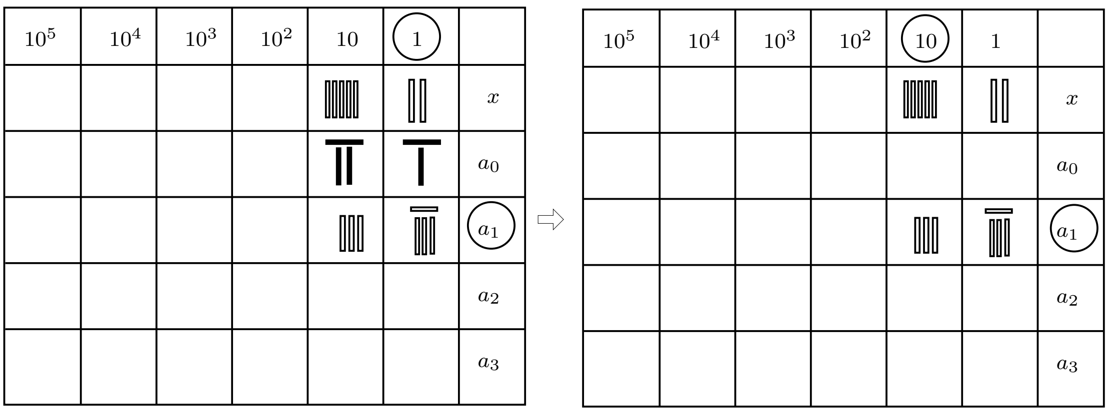
The division $1976 \div 38$ is the solution of the equation

$$38x - 1976 = 0.$$

We set the coefficients in the table:



Guess the solution and put 5 in the top of 10-column. Apply the operation to get 76 at the a_0-row.



Then shift the a_1-row to the right and apply the operation. Then a_0-row becomes zero. Therefore, $x = 52$ is the solution and it is the result of the division $1976 \div 38$.

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