



محاضرات Math553

نظرية القياس

Measure Theory

(دراسات عليا / ماجستير)

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محاضرات
الأستاذ الدكتور نوري فرحان المياحي
في
نظرية القياس Measure Theory

What is Measure Theory ?

Measure theory is a branch of mathematical analysis which studies the notion of size of an abstract set and aggregates of a function. Measure theory guides to the rigorous concepts of an event for which a reasonable of chance can be defined and computed.

Measure theory not only answers existential questions, but also makes a unified treatment of random events and random variables possible. For instance, the nuisance of separate treatment of discrete and continuous variables, p.m.f and p.d.f, is completely eliminated. In addition, it lets us treat much more general events and random variables.

Measure theory is absolutely indispensable while dealing with many, in particular infinite number of, random variables Existential issues can be really nontrivial in such cases(example, spatial regression)

Measure theory broadens our vision about probability, random events, random variables and integration. Naturally, measure theory interacts with many other branches of mathematics, such that as topology, real analysis, functional analysis, probability theory, algebra, number theory etc.

Measure theory , to a large extent, is self contained, but preliminary concepts on set theory, functions and real and real analysis is essential.



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1. Fundamental Concepts

Set theory is fundamental to the topics discussed in this section, and thus we begin by collecting and presenting some well know concepts and notional conventions in order to make this thesis more self contained. It must be emphasized that we make no attempt to discuss this elementary material thoroughly. We assume a familiarity with the most well-known and basic mathematical symbols and abbreviations.

Notation

Sets will usually be denoted by capital letters $A, B, C, \dots$. the elements in our sets will usually be represented by small case letters $a, b, c, \dots$. We will always be considering sets that are subsets of a universal set U . If x is an element and A is a set then we write $x \in A$ which is read " x belongs to A " or x in A . When x not a member of A , we write $x \notin A$. The collection of all elements that satisfy a condition $P(x)$ will be written as $\{x \in U : P(x)\}$. We denote by ϕ the empty set, that is the set that contain no elements. As usual, we denote the natural numbers by $\mathbb{N}$, the integers by $\mathbb{Z}$, the rational numbers by $\mathbb{Q}$, the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. We will denote the extended real line by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. to denote intervals of the extended real line, we use the following standard notation for $a, b \in \overline{\mathbb{R}}$, $a \leq b$:

$$[a, b] = \{x \in \overline{\mathbb{R}} : a \leq x \leq b\}, \quad (a, b) = \{x \in \overline{\mathbb{R}} : a < x < b\}$$

$$[a, b) = \{x \in \overline{\mathbb{R}} : a \leq x < b\}, \quad (a, b] = \{x \in \overline{\mathbb{R}} : a < x \leq b\}$$

Throughout the lector, $\mathbb{R}^+$ will denote the set $(0, \infty)$ of positive real numbers

Definition(1.1)

A set is said to be finite if it consists of a finite elements. Otherwise a set is infinite.



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i.e. a set A is finite if $\#(A)$ is finite, where $\#(A)$ is the number of elements in A .

For example $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are infinite sets.

Definition (1.2)

Let A and B be two sets, we set that

1. A is a subset of B , denoted $A \subseteq B$, if every element of A is also an element of B . More specifically, $A \subseteq B$, if $x \in A$ implies $x \in B$.
2. A is equal to B , denoted $A = B$, if they both have the same elements, i.e. if

$$A \subseteq B \text{ and } B \subseteq A.$$

3. A is a Proper subset of B , denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.
4. A and B are comparable if $A \subset B$ or $B \subset A$.

Remarks

1. The empty set ϕ is a unique set
2. Every set A is a subset of itself, i.e. $A \subseteq A$
3. The empty set ϕ is a subset of any set, i.e. $\phi \subset A$ for any set A .
4. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
5. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Definition(1.3)

Let A, B be subsets of the universal set U

1. The set of all elements which either belong to A or to B is called the union of the two sets A and B and is denoted by $A \cup B$. Thus

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

- If $x \in A \cup B$, then $x \in A$ or $x \in B$, or both,



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- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$
- 2. The set of all elements which belong to both A and B is called the intersection of the two sets A and B and is denoted by $A \cap B$. Thus

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

- If $x \in A \cap B$, then $x \in A$ and $x \in B$,
- If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$ or both
- 3. A and B are said to be disjoint sets if no element of A is in B and no element of B , i.e. if $A \cap B = \emptyset$

For example : If $A = \{1, 3, 8\}$, $B = \{1, 2, 5, 7\}$, then $A \cup B = \{1, 2, 3, 5, 7, 8\}$, $A \cap B = \{1\}$

Theorem(1.4)

Let A, B, C be subsets of the universal set U

1. $A \subset A \cup B$ and $A \cap B \subset A$
2. $A \cup B = B$ iff $A \subseteq B$, and $A \cap B = A$ iff $A \subseteq B$
3. $A \cup A = A$, $A \cup \emptyset = A$, $A \cup U = U$ and $A \cap A = A$, $A \cap \emptyset = \emptyset$, $A \cap U = A$
4. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Definition(1.5)

Let A, B be subsets of the universal set U . The set of all elements which belong to A but which do not belong to B and denoted by $A \setminus B$ or $A - B$ is called the difference of A and B . Thus $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

- $A \setminus B$ is called the complement of B in A if $B \subset A$



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- The complement of a set A relative to universal set U is the set of all elements which do not belong to A and which, of course, belong to U . i.e. $U \setminus A$ and it is denoted by A^c .

Thus

$$A^c = U \setminus A = \{x : x \in U \text{ and } x \notin A\} \text{ or simply } A^c = \{x : x \notin A\}$$

- The symmetric difference of A and B is denoted by $A \Delta B$ and defined as

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

Theorem(1.6)

If A, B, C are any subsets of the universal set U , then

- $A \setminus B = A \cap B^c$
- If $A \subseteq B$, then $B^c \subseteq A^c$
- $(A^c)^c = A$
- $U^c = \phi, \phi^c = U$
- $A \cup A^c = U, A \cap A^c = \phi$
- $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$
- $A \Delta B = B \Delta A$
- $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
- $A \Delta \phi = A, A \Delta A = \phi$

Theorem(1.7)

If A, B, C are any subsets of the universal set U , then

- $A \setminus B = A \cap B^c$
- If $A \subseteq B$, then $B^c \subseteq A^c$



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3. $(A^c)^c = A$
4. $U^c = \phi, \phi^c = U$
5. $A \cup A^c = U, A \cap A^c = \phi$
6. $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$
7. $A \Delta B = B \Delta A$
8. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
9. $A \Delta \phi = A, A \Delta A = \phi$.

Definition(1.8)

Let X and Y be two sets. A function f from X into Y , denoted by $f : X \rightarrow Y$ is a rule that assigns to each $x \in X$ a unique element $y \in Y$. we write $y = f(x)$ to denote that f assigns the element $x \in X$ to the element $y \in Y$. We refer to the set X as the domain, Y as the Co domain, and the set $f(X) = \{f(x) : x \in X\}$ as the range of f respectively.

Definition(1.9)

Let X and Y be two sets. A function $f : X \rightarrow Y$ is said to be

1. Injective(or one-to-one), if for each $y \in Y$ there is at most one $x \in X$ such that

$f(x) = y$, that is, if no two different elements in X have the same image.

More briefly, $f : X \rightarrow Y$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ or, equivalently, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

2. Surjective (or onto), if for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$,

i.e. if $f(X) = Y$

3. Bijective if f is both injective and onto.

Definition(1.10)

Let X and Y be two sets. A function $f : X \rightarrow Y$ is said to be

1. Constant function if the range of f consists of only one element, i.e. if there is $c \in Y$ such that $f(x) = c$ for all $x \in X$.
2. Identity function if $Y = X$ and $f(x) = x$ for all $x \in X$ and denoted by I_X .
3. Inclusion function if $X \subseteq Y$ and $f(x) = x$ for all $x \in X$.

Definition(1.11)

Let X and Y be two sets, $A \subset X$, and let $f : X \rightarrow Y$ be function. we denote by $f|_A$, the restriction of f to A , defined by $f|_A : A \rightarrow Y$, $f|_A(x) = f(x)$ for $x \in A$.

Definition(1.12)

Let X and Y be two sets, and let $f : X \rightarrow Y$ be function

1. The image of $A \subseteq X$ under f , denoted by $f(A)$, consists of those elements in Y Which appear as the image of at least one element in A . More briefly,

$$f(A) = \{f(x) : x \in A\}$$

2. The inverse of $B \subseteq Y$ under f , denoted by $f^{-1}(B)$, consists of those elements in X which are mapped onto some element in B . More briefly,

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Theorem(1.13)

Let X and Y be two sets, $f : X \rightarrow Y$ be function, and let $A_1, A_2 \subseteq X$, $B_1, B_2 \subseteq Y$

1. If $A_1 = A_2$, then $f(A_1) = f(A_2)$
2. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$
3. If $B_1 = B_2$, then $f^{-1}(B_1) = f^{-1}(B_2)$ (4) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$

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Proof :

$$(1) y \in f(A_1) \leftrightarrow \exists x \in A_1 \ni y = f(x) \leftrightarrow \exists x \in A_2 \ni y = f(x) \leftrightarrow y \in f(A_2)$$

$$\text{so } f(A_1) = f(A_2)$$

$$(2) y \in f(A_1) \rightarrow \exists x \in A_1 \ni y = f(x) \rightarrow \exists x \in A_2 \ni y = f(x) \rightarrow y \in f(A_2)$$

$$\text{so } f(A_1) \subseteq f(A_2)$$

$$(3) x \in f^{-1}(B_1) \leftrightarrow f(x) \in B_1 \leftrightarrow f(x) \in B_2 \leftrightarrow x \in f^{-1}(B_2)$$

$$\text{so } f^{-1}(B_1) = f^{-1}(B_2)$$

$$(4) x \in f^{-1}(B_1) \rightarrow f(x) \in B_1 \rightarrow f(x) \in B_2 \rightarrow x \in f^{-1}(B_2)$$

$$\text{so } f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

Theorem(1.14)

Let $f : X \rightarrow Y$ be a function

1. If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$. In particular, if f is one-to-one, then $f^{-1}(f(A)) = A$

2. If $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$. In particular, if f is onto, then $f(f^{-1}(B)) = B$

3. If $A_1, A_2 \subseteq X$, then

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2), f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \text{ and } f(A_1 | A_2) \supseteq f(A_1) | f(A_2)$$

4. If $B_1, B_2 \subseteq Y$, then

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2), f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \text{ and } f^{-1}(B_1 | B_2) = f^{-1}(B_1) | f^{-1}(B_2)$$

Proof :

$$(1) x \in A \rightarrow f(x) \in f(A) \rightarrow x \in f^{-1}(f(A)), \text{ so } A \subseteq f^{-1}(f(A))$$

$$(2) y \in f(f^{-1}(B)) \rightarrow \exists x \in f^{-1}(B) \ni y = f(x) \rightarrow \exists f(x) \in B \ni y = f(x) \rightarrow y \in B$$

$$\text{so } f(f^{-1}(B)) \subseteq B$$

(3)



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$$\begin{aligned} y \in f(A_1 \cup A_2) &\leftrightarrow \exists x \in A_1 \cup A_2 \exists y = f(x) \leftrightarrow \exists x \in A_1 \vee x \in A_2 \exists y = f(x) \\ (i) \quad &\leftrightarrow (\exists x \in A_1 \exists y = f(x)) \vee (\exists x \in A_2 \exists y = f(x)) \\ &\leftrightarrow y \in f(A_1) \vee y \in f(A_2) \leftrightarrow y \in f(A_1) \cup f(A_2) \end{aligned}$$

$$\text{so } f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

$$\begin{aligned} y \in f(A_1 \cap A_2) &\rightarrow \exists x \in A_1 \cap A_2 \exists y = f(x) \rightarrow (x \in A_1 \wedge x \in A_2) \exists y = f(x) \\ (ii) \quad &\rightarrow (x \in A_1 \exists y = f(x)) \wedge (x \in A_2 \exists y = f(x)) \\ &\rightarrow y \in f(A_1) \wedge y \in f(A_2) \rightarrow y \in f(A_1) \cap f(A_2) \end{aligned}$$

$$\text{so } f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$\begin{aligned} y \in (f(A_1) | f(A_2)) &\rightarrow y \in f(A_1) \wedge y \notin f(A_2) \rightarrow \exists x \in A_1 \exists y = f(x), f(x) \notin f(A_2) \\ (iii) \quad &\rightarrow \exists x \in A_1 \wedge x \in f^{-1}(f(A_2)) \supseteq A_2 \\ &x \in A_1 | A_2 \rightarrow y = f(x) \in f(A_1 | A_2) \end{aligned}$$

$$\text{so } f(A_1 | A_2) \supseteq f(A_1) | f(A_2)$$

(4)

$$\begin{aligned} (i) \quad x \in f^{-1}(B_1 \cup B_2) &\leftrightarrow f(x) \in B_1 \cup B_2 \leftrightarrow f(x) \in B_1 \vee f(x) \in B_2 \leftrightarrow x \in f^{-1}(B_1) \vee x \in f^{-1}(B_2) \\ &\leftrightarrow x \in f^{-1}(B_1) \cup f^{-1}(B_2) \end{aligned}$$

$$\text{so } f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

$$\begin{aligned} (ii) \quad x \in f^{-1}(B_1 \cap B_2) &\leftrightarrow f(x) \in B_1 \cap B_2 \leftrightarrow f(x) \in B_1 \wedge f(x) \in B_2 \leftrightarrow x \in f^{-1}(B_1) \wedge x \in f^{-1}(B_2) \\ &\leftrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

$$\text{so } f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$\begin{aligned} (iii) \quad x \in f^{-1}(B_1 | B_2) &\leftrightarrow f(x) \in (B_1 | B_2) \leftrightarrow f(x) \in B_1 \wedge f(x) \notin B_2 \leftrightarrow x \in f^{-1}(B_1) \wedge x \notin f^{-1}(B_2) \\ &\leftrightarrow x \in (f^{-1}(B_1) | f^{-1}(B_2)) \end{aligned}$$

$$\text{so } f^{-1}(B_1 | B_2) = f^{-1}(B_1) | f^{-1}(B_2)$$

Definition(1.15)

Let X, Y and Z be sets ; $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. The composition of f and g ,



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denoted by $g \circ f$, is the function $g \circ f : X \rightarrow Z$ defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

Theorem(1.16)

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions

1. If f and g are one-to-one, then $g \circ f$ is also one-to-one.
2. If f and g are onto, then $g \circ f$ is also onto.
3. If f and g are bijective, then $g \circ f$ is also bijective.
4. If $g \circ f$ is one-to-one, then f is one-to-one.
5. If $g \circ f$ is onto, then g is onto.

Proof :

(1) let $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2))$$

Since g is one-to-one, then $f(x_1) = f(x_2)$

Since f is one-to-one, then $x_1 = x_2$. So $g \circ f$ is one-to-one.

(2) Let $z \in Z$

Since g is onto, then there is $y \in Y$ such that $g(y) = z$

Since f is onto, then there is $x \in X$ such that $f(x) = y$

$$\Rightarrow (g \circ f)(x) = g(f(x)) = g(y) = z. \text{ So } g \circ f \text{ is onto.}$$

(3) since f and g are bijective, then f and g are one-to-one and onto

$$\Rightarrow g \circ f \text{ is one-to-one and onto} \Rightarrow g \circ f \text{ is bijective.}$$

(4) let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$



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Since $g \circ f$ is one-to-one, then $x_1 = x_2$. So f is one-to-one.

(5) Let $z \in Z$

Since $g \circ f$ is onto, then there is $x \in X$ such that $(g \circ f)(x) = z$

Since $(g \circ f)(x) = g(f(x)) \Rightarrow g(f(x)) = z$

Since $f(x) \in Y$. Put $f(x) = y \Rightarrow y \in Y$ such that $g(y) = z$

So g is onto.

Definition(1.17)

Let X and Y be two sets. A function $f : X \rightarrow Y$ is said to be Invertible if $f^{-1} : Y \rightarrow X$ is a function. f^{-1} is called the inverse of f ,

Remark

If $f : X \rightarrow Y$ is an invertible, then $y = f(x)$ iff $x = f^{-1}(y)$.

Theorem(1.18)

Let $f : X \rightarrow Y$ be a function

1. f is an invertible iff it is bijective
2. f is one-to-one iff there is a function $g : Y \rightarrow X$ such that $g \circ f = I_X$
3. If f is an invertible, then f^{-1} is bijective and $f^{-1} \circ f = I_X$, and $f \circ f^{-1} = I_Y$
4. f is an invertible iff there is a function $g : Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$

Proof :

(1) suppose that f is an invertible

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

Put $f(x_1) = f(x_2) = y \Rightarrow x_1 = f^{-1}(y)$ and $x_2 = f^{-1}(y)$

Since f^{-1} is a function, then $x_1 = x_2 \Rightarrow f$ is one-to-one



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Let $y \in Y$

Since $f^{-1}: Y \rightarrow X$ is a function, then there exists $x \in X$ such that $x = f^{-1}(y)$

$\Rightarrow f$ is onto. So f is bijective.

Conversely, suppose that f is bijective

Let $y \in Y$

Since f is onto, then there exists $x \in X$ such that $f(x) = y \Rightarrow x = f^{-1}(y)$.

To prove uniqueness: suppose that $x_1 = f^{-1}(y)$ and $x_2 = f^{-1}(y)$

$\Rightarrow f(x_1) = y$ and $f(x_2) = y \Rightarrow f(x_1) = f(x_2)$

Since f is one-to-one, then $x_1 = x_2$.

(2) Suppose that $f: X \rightarrow Y$ is one to one

Let $Z = \text{ran}(f) \Rightarrow f: X \rightarrow Z$ is a function also onto, so $f: X \rightarrow Z$ is bijective.

$\Rightarrow f^{-1}: Z \rightarrow X$ is a function.

Let $a \in X$. Define $h: Y|Z \rightarrow X$ by $h(x) = a$ for all $x \in Y|Z$

Let $g = f^{-1} \cup h$, then $g: Y \rightarrow X$ is a function.

Let $x \in X$, $y = f(x) \Rightarrow (g \circ f)(x) = g(f(x)) = g(y)$

Since $g(y) = f^{-1}(y) \Rightarrow x = g(y) \Rightarrow (g \circ f)(x) = x = I_X(x)$

Conversely, suppose that there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow I(x_1) = I(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one to one

(3) Let $y_1, y_2 \in Y$ such that $f^{-1}(y_1) = f^{-1}(y_2)$

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Assume that $f^{-1}(y_1) = f^{-1}(y_2) = x \Rightarrow y_1 = f(x)$ and $y_2 = f(x)$

By definition of function, we have $y_1 = y_2$. So f^{-1} is one to one .

Let $x \in X$, by definition of function, there is $y \in Y$ such that $f(x) = y$.

Since f is invertible $\Rightarrow f^{-1}(y) = x \Rightarrow f^{-1}$ is onto function. So f^{-1} is bijective .

Let $x \in X$ and $y = f(x) \Rightarrow f^{-1}(y) = x$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_X(x)$$

$$\Rightarrow (f^{-1} \circ f)(x) = I_X(x) \text{ for all } x \in X \Rightarrow f^{-1} \circ f = I_X$$

Let $y \in Y$ and $x = f^{-1}(y) \Rightarrow y = f(x)$

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y = I_Y(y)$$

$$\Rightarrow (f \circ f^{-1})(y) = I_Y(y) \text{ for all } y \in Y \Rightarrow f \circ f^{-1} = I_Y$$

(4) Suppose that f is an invertible $\Rightarrow f$ is bijective $\Rightarrow f$ is one-to-one , then there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$. It is clear to show that $f \circ g = I_Y$.

Conversely , suppose that exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow I(x_1) = I(x_2) \Rightarrow x_1 = x_2$$

$\Rightarrow f$ is one-to-one

Let $y \in Y \Rightarrow y = I_Y(y) = (f \circ g)(y) = f(g(y))$

If we put $g(y) = x$, then $y = f(x)$, so that f is onto $\Rightarrow f$ is bijective. Hence f is invertible.

Theorem(1.19)

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two Invertible functions, then $g \circ f$ is invertible function and

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$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof :

Since $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are invertible function, then $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are bijective function, so $g \circ f$ is bijective function. Hence $g \circ f$ is invertible function.

Definition (1.20)

Let A be a subset of a set Ω . A function $I_A : \Omega \rightarrow \mathbb{R}$ defined by

$$I_A(x) = \begin{cases} 1 & , \quad x \in A \\ 0 & , \quad x \notin A \end{cases}$$

is called the indicator (or characteristic) of A . Note that: $I_\phi = 0$, $I_\Omega = 1$

Theorem (1.21)

Let A, B be subsets of a set Ω

1. $A = B$ iff $I_A = I_B$
2. $A \subseteq B$ iff $I_A \leq I_B$
3. $I_{A \cap B} = I_A \cdot I_B$
4. $I_{A \cup B} = I_A + I_B - I_{A \cap B}$
5. $I_{A^c} = 1 - I_A$
6. $I_{A \Delta B} = |I_A - I_B|$

Proof :

(1) Suppose $A = B$

$$I_A(x) = \begin{cases} 1 & , \quad x \in A \\ 0 & , \quad x \notin A \end{cases} = \begin{cases} 1 & , \quad x \in B \\ 0 & , \quad x \notin B \end{cases} = I_B(x) \Rightarrow I_A(x) = I_B(x) \text{ for all } x \in \Omega \Rightarrow I_A = I_B$$

Conversely, suppose that $I_A = I_B$



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Let $x \in A \Rightarrow I_A(x)=1$, but $I_A=I_B$ $I_B(x)=1 \Rightarrow x \in B \Rightarrow A \subseteq B$

Similarly to prove $B \subseteq A \Rightarrow A=B$

(3) Let $x \in \Omega$

If $x \in A \cap B \Rightarrow I_{A \cap B}(x)=1$ and $x \in A \cap B \Rightarrow x \in A, x \in B$

$$\Rightarrow I_A(x)=1, I_B(x)=1 \Rightarrow I_A(x) \times I_B(x)=1 \Rightarrow I_{A \cap B}(x)=I_A(x) \times I_B(x)$$

or if $x \notin A \cap B \Rightarrow I_{A \cap B}(x)=0$ and $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

$$\Rightarrow I_A(x)=0 \text{ or } I_B(x)=0 \Rightarrow I_A(x) \times I_B(x)=0 \Rightarrow I_{A \cap B}(x)=I_A(x) \times I_B(x)$$

So $I_{A \cap B}(x)=I_A(x) \times I_B(x)$ for all $x \in \Omega \Rightarrow I_{A \cap B}=I_A \times I_B$

(4) Let $x \in \Omega$

If $x \in A \cup B \Rightarrow I_{A \cup B}(x)=1$ and $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

There are three cases

(a) $x \in A$ and $x \in B \Rightarrow x \in A \cap B \Rightarrow I_A(x)=1, I_B(x)=1, I_{A \cap B}(x)=1$

$$\Rightarrow I_A(x)+I_B(x)-I_{A \cap B}(x)=1+1-1=1$$

(b) $x \in A$ and $x \notin B \Rightarrow x \notin A \cap B \Rightarrow I_A(x)=1, I_B(x)=0, I_{A \cap B}(x)=0$

$$\Rightarrow I_A(x)+I_B(x)-I_{A \cap B}(x)=1+0-0=1$$

(c) $x \notin A$ and $x \in B \Rightarrow x \notin A \cap B \Rightarrow I_A(x)=0, I_B(x)=1, I_{A \cap B}(x)=0$

$$\Rightarrow I_A(x)+I_B(x)-I_{A \cap B}(x)=0+1-0=1 \Rightarrow I_{A \cup B}(x)=I_A(x)+I_B(x)-I_{A \cap B}(x)$$

Now if $x \notin A \cup B \Rightarrow I_{A \cup B}(x)=0$ and $x \notin A \cup B \Rightarrow x \notin A, x \notin B \Rightarrow x \notin A \cap B$

$$I_A(x)=0, I_B(x)=0, I_{A \cap B}(x)=0 \Rightarrow I_A(x)+I_B(x)-I_{A \cap B}(x)=0+0-0=0$$

So $I_{A \cup B}(x)=I_A(x)+I_B(x)-I_{A \cap B}(x)$ for all $x \in \Omega \Rightarrow I_{A \cup B}=I_A+I_B-I_{A \cap B}$

(5) Let $x \in \Omega$

If $x \in A^c \Rightarrow I_{A^c}(x)=1$ and $x \in A^c \Rightarrow x \notin A \Rightarrow I_A(x)=0 \Rightarrow 1-I_A(x)=1-0=1$

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If $x \notin A^c \Rightarrow I_{A^c}(x) = 0$ and $x \in A^c \Rightarrow x \notin A \Rightarrow I_A(x) = 0 \Rightarrow 1 - I_A(x) = 1 - 0 = 1$

So $I_{A^c}(x) = 1 - I_A(x)$ for $x \in \Omega \Rightarrow I_{A^c} = 1 - I_A$

Definition (1.22)

Let Ω be any set and $f : \Omega \rightarrow \mathbb{R}$ a function, we define the positive and negative parts f^+ and f^- by $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\} = \max\{-f, 0\}$, i.e.

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

It follows that f^+ and f^- are non-negative

Theorem (1.23)

Let Ω be any set and $f : \Omega \rightarrow \mathbb{R}$ be a function

1. $f = f^+ - f^-$ and $|f| = f^+ + f^-$
2. $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$
3. $(-f)^+ = f^-$ and $(-f)^- = f^+$
4. If $\lambda > 0$, then $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- = \lambda f^-$

Proof :

$$(1) \quad f^+ - f^- = \max\{f, 0\} - \max\{-f, 0\} = \max\{f, 0\} + \min\{f, 0\} = f + 0 = f$$

Let $x \in \Omega$. If $f(x) \geq 0$, then $f^+(x) = f(x)$ and $f^-(x) = 0$

If $f(x) \leq 0$, then $f^+(x) = 0$ and $f^-(x) = -f(x)$

In any case, $f^+(x) + f^-(x) = |f|(x)$. So $|f| = f^+ + f^-$



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Exercises(1)

Let A, B, C, D be subsets of a set U . Prove or disprove

1. $\phi \subseteq A \cap B \subseteq A \cup B$
2. If $A \cap C = \phi$. Then $A \cap (B \cup C) = A \cap B$
3. If $A \cap (B \cup C) = A \cap B$. Then $C \subseteq B$
4. If $A \cap C = B \cap C$. Then $A = C$
5. If $B \subseteq A$ and $B \subseteq C$. Then $B \subseteq A \cap C$
6. If $A \cup B = B \cap A$. Then $A = B$
7. If $A \cap B = \phi$. Then $A \subseteq B^c$
8. $(A \cap B) \cup (A \cap B^c) = A$
9. If $A | B = \phi$ and $B | A = \phi$. Then $A = B$
10. $A^c | B^c = B | A$
11. $A \cap (B | C) = (A \cap B) | (A \cap C)$
12. $A \cup (B | C) = (A \cup B) | (A \cup C)$
13. $A \cap (A \cap B^c) = \phi$
14. $A \cup B = (A | B) \cup B$
15. If $A \cup B = U$. Then $A^c \subseteq B$
16. $(A \cap B) \cup (A \cap B)^c = A$
17. If $A \subseteq B$. Then $A \cup (B | A) = A$
18. If $A \Delta C = B \Delta C$. Then $A = B$.
19. $A \cap B = (A \cup B) | (A \Delta B)$
20. $A \cup B = (A \Delta B) \Delta (A \cap B)$



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21. $A | B = (A \Delta (A \cap B))$
22. $A \cup B = (A | (A \cap B)) \cup (B | (A \cap B)) \cup (A \cap B)$
23. $A \cup B = (A \Delta B) \Delta (A \cap B)$
24. $A | B = (A \Delta (A \cap B))$
25. $A \Delta B = (A | (A \cap B)) \cup (B | (A \cap B))$
26. $A \cap B = (A \cup B) | (A \Delta B)$
27. Let $A_1, A_2, A_3, \dots$ be subsets of a set Ω . Show that there are subsets $B_1, B_2, B_3, \dots$ of Ω such that $B_n \subseteq A_n$ for all $n = 1, 2, \dots$, $B_n \cap B_m = \emptyset$ for all $n \neq m$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.
28. Show that : The union of a countable family of countable sets is countable set.
29. Let $f : X \rightarrow Y$ be an injective function and let $A \subseteq X$. Show that f_A is injective function.
30. Let $f : X \rightarrow Y$ be a function. Show that if $f \subseteq I_X$ or $I_X \subseteq f$, then $f = I_X$.
31. Let $f : X \rightarrow Y$ be a function and let $A \subseteq X$. Show that $f(A) = \emptyset$ iff $A = \emptyset$.
32. Let $f : X \rightarrow Y$ be a function. Show that f is onto iff $Y | f(A) \subseteq f(X | A)$ for all $A \subseteq X$
33. Let $f : X \rightarrow Y$ be a function and $B \subseteq Y$. Show that $f^{-1}(B) = \emptyset$ iff $B \cap f(X) = \emptyset$
34. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be two functions and let $g \circ f = I_X$.
- Show that f is one-to-one and g is onto.
35. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be two functions and let $g \circ f = I_X$, $f \circ g = I_Y$.
- Show that f and g are invertible functions and $g = f^{-1}$
36. Let $f : X \rightarrow Y$ be a function. Show that $f \circ I_X = f$ and $I_Y \circ f = f$
37. Let $g : Y \rightarrow Z$ and $h : Y \rightarrow Z$ be two functions and $g \circ f = h \circ f$ for every function $f : X \rightarrow Y$. Show that $g = h$.

2. Family of sets

It sometimes will happen that the objects of set are sets themselves; for examples, the set of all subsets of Ω . In order to avoid saying “set of sets”, it is common practice to say “family of sets” or “Class of sets”. We shall denote families of sets by the script letters $A, B, C, D, E, F, G, \dots$. Let A be a nonempty set, each $x \in A$, we find that $x \in \{x\} \subseteq A$, $x \neq \{x\}$. The set, consisting of only one element is called a single set.

Definition(2.1)

A set $\mathcal{F}$ is called a family of sets if each element of $\mathcal{F}$ is also set

Example(2.2)

1. In geometry we usually say “a family of lines” or “a family of Curves” since lines and curves are themselves sets of points.
2. The set $\{\{2,3\}, \{2\}, \{5,6\}\}$ is a family of sets.
3. $\mathcal{F} = \{\{2,3\}, \{2\}, 3, \{2,5\}\}$ is not family of sets

Remark

Let $\mathcal{F}$ be a nonempty family $\mathcal{F}$ of subsets of a set Ω . Usually, we also use the term “space” to mean $(\Omega, \mathcal{F})$. We say $(\Omega, \mathcal{F})$ is said be finite, countable, or uncountable if Ω is finite, countable, or uncountable, respectively.

Definition(2.3)

The family of all the subsets of any set Ω is called the power set of Ω . We denote the power set of Ω by $\mathcal{P}(\Omega)$. i.e. $\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}$

Example(2.4)

1. If $\Omega = \{a\}$, then $\mathcal{P}(\Omega) = \{\emptyset, \Omega\}$, note that $\#(\mathcal{P}(\Omega)) = 2$

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2. If $\Omega = \{a, b\}$, then $\mathcal{P}(\Omega) = \{\emptyset, \{a\}, \{b\}, \Omega\}$, note that $\#(\mathcal{P}(\Omega)) = 4 = 2^2$

3. If $\Omega = \{a, b, c\}$, then $\mathcal{P}(\Omega) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \Omega\}$, note that $\#(\mathcal{P}(\Omega)) = 8 = 2^3$

In general we can prove the following

Theorem(2.5)

Let Ω be any set . if $\#(\Omega) = n$, then $\#(\mathcal{P}(\Omega)) = 2^n$

Proof :

We shall prove by induction

If $n = 0$, then $\#(\Omega) = 0 \Rightarrow \Omega = \emptyset \Rightarrow \mathcal{P}(\Omega) = \{\Omega\}$

$\Rightarrow \#(\mathcal{P}(\Omega)) = 1 = 2^0$, i.e. the statement is true when $n = 0$

If $n = 1$, then $\#(\Omega) = 1 \Rightarrow \Omega = \{a\} \Rightarrow \mathcal{P}(\Omega) = \{\emptyset, \Omega\}$

$\Rightarrow \#(\mathcal{P}(\Omega)) = 2 = 2^1$, i.e. the statement is true when $n = 1$

Now suppose statement is true when $n = m$. To prove the statement is true when $n = m + 1$

i.e. we assume $\#(\Omega) = m \rightarrow \#(\mathcal{P}(\Omega)) = 2^m$. To prove $\#(\Omega) = m + 1 \rightarrow \#(\mathcal{P}(\Omega)) = 2^{m+1}$

Assume $\#(\Omega) = m + 1 \Rightarrow \Omega = \{x_1, x_2, \dots, x_m, x_{m+1}\}$

Let $\Omega' = \{x_1, x_2, \dots, x_m\} \Rightarrow \Omega' \subset \Omega$, Since $\#(\Omega') = m \Rightarrow \#(\mathcal{P}(\Omega')) = 2^m$

Let $\Omega'' \subseteq \Omega$

(1) If $x_{m+1} \notin \Omega'' \Rightarrow \Omega'' \subseteq \Omega'$

(2) If $x_{m+1} \in \Omega'' \Rightarrow \Omega'' = \Omega''' \cup \{x_{m+1}\}$, where $\Omega''' \subseteq \Omega'$

$\Omega'' \subseteq \Omega$ implies $\Omega'' \subseteq \Omega'$ or $\Omega'' = \Omega''' \cup \{x_{m+1}\}$, where $\Omega''' \subseteq \Omega'$

$\Rightarrow \#(\mathcal{P}(\Omega)) = 2^m + 2^m = 2 \times 2^m = 2^{m+1}$, i.e. i.e. the statement is true when $n = m + 1$

$\Rightarrow$ The statement is true for all natural numbers.



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Theorem(2.6)

Let Ω, Ω' be two sets.

1. $\Omega' \subseteq \Omega$ iff $\mathcal{P}(\Omega) \subseteq \mathcal{P}(\Omega')$
2. If $\Omega \cap \Omega' = \emptyset$, then $\mathcal{P}(\Omega) \cap \mathcal{P}(\Omega') = \{\emptyset\}$
3. $\mathcal{P}(\Omega) \cup \mathcal{P}(\Omega') \subseteq \mathcal{P}(\Omega \cup \Omega')$
4. $\mathcal{P}(\Omega \cap \Omega') = \mathcal{P}(\Omega) \cap \mathcal{P}(\Omega')$

Proof :

(1) Assume $\Omega' \subseteq \Omega$

Let $A \in \mathcal{P}(\Omega') \Rightarrow A \subseteq \Omega'$

But $\Omega' \subseteq \Omega \Rightarrow A \subseteq \Omega \Rightarrow A \in \mathcal{P}(\Omega) \Rightarrow \mathcal{P}(\Omega') \subseteq \mathcal{P}(\Omega)$

Conversely, assume $\mathcal{P}(\Omega') \subseteq \mathcal{P}(\Omega)$

Let $x \in \Omega' \Rightarrow \{x\} \subseteq \Omega' \Rightarrow \mathcal{P}(\{x\}) \in \mathcal{P}(\Omega')$

But $\mathcal{P}(\Omega') \subseteq \mathcal{P}(\Omega) \Rightarrow \{x\} \in \mathcal{P}(\Omega) \Rightarrow \{x\} \subseteq \Omega \Rightarrow x \in \Omega \Rightarrow \Omega' \subseteq \Omega$

(3) $A \in \mathcal{P}(\Omega) \cup \mathcal{P}(\Omega') \rightarrow A \in \mathcal{P}(\Omega) \vee A \in \mathcal{P}(\Omega') \rightarrow A \subseteq \Omega \vee A \subseteq \Omega' \rightarrow A \subseteq \Omega \cup \Omega' \rightarrow A \in \mathcal{P}(\Omega \cup \Omega')$
 $\Rightarrow \mathcal{P}(\Omega) \cup \mathcal{P}(\Omega') \subseteq \mathcal{P}(\Omega \cup \Omega')$

(4) $A \in \mathcal{P}(\Omega \cap \Omega') \leftrightarrow A \subseteq \Omega \cap \Omega' \leftrightarrow A \subseteq \Omega \wedge A \subseteq \Omega' \leftrightarrow A \in \mathcal{P}(\Omega) \wedge A \in \mathcal{P}(\Omega') \leftrightarrow A \in \mathcal{P}(\Omega) \cap \mathcal{P}(\Omega')$
 $\Rightarrow \mathcal{P}(\Omega \cap \Omega') = \mathcal{P}(\Omega) \cap \mathcal{P}(\Omega')$

Definition(2.7)

Let Λ be any set. The collection of sets such that to each member of Λ , there corresponds a member of the collection of the sets, is called an indexed family of sets. The set Λ is called an index set. More concisely, we write $\mathcal{F} = \{A_\lambda : \lambda \in \Lambda\}$ or $\mathcal{F} = \{A_\lambda\}_{\lambda \in \Lambda}$ Which reads “ $\mathcal{F}$ is a

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set of all sets A_λ such that $\lambda \in \Lambda$ ". Here $\mathcal{F}$ is the indexed family of sets, the sets A_λ are the indexed sets, $\lambda \in \Lambda$ is an index and Λ is the index set.

Definition(2.8)

Let $\mathcal{F} = \{A_\lambda\}_{\lambda \in \Lambda}$ be a Family of subsets of a set Ω . The union of the family of sets is a set such that an object belongs to it iff there exists some $\lambda \in \Lambda$ such that the object in question belongs to A_λ . Thus the union of the sets $\{A_\lambda : \lambda \in \Lambda\}$ denoted by $\bigcup_{\lambda \in \Lambda} A_\lambda$ or

$\bigcup \{A_\lambda : \lambda \in \Lambda\}$ is defined by $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda, \text{ for some } \lambda \in \Lambda\}$

The intersection of the family of sets is a set such that an object belongs to it iff it belongs to all the sets of the family. This intersection is denoted by $\bigcap_{\lambda \in \Lambda} A_\lambda$ or

$\bigcap \{A_\lambda : \lambda \in \Lambda\}$ and is defined by $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda, \text{ for all } \lambda \in \Lambda\}$

Example(2.9)

Let $\Omega = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$, $\Lambda = \mathbb{N}$ and $A_\lambda = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{\lambda}\}$, then

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{\lambda}, \frac{1}{\lambda+1}, \dots\}, \quad \bigcap_{\lambda \in \Lambda} A_\lambda = \{1\}$$

Remark

If $\Lambda = \emptyset$, then $\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset$ and $\bigcap_{\lambda \in \Lambda} A_\lambda = \Omega$

To prove this, Let x be an arbitrary element of Ω , i.e. $x \in \Omega$

1. since $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$, if $x \in A_\lambda$ for some $\lambda \in \Lambda$ but this is impossible because there is no $\lambda \in \Lambda$.

Hence $x \notin \bigcup_{\lambda \in \Lambda} A_\lambda$. Since x was an arbitrary element of Ω , we have $\bigcup_{\lambda \in \Lambda} A_\lambda = \emptyset$

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2. since $x \in \bigcap_{\lambda \in \Lambda} A_\lambda$, if $x \in A_\lambda$ for all $\lambda \in \Lambda$. But there are no element in Λ because $\Lambda = \emptyset$.

Hence the requirement that x should belong to $x \in A_\lambda$ for all $\lambda \in \Lambda$ is fulfilled.

Consequently $x \in \bigcap_{\lambda \in \Lambda} A_\lambda$. Therefore $\Omega \subseteq \bigcap_{\lambda \in \Lambda} A_\lambda$

But every set is a subset of Ω so that $\bigcap_{\lambda \in \Lambda} A_\lambda \subseteq \Omega$. It follows that $\bigcap_{\lambda \in \Lambda} A_\lambda = \Omega$

Theorem(2.10)

Let $\mathcal{F} = \{A_\lambda\}_{\lambda \in \Lambda}$ be a Family of subsets of a set Ω and let $A \subseteq \Omega$.

1. If $A_\lambda \subseteq A$ for all $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} A_\lambda \subseteq A$

2. If $A_\lambda \subseteq A$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} A_\lambda \subseteq A$

3. $A \cup (\bigcap_{\lambda \in \Lambda} A_\lambda) = \bigcap_{\lambda \in \Lambda} \{A \cup A_\lambda\}$

4. $A \cap (\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} (A \cap A_\lambda)$

5. $(\bigcup_{\lambda \in \Lambda} A_\lambda)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$

6. $(\bigcap_{\lambda \in \Lambda} A_\lambda)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$

7. $(\bigcup_{\lambda \in \Lambda} A_\lambda) \cap (\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{(\lambda, \alpha) \in \Lambda \times \Gamma} (A_\lambda \cap B_\alpha)$

8. $(\bigcap_{\lambda \in \Lambda} A_\lambda) \cup (\bigcap_{\alpha \in \Gamma} B_\alpha) = \bigcap_{(\lambda, \alpha) \in \Lambda \times \Gamma} (A_\lambda \cup B_\alpha)$

Proof:

(1) Let $x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow x \in A_\lambda$ for some $\lambda \in \Lambda$

Since $A_\lambda \subseteq A$ for all $\lambda \in \Lambda \Rightarrow x \in A$, so $\bigcup_{\lambda \in \Lambda} A_\lambda \subseteq A$

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$$(5) \quad x \in \left(\bigcup_{\lambda \in \Lambda} A_{\lambda} \right)^c \leftrightarrow x \notin \bigcup_{\lambda \in \Lambda} A_{\lambda} \leftrightarrow x \notin A_{\lambda}, \quad \forall \lambda \in \Lambda \leftrightarrow x \in A_{\lambda}^c, \quad \forall \lambda \in \Lambda \leftrightarrow x \in \bigcap_{\lambda \in \Lambda} A_{\lambda}^c$$

$$\text{So } \left(\bigcup_{\lambda \in \Lambda} A_{\lambda} \right)^c = \bigcap_{\lambda \in \Lambda} A_{\lambda}^c$$

$$(7) \quad x \in \left(\bigcup_{\lambda \in \Lambda} A_{\lambda} \right) \cap \left(\bigcup_{\alpha \in \Gamma} B_{\alpha} \right) \leftrightarrow x \in \bigcup_{\lambda \in \Lambda} A_{\lambda} \wedge x \in \bigcup_{\alpha \in \Gamma} B_{\alpha} \leftrightarrow \exists \lambda \in \Lambda \ni x \in A_{\lambda} \wedge \exists \alpha \in \Gamma \ni x \in B_{\alpha}$$

$$\leftrightarrow \exists (\lambda, \alpha) \in \Lambda \times \Gamma \ni x \in A_{\lambda} \cap B_{\alpha} \leftrightarrow x \in \bigcup_{(\lambda, \alpha) \in \Lambda \times \Gamma} (A_{\lambda} \cap B_{\alpha}). \text{ So } \left(\bigcup_{\lambda \in \Lambda} A_{\lambda} \right) \cap \left(\bigcup_{\alpha \in \Gamma} B_{\alpha} \right) = \bigcup_{(\lambda, \alpha) \in \Lambda \times \Gamma} (A_{\lambda} \cap B_{\alpha})$$

Definition(2.11)

1. A disjoint family $\mathcal{F}$ of sets such that any two distinct sets of $\mathcal{F}$ are disjoint.
2. The union of a disjoint family is sometimes called a disjoint union.

Theorem (2.12)

Let $A_1, A_2, A_3, \dots$ be subsets of a set Ω , then there are subsets $B_1, B_2, B_3, \dots$ of Ω such that

$$B_n \subseteq A_n \text{ for all } n = 1, 2, \dots, \quad B_n \cap B_m = \emptyset \text{ for all } n \neq m \text{ and } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Proof :

$$\text{Define } B_1 = A_1, \quad B_2 = A_2 \setminus A_1, \quad B_3 = A_3 \setminus (A_1 \cup A_2), \quad \dots, \quad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k, \quad n = 2, 3, \dots$$

Then $B_n \subseteq A_n$ for all $n = 1, 2, \dots$, $B_n \cap B_m = \emptyset$ for all $n \neq m$.

$$\text{It is clear to show that } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

Theorem(2.13)

The union of a countable family of countable sets is countable set

Proof :

By the process of above theorem we can replace of countable union by a countable disjoint union of sets which are subsets of those in the original family, each of which is therefore

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countable. Each countable set can be enumerated as a finite or infinite sequence.

So we have $A = \bigcup_{n=1}^{\infty} A_n$ a disjoint union, $A_i = \{x_{ij}\}$, $j=1,2,\dots$ where the infinite union be a finite one and some (or all) of the sequences $\{x_{ij}\}$ may be finite. Put $B_n = \{x_{ij} : i+j=n\}$, then B_n is a finite set containing at most $(n+1)$ elements. The sets B_n are disjoint, and A can be enumerated by first enumerating B_1 , then B_2 , and so on.

If $B_n = \emptyset$ for $n \geq N$ then A is finite ; otherwise it is enumerable.

Definition(2.14)

A Family $\mathcal{F} = \{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of a set Ω is called a Partition of Ω if

1. $A_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$
2. $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha \neq \beta$
3. $\bigcup_{\lambda \in \Lambda} A_\lambda = \Omega$

Example(2.15)

1. $\mathcal{F} = \{\mathbb{Z}_0, \mathbb{Z}_e\}$ is a Partition of $\mathbb{Z}$
2. Let $\Omega = \{1,2,3,4,5\}$

(i) If $A_1 = \{1\}, A_2 = \{2,3\}, A_3 = \{4,5\}$, then $\mathcal{F} = \{A_1, A_2, A_3\}$ is a Partition of Ω

(ii) If $A_1 = \{1,2\}, A_2 = \{2\}, A_3 = \{3,4,5\}$, then $\mathcal{F} = \{A_1, A_2, A_3\}$ is not a Partition of Ω

Definition(2.16)

Let $\mathcal{F}$ be a Family of subsets of a set Ω . A set $A \in \mathcal{F}$ is called the unit of $\mathcal{F}$, if $A \cap B = B$ for every $B \in \mathcal{F}$.

Cleary A is unique. Thus the unit of $\mathcal{F}$ is just the maximum set of $\mathcal{F}$, i.e. the set containing all other sets of $\mathcal{F}$.

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Definition(2.17)

Let $\{A_n\}$ be a sequence of subsets of a set Ω . The set of all those points of Ω that belong to A_n for infinitely many values of n is called the upper limit (or limit superior) of $\{A_n\}$ and denoted by $\limsup_{n \rightarrow \infty} A_n$ or $\overline{\lim}_{n \rightarrow \infty} A_n$ or A^* and defined by

$$A^* = \overline{\lim}_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n = \{x \in A_n : \text{for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k$$

Thus $x \in A^*$ iff for all n , then $x \in A_k$ for some $k \geq n$

The lower limit (or limit inferior) of $\{A_n\}$, denoted by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and defined by

$$A_* = \liminf_{n \rightarrow \infty} A_n = \{x \in A_n : \text{for all but finitely many } n\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k$$

Thus $x \in A_*$ iff for some n , then $x \in A_k$ for all $k \geq n$

Example(2.18)

1. If $A_n = \begin{cases} A, & z \in \mathbb{Z}_e \\ B, & n \in \mathbb{Z}_o \end{cases}$, then $\limsup_{n \rightarrow \infty} A_n = A \cup B$ and $\liminf_{n \rightarrow \infty} A_n = A \cap B$

2. If $\Omega = \mathbb{R}$ and $A_n = \begin{cases} (-\frac{1}{n}, 1], & n \in \mathbb{Z}_o \\ (-1, \frac{1}{n}], & n \in \mathbb{Z}_e \end{cases}$, then $\limsup_{n \rightarrow \infty} A_n = (-1, 1]$ and $\liminf_{n \rightarrow \infty} A_n = \{0\}$

Solution :

$$(1) \bigcup_{k=n}^{\infty} A_k = A \cup B, \quad \bigcap_{k=n}^{\infty} A_k = A \cap B,$$

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right) = A \cup B, \quad \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right) = A \cap B \Rightarrow \limsup_{n \rightarrow \infty} A_n = A \cup B, \quad \liminf_{n \rightarrow \infty} A_n = A \cap B$$

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Theorem(2.19)

Let $\{A_n\}$ be a sequence of subsets of a set Ω .

1. $(\limsup_{n \rightarrow \infty} A_n)^c = \liminf_{n \rightarrow \infty} A_n^c$
2. $(\liminf_{n \rightarrow \infty} A_n)^c = \limsup_{n \rightarrow \infty} A_n^c$
3. $\liminf_{n \rightarrow \infty} A_n \subseteq \limsup_{n \rightarrow \infty} A_n$

Proof :

(1) Since $\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, then

$$(\limsup_{n \rightarrow \infty} A_n)^c = \left(\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right) \right)^c = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k^c \right) = \liminf_{n \rightarrow \infty} A_n^c$$

(3) Since $\bigcap_{k=n}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k$ for all $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k \subseteq \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k$, so $\liminf_{n \rightarrow \infty} A_n \subseteq \limsup_{n \rightarrow \infty} A_n$

Definition(2.20)

A sequence $\{A_n\}$ of subsets of a set Ω is said to converge if $\limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = A$ (say),

and A is said to be the limit of $\{A_n\}$, we write $A = \lim_{n \rightarrow \infty} A_n$ or $A_n \rightarrow A$

Example(2.21)

Let $\Omega = \{1, 2, 3, \dots\}$. Define $A_n = \{n\}$ for all $n = 1, 2, 3, \dots$. Then we have

$$\limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = \emptyset$$

Hence $\lim_{n \rightarrow \infty} A_n = \emptyset$

Definition(2.22)

A sequence $\{A_n\}$ of subsets of a set Ω is said to be

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1. Increasing if $A_n \subset A_{n+1}$ for $n = 1, 2, \dots$.
2. Decreasing if $A_{n+1} \subset A_n$ for $n = 1, 2, \dots$.
3. A monotone sequence of sets is one which either increasing or decreasing.

Theorem(2.23)

Any monotone sequence is converge. But the converse is not true.

Proof:

Let $\{A_n\}$ be a monotone sequence of subsets of a set Ω .

If $\{A_n\}$ is an increasing $\Rightarrow A_n \subset A_{n+1}$ for $n = 1, 2, \dots$

$$\Rightarrow \bigcup_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n \text{ and } \bigcap_{k=n}^{\infty} A_k = A_n \text{ for } n = 1, 2, \dots$$

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right) = \bigcap_{n=1}^{\infty} \left(\bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} A_n, \quad \liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right) = \bigcup_{n=1}^{\infty} A_n$$

$$\text{Thus } \limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

Therefore the sequence $\{A_n\}$ is converge, while, if $\{A_n\}$ is decreasing

$$\Rightarrow A_{n+1} \subset A_n \text{ for } n = 1, 2, \dots \Rightarrow \bigcup_{k=n}^{\infty} A_k = A_n \text{ and } \bigcap_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n \text{ for } n = 1, 2, \dots$$

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right) = \bigcap_{n=1}^{\infty} \left(\bigcup_{n=1}^{\infty} A_n \right) = \bigcap_{n=1}^{\infty} A_n, \quad \liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right) = \bigcap_{n=1}^{\infty} A_n$$

$$\text{Thus } \limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Therefore the sequence $\{A_n\}$ is converge

• Example

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Let $\Omega = \mathbb{R}$ and $A_n = \begin{cases} (0, 1 - \frac{1}{n}], & n \in \mathbb{Z}_o \\ [\frac{1}{n}, 1), & n \in \mathbb{Z}_e \end{cases}$, then the $\{A_n\}$ is converge but not monotone.

Definition(2.24)

Let A be a subset of a set Ω . A sequence $\{A_n\}$ of subsets of Ω is said to be

1. Increase to A , " we write $A_n \uparrow A$ ", if $\{A_n\}$ is an increasing and $\bigcup_{n=1}^{\infty} A_n = A$
2. Decrease to A , " we write $A_n \downarrow A$ ", if $\{A_n\}$ is a decreasing and $\bigcap_{n=1}^{\infty} A_n = A$

Example(2.25)

Let $\Omega = \mathbb{R}$. Define

1. $A_n = [\frac{1}{n}, 1]$ for all $n = 1, 2, \dots$. Then $\{A_n\}$ is an increasing and $\bigcup_{n=1}^{\infty} A_n = (0, 1]$. Hence $A_n \uparrow A$
2. $B_n = (-1 + \frac{1}{n}, 1 + \frac{1}{n})$ for all $n = 1, 2, \dots$. Then $\{B_n\}$ is a decreasing and $\bigcap_{n=1}^{\infty} B_n = [-1, 1]$. Hence $B_n \downarrow B$

Theorem(2.26)

Let $\{A_n\}$ be a sequence of subsets of a set Ω and let $A \subset \Omega$

1. If $A_n \uparrow A$, then $A_n^c \downarrow A^c$
2. If $A_n \downarrow A$, then $A_n^c \uparrow A^c$

Proof :

(1) Since $A_n \uparrow A \Rightarrow A_n \subset A_{n+1}$ for $n = 1, 2, \dots$ and $\bigcup_{n=1}^{\infty} A_n = A$

$$\Rightarrow A_{n+1}^c \subset A_n^c \text{ for } n = 1, 2, \dots \text{ and } \bigcap_{n=1}^{\infty} A_n^c = (\bigcup_{n=1}^{\infty} A_n)^c = A^c \Rightarrow A_n^c \downarrow A^c$$

Definition(2.27)

A nonempty family $\mathcal{F}$ of subsets of a set Ω is called a monotone family if for every monotone sequence $\{A_n\}$ of sets in $\mathcal{F}$, we have $\lim_{n \rightarrow \infty} A_n \in \mathcal{F}$

- Let $\mathcal{G}$ be family of subsets of a set Ω . The smallest monotone family $\mathcal{F}$ containing $\mathcal{G}$ is called the monotone family generated by $\mathcal{G}$, and it is denoted by $\mathcal{M}(\mathcal{G})$.

Example(2.28)

It is clear to show that

1. The family of all subsets of a set Ω is a monotone family.
2. The intersection of any collection of monotone families is a monotone family.

Exercises (2)

1. Let $\Omega = \{a, b\}$. Define $A_n = \begin{cases} \{a\}, & n \in \mathbb{Z}_e \\ \{b\}, & n \in \mathbb{Z}_0 \end{cases}$. Show that $\limsup_{n \rightarrow \infty} A_n = \Omega$ and $\liminf_{n \rightarrow \infty} A_n = \phi$
2. Let $\Omega = \mathbb{R}$. Define $A_1 = [0, 1), A_2 = [0, \frac{1}{2}), A_3 = [\frac{1}{2}, 1), A_4 = [0, \frac{1}{4}), A_5 = [\frac{1}{4}, \frac{1}{2}), A_6 = [\frac{1}{2}, \frac{3}{4}), A_7 = [\frac{3}{4}, 1), A_8 = [0, \frac{1}{8}), \dots$
Show that $\limsup_{n \rightarrow \infty} A_n = [0, 1)$ and $\liminf_{n \rightarrow \infty} A_n = \phi$

3. Show that any sequence of disjoint sets converges to ϕ .
4. Let $\{A_n\}$ be a sequence of subsets of a sets Ω . Show that $\{A_n\}$ is a convergent iff there is no point $x \in \Omega$ such that each $x \in A_n$, $x \in \Omega \mid A_n$ holds for infinitely many.

5. Let $\{A_n\}$ be a sequence of subsets of a sets Ω . Prove that

$$(a) A \mid (\limsup_{n \rightarrow \infty} A_n) = \liminf_{n \rightarrow \infty} (A \mid A_n) \quad (b) A \mid (\liminf_{n \rightarrow \infty} A_n) = \limsup_{n \rightarrow \infty} (A \mid A_n)$$

6. If $\{A_n\}$ is a sequence of a sets and $B_1 = A_1$, $B_n = B_{n-1} \Delta A_n$ for $n = 1, 2, \dots$. Show that the sequence $\{B_n\}$ convergent iff $\lim_{n \rightarrow \infty} A_n = \phi$.

3. Some Special Types of Family of Sets

In set theory and related branches of mathematics, a collection $\mathcal{F}$ of subsets of a given set Ω is called a family of subsets of Ω , or a family of sets over Ω . More generally, a collection of any sets whatsoever is called a family of sets or a set-family or a set-system.

Definition(3.1)

A nonempty family $\mathcal{R}$ of subsets of a set Ω is called

1. A **semiring** on Ω if

I. If $A, B \in \mathcal{R}$, then $A \setminus B = \bigcup_{i=1}^n A_i$, where A_i are disjoint sets in $\mathcal{R}$.

II. If $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$

Note that

If $n = 0$ it is understood that the corresponding union is equal to $\emptyset$, (in which case $A \subseteq B$)

2. A **ring** on Ω if,

I. If $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$

II. If $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$,

3. A **σ -ring** on a set Ω if,

I. A If $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$

II. If $A_n \in \mathcal{R}$, $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$

Example(3.2)

1. The family of all singletons of set Ω and the empty set is a semiring.



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2. The set $\mathcal{R}$ of all open intervals (a,b) , closed of intervals $[a,b]$ and half open intervals $[a,b), (a,b]$, including the empty interval $(a,a) = \emptyset$ and the single-element set $[a,a] = \{a\}$, is a semiring on $\mathbb{R}$

Example(3. 3)

1. The family $\mathcal{R} = \{\emptyset, \Omega\}$ is a ring on Ω .
2. The family $\mathcal{R}$ of all subsets of a set Ω is a ring on Ω .
3. The family $\mathcal{R}$ of all finite subsets of $\mathbb{R}$ is a ring on $\mathbb{R}$.
4. The family $\mathcal{R}$ of all bounded subsets of $\mathbb{R}$ is a ring on $\mathbb{R}$.
5. The family $\mathcal{R}$ of all countable subsets of a set Ω is a σ -ring on Ω .

Theorem(3.4)

1. The empty set belongs to any semiring.
2. Let $\mathcal{R}$ be any ring on a set Ω , then
 - a. If $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$ and $A \Delta B \in \mathcal{R}$
 - b. If $A_1, A_2, \dots, A_n \in \mathcal{R}$, then $\bigcup_{i=1}^n A_i \in \mathcal{R}$ and $\bigcap_{i=1}^n A_i \in \mathcal{R}$
3. Every ring is a semiring but the converse not true
4. Every σ -ring is a ring but the converse not true
5. Let $\mathcal{R}$ be any σ -ring on a set Ω . If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$
6. A nonempty family that is closed under the formation of intersection, proper difference. and disjoint unions is a ring.

Proof :

1. Let $\mathcal{R}$ be a semiring on a set Ω , then $\mathcal{R} \neq \emptyset \Rightarrow \exists A \in \mathcal{R}$

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$\Rightarrow A | A = \bigcup_{i=1}^n A_i$, where A_i are disjoint sets in $\mathcal{R}$.

Since $A | A = \phi$, then $A_i = \phi$, since $A_i \in \mathcal{R}$, then $\phi \in \mathcal{R}$

2. a. Since $A, B \in \mathcal{R}$, then $A | (A | B) \in \mathcal{R}$ and $(A | B) \cup (B | A) \in \mathcal{R}$

since $A \cap B = A | (A | B)$ and $A \Delta B = (A | B) \cup (B | A)$, then $A \cap B \in \mathcal{R}$ and $A \Delta B \in \mathcal{R}$

b. An application of mathematical induction and the associative laws for unions and

intersection show that if $A_1, A_2, \dots, A_n \in \mathcal{R}$, then $\bigcup_{i=1}^n A_i \in \mathcal{R}$ and $\bigcap_{i=1}^n A_i \in \mathcal{R}$

3. Let $\mathcal{R}$ be a ring on a set Ω and if $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$

Now : Let $A, B \in \mathcal{R}$, then $A | B \in \mathcal{R}$ because $\mathcal{R}$ is a ring

Take $n=1$ and $A_1 = A | B \Rightarrow A_1 \in \mathcal{R} \Rightarrow A | B = \bigcup_{i=1}^n A_i \Rightarrow \mathcal{R}$ be a semiring on Ω .

The converse : In example (3.2), $\mathcal{R}$ is a semiring on $\mathbb{R}$ but not a ring for it is not closed under the operation of difference.

5. Let $\mathcal{R}$ be any σ -ring on a set Ω . If $A_n \in \mathcal{R}$, $n=1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$

Let $A = \bigcup_{n=1}^{\infty} A_n \Rightarrow A \in \mathcal{R} \Rightarrow A | (\bigcup_{n=1}^{\infty} (A | A_n)) A \in \mathcal{R}$

Since $\bigcap_{n=1}^{\infty} A_n = A | (\bigcup_{n=1}^{\infty} (A | A_n)) A \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$

6. Let $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$, $A | B \in \mathcal{R}$ and $B | A \in \mathcal{R}$

Since $A | B$ and $B | A$ are disjoint sets, then $(A | B) \cup (B | A) \in \mathcal{R}$

Since $A \Delta B = (A | B) \cup (B | A) \Rightarrow A \Delta B \in \mathcal{R} \Rightarrow (A \Delta B) \Delta (A \cap B) \in \mathcal{R}$ and $A \Delta (A \cap B) \in \mathcal{R}$

Since $A \cup B = (A \Delta B) \Delta (A \cap B)$ and $A | B = (A \Delta (A \cap B))$, then $A \cup B \in \mathcal{R}$ and $A | B \in \mathcal{R}$

Hence $\mathcal{R}$ is a ring.

Remark

The intersection of a collection of semirings need not be semiring

Example(3.5)

Let $\Omega = \{a, b, c\}$, $\mathcal{R}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \Omega\}$ and $\mathcal{R}_2 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$, then $\mathcal{R}_1$ and $\mathcal{R}_2$ are semiring on Ω but that $\mathcal{R}_1 \cap \mathcal{R}_2$ fails to be a semiring on Ω .

Theorem(3.6)

If $\{\mathcal{R}_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of ring (σ -ring) on a set Ω with $\Lambda \neq \emptyset$, then $\mathcal{R} = \bigcap_{\lambda \in \Lambda} \mathcal{R}_\lambda$ is a ring (σ -ring) on Ω .

Theorem(3.7)

1. If $\mathcal{G}$ is any family of subsets of a set Ω there is a unique ring $\mathcal{R}_0$ such that $\mathcal{R}_0$ containing $\mathcal{G}$ and such that, if $\mathcal{H}$ is any ring containing $\mathcal{G}$ we must have $\mathcal{R}_0 \subset \mathcal{H}$.
2. If $\mathcal{G}$ is any family of subsets of a set Ω there is a unique σ -ring $\mathcal{R}$ such that $\mathcal{R}$ containing $\mathcal{G}$ and such that, if $\mathcal{H}$ is any σ -ring containing $\mathcal{G}$ we must have $\mathcal{R} \subset \mathcal{H}$.

Proof:

1. Since The family of all subsets of Ω is a ring

It is clear at least one ring containing $\mathcal{G}$ always exists.

Since moreover the intersection of any collection of rings is also a ring, the intersection of all rings containing $\mathcal{G}$ is easily seen to be the desired ring $\mathcal{R}_0$.

Remarks

Let $\mathcal{G}$ be a family of subsets of a set Ω



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1. The ring $\mathcal{R}_0$, the smallest ring containing $\mathcal{G}$, is called the ring generated by $\mathcal{G}$; it will be denoted by $\mathcal{R}(\mathcal{G})$
2. The σ -ring $\mathcal{R}_0$, the smallest σ -ring containing $\mathcal{G}$, is called the σ -ring generated by $\mathcal{G}$; it will be denoted by $\sigma_{\mathcal{R}}(\mathcal{G})$
3. $\mathcal{R}(\mathcal{G})$ = intersection of all ring of subsets, which contain $\mathcal{G}$. and $\sigma_{\mathcal{R}}(\mathcal{G})$ = intersection of all σ -ring of subsets, which contain $\mathcal{G}$.
4. $\mathcal{R}(\mathcal{G}) = \mathcal{G}$ iff $\mathcal{G}$ is a ring on Ω , and $\sigma_{\mathcal{R}}(\mathcal{G}) = \mathcal{G}$ iff $\mathcal{G}$ is a σ -ring on Ω .
5. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{R}(\mathcal{G}_1) \subseteq \mathcal{R}(\mathcal{G}_2)$ and $\sigma_{\mathcal{R}}(\mathcal{G}_1) \subseteq \sigma_{\mathcal{R}}(\mathcal{G}_2)$, where $\mathcal{G}_1, \mathcal{G}_2$ are two families of subsets of a set Ω .

Example(3.8)

Let Ω be an infinite set. If is $\mathcal{G}$ the family of all singletons, then $\mathcal{R}(\mathcal{G})$ is the family of all finite sets, and $\sigma_{\mathcal{R}}(\mathcal{G})$ is the family of all countable sets.

Theorem(3.9)

1. If $\mathcal{G}$ is any family of subsets of a set Ω then every set in $\mathcal{R}(\mathcal{G})$ may be covered by a finite union of sets in $\mathcal{G}$.
2. If $\mathcal{G}$ is a countable family of subsets of a set Ω then $\mathcal{R}(\mathcal{G})$ is countable.
3. If $\mathcal{G}$ is any family of subsets of a set Ω and let $A \in \sigma_{\mathcal{R}}(\mathcal{G})$, then there exists a countable subfamily $\mathcal{F}$ of $\mathcal{G}$ such that $A \in \sigma_{\mathcal{F}}(\mathcal{G})$.

Proof :

1. The family of all sets which may be covered by a finite union of sets in $\mathcal{G}$

Since the ring contains $\mathcal{G}$, it also contains $\mathcal{R}(\mathcal{G})$.

2. For any family $\mathcal{F}$ of sets, we write $\mathcal{F}^*$ for the family of all finite unions of differences

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of sets of $\mathcal{F}$. It is clear that if $\mathcal{F}$ is countable, then so is $\mathcal{F}^*$, and if $\phi \in \mathcal{F}$, then $\mathcal{F} \subset \mathcal{F}^*$.

To prove the theorem we assume, as we may without may loss of generality, that $\phi \in \mathcal{G}$ and we write $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_n = \mathcal{G}_{n-1}^*$, $n = 1, 2, \dots$

It is clear that $\mathcal{G} \subset \bigcup_{n=0}^{\infty} \mathcal{G}_n \subset \mathcal{R}(\mathcal{G})$ and that the family $\bigcup_{n=0}^{\infty} \mathcal{G}_n$ is countable; we shall complete the

prove by showing that $\bigcup_{n=0}^{\infty} \mathcal{G}_n$ is a ring.

Since $\mathcal{G} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$ it follows that if A and B are any two sets in $\bigcup_{n=0}^{\infty} \mathcal{G}_n$, then there exists

a positive integer n such that both A and B are belong to $\mathcal{G}_n$, we have $A|B \in \mathcal{G}_{n+1}$

Since $\phi \in \mathcal{G} \subset \mathcal{G}_n$, it follows that $A \cup B = (A|B) \cup (B|\phi) \in \mathcal{G}_{n+1}$

We have prove that both $A|B$ and $A \cup B$ belong to $\bigcup_{n=0}^{\infty} \mathcal{G}_n$, i.e. $\bigcup_{n=0}^{\infty} \mathcal{G}_n$ is a ring.

Definition(3.10)

Let $\mathcal{G}$ be a family of subsets of a set Ω , and let $A \subset \Omega$. The restriction (or trace) of $\mathcal{G}$ on A is the collection of all sets of the form $A \cap B$, were $B \in \mathcal{G}$, and it is denoted by $\mathcal{G}_A$ (or $A \cap \mathcal{G}$)

$$\mathcal{G}_A = A \cap \mathcal{G} = \{A \cap B : B \in \mathcal{G}\}$$

$\mathcal{G}_A$ is a family of subsets of A .

Example (3.11)

1. If $\Omega = \{a, b, c, d\}$, $\mathcal{G} = \{\{a\}, \{b, c\}, \{c, d\}\}$, $A = \{a, b\}$, then $\mathcal{G}_A = A \cap \mathcal{G} = \{\emptyset, \{a\}, \{b\}\}$
2. If $\Omega = \mathbb{R}$, $\mathcal{G} = \{[a, b] : -\infty < a \leq b < \infty\}$, $A = [0, 1]$, then $\mathcal{G}_A = A \cap \mathcal{G} = \{[a, b] : 0 \leq a \leq b \leq 1\}$

Theorem(3.12)

Let $\mathcal{G}$ be a family of subsets of a set Ω , and let $A \subset \Omega$. Then $A \cap \sigma_{\mathcal{R}}(\mathcal{G}) = \sigma_{\mathcal{R}}(A \cap \mathcal{G})$

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Proof :

$$\text{Let } \mathcal{F} = \{B \cup (C \mid A) : B \in \sigma_{\mathcal{R}}(A \cap \mathcal{G}), C \in \sigma_{\mathcal{R}}(\mathcal{G})\}$$

It is easy to verify that $\mathcal{F}$ is a σ - ring .

If $A \in \mathcal{G}$, then the relation $B = (B \cap A) \cup (B \mid A)$, together with $B \cap A \in A \cap \mathcal{G} \subset \sigma_{\mathcal{R}}(A \cap \mathcal{G})$, shows that $B \in \mathcal{F}$, and therefore that $\mathcal{G} \subset \mathcal{F}$.

It follows that $\sigma_{\mathcal{R}}(\mathcal{G}) \subset \mathcal{F}$ and therefore that $A \cap \sigma_{\mathcal{R}}(\mathcal{G}) \subset A \cap \mathcal{F}$

Since, however, it is obvious that $A \cap \mathcal{F} = \sigma_{\mathcal{R}}(A \cap \mathcal{G})$

It follows that $A \cap \sigma_{\mathcal{R}}(\mathcal{G}) \subseteq \sigma_{\mathcal{R}}(A \cap \mathcal{G})$

The reverse inequality, $\sigma_{\mathcal{R}}(A \cap \mathcal{G}) \subseteq A \cap \sigma_{\mathcal{R}}(\mathcal{G})$, follows from the facts that $A \cap \sigma_{\mathcal{R}}(\mathcal{G})$ is a σ - ring and $A \cap \mathcal{G} \subseteq A \cap \sigma_{\mathcal{R}}(\mathcal{G})$

Theorem(3.13)

Let $\mathcal{G}$ be a family of subsets of a set Ω , and let $\mathcal{D} = \{ \text{finite disjoint unions of sets in } \mathcal{G} \}$.

If $\mathcal{G}$ is a semiring on Ω , then $\mathcal{D}$ is a ring, and $\mathcal{R}(\mathcal{G}) = \mathcal{D}$.

Proof :

Clearly, $\mathcal{G} \subseteq \mathcal{D}$, $\mathcal{D} \neq \emptyset$? why

Let $A, B \in \mathcal{D} \Rightarrow A = \bigcup_{i=1}^n A_i$, $B = \bigcup_{j=1}^m B_j$ where A_i , $i = 1, 2, \dots, n$ and B_j , $j = 1, 2, \dots, m$ are

disjoint sets $\mathcal{G}$, we have $A \cap B = (\bigcup_{i=1}^n A_i) \cap (\bigcup_{j=1}^m B_j) = \bigcup_{i=1}^n \bigcup_{j=1}^m (A_i \cap B_j)$ and, moreover, we have

$\{A_i \cap B_j : i = 1, 2, \dots, n; j = 1, 2, \dots, m\}$ is a disjoint family.

Since $\mathcal{G}$ is a semiring $\Rightarrow A_i \cap B_j \in \mathcal{G}$, $i = 1, 2, \dots, n; j = 1, 2, \dots, m$

Hence, we have $A \cap B \in \mathcal{D}$

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If $B \subseteq A$, the difference $A \setminus B$ may be expressed by a finite, disjoint union of a sets having the form $A_i \setminus \bigcup_{j=1}^m B_j$.

Each $A_i \setminus \bigcup_{j=1}^m B_j$ may also be expressed by a finite, disjoint union of the sets in $\mathcal{G}$.

Thus, we have $A \setminus B \in \mathcal{D}$. By using theorem(3.4), part(6), we know that $\mathcal{D}$ is a ring.

Since $\mathcal{R}(\mathcal{G})$ is the smallest ring containing $\mathcal{G}$ and $\mathcal{G} \subseteq \mathcal{D} \Rightarrow \mathcal{R}(\mathcal{G}) \subseteq \mathcal{D}$

Let $A \in \mathcal{D} \Rightarrow A = \bigcup_{j=1}^n A_i$ where A_i are disjoint sets in $\mathcal{G}$

Since $\mathcal{G} \subseteq \mathcal{R}(\mathcal{G}) \Rightarrow A_i \in \mathcal{R}(\mathcal{G})$

Since $\mathcal{R}(\mathcal{G})$ a ring $\Rightarrow A = \bigcup_{j=1}^n A_i \in \mathcal{R}(\mathcal{G}) \Rightarrow A \in \mathcal{R}(\mathcal{G}) \Rightarrow \mathcal{D} \subseteq \mathcal{R}(\mathcal{G}) \Rightarrow \mathcal{R}(\mathcal{G}) = \mathcal{D}$,

Theorem(3.14)

Let $\mathcal{G}$ be a family of subsets of a set Ω . Then $\mathcal{R}_\sigma(\mathcal{G}) = \mathcal{R}_\sigma(\mathcal{R}(\mathcal{G}))$

Proof :

Since $\mathcal{G} \subseteq \mathcal{R}(\mathcal{G}) \Rightarrow \mathcal{R}_\sigma(\mathcal{G}) \subseteq \mathcal{R}_\sigma(\mathcal{R}(\mathcal{G}))$

On the other hand, since $\mathcal{G} \subseteq \mathcal{R}_\sigma(\mathcal{G})$ and $\mathcal{R}_\sigma(\mathcal{G})$ is a ring, we have $\mathcal{R}(\mathcal{G}) \subseteq \mathcal{R}_\sigma(\mathcal{G})$.

Furthermore, since $\mathcal{R}_\sigma(\mathcal{G})$ is σ -ring, we have $\mathcal{R}_\sigma(\mathcal{R}(\mathcal{G})) \subseteq \mathcal{R}_\sigma(\mathcal{G})$

Consequently, we have $\mathcal{R}_\sigma(\mathcal{G}) = \mathcal{R}_\sigma(\mathcal{R}(\mathcal{G}))$

Theorem(3.15)

1. A σ -ring is a monotone Family.
2. A monotone Family which is a ring is also a σ -ring.

Proof:



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(1) let $\mathcal{R}$ be a σ -ring

If $\{A_n\}$ is a monotone sequence of sets in $\mathcal{R} \Rightarrow \lim_{n \rightarrow \infty} A_n$ is either $\cup A_n$ or $\cap A_n$

Since $\mathcal{R}$ is a σ -ring and $A_n \in \mathcal{R}$ for all $n \geq 1$, $\cup A_n \in \mathcal{R}$ and $\cap A_n \in \mathcal{R}$

$\Rightarrow \lim_{n \rightarrow \infty} A_n \in \mathcal{R} \Rightarrow \mathcal{R}$ is a monotone Family

(2) Let $\mathcal{R}$ is a monotone ring

We want to show that $\mathcal{R}$ is closed under the countable union.

Let $A_n \in \mathcal{R}$ for all $n = 1, 2, \dots$

Since $\mathcal{R}$ is monotone family $\Rightarrow \lim_{n \rightarrow \infty} A_n \in \mathcal{R}$

Since $\mathcal{R}$ is ring, then $\bigcup_{i=1}^m A_i \in \mathcal{R}$, $m = 1, 2, \dots$ $A_n \in \mathcal{R}$

Since $\{\bigcup_{i=1}^m A_i\}$ is an increasing sequence of sets whose union is $\bigcup_{i=1}^{\infty} A_i$, the fact that M is a

monotone family implies that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.

Theorem(3.16) Monotone Family Theorem

If $\mathcal{R}$ is any ring, the monotone family $\mathcal{M}(\mathcal{R})$ generated by $\mathcal{R}$ is the same as the σ -ring $\mathcal{R}_{\sigma}(\mathcal{R})$ generated by $\mathcal{R}$, i.e. $\mathcal{M}(\mathcal{R}) = \mathcal{R}_{\sigma}(\mathcal{R})$.

Proof:

Since any σ -ring is a monotone family (from theorem 3.15)) and $\mathcal{R} \subset \mathcal{R}_{\sigma}(\mathcal{R})$, we have $\mathcal{M}(\mathcal{R}) \subseteq \mathcal{R}_{\sigma}(\mathcal{R})$.

If $\mathcal{M}(\mathcal{R})$ is a ring, since $\mathcal{M}(\mathcal{R})$ is a monotone family, then by theorem(3.15), we have $\mathcal{M}(\mathcal{R})$ is a σ -ring,

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Since $\mathcal{R}_\sigma(\mathcal{R})$ is the smallest σ -ring containing $\mathcal{R}$, then we have $\mathcal{R}_\sigma(\mathcal{R}) \subseteq \mathcal{M}(\mathcal{R})$ and, therefore, the proof would be complete.

To complete the proof, we need to prove that $\mathcal{M}(\mathcal{R})$ is a ring.

Let $\mathfrak{Z}(D)$ be the family of sets A for which $A \mid D, D \mid A, A \cup D$ are all in $\mathcal{M}(\mathcal{R})$. It is easy to see, by the symmetry of the positions of A and D in the definition of $\mathfrak{Z}(D)$, that

$$A \in \mathfrak{Z}(D) \text{ iff } D \in \mathfrak{Z}(A)$$

If $\{A_n\}$ is a monotone sequence of sets in $\mathfrak{Z}(D)$, then $(\lim_{n \rightarrow \infty} A_n) \mid D = \lim_{n \rightarrow \infty} (D \mid A_n) \in \mathfrak{Z}(D)$,

$$D \mid (\lim_{n \rightarrow \infty} A_n) = \lim_{n \rightarrow \infty} (D \mid \lim_{n \rightarrow \infty} A_n) \in \mathfrak{Z}(D), \quad D \cup (\lim_{n \rightarrow \infty} A_n) = \lim_{n \rightarrow \infty} (D \cup A_n) \in \mathfrak{Z}(D)$$

That is, $\lim_{n \rightarrow \infty} A_n \in \mathfrak{Z}(D)$. So if $\mathfrak{Z}(D)$ is not empty, then it is a monotone family.

For all $D \in \mathcal{R}$, if $A \in \mathcal{R}$, then by definition of a ring, $A \in \mathfrak{Z}(D)$,

So that $\mathcal{R} \subset \mathfrak{Z}(D)$ for any $D \in \mathcal{R}$.

Since $\mathcal{M}(\mathcal{R})$ is the smallest monotone family containing $\mathcal{R}$, it follows that

$$\mathcal{M}(\mathcal{R}) \subseteq \mathfrak{Z}(D) \text{ for all } D \in \mathcal{R}$$

Hence, for all $A \in \mathcal{M}(\mathcal{R})$, for all $D \in \mathcal{R}$, we have $A \in \mathfrak{Z}(D)$; therefore, by symmetry,

$D \in \mathfrak{Z}(A)$; that is, $\mathcal{R} \subset \mathfrak{Z}(A)$ for any $A \in \mathcal{M}(\mathcal{R})$.

Noting that again $\mathfrak{Z}(A)$ is a monotone family, we have $\mathcal{M}(\mathcal{R}) \subset \mathfrak{Z}(A)$ for all $A \in \mathcal{M}(\mathcal{R})$

This shows that $\mathcal{M}(\mathcal{R})$ is a ring.

Corollary(3.17)

Any monotone family $\mathcal{F}$ which contains a ring $\mathcal{R}$ contains the σ -ring $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.

Proof :

$$\mathcal{M}(\mathcal{R}) \subseteq \mathcal{F}$$



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Since $\mathcal{M}(\mathcal{R}) = \mathcal{R}_\sigma(\mathcal{R}) \Rightarrow \mathcal{R}_\sigma(\mathcal{R}) \subseteq \mathcal{F}$

Definition(3.18)

A nonempty family $\mathcal{F}$ of subsets of a set Ω is called

1. A **semifield (or semialgebra)** on Ω if,

I. If $A \in \mathcal{F}$, then $A^c = \bigcup_{i=1}^n A_i$, where A_i are disjoint sets in $\mathcal{F}$.

II. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$

2. A **field (or algebra)** on Ω if,

I. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

II. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$

3. A **σ -field (σ -algebra)** on Ω , if

I. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

II. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

4. A **α -field** on Ω if,

I. $\phi \in \mathcal{F}$

II. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$

5. A **$\alpha\sigma$ -field** on Ω if,

I. $\phi, \Omega \in \mathcal{F}$

II. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

6. A **β -field** on Ω if,



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I. $\phi \in \mathcal{F}$

II. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcap_{i=1}^n A_i \in \mathcal{F}$

7. A $\beta\sigma$ -field on Ω if,

I. $\phi, \Omega \in \mathcal{F}$

II. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

8. A δ -field on Ω if,

I. $\phi \in \mathcal{F}$, $\mathcal{F} \neq \{\phi\}$.

II. If A is a nonempty set in $\mathcal{F}$ and $A \subseteq B \subseteq \Omega$, then $B \in \mathcal{F}$

III. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

9. A λ -field on Ω if,

I. $\Omega \in \mathcal{F}$, $\mathcal{F} \neq \{\Omega\}$.

II. If $B \in \mathcal{F}$ and $A \subseteq B \subseteq \Omega$, then $A \in \mathcal{F}$

III. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Example(3.19)

1. The family of all singletons of set Ω and the empty set is a semiring.

2. The set $\mathcal{R}$ of all open intervals (a, b) , closed of intervals $[a, b]$ and half open intervals $[a, b), (a, b]$, including the "empty interval" $(a, a) = \phi$ and the single-element set $[a, a] = \{a\}$, is a semiring on $\mathbb{R}$

Example(3.20)

1. The family $\mathcal{F} = \{\phi, \Omega\}$ is a field on Ω .

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2. The family $\mathcal{F}$ of all subsets of a set Ω is a field on Ω .
3. The family $\mathcal{F}$ of all finite subsets of Ω is a field on Ω iff Ω is finite.
4. The family $\mathcal{F}$ of all bounded subsets of $\mathbb{R}$ is not a field on $\mathbb{R}$.
5. The family $\mathcal{F} = \{(a, b] : -\infty \leq a < b < \infty\}$ is a semifield but not field on Ω .
- 6.. The family $\mathcal{R}$ of all countable subsets of $\mathbb{R}$ is not a σ -field on $\mathbb{R}$.
7. The family $\mathcal{F}$ of all countable subsets of a set Ω and their complements is a σ -field on Ω .

Theorem(3.21)

1. Every semifield is a semiring, but the converse is not true
2. If $\mathcal{F}$ is a semifield on Ω , then $\phi, \Omega \in \mathcal{F}$
3. Let $\mathcal{F}$ be a field on a set Ω .
 - a. $\phi, \Omega \in \mathcal{F}$. $\phi \in \mathcal{F} : \text{for } \Omega \in \mathcal{F} \text{ and } \phi = \Omega^c \in \mathcal{F}$
 - b. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcap_{i=1}^n A_i \in \mathcal{F}$
4. Any field is a ring, but the converse is not true
5. A ring on Ω which contain Ω is a field on Ω .
6. Every field is also a semifield, but the converse is not true
7. Every σ -field is also a field, but the converse is not true
8. Let $\mathcal{F}$ be a σ -field on a set Ω . If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$

Proof :

4. Let $\mathcal{F}$ be any field and let $A, B \in \mathcal{F}$, then $A, B^c \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$

Since $A \setminus B = A \cap B^c \Rightarrow A \setminus B \in \mathcal{F}$. Hence $\mathcal{F}$ is a ring.

The converse : The family $\mathcal{F}$ of all bounded subsets of $\mathbb{R}$ is a ring but not a field on $\mathbb{R}$.

5. Let $\mathcal{F}$ be ring on Ω which contain Ω

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Let $A \in \mathcal{F}$, since $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is a ring, then $\Omega \setminus A \in \mathcal{F}$

Since $A^c = \Omega \setminus A$, then $A^c \in \mathcal{F}$. Hence $\mathcal{F}$ is a field.

7. **The converse :** Let $\Omega = \mathbb{R}$, and take $\mathcal{F}$ to be the family of all finite disjoint union of intervals of the form $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. By convention we also count (a, ∞) as right semi closed. $\mathcal{F}$ is an a field but not σ -field on Ω . Because $A_n = (0, 1 - \frac{1}{n}] \in \mathcal{F}$ for all

$n = 1, 2, \dots$, but $\bigcup_{n=1}^{\infty} A_n = (0, 1) \notin \mathcal{F}$

8. Let $\mathcal{F}$ be a σ -field on a set Ω . If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $A_n^c \in \mathcal{F}$, $n = 1, 2, \dots$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{F} \Rightarrow \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$$

$$\text{Since } \bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$$

Definition(3.22)

A measurable Space is a pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ -field on Ω . A subset A of Ω is called measurable (measurable with respect to the σ -field $\mathcal{F}$), if $A \in \mathcal{F}$, i.e. any member of $\mathcal{F}$ is called a measurable set.

Remark

The intersection of a collection of semifields need not be semifields

Theorem(3.23)

If $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of field (σ -field) on a set Ω with $\Lambda \neq \emptyset$, then $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a field (σ -field) on Ω .



Theorem(3.24)

1. If $\mathcal{G}$ is any family of subsets of a set Ω there is a unique ring $\mathcal{F}_0$ such that $\mathcal{F}_0$ containing $\mathcal{G}$ and such that, if $\mathcal{H}$ is any ring containing $\mathcal{G}$ we must have $\mathcal{F}_0 \subset \mathcal{H}$.
2. If $\mathcal{G}$ is any family of subsets of a set Ω there is a unique σ -ring $\mathcal{F}_0$ such that $\mathcal{F}_0$ containing $\mathcal{G}$ and such that, if $\mathcal{H}$ is any σ -ring containing $\mathcal{G}$ we must have $\mathcal{F}_0 \subset \mathcal{H}$.

Proof:

1. Since The family of all subsets of Ω is a ring

It is clear at least one ring containing $\mathcal{G}$ always exists .

Since moreover the intersection of any collection of rings is also a ring , the intersection of all rings containing $\mathcal{G}$ is easily seen to be the desired ring $\mathcal{F}_0$.

Remarks

Let $\mathcal{G}$ be a family of subsets of a set Ω

1. The field $\mathcal{F}_0$, the smallest field containing $\mathcal{G}$, is called the field generated by $\mathcal{G}$; it will be denoted by $\mathcal{F}(\mathcal{G})$
2. The σ -field $\mathcal{F}_0$, the smallest σ -field containing $\mathcal{G}$, is called the σ -field generated by $\mathcal{G}$; it will be denoted by $\sigma(\mathcal{G})$
3. $\mathcal{F}(\mathcal{G})$ = intersection of all fields on Ω , which contain $\mathcal{G}$. and $\sigma(\mathcal{G})$ = intersection of all σ -fields on Ω , which contain $\mathcal{G}$.
4. $\mathcal{F}(\mathcal{G}) = \mathcal{G}$ iff $\mathcal{G}$ is a field on Ω , and $\sigma(\mathcal{G}) = \mathcal{G}$ iff $\mathcal{G}$ is a σ -field on Ω .
5. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{F}(\mathcal{G}_1) \subseteq \mathcal{F}(\mathcal{G}_2)$ and $\sigma(\mathcal{G}_1) \subseteq \sigma(\mathcal{G}_2)$, where $\mathcal{G}_1, \mathcal{G}_2$ are two families of subsets of a set Ω .

Definition(3.25)

Let (Ω, τ) be a topological space. The σ -field generated by τ is called the Borel σ -field and it is denoted by $\beta(\Omega)$, i.e. $\beta(\Omega) = \sigma(\tau)$. The member of $\beta(\Omega)$ are called Borel sets of Ω .

Remark

The σ -field $\sigma(\mathcal{G}_A)$ generated by $\mathcal{G}_A$ some time denoted by $\sigma_A(A \cap \mathcal{G})$, i.e. $\sigma(\mathcal{G}_A) = \sigma_A(A \cap \mathcal{G})$

Theorem(3.26)

Let $\mathcal{G}$ be a family of subsets of a set Ω , and let $A \subset \Omega$

1. $A \cap \sigma(\mathcal{G})$ is the σ -field on A .
2. $\sigma(\mathcal{G}_A) = A \cap \sigma(\mathcal{G})$
3. If $\mathcal{G}$ is closed under finite intersection and $A \in \mathcal{G}$, then $\mathcal{G}_A = \{B \in \mathcal{G} : B \subset A\}$
4. If $\mathcal{G}$ is a σ -field on Ω , then $\mathcal{G}_A$ is a σ -field on A

Theorem(3.27)

Let $\mathcal{G}$ be a family of subsets of a set Ω , and let $\mathcal{D} = \{ \text{finite disjoint unions of sets in } \mathcal{G} \}$.

If $\mathcal{G}$ is a semifield on Ω , then $\mathcal{D}$ is a field, and $\mathcal{F}(\mathcal{G}) = \mathcal{D}$.

Proof :

Clearly, $\mathcal{G} \subseteq \mathcal{D}$, $\mathcal{D} \neq \emptyset$? why

1. Let $A \in \mathcal{D} \Rightarrow A = \bigcup_{j=1}^m B_j$ where B_i are disjoint sets $\mathcal{G}$

Since $\mathcal{G}$ is a semifield on Ω and $B_j \in \mathcal{G}$ for $j = 1, 2, \dots, m$

$$\Rightarrow B_j^c = \bigcup_{r=1}^k C_r \text{ where } C_r \text{ are disjoint sets} \quad \Rightarrow A^c = \left(\bigcup_{j=1}^m B_j \right)^c = \bigcap_{j=1}^m B_j^c = \bigcap_{j=1}^m \left(\bigcup_{r=1}^k C_r \right) \in \mathcal{D}$$

2. Let $A_1, A_2, \dots, A_n \in \mathcal{D}$

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$$\Rightarrow A_i = \bigcup_{j=1}^m B_j \text{ where } B_j \in \mathcal{G} \text{ for } j = 1, 2, \dots, m \text{ and } B_i \cap B_j = \emptyset \text{ for all } i \neq j$$

Since $\mathcal{G}$ is a semifield on Ω and $B_j \in \mathcal{G}$ for $j = 1, 2, \dots, m$

$$\Rightarrow \bigcap_{j=1}^m B_j \in \mathcal{G} \Rightarrow \bigcup_{i=1}^n (\bigcap_{j=1}^m B_j) \in \mathcal{D} \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{D}$$

Product σ -fields

Let $\Omega_1, \Omega_2, \dots, \Omega_n$ be any sets, the Cartesian product $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ is denoted by $\prod_{i=1}^n \Omega_i$

and defined as : $\prod_{i=1}^n \Omega_i = \{(x_1, x_2, \dots, x_n) : x_i \in \Omega_i, i = 1, 2, \dots, n\}$

Definition(3.28)

Let $\mathcal{F}_i$ be a σ -field of subsets of a set Ω_i , $i = 1, 2, \dots, n$ and let $\Omega = \prod_{i=1}^n \Omega_i$.

1. A rectangle set in Ω is a set $A = \prod_{i=1}^n A_i$ where $A_i \subseteq \Omega_i$ for each $i = 1, 2, \dots, n$.
2. A measurable rectangle in Ω is a set $A = \prod_{i=1}^n A_i$ where $A_i \in \mathcal{F}_i$ for each $i = 1, 2, \dots, n$.
3. A set of the form $D = \bigcup_{i=1}^n D_i$ where the D_i are disjoint measurable rectangles, is called an elementary sets. We denote this family by $\mathcal{E}$.

Theorem (3.29)

Let $\mathcal{F}_i$ be a σ -field of subsets of a set Ω_i , $i = 1, 2, \dots, n$ and let $\Omega = \prod_{i=1}^n \Omega_i$.

1. The family of all measurable rectangle sets in Ω is a semi-field on Ω .
2. The family of finite disjoint unions of measurable rectangles is a field on Ω .

Definition(3.30)

Let $\mathcal{F}_i$ be a σ -field of subsets of a set Ω_i , $i = 1, 2, \dots, n$ and let $\Omega = \prod_{i=1}^n \Omega_i$. The smallest σ -field containing the measurable rectangle in Ω is called the product σ -field, written $\prod_{i=1}^n \mathcal{F}_i$, i.e. $\prod_{i=1}^n \mathcal{F}_i = \sigma(\prod_{i=1}^n A_i : A_i \in \mathcal{F}_i, i = 1, 2, \dots, n)$. The pair $(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{F}_i)$ is called the product measurable space. If all $\mathcal{F}_i$ coincide with a fixed σ -field $\mathcal{F}$, the product σ -field is denoted by $\mathcal{F}^n$.

Note that : $\prod_{i=1}^n \mathcal{F}_i$ is not the Cartesian product of the $\mathcal{F}_i$; the Cartesian product is the set of measurable rectangles, while the product σ -field is the minimal σ -field over the measurable rectangles.

Definition(3.31)

Let $\mathcal{F}_i$ be a σ -field of subsets of a set Ω_i , $i = 1, 2$, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, and let $D \subseteq \Omega_1 \times \Omega_2$. For any $x \in \Omega_1$, the set $D_x = \{y \in \Omega_2 : (x, y) \in D\}$ is called the x -section of D , and for any $y \in \Omega_2$, the set $D^y = \{x \in \Omega_1 : (x, y) \in D\}$ is called the y -section of D .

Theorem(3.32)

Let $\mathcal{F}_i$ be a σ -field of subsets of a set Ω_i , $i = 1, 2$, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. If $D \in \mathcal{F}$, then $D_x \in \mathcal{F}_2$ and $D^y \in \mathcal{F}_1$ for all $x \in \Omega_1$ and $y \in \Omega_2$.

Proof :

Let $E = \{D \in \mathcal{F} : D_x \in \mathcal{F}_2\}$. then E is a σ -field on Ω . To prove this

(i) Since $\Omega_x = \{y \in \Omega_2 : (x, y) \in \Omega\} \in \mathcal{F}_2 \Rightarrow \Omega \in E$

(ii) Let $D \in E \Rightarrow D_x \in \mathcal{F}_2$



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Since $\mathcal{F}_2$ is a σ -field on Ω_2 , then $(D_x)^c \in \mathcal{F}_2$, but $(D^c)_x = (D_x)^c \in \mathcal{F}_2 \Rightarrow D^c \in E$

(iii) Let $D_n \in E$ for all $n \Rightarrow (D_n)_x \in \mathcal{F}_2$ for all n

Since $\mathcal{F}_2$ is a σ -field on Ω_2 , then $\bigcup_{n=1}^{\infty} (D_n)_x \in \mathcal{F}_2$,

But $(\bigcup_{n=1}^{\infty} D_n)_x = \bigcup_{n=1}^{\infty} (D_n)_x \in \mathcal{F}_2 \Rightarrow \bigcup_{n=1}^{\infty} D_n \in E$. Thus E is a σ -field on Ω .

Let $D = A \times B$, where $A \subseteq \Omega_1$ and $B \subseteq \Omega_2$, then $D_x = \begin{cases} B, & x \in A \\ \phi, & x \notin A \end{cases}$

Since $D \in \mathcal{F}_1 \times \mathcal{F}_2$, then $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Thus $D \in E \Rightarrow D_x \in \mathcal{F}_2$.

π -system and λ -system

Definition(3.33)

A family $\mathcal{C}$ of subsets of a set Ω is called a π -system (or π class) on a set Ω , if $A, B \in \mathcal{C}$, then $A \cap B \in \mathcal{C}$.

Example (3.34)

1. The family of intervals in $\mathbb{R}$ is a π -system whereas the family of all open disk in $\mathbb{R}^2$ is not.
2. Every semifield is a π -system

Definition(3.35)

A family $\mathcal{D}$ of subsets of a set Ω is called a λ -system (or Dynkin system or λ -class) on a set Ω , if

1. $\Omega \in \mathcal{D}$
2. If $A, B \in \mathcal{D}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{D}$

3. If $\{A_n\}$ is an increasing sequence of sets in $\mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$, i.e. if $A_n \in \mathcal{D}$ for all, n such



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that $A_n \uparrow A$, then $A \in \mathcal{D}$

Example (3.36)

Every σ -field $\mathcal{F}$ is a λ -system, but the field $\mathcal{F}$ need not be a λ -system.

Theorem (3.37)

A family $\mathcal{D}$ of subsets of a set Ω is λ -system (Dynkin system) on a set Ω , iff

1. $\Omega \in \mathcal{D}$
2. If $A \in \mathcal{D}$ then $A^c \in \mathcal{D}$
3. If $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$

Proof :

Suppose that $\mathcal{D}$ is Dynkin system on a set Ω ,

Let $A, B \in \mathcal{D}$ such that $A \cap B = \emptyset$, then $A \cup B = \Omega \setminus ((\Omega \setminus A) \cap B) \in \mathcal{D}$

Conversely, If (1),(2),(3) hold, then for $A, B \in \mathcal{D}$ with $A \subseteq B$ then $B \setminus A = \Omega \setminus (A \cup (\Omega \setminus B)) \in \mathcal{D}$

Theorem (3.38)

A family $\mathcal{F}$ of subsets of a set Ω is σ -field iff it is both a π -system and a λ -system on Ω

Proof :

Suppose that $\mathcal{F}$ is σ -field it is obvious to show that $\mathcal{F}$ is both a π -system and a λ -system Ω .

Conversely, assume $\mathcal{F}$ is both a π -system and a λ -system Ω

Let $A \in \mathcal{F}$, by (2) of theorem of (3.37), we have $A^c \in \mathcal{F}$

Let $A, B \in \mathcal{F}$, then $A^c, B^c \in \mathcal{F}$

Since $\mathcal{F}$ is a π -system, then $A^c \cap B^c \in \mathcal{F}$

Since $(A \cup B)^c = A^c \cap B^c$, then $(A \cup B)^c \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$

Let $\{A_n\}$ is a sequence of sets in $\mathcal{F}$,

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Put $B_1 = A_1$ and $B_n = A_n \mid (\bigcup_{i=1}^{n-1} A_i)$ for all $n \geq 2 \Rightarrow B_n \in \mathcal{F}$ for all n

Since $B_n \cap B_m = \emptyset$ for all $n \neq m$, then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$

Since $B_n \subseteq A_n$ for all n , $B_n \cap B_m = \emptyset$ for all $n \neq m$, and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Theorem (3.39)

1. If $\{\mathcal{C}_\alpha\}_{\alpha \in \Lambda}$ is a family of π -system, then $\mathcal{C} = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha$ is also π -system
2. If $\{\mathcal{D}_\alpha\}_{\alpha \in \Lambda}$ is a family of λ -system, then $\mathcal{D} = \bigcap_{\alpha \in \Lambda} \mathcal{D}_\alpha$ is also λ -system

Definition(3.40)

Let $\mathcal{G}$ be a family of subsets of a set Ω .

1. The smallest π -system containing $\mathcal{G}$ called the π -system generated by $\mathcal{G}$ and it is denoted by $\pi(\mathcal{G})$
2. The smallest λ -system containing $\mathcal{G}$ called the λ -system generated by $\mathcal{G}$ and it is denoted by $\lambda(\mathcal{G})$

Definition(3.41)

Let $\mathcal{G}$ be a family of subsets of a set Ω . For any point $x \in \Omega$, the set $\bigcap \{A : x \in A \in \mathcal{G}\}$ is called the atom of $\mathcal{G}$ at x , and denoted by $A(x / \mathcal{G})$, i.e. $A(x / \mathcal{G}) = \bigcap \{A : x \in A \in \mathcal{G}\}$

- If there is no confusion, it will be called the atom at x , or atom for short, and denoted by $A(x)$. i.e. $A(x) = A(x / \mathcal{G})$
- The family of all atoms of $\mathcal{G}$ is denoted by $\mathcal{A}[\mathcal{G}]$, i.e. $\mathcal{A}[\mathcal{G}] = \{A(x / \mathcal{G}) : x \in \Omega\}$
- Clearly, for every $x \in \Omega$, $x \in A(x)$, So, every atom is nonempty.



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- When $\bigcup \mathcal{G} \neq \Omega$, then $A(x / \mathcal{G}) = \Omega$ for any $x \notin \bigcup \mathcal{G}$.
- Thus, if we write $\mathcal{A}^-[\mathcal{G}] = \{A(x / \mathcal{G}) : x \in \bigcup \mathcal{G}\}$, we have $\mathcal{A}[\mathcal{G}] - \mathcal{A}^-[\mathcal{G}] \subset \{x\}$
- If $x \in A \in \mathcal{G}$, then $A(x) \subseteq A$
- If $\mathcal{G} = P(\Omega)$, then $\mathcal{A}[\mathcal{G}] = \{\{x\} : x \in \Omega\}$
- $\bigcup \mathcal{A}^-[\mathcal{G}] = \bigcup \mathcal{G}$

Example (3.42)

Let $\Omega = \{a, b, c\}$, $\mathcal{G} = \{\{a\}, \{a, b\}, \{b, c\}\}$. Then $\{a\}, \{b\}$, and $\{b, c\}$ are atoms.

i.e. $\{a\} = A(a)$, $\{b\} = A(b)$, $\{b, c\} = A(c)$

From this example, we can see that it is not necessary that all sets in $\mathcal{G}$ be atoms of $\mathcal{G}$, and that all atoms of $\mathcal{G}$ belong to $\mathcal{G}$. But, if $\mathcal{G}$ is closed under the formation of arbitrary intersections, then we have $\mathcal{A}[\mathcal{G}] \subseteq \mathcal{G}$; i.e. in this case, $\forall A(x)$, $A(x) = \bigcap \{A : x \in A \in \mathcal{G}\} \in \mathcal{G}$

Theorem(3.43)

Let $\mathcal{G}$ be a family of subsets of a set Ω .

1. Any set in $\mathcal{G}$ may be expressed by a union of atoms of $\mathcal{G}$.
2. Any intersection of sets in $\mathcal{G}$ may be expressed by a union of atoms of $\mathcal{G}$.

Proof :

(2) Let $\{A_t : t \in T\}$ be a family of sets in $\mathcal{G}$. To prove $\bigcap_{t \in T} A_t = \bigcup_{t \in T} \{A(x) : x \in \bigcap_{t \in T} A_t\}$

For any $x \in \bigcap_{t \in T} A_t$, and any $t \in T$, then $A(x) \subset A_t$

So, for any $x \in \bigcap_{t \in T} A_t$, then $A(x) \subseteq \bigcap_{t \in T} A_t$ and it follows that $\bigcup_{t \in T} \{A(x) : x \in \bigcap_{t \in T} A_t\} \subseteq \bigcap_{t \in T} A_t$

On the other hand, since $x \in A(x)$, we have $\bigcap_{t \in T} A_t = \{x : x \in \bigcap_{t \in T} A_t\} \subseteq \bigcup_{t \in T} \{A(x) : x \in \bigcap_{t \in T} A_t\}$



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$$\Rightarrow \bigcap_{t \in T} A_t = \bigcup \{A(x) : x \in \bigcap_{t \in T} A_t\}$$

Theorem(3.44)

Any intersection of atoms may be expressed by a union of atoms .

Proof :

Since any atom of $\mathcal{G}$ is an intersection of sets in $\mathcal{G}$, by using theorem (3.33) and the associativity of intersections, we obtain the conclusion.

Example (3.45)

Let $\Omega = \{a, b, c, d\}$, $\mathcal{G} = \{A, B, C, D\}$, where $A = \{a, c, d\}$, $B = \{b, c, d\}$, $C = \{c\}$, $D = \{d\}$. Then

$A(a) = A$, $A(b) = B$, $A(c) = C$, $A(d) = D$, we have $A(a) \cap A(b) = A(c) \cup A(d)$

Theorem(3.46)

If $A \in \mathcal{A}[\mathcal{G}]$, $x \in B$, then $A(x) \subseteq B$.

Proof :

Let $B = A(y) = \bigcap \{y \in A : A \in \mathcal{G}\} = \bigcap_{t \in T} A_t$ where $A_t \in \mathcal{G}$, T is an index set. Since $x \in B$, we have $x \in A_t$ for all $t \in T$. Thus $A(x) \subset A_t$ for all $t \in T$. consequently, we have $A(x) \subseteq B$.

Theorem(3.47)

Let $\mathcal{G}$ be a family of subsets of a set Ω . Then $A(x / \mathcal{G}) = A(x / \mathcal{A}[\mathcal{G}])$ for all $x \in \Omega$, and

$\mathcal{A}[\mathcal{G}] = \mathcal{A}[\mathcal{A}[\mathcal{G}]]$.

Proof :

For all $x \in \Omega$, if $x \in D$ for some $D \in \mathcal{A}[\mathcal{G}]$, we have, by using theorem (2.49),

$A(x / \mathcal{G}) \subseteq D$ and, therefore, $A(x / \mathcal{G}) \subseteq \bigcap \{D : x \in D \in \mathcal{A}[\mathcal{G}]\}$.

Reviewing $x \in A(x / \mathcal{G}) \in \mathcal{A}[\mathcal{G}]$, we have $\bigcap \{D : x \in D \in \mathcal{A}[\mathcal{G}]\} \subseteq A(x / \mathcal{G})$



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Thus $A(x / \mathcal{G}) = \bigcap \{D : x \in D \in \mathcal{A}[\mathcal{G}]\} = A(x / \mathcal{A}[\mathcal{G}])$. Consequently, we have $\mathcal{A}[\mathcal{G}] = \mathcal{A}[\mathcal{A}[\mathcal{G}]]$.

Theorem(3.48)

Let $\mathcal{G}$ be a family of subsets of a set Ω . Then $\mathcal{A}[\mathcal{G} \cup \mathcal{A}[\mathcal{G}]] = \mathcal{A}[\mathcal{G}]$

Proof :

for all $x \in \Omega$,

$$\begin{aligned} A(x / \mathcal{G} \cup \mathcal{A}[\mathcal{G}]) &= \bigcap \{A : x \in A \in \mathcal{G} \cup \mathcal{A}[\mathcal{G}]\} = \left(\bigcap \{A : x \in A \in \mathcal{G}\} \right) \cap \left(\bigcap \{A : x \in A \in \mathcal{A}[\mathcal{G}]\} \right) \\ \Rightarrow A(x / \mathcal{G} \cup \mathcal{A}[\mathcal{G}]) &= A(x / \mathcal{G}) \cap A(x / \mathcal{A}[\mathcal{G}]) = A(x / \mathcal{G}). \text{ Thus } \mathcal{A}[\mathcal{G} \cup \mathcal{A}[\mathcal{G}]] = \mathcal{A}[\mathcal{G}] \end{aligned}$$

Theorem(3.49)

Let $\mathcal{G}$ be a family of subsets of a set Ω . If $\mathcal{G}' = \{\bigcup_{t \in T} A_t : A_t \in \mathcal{G}, t \in T\}$, where T is an arbitrary index set, then $\mathcal{A}[\mathcal{G}'] = \mathcal{A}[\mathcal{G}]$

Proof :

$\bigcup \mathcal{G}' = \bigcup \mathcal{G}$, for all $x \in \bigcup \mathcal{G}$, by using absorption, we have

$$A(x / \mathcal{G}') = \bigcap \{A : x \in A \in \mathcal{G}'\} = \bigcap \left\{ \bigcup_{t \in T} A_t : A_t \in \mathcal{G}, t \in T \right\}, \text{ where } T \text{ is an arbitrary index set}$$

$$\Rightarrow A(x / \mathcal{G}') = \bigcap \{A : x \in A \in \mathcal{G}\} = A(x / \mathcal{G}). \text{ Thus, we have } \mathcal{A}[\mathcal{G}'] = \mathcal{A}[\mathcal{G}]$$

Theorem(3.50)

Let $\mathcal{G}$ be a family of subsets of a set Ω . If $\mathcal{G}$ is closed under the difference, then $\mathcal{A}^-[\mathcal{G}]$ is a partition of $\bigcup \mathcal{G}$

Proof :

Since $\bigcup \mathcal{A}^-[\mathcal{G}] = \bigcup \mathcal{G}$, we only need to prove that that different atoms in $\mathcal{A}^-[\mathcal{G}]$ must be disjoint, that is for all $A(x), A(y) \in \mathcal{A}^-[\mathcal{G}]$, $A(x) \neq A(y) \Rightarrow A(x) \cap A(y) = \emptyset$



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If both $x \in A(y)$ and $y \in A(x)$, then by using (3.36), we have $A(x) = A(y)$. So, when , we can suppose $x \notin A(y)$ without any loss of generality. In this case, if there exists $z \in A(x) \cap A(y)$, we have the result that, form $x \notin A(y)$ and $z \in A(y)$, there exists $A \in \mathcal{G}$ such that $x \notin A$, but $z \in A$. Thus, if we take $B \in \mathcal{G}$ satisfying $x \in B$ and set $D = B \setminus A$, then $x \in D \in \mathcal{G}$, but $z \notin D$.

This contradicts the fact $z \in A(x)$. Therefore, we have $A(x) \cap A(y) = \emptyset$.

Corollary (3.51)

If $\mathcal{F}$ is a field on a set Ω , then $\mathcal{A}[\mathcal{F}]$ is a partition of Ω

Exercises (3)

1. Prove that any finite field is also a σ -field.
2. If $\mathcal{R}$ is a ring and $\mathcal{F}$ is the family of subsets A of a set Ω such that either A or A^c in $\mathcal{R}$, show that $\mathcal{F}$ is a field.
3. What is the ring $\mathcal{R}(\mathcal{G})$ generated by each of the following families:
 - (a) for a single fixed A , $\mathcal{G} = \{A\}$
 - (b) for a single fixed A , $\mathcal{G}$ is family of all subsets of A
 - (c) $\mathcal{G}$ is family of all sets which precisely two points.
4. Prove that if A is any subset of a set Ω , $A \neq \emptyset$ or Ω , then the σ -field $\mathcal{F}(A)$ generated by A is the family $\{\emptyset, A, A^c, \Omega\}$
5. If $\mathcal{G}$ is a non-empty family of sets. Show that every set in the σ -ring generated by $\mathcal{G}$ is a subset of a countable union of sets of $\mathcal{G}$.
6. If $\mathcal{G}$ is a countable family of subsets of a set Ω , then the ring $\mathcal{R}(\mathcal{G})$ generated by $\mathcal{G}$ is countable.



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7. Suppose $\mathcal{G}$ is a $\mathcal{Z}$ -family of subsets of a set Ω_2 , $f : \Omega_1 \rightarrow \Omega_2$ is any function and $f^{-1}(\mathcal{G}) = \{f^{-1}(A) : A \in \mathcal{G}\}$. Show that $f^{-1}(\mathcal{G})$ is a $\mathcal{Z}$ -family of subsets of Ω_1 .
8. Given an example of a function $f : \Omega_1 \rightarrow \Omega_2$ and a ring $\mathcal{R}$ on Ω_2 such that $f^{-1}(\mathcal{R})$ is not a ring on Ω_1 .
9. Given an example of a function $f : \Omega_1 \rightarrow \Omega_2$ and a semiring $\mathcal{R}$ on Ω_2 such that $f^{-1}(\mathcal{R})$ is not a semiring on Ω_1 .
10. Given a function $f : \Omega_1 \rightarrow \Omega_2$ and a family $\mathcal{G}$ of subsets of Ω_2 . Show that
 - (1) $\mathcal{R}(f^{-1}(\mathcal{G})) = f^{-1}(\mathcal{R}(\mathcal{G}))$, where $\mathcal{R}(\mathcal{H})$ is the σ -ring generated by $\mathcal{H}$.
 - (2) $\mathcal{F}(f^{-1}(\mathcal{G})) = f^{-1}(\mathcal{F}(\mathcal{G}))$, where $\mathcal{F}(\mathcal{H})$ is the σ -field generated by $\mathcal{H}$.
11. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces, and let $f : \Omega_1 \rightarrow \Omega_2$ be a function. Define $\mathcal{F} = \{B \in \mathcal{F}_2 : f^{-1}(B) \in \mathcal{F}_1\}$ show that $\mathcal{F}$ is a σ -field on Ω_1 .
12. Given a function $f : \Omega_1 \rightarrow \Omega_2$ and a family $\mathcal{G}$ of subsets of Ω_2 . Show that $\sigma(\mathcal{F}) = \{f^{-1}(A) : A \in \sigma(\mathcal{G})\}$, where $\mathcal{F} = \{f^{-1}(B) : B \in \mathcal{G}\}$.
13. Let Ω be a nonempty set and let $\mathcal{G} = \{A_n : n \in \mathbb{N}\}$ be a partition of Ω , show that $\mathcal{F} = \{\bigcup_{n \in J} A_n : J \subset \mathbb{N}\}$ is a σ -field on Ω .
14. Let $\mathcal{R}$ is a σ -ring on a set Ω , show that $\mathcal{F} = \{A \subseteq \Omega : A \in \mathcal{R} \text{ or } A^c \in \mathcal{R}\}$ is a σ -field on Ω .
15. Let Ω be a nonempty set and $A \subset \Omega$, show that $\sigma(\{I_A\}) = \sigma(\{A\}) = \{\emptyset, A, A^c, \Omega\}$.
16. Let $A_n = \{m \in \mathbb{N} : m \geq n\}$, $n \in \mathbb{N}$, show that $\sigma(\mathcal{G}) = P(\mathbb{N})$, where $\mathcal{G} = \{A_n : n \in \mathbb{N}\}$.
17. Let $\mathcal{G}$ be a semifield on a set Ω and let $\mathcal{F}(\mathcal{G}) = \{\bigcup_{i=1}^n A_i : A_i \in \mathcal{G}, i = 1, 2, \dots, n, A_i \cap A_j = \emptyset, i \neq j\}$, show that $\sigma(\mathcal{G}) = \sigma(\mathcal{F}(\mathcal{G}))$.



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18. If A is a Borel subsets of $\mathbb{R}$, show that the smallest σ -field of subsets of A Containing the sets open in A is $\{B \in \beta(\mathbb{R}) : B \in A\}$

19. Let $\mathcal{R}$ be a σ -ring on a set Ω , if $A_n \in \mathcal{R}$, $n = 1, 2, \dots$, show that

$$\limsup_{n \rightarrow \infty} A_n \in \mathcal{R} \quad \text{and} \quad \liminf_{n \rightarrow \infty} A_n \in \mathcal{R}$$

20. Let $\mathcal{F}$ be a σ -field on a set Ω , if $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, show that

$$\limsup_{n \rightarrow \infty} A_n \in \mathcal{F} \quad \text{and} \quad \liminf_{n \rightarrow \infty} A_n \in \mathcal{F}$$

21. Show that any σ -field is a monotone family.

22. Let $\mathcal{D}$ be a λ -system on a set Ω . Fix $D \in \mathcal{D}$, show that $\mathcal{F} = \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$ is λ -system.

23. Let $\mathcal{C}$ be a λ -system, show that $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$

24. Let $\mathcal{C}$ be a λ -system, $\mathcal{D}$ is a λ -system such that $\mathcal{C} \subset \mathcal{D}$. Show that $\sigma(\mathcal{C}) \subset \mathcal{D}$.

25. Let $\mathcal{L}$ be a family of subset of a set Ω , and assume $\mathcal{L}$ closed under finite intersection.

If $\mathcal{D}$ is a λ -system such that $\mathcal{L} \subset \mathcal{D}$. Show that $\mathcal{D}$ includes the minimal σ -field $\mathcal{F} = \sigma(\mathcal{L})$.

26. For any λ -field on a set Ω , show that

a. $\Omega \in \mathcal{F}$

b. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$ and $\bigcap_{i=1}^n A_i \in \mathcal{F}$

c. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

27. For any λ -field on a set Ω , show that

a. $\phi \in \mathcal{F}$ b. If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$ and $\bigcap_{i=1}^n A_i \in \mathcal{F}$

c. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

4. Set Functions

Let $\mathbb{R}$ be the set of real numbers. The extended real numbers system consists of the real numbers system to which two symbols, $+\infty$ and $-\infty$, and it is denoted by $\overline{\mathbb{R}}$, i.e.

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\} = [-\infty, \infty]$$

$\beta(\overline{\mathbb{R}}) = \sigma(\{-\infty\}, \beta(\mathbb{R}), \{\infty\}) = \{A, A \cup \{-\infty\}, A \cup \{\infty\}, A \cup \{-\infty, \infty\} : A \in \beta(\mathbb{R})\}$, $\mathbb{R}^*$ is compact space, and $\sup, \inf$ exists in $\mathbb{R}^*$. The symbols $+\infty$ and $-\infty$, and the following algebraic relations among them and real numbers x : $-\infty < x < \infty$

$$\begin{aligned} 1. (\pm\infty) + (\pm\infty) = x + (\pm\infty) = (\pm\infty) + x = \pm\infty \quad 2. x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty, & x > 0 \\ 0, & x = 0 \\ \mp\infty, & x < 0 \end{cases} \quad 3. (\pm\infty)(\pm\infty) = +\infty \\ 4. (\pm\infty)(\mp\infty) = -\infty \quad 5. \frac{x}{\pm\infty} = 0, \quad \infty - \infty, -\infty + \infty, \frac{0}{0} \text{ and } \frac{\infty}{\infty} \text{ meaning less.} \end{aligned}$$

Definition(4.1)

A set function is a function whose domain of definition is a family of sets. We Consider only functions $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ where $\mathcal{F}$ is a non-empty family of subsets of a set Ω . Thus μ is a rule which determines, for each $A \in \mathcal{F}$, a unique elements $\mu(A)$ which either a real number or $\pm\infty$, we always $\mathcal{F}$ contains the empty set ϕ . and we always assume that $-\infty$ and ∞ cannot both belong to the rang of μ , because $\infty - \infty$ and $-\infty + \infty$ meaning less.

Definition(4.2)

Let $\mathcal{F}$ is a non-empty family of subsets of a set Ω . A set function $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is said to be

1. Finite if, $\mu(A) < \infty$ for each $A \in \mathcal{F}$
2. Semi-finite, if for each $A \in \mathcal{F}$ with $\mu(A) = \infty$, there exists $B \in \mathcal{F}$ with $B \subseteq A$ and $0 < \mu(B) < \infty$
3. Bounded if, $\sup\{|\mu(A)| : A \in \mathcal{F}\} < \infty$

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4 σ - finite if, for each $A \in \mathcal{F}$, there is a sequence $\{A_n\}$ of sets in $\mathcal{F}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all n .

Example(4.3)

Let Ω be any set, $\#(A)$ denote the number elements of a set A , and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \#(A)$, if A is finite and $\mu(A) = \infty$, if A is infinite, then μ is σ -finite iff Ω is countable.

Solution :

Suppose that Ω is countable

Let $A \in \mathcal{F} \Rightarrow A \subseteq \Omega \Rightarrow A$ is countable. Take $A = \{x_1, x_2, \dots\}$

Put $A_n = \{x_n\} \Rightarrow \mu(A_n) = 1 < \infty$ for all n and $A \subseteq \bigcup_{n=1}^{\infty} A_n \Rightarrow \mu$ is σ -finite

Conversely : Suppose that μ is σ -finite

Since $\Omega \subseteq \Omega \Rightarrow$ there is a sequence $\{A_n\}$ of sets in $\mathcal{F}$ such that $\Omega = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for all n .

Since $\mu(A_n) < \infty$ for all $n \Rightarrow A_n$ is finite set for all $n \Rightarrow A_n$ is countable set for all n

$\Rightarrow \bigcup_{n=1}^{\infty} A_n$ is countable, but $\Omega = \bigcup_{n=1}^{\infty} A_n$, then Ω is countable

Definition(4.4)

Let $\mathcal{F}$ is a non-empty family of subsets of a set Ω . A set function $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is said to be

1. Additive if, $A, B \in \mathcal{F}$, $A \cup B \in \mathcal{F}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

2. Finitely additive if, $\mu(\bigcup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$, whenever $A_1, A_2, \dots, A_n$ are disjoint sets in $\mathcal{F}$ and



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$$\bigcup_{k=1}^n A_k \in \mathcal{F}.$$

Remarks

1. The natural domain of definition for an finitely additive set function μ is a ring,
2. Every finitely additive set function is additive, but converse is not true.

Theorem (4.5)

Let μ be finitely additive set function on a ring $\mathcal{R}$, and let $A, B \in \mathcal{R}$

1. $\mu(\phi) = 0$
2. $\mu(A) = \mu(A \cap B) + \mu(A \setminus B)$
3. $\mu(A \cup B) = \mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A)$
4. $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
5. $\mu(A \Delta B) = \mu(A \setminus B) + \mu(B \setminus A)$
6. If $A \subseteq B$, then
 - (a) $\mu(B) = \mu(A) + \mu(B \setminus A)$
 - (b) If $\mu(A)$ is finite, then $\mu(B \setminus A) = \mu(B) - \mu(A)$
 - (c) If $\mu(A)$ is finite and $\mu(B \setminus A) \geq 0$, then $\mu(A) \leq \mu(B)$.
- (7) If $A \subset B$ and $\mu(A)$ is infinite, then $\mu(B) = \mu(A)$
- (8) If $\mu(B) = \infty$, then $\mu(A) \neq -\infty$

Proof:

(1) Let $A \in \mathcal{F}$ such that $\mu(A) < \infty \Rightarrow A = A \cup \phi \Rightarrow \mu(A) = \mu(A \cup \phi)$
 $\Rightarrow \mu(A) = \mu(A) + \mu(\phi) \Rightarrow \mu(\phi) = 0$

(2) Let $A, B \in \mathcal{R}$ then $A \cap B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$

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Since $A \cap B$ and $A \setminus B$ are disjoint set , $(A \cap B) \cup (A \setminus B) = A$

$$\Rightarrow \mu(A) = \mu((A \cap B) \cup (A \setminus B)) = \mu(A \cap B) + \mu(A \setminus B)$$

$$(3) A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R} , A \setminus B \in \mathcal{R} \text{ and } B \setminus A \in \mathcal{R}$$

Since $A \cap B$, $A \setminus B$ and $B \setminus A$ are disjoint set , and $(A \cap B) \cup (A \setminus B) \cup (B \setminus A) = A \cup B$

$$\mu(A \cup B) = \mu(A \cap B) \cup (A \setminus B) \cup (B \setminus A) = \mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A)$$

$$(4) \mu(A) = \mu(A \cap B) + \mu(A \setminus B) , \mu(B) = \mu(A \cap B) + \mu(B \setminus A)$$

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A)$$

$$= (\mu(A \cap B) + \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B)) = \mu(A \cup B) + \mu(A \cap B)$$

$$(5) A, B \in \mathcal{R} \Rightarrow A \setminus B, B \setminus A \in \mathcal{R}$$

$A \setminus B$ and $B \setminus A$ are disjoint sets , and $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$$\mu(A \Delta B) = \mu((A \setminus B) \cup (B \setminus A)) = \mu(A \setminus B) + \mu(B \setminus A)$$

$$(6) B = (A \cap B) \cup (B \setminus A)$$

$$A \subset B \Rightarrow A \cap B = A \Rightarrow B = A \cup (B \setminus A) \Rightarrow \mu(B) = \mu(A \cup (B \setminus A))$$

Since $A, B \setminus A \in \mathcal{R}$ and $A, B \setminus A$ are disjoint set

$$\mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A)$$

(7) If $\mu(A) = \infty$, then $\mu(B) = \mu(A) + \mu(B \setminus A)$ can only have a meaning if $\mu(A) \neq -\infty$, and

this implies $\mu(B) = \infty$. The case $\mu(A) = -\infty$ is similar.

(8) If $\mu(A) = -\infty$

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = -\infty , \mu(B) = \mu(A \cap B) + \mu(B \setminus A) = \infty$$

Could only have meaning if $\mu(A \cap B)$ is finite. But this would imply $\mu(A \setminus B) = -\infty , \mu(B \setminus A) = \infty$,

and then since $A \Delta B \in \mathcal{F}$, then $\mu(A \Delta B) = \mu(B \setminus A) + \mu(A \setminus B) = \infty + (-\infty)$ which is impossible.



Definition(4.6)

Let $\mathcal{F}$ is a non-empty family of subsets of a set Ω . A set function $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is said to be

1. σ -additive (sometimes called completely additive, or Countably additive) if,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

whenever $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

2. Measure, if μ is σ -additive and $\mu(A) \geq 0$ for all $A \in \mathcal{F}$.

3. Probability, if μ is a measure and $\mu(\Omega) = 1$

A measure space is a triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space, and μ is a measure on $\mathcal{F}$. If μ is a probability measure on $\mathcal{F}$, $(\Omega, \mathcal{F}, \mu)$ is a probability space.

Remarks

1. The natural domain of definition for a σ -additive set function μ is a σ -ring.

2. Every σ -additive set function is finitely additive, but converse is not true.

Example (4.7)

Let $\Omega = (0,1]$, and $\mathcal{F}$ the family of half-open intervals $(a,b]$ where $0 \leq a \leq b \leq 1$. Define

$\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by

$$\mu(a,b] = \begin{cases} b-a, & a \neq 0 \\ \infty, & a = 0 \end{cases}$$

It is clear to show that μ is finitely additive.

Now, to see μ is not σ -additive: Let $A_n = (\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, 3, \dots$

Then $\{A_n\}$ is disjoint sequence in $\mathfrak{R}$ and $\bigcup_{n=1}^{\infty} A_n = (0,1]$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(0,1] = \infty$



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$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu\left(\frac{1}{n+1}, \frac{1}{n}\right] = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \neq \sum_{n=1}^{\infty} \mu(A_n) \Rightarrow \mu \text{ is not } \sigma\text{-additive}$$

Example(4.8)

1. Let $\mathcal{F} = \{\phi, \Omega\}$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(\phi) = 0$ and $\mu(\Omega) = \lambda$ where $0 \leq \lambda \leq \infty$, then μ is a measure on $\mathcal{F}$.
2. Let Ω be any set, and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = 0$ for all $A \in \mathcal{F}$, then μ is a measure on $\mathcal{F}$. This measure is called Null Measure
3. Let Ω be any set, $\#(A)$ denote the number elements of a set A , and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \#(A)$, if A is finite and $\mu(A) = \infty$, if A is infinite, then μ is a measure on $\mathcal{F}$. This measure is called Counting Measure.
4. Let Ω be any set, let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \begin{cases} 0, & A = \phi \\ \infty, & A \neq \phi \end{cases}$, then μ is a measure on $\mathcal{F}$.
5. Let Ω be an uncountable set, and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = 0$ if A is countable set and $\mu(A) = \infty$ if A is an uncountable set, then μ is a measure on $\mathcal{F}$
6. Let Ω be an uncountable set, $0 \leq \lambda \leq \infty$, and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = 0$ if A is countable set and $\mu(A) = \lambda$ if A is an uncountable set, then μ is a measure on $\mathcal{F}$
7. Let Ω be any set, $x_0 \in \Omega$, and let $\mathcal{F} = \mathcal{P}(\Omega)$. Define $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \begin{cases} \lambda, & x_0 \in A \\ 0, & x_0 \notin A \end{cases}$ where $0 \leq \lambda \leq \infty$, then μ is a measure on $\mathcal{F}$. This measure is called Dirac Measure. If $\lambda = 1$, μ is a probability measure
8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\mathcal{G}$ be a sub σ -field of $\mathcal{F}$. Define $\mu_{\mathcal{G}}: \mathcal{G} \rightarrow \overline{\mathbb{R}}$ by $\mu_{\mathcal{G}}(A) = \mu(A)$ for all $A \in \mathcal{G}$, then $\mu_{\mathcal{G}}$ is a measure on $\mathcal{G}$.

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9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(\Omega, \mathcal{F}, \mu)$. Define $\mu_A : \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu_A(B) = \mu(A \cap B)$

For all $A \in \mathcal{F}$, then μ_A is a measure on $\mathcal{F}$. This measure is called trace μ on A .

10. Let $\Omega = \mathbb{R}$, and $\mathcal{F}$ the family of all bounded intervals of $\mathbb{R}$ (For $A = [a, b]$ or $[a, b)$ or $(a, b]$ or (a, b)). Define $\mu : \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = b - a$. Then μ is a finite measure

11. Let Ω be any set with at least two distinct points a, b ; $\mathcal{F}$ family of all subsets of Ω .

Define $\mu : \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \begin{cases} -1, & a \notin A, b \in A \\ 0, & a \notin A, b \notin A \\ 1, & a \in A, b \in A \end{cases}$, then μ is finite set function, but not measure

12. Let Ω be any infinite set, $\mathcal{F} = \mathcal{P}(\Omega)$. Let $\{x_n\}$ be a sequence of distinct points of Ω , and

suppose $\{P_n\}$ is of real numbers such that $\sum_{i=1}^{\infty} P_i$ either convergent absolutely or is properly

divergent to ∞ or $-\infty$ (the case $\sum_{i=1}^{\infty} P_i$ convergent, $\sum_{i=1}^{\infty} |P_i|$ is not allowed: why?). Define

$\mu : \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\mu(A) = \sum_{x_i \in A} P_i$, where the sum extends over all integers $i = 1, 2, \dots$ for which

$x_i \in A$. Then a. μ is finite set function if $\sum_{i=1}^{\infty} |P_i|$ converges and otherwise it is σ -finite

b. μ is not measure

Definition(4.9)

Let (Ω, τ) be a topological space. A subset A of Ω is said to be

1. Nowhere dense of Ω if $\text{int}(\overline{A}) = \emptyset$.

2. First category in Ω if there is a sequence $\{A_n\}$ of nowhere dense subsets of Ω such That

$$A = \bigcup_{n=1}^{\infty} A_n.$$

3. Second category in Ω if it is not First category .

Example (4.10)

Let Ω be any topological space, $\mathcal{F}$ the family of all subsets of Ω . Define $\mu: \mathcal{F} \rightarrow \bar{\mathbb{R}}$ by $\mu(A) = 0$ if A is the first category in Ω , and $\mu(A) = \infty$, if A is of the second category in Ω , then μ is
(1) A measure (2) not σ -finite if Ω is of the second category.

Example(4.11)

Let $\Omega = \{x_1, x_2, \dots\}$ be a Countably set, $P_1, P_2, \dots$ be non-negative real numbers, and let $\mathcal{F} = \mathcal{P}(\Omega)$.

Define $\mu: \mathcal{F} \rightarrow \bar{\mathbb{R}}$ by $\mu(A) = \sum_{x_i \in A} P_i$ for all $A \in \mathcal{F}$. Thus, if $A = \{x_{i_1}, x_{i_2}, \dots\}$, then $\mu(A) = \sum_{j=1}^{\infty} P_{i_j}$.

The set function μ is a measure on $\mathcal{F}$ and $\mu(\{x_i\}) = P_i$, $i = 1, 2, \dots$. A probability measure will be obtained iff $\sum_i P_i = 1$.

Definition(4.12)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A condition (or property) is said to be hold almost everywhere with respect to the measure μ , written *a.e.* $[\mu]$ (or simply *a.e.* if μ is understood), if there is a set $B \in \mathcal{F}$ of μ -measure 0, (i.e. $\mu(B) = 0$) such that the condition holds outside of B .

In particular: Let $f, g: \Omega \rightarrow \bar{\mathbb{R}}$ be two functions, we say that $f = g$ *a.e.*, if there is $B \in \mathcal{F}$ with $\mu(B) = 0$ such that $f(x) = g(x)$ for all $x \notin B$, i.e. $\mu\{x \in \Omega: f(x) \neq g(x)\} = 0$. Also we say f is finite *a.e.*, if for every $\varepsilon > 0$, there is k such that $\mu\{x \in \Omega: f(x) > k\} \leq \varepsilon$.

Theorem (4.13)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $A, B \in \mathcal{F}$ and $\{A_n\}$ be a sequence of sets in $\mathcal{F}$.

1. If $A \subset B$ then $\mu(A) \leq \mu(B)$

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2. If $A \subseteq \bigcup_{n=1}^{\infty} A_n$, then $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$

3. $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$

Proof :

(1) Let $A, B \in \mathcal{F}$ such that $A \subset B$

Since $B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \emptyset$, then $\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$

(2) Since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_1^c \cap \dots \cap A_{n-1}^c \cap A_n)$ where the sets in the last union are disjoint;

therefore, $\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} (A_1^c \cap \dots \cap A_{n-1}^c \cap A_n)) = \sum_{n=1}^{\infty} \mu(A_1^c \cap \dots \cap A_{n-1}^c \cap A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$

(3) Put $B_1 = A_1$ and $B_n = A_n \setminus (\bigcup_{k=1}^{n-1} A_k)$ for all $n \geq 2$

$\Rightarrow B_n \subseteq A_n$ for all n , $B_n \cap B_m = \emptyset$ for all $n \neq m$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Theorem (4.14)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $A \in \mathcal{F}$ and $\{A_n\}$ be a sequence of sets in $\mathcal{F}$.

1. If $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$

2. If $A_n \downarrow A$ and $\mu(A_n) < \infty$ for some n , then $\mu(A_n) \downarrow \mu(A)$

Proof :

(1) Put $B_1 = A_1$ and $B_n = A_n \setminus (\bigcup_{k=1}^{n-1} A_k)$ for all $n \geq 2$



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$$\Rightarrow B_n \cap B_m = \emptyset \text{ for all } n \neq m, \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \text{ and } \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A$$

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \rightarrow \infty} \left(\sum_{n=1}^n \mu(B_n)\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{n=1}^n B_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

$$(2) A_1 | A = A_1 \cap A^c = A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right) = \bigcup_{n=1}^{\infty} (A_1 \cap A_n^c) = \bigcup_{n=1}^{\infty} (A_1 | A_n^c)$$

Since $A_n \downarrow A$, then $A_1 | A_n \uparrow A_1 | A$, so $\mu(A_1 | A_n) \uparrow \mu(A_1 | A)$, then $\mu(A_1) - \mu(A_n) \uparrow \mu(A_1) - \mu(A)$.
Therefore $\mu(A_n) \downarrow \mu(A)$.

Note : In particular, if $A_n \downarrow \emptyset$ and $\mu(A_n) < \infty$ for some n , then $\mu(A_n) \downarrow 0$. This property is called continuity at $\emptyset$. The property fails if $\mu(A_n) = \infty$ for all n .

Example (4.15)

Let $\Omega = \{1, 2, 3, \dots\}$, μ the counting measure. If $A_n = \{n, n+1, n+2, \dots\}$, then $A_n \downarrow \emptyset$, but $\mu(A_n) = \infty$.

For all n , so $\mu(A_n) \not\rightarrow 0$.

Definition (4.16)

Let $\mathcal{F}$ be a ring on a set Ω . A set function $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is said to be

1. Continuous from below at $A \in \mathcal{F}$, if $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ with $A_n \uparrow A$.
2. Continuous from above at $A \in \mathcal{F}$, if $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ with $A_n \downarrow A$.
3. Semicontinuous at $A \in \mathcal{F}$, if it is continuous from below or from above at A .
4. Continuous at $A \in \mathcal{F}$, if it is continuous both from below and from above at A .

Theorem (4.17)

Let $\mathcal{F}$ be a ring on a set Ω , and $\mu: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is finitely additive with $\mu(A) > -\infty$ for all $A \in \mathcal{F}$.

1. If μ is σ -additive, then μ is continuous at A for all $A \in \mathcal{F}$.
2. If μ is continuous from below at every set $A \in \mathcal{F}$, then μ is σ -additive.
3. If μ is, finite and continuous from above at ϕ , then μ is σ -additive.

Proof :

(1) If $\mu(A_n) = \infty$ for $n = N$ and $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ with $A_n \uparrow A \Rightarrow \{A_n\}$ is an increasing sequence and $\bigcup_{n=1}^{\infty} A_n = A$

$A = A_1 \cup (\bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n))$ is a disjoint decomposition and

$$\mu(A) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(A_{n+1} \setminus A_n) = \mu(A_1) + \lim_{k \rightarrow \infty} \sum_{n=1}^k \mu(A_{n+1} \setminus A_n) = \lim_{k \rightarrow \infty} \mu(A_k)$$

Since μ is a finitely additive on $\mathcal{F}$. Thus μ is continuous from below at A

Now suppose $\{A_n\}$ decreases sequence and $\bigcap_{n=1}^{\infty} A_n = A$

Put $B_n = A_k \setminus A_n$ for $n \geq k \Rightarrow \{B_n\}$ increasing to $A_k \setminus A$. Hence, as $n \rightarrow \infty$

$$\mu(B_n) \rightarrow \mu(A_k \setminus A) = \mu(A_k) - \mu(A)$$

But $\mu(B_n) = \mu(A_k) - \mu(A_n)$ so that $\mu(A_n) \rightarrow \mu(A)$ as $n \rightarrow \infty$ since $\mu(A_n)$ is finite, and μ is also continuous from above at A .

(2) Suppose $A \in \mathcal{F}$, $A_n \in \mathcal{F}$, $n = 1, 2, \dots$ are such that $A = \bigcup_{n=1}^{\infty} A_n$ and the sets A_n are disjoint.



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Put $B_n = \bigcup_{i=1}^n A_i \in \mathcal{F}$, $n = 1, 2, \dots$, and $\{B_n\}$ is a monotone sequence of sets in $\mathcal{F}$ which converges to $A \in \mathcal{F}$. If μ is continuous from below at A .

$$\sum_{i=1}^n \mu(A_i) = \mu(B_n) \rightarrow \mu(A) \text{ as } n \rightarrow \infty$$

So that again $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ and μ is σ -additive.

(3) In the notation of (2), put $G_n = A - B_n \in \mathcal{F}$, $n = 1, 2, \dots$. Then $\{G_n\}$ is a monotone decreasing sequence converging to ϕ and, for $n = 1, 2, \dots$

$$\mu(A) = \sum_{i=1}^n \mu(A_i) + \mu(G_n)$$

If μ is finite and continuous from above at ϕ we must have $\mu(G_n) \rightarrow 0$ as $n \rightarrow \infty$. So that again $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$.

Remarks

1. In our definition of continuity from above we only require to have $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ for those sequences $\{A_n\}$ decreases to A for which $\mu(A_n)$ is finite for some n . To see that we could not relax this finiteness condition, consider example (2.1.10) which we have already seen to be σ -additive with $\Omega = (0, 1)$. Then if $A_n = (0, \frac{1}{n})$, $n = 1, 2, \dots$, we have a sequence decreasing to ϕ such that $\mu(A_n) = \infty$ for all n since A_n is of the second category.
2. The condition that μ be finite cannot be omitted in theorem (4.17)



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The Probability

The starting point for probability theory is a set Ω called the sample space whose points are in one-to-one correspondence with the possible outcomes of a given performance of a random experiment.

Definition(4.18)

A measure on a σ -field $\mathcal{F}$ is a non-negative, extended real valued function P on $\mathcal{F}$ such that whenever $A_1, A_2, \dots$ form a finite or Countably infinite collection of disjoint sets in $\mathcal{F}$, we have $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

1. If $P(\Omega) = 1$, P is called a probability measure.
2. A measure space is a triple $(\Omega, \mathcal{F}, P)$ where Ω is a set, $\mathcal{F}$ is a σ -field on Ω , P is a measure on $\mathcal{F}$. If P is a probability measure, $(\Omega, \mathcal{F}, P)$ is called probability space.
3. If $(\Omega, \mathcal{F}, P)$ is a probability space, the set Ω is called the sample space, the subsets of Ω which belong to $\mathcal{F}$ are called events.

The first basic properties of a probability measure are collected in the following theorem.

Theorem(4.19)

Let $(\Omega, \mathcal{F}, P)$ be a probability space

1. $P(\emptyset) = 0$
2. If $A_1, A_2, \dots, A_n \in \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then $P(\bigcup_{k=1}^n A_k) = \sum_{k=1}^n P(A_k)$
3. $P(A^c) = 1 - P(A)$ for all $A \in \mathcal{F}$
4. $0 \leq P(A) \leq 1$ for all $A \in \mathcal{F}$



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5. $P(A) = P(A \cap B) + P(A \cap B^c)$ for all $A, B \in \mathcal{F}$

6. $P(A \cup B) + P(A \cap B) = P(A) + P(B)$ for all $A, B \in \mathcal{F}$

7. If $A, B \in \mathcal{F}$ and $A \subset B$, then

(a) $P(A) \leq P(B)$ (b) $P(B|A) = P(B) - P(A)$

8. $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$ for all $A, B \in \mathcal{F}$

9. If $A_n \in \mathcal{F}$, $n \geq 1$, then $P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$

Proof :

(1) Put $A_1 = \Omega$, $A_n = \phi$ for all $n \geq 2 \Rightarrow \Omega = \bigcup_{n=1}^{\infty} A_n$, $\phi = \bigcup_{n=2}^{\infty} A_n$

$P(\phi) = P(\bigcup_{n=2}^{\infty} A_n) = \sum_{n=2}^{\infty} P(A_n)$ and $P(\Omega) = P(\bigcup_{n=1}^{\infty} A_n) = P(A_1) + \sum_{n=2}^{\infty} P(A_n) = P(\Omega) + P(\phi)$

$\Rightarrow P(\phi) = P(\Omega) - P(\Omega) = 0$

(2) Put $A_k = \phi$ for all $k \geq n+1 \Rightarrow \{A_k\}$ is sequence of disjoint sets in $\mathcal{F}$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^n A_k$

$P(\bigcup_{k=1}^n A_k) = P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k) = \sum_{k=1}^n P(A_k) + \sum_{k=n+1}^{\infty} P(A_k) = \sum_{k=1}^n P(A_k) + P(\phi) = \sum_{k=1}^n P(A_k)$

(3) Let $A \in \mathcal{F}$

Since $A \cup A^c = \Omega \Rightarrow P(A \cup A^c) = P(\Omega) = 1$

Since $A \cap A^c = \phi \Rightarrow P(A \cup A^c) = P(A) + P(A^c) \Rightarrow P(A) + P(A^c) = 1$

Thus $P(A^c) = 1 - P(A)$.

(4) Let $A \in \mathcal{F} \Rightarrow P(A) \geq 0$



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$$A^c \in \mathcal{F} \Rightarrow P(A^c) \geq 0 \Rightarrow 1 - P(A) \geq 0 \Rightarrow P(A) \leq 1 \Rightarrow 0 \leq P(A) \leq 1$$

(5) Let $A, B \in \mathcal{F}$

$$\text{Since } A = (A \cap B) \cup (A \cap B^c), (A \cap B) \cap (A \cap B^c) = \phi$$

$$P(A) = P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c)$$

(6) Let $A, B \in \mathcal{F}$

$$\text{Since } A \cup B = (A \cap B^c) \cup B, (A \cap B^c) \cap B = \phi$$

$$P(A \cup B) = P((A \cap B^c) \cup B) = P(A \cap B^c) + P(B)$$

$$\text{Put } P(A) = P(A \cap B) + P(A \cap B^c)$$

$$\text{Therefore } P(A \cup B) + P(A \cap B) = P(A) + P(B).$$

$$(7) \text{ Since } \Rightarrow A \subseteq B \Rightarrow B = A \cup (B \cap A^c) \Rightarrow P(B) = P(A) + P(B \cap A^c)$$

$$(a) \text{ Since } P(A \cap B^c) \geq 0 \Rightarrow P(A) \leq P(B)$$

$$(b) \text{ From (a), we have } P(B \cap A^c) = P(B) - P(A)$$

$$\text{Since } B \cap A^c = B \setminus A \Rightarrow P(B \setminus A) = P(B) - P(A)$$

$$(8) \text{ since } A \cap B \subseteq A \Rightarrow P(A \cap B) \leq P(A) \text{ and } A \subseteq A \cup B \Rightarrow P(A) \leq P(A \cup B)$$

$$\text{since } P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq 0 \Rightarrow P(A \cup B) \leq P(A) + P(B)$$

$$\text{Thus } P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

$$(9) \bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \cup \dots$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \dots$$

$$\text{Thus } P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

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Theorem(4.20)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of sets in $\mathcal{F}$. If $A_n \uparrow A$ or $A_n \downarrow A$, then $P(A_n) \rightarrow P(A)$

Proof :

(1) If $A_n \uparrow A$. Put $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$

$$\Rightarrow B_n \cap B_m = \emptyset \text{ for all } n \neq m \text{ and } \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A$$

$$\begin{aligned} P(A) &= P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n) = P(B_1) + \sum_{n=2}^{\infty} P(B_n) \\ &= P(B_1) + \sum_{n=2}^{\infty} P(A_n \setminus A_{n-1}) = P(A_1) + \sum_{n=2}^{\infty} (P(A_n) - P(A_{n-1})) \Rightarrow P(A) = \lim_{n \rightarrow \infty} P(A_n) \end{aligned}$$

(2) If $A_n \downarrow A$, then

$$A_1 \setminus A = A_1 \cap A^c = A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right) = \bigcup_{n=1}^{\infty} (A_1 \cap A_n^c) = \bigcup_{n=1}^{\infty} (A_1 \setminus A_n)$$

Since $A_n \downarrow A$, then $A_1 \setminus A_n \uparrow A_1 \setminus A \Rightarrow P(A_1 \setminus A_n) \rightarrow P(A_1 \setminus A)$, so $P(A_n) \rightarrow P(A)$.

Theorem(4.21) Borel-Cantelli Lemma

Let $(\Omega, \mathcal{F}, P)$ be a measure space and let $\{A_n\}$ be a sequence of sets in $\mathcal{F}$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$,

then $P(\limsup_{n \rightarrow \infty} A_n) = 0$

Proof :

$$\text{Since } \limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k \Rightarrow P(\limsup_{n \rightarrow \infty} A_n) \leq P\left(\bigcup_{k=n}^{\infty} A_k\right) \text{ for all } n$$

$$\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) \leq \sum_{k=n}^{\infty} P(A_k) \rightarrow 0 \text{ as } n \rightarrow \infty.$$



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Definition(4.22)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A, B \in \mathcal{F}$. We say that A and B are independent if,

$$P(A \cap B) = P(A)P(B).$$

Theorem(4.23)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A, B \in \mathcal{F}$. Then the following conditions are equivalent:

- (1) A, B are independent
- (2) A, B^c are independent
- (3) A^c, B^c are independent

Proof :

$$(1) \Rightarrow (2) \quad \text{Since } A, B \text{ are independent} \Rightarrow P(A \cap B) = P(A)P(B)$$

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

$$\Rightarrow A, B^c \text{ are independent}$$

$$(2) \Rightarrow (3) \quad \text{Since } A, B^c \text{ are independent} \Rightarrow P(A \cap B^c) = P(A)P(B^c)$$

$$P(A^c \cap B^c) = P(B^c) - P(A \cap B^c) = P(B^c) - P(A)P(B^c) = (1 - P(A))P(B^c) = P(A^c)P(B^c)$$

$$\Rightarrow A^c, B^c \text{ are independent}$$

$$(3) \Rightarrow (1) \quad \text{Since } A^c, B^c \text{ are independent} \Rightarrow P(A^c \cap B^c) = P(A^c)P(B^c)$$

$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - P(A^c)P(B^c)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = P(A) + P(B) - 1 + P(A^c)P(B^c)$$

$$= -(1 - P(A)) + P(B) + P(A^c)P(B^c) = -P(A^c) + P(A^c)P(B^c) + P(B)$$

$$= -P(A^c)(1 - P(B^c)) + P(B) = -P(A^c)P(B) + P(B) = (-1 + P(A))P(B) = P(A)P(B)$$

$$\Rightarrow A, B \text{ are independent.}$$



Definition(4.24)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots, A_n \in \mathcal{F}$. We say that $A_1, A_2, \dots, A_n$ are

(1) Pair wise Independent, if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$

(2) Mutually Independent, if for all integers $\{i_1, i_2, \dots, i_k\}$ such that $1 \leq i_1 < i_2 < \dots < i_k \leq n$

the relations $P(\bigcap_{j=1}^k A_{i_j}) = \prod_{j=1}^k P(A_{i_j})$ hold.

Definition(4.25)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{A_i : i \in I\}$ of events in $\mathcal{F}$ is called an independent family (w.r.s.p) if for every $n \geq 2$ the event $A_1, A_2, \dots, A_n$ are Mutually Independent.

In other words : A family $\{A_i : i \in I\}$ of events in $\mathcal{F}$ is called an independent family (w.r.s.p) if for every finite subset $\{i_1, \dots, i_k\} \neq \emptyset$ of I , we have

$$P(\bigcap_{j=1}^k A_{i_j}) = \prod_{j=1}^k P(A_{i_j})$$

Definition(4.26)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let I be an arbitrary set. The family $\{\mathcal{F}_i : i \in I\}$ of sets in $\mathcal{F}$ are said to be independent if given any choice of $A_i \in \mathcal{F}_i, i \in I$, the events A_i are independent.

In other words : A family $\{\mathcal{F}_i : i \in I\}$ of sub σ -field of $\mathcal{F}$ is said to be independent if for every finite subset $\{i_1, \dots, i_k\}$ of I and $A_{i_j} \in \mathcal{F}_{i_j}$, we have

$$P(\bigcap_{j=1}^k A_{i_j}) = \prod_{j=1}^k P(A_{i_j})$$



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Theorem(4.27) Second Borel-Cantelli Lemma

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of independent events in $\mathcal{F}$.

If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 1$

Proof :

$$\text{Since } \limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k$$

$$\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} P\left(\bigcup_{k=n}^m A_k\right)\right)$$

Now

$$P\left(\bigcup_{k=n}^m A_k\right)^c = P\left(\bigcap_{k=n}^m A_k^c\right) = \prod_{k=n}^m P(A_k^c)$$

$$\text{Since } P(A_k^c) = 1 - P(A_k) \leq e^{-P(A_k)} \Rightarrow P\left(\bigcup_{k=n}^m A_k\right)^c \leq \prod_{k=n}^m e^{-P(A_k)} = e^{-\sum_{k=n}^m P(A_k)}$$

$$\text{Since } \sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P\left(\bigcup_{k=n}^m A_k\right)^c \rightarrow 0 \text{ as } m \rightarrow \infty \Rightarrow P\left(\bigcup_{k=n}^m A_k\right) \rightarrow 1 \text{ as } m \rightarrow \infty$$



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Exercises(4)

1. Show that if $\mathcal{R}$ is any finite ring of subsets of a set Ω and μ is finitely additive on $\mathcal{R}$ then μ is σ -additive on $\mathcal{R}$.
2. Let $\mathbb{Z}$ be the set of all integer numbers, $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms.
Put $\mu(A) = \sum_{n \in A} a_n$, if A is a finite subset of $\mathbb{Z}$, $\mu(A) = \infty$, if A is an infinite subset of $\mathbb{Z}$.
Show that μ is finitely additive, but not σ -additive on the family of all subsets of $\mathbb{Z}$.
3. Let $\mathbb{Z}$ be the set of all integer numbers, for $A \subset \mathbb{Z}$ let $\mu_n(A)$ be the number of integer in A which are not greater than n , let $\mathcal{F}$ be the family of subsets A for which $\lim_{n \rightarrow \infty} \frac{\mu_n(A)}{n} = \mu(A)$ exists, show that μ is finitely additive, but not σ -additive on $\mathcal{F}$, but that $\mathcal{F}$ is not even a semi-ring.
4. If μ is faintly additive on a ring $\mathcal{R}$, $A, B, C \in \mathcal{R}$ show
$$\mu(A) + \mu(B) + \mu(C) + \mu(A \cap B \cap C) = \mu(A \cup B \cup C) + \mu(A \cap B) + \mu(A \cap C) + \mu(B \cap C)$$

State prove a relationship of this kinds for n subsets of $\mathcal{R}$.
5. Suppose $\mathcal{R}$ is a σ -ring of subsets of Ω , μ is a measure on $\mathcal{R}$. Show that the family of sets $A \in \mathcal{F}$ with $\mu(A)$ finite forms a ring, and the family of sets $A \in \mathcal{F}$ with $\mu(A)$ σ -finite forms a σ -ring.
6. If A is a set in $\mathcal{F}$ of σ -finite μ measure (where μ measure on $\mathcal{F}$) and $\mathcal{G} \subset \mathcal{F}$ where $\mathcal{G}$ is a family of disjoint subset of A . Show that the sub-family of these $B \in \mathcal{G}$ for which $\mu(B) > 0$ is countable.
7. Let μ be a non-negative finitely set function on the field $\mathcal{F}$. Show that



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(a) $\mu(\bigcup_{n=1}^n A_n) \leq \sum_{n=1}^n \mu(A_n)$ for all $A_1, A_2, \dots, A_n \in \mathcal{F}$

(b) $\mu(\bigcup_{n=1}^{\infty} A_n) \geq \sum_{n=1}^{\infty} \mu(A_n)$ for all disjoint sets $A_1, A_2, \dots \in \mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

(c) μ is finite iff it is bounded.

8. Let Ω be a countable infinite set, and let consist of all subsets of Ω sequence of subsets of a sets Ω . Define $\mu(A) = 0$ if A is finite, $\mu(A) = \infty$ if A is infinite. Show that

(a) M is finitely additive but not σ -additive on $\mathcal{F}$

(b) Ω is the limit of all increasing sequence of sets $A_n \in \mathcal{F}$ with $\mu(A_n) = 0$ for all n , but $\mu(\Omega) = \infty$

9. Let μ be a counting measure on Ω , where Ω is an infinite set. Show that there is a sequence of sets $A_n \downarrow \phi$ with $\lim_{n \rightarrow \infty} \mu(A_n) \neq 0$

10. Let Ω be a Countably infinite set, and let $\mathcal{F}$ be the field consisting of all finite subsets of Ω and their complements, if A is finite set, $\mu(A) = 0$, if A^c is finite, set $\mu(A) = 1$. Show that

(a) M is finitely additive but not σ -additive on $\mathcal{F}$

(b) Ω is the limit of all increasing sequence of sets $A_n \in \mathcal{F}$ with $\mu(A_n) = 0$ for all n , but $\mu(\Omega) = 1$

11. Let $\mathcal{F}$ be the field of finite disjoint unions of right-semi-closed intervals of $\mathbb{R}$, and define the set function μ on $\mathcal{F}$ as follow:

$$\mu(A) = \begin{cases} a, & A = (-\infty, a] \\ b-a, & A = \{a, b\} \\ -b, & A = (b, \infty) \\ 0, & A = \mathbb{R} \end{cases}, \quad \mu(\bigcup_{i=1}^n I_i) = \sum_{i=1}^n \mu(I_i) \text{ if } I_1, I_2, \dots, I_n \text{ are disjoint right-semi-closed intervals.}$$

Show that :



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- (a) M is finitely additive but not σ -additive on $\mathcal{F}$
- (b) M is finite but unbounded.
12. Let μ be a finite measure on the σ -field $\mathcal{F}$. If $A_n \in \mathcal{F}$, $n = 1, 2, \dots$ and $A = \lim_{n \rightarrow \infty} A_n$, show that $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.
13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\lambda(A) = \sup\{\mu(B) : B \subset A\}$, show that for any set $A \subset \Omega$, there is a measurable set $B \subset A$ with $\mu(B) = \lambda(B)$.
14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\lambda(A) = \sup\{\mu(B) : B \subseteq A, \mu(B) < \infty\}$, show that λ is a measure on $(\Omega, \mathcal{F})$.
15. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. If $A, B \in \mathcal{F}$ such that $\mu(A \Delta B) = 0$, show that $\mu(A) = \mu(B)$.
16. Let $(\Omega, \mathcal{F})$ be a measurable space, and let $x \in \Omega$ be some point. Define $\mu_x : \mathcal{F} \rightarrow \{0, 1\}$ by $\mu_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ for all $A \in \mathcal{F}$, show that μ is a measure.
17. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that
- (a) If $\mathcal{F}^* = \{A \cup B : A, B \in \mathcal{F}, \mu(B) = 0\}$, then $\mathcal{F}^*$ is a σ -field on Ω
- (b) Define $\mu^* : \mathcal{F}^* \rightarrow \mathbb{R}^*$ by $\mu^*(A \cup B) = \mu(A)$, then μ^* is a measure.
18. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{A_n\}$ be a sequence of set in Ω . Show that
- (a) $\mu(\liminf_{n \rightarrow \infty} A_n) \leq \liminf_{n \rightarrow \infty} \mu(A_n)$
- (b) $\mu(\limsup_{n \rightarrow \infty} A_n) \geq \limsup_{n \rightarrow \infty} \mu(A_n)$ provided $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$
19. Let $\mathcal{R}$ be a σ -ring on a set Ω and μ be a measure on $\mathcal{R}$, show that $\mathcal{F} = \{A \in \mathcal{R} : \mu(A) \text{ is } \sigma\text{-finite}\}$ is a σ -ring



5. Extension of Measures

Definition(5.1)

Let $\mathcal{F}$ be a family of subsets of a set Ω . A set function $\mu: \mathcal{F} \rightarrow \mathbb{R}^*$ is said to be monotone if

1. $\mu(\phi) = 0$
2. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

Example(5.2)

Let $\mu: \mathcal{F} \rightarrow \mathbb{R}^*$ be a set function. Show that

1. If μ is monotone, then it is non-negative
2. If $\mathcal{F}$ is a ring and μ is additive non-negative, then μ is monotone.

Proof :

1. Let $A \in \mathcal{F}$ since $\phi \subseteq A$ and μ is monotone, then $\mu(\phi) \leq \mu(A)$

Since μ is monotone, then $\mu(\phi) = 0$. So $0 \leq \mu(A)$ for all $A \in \mathcal{F}$, i.e. μ is non negative.

2. Let $A, B \in \mathcal{F}$ such that $A \subseteq B$

Since $A \subseteq B$, then $B = A \cup (B \setminus A)$

Since $\mathcal{F}$ is a ring, then $B \setminus A \in \mathcal{F}$ and $A \cup (B \setminus A) \in \mathcal{F}$

Since A and $A, (B \setminus A)$ are disjoint sets, then $\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$

Since μ is non negative, $\mu(B \setminus A) \geq 0$. So $\mu(A) \leq \mu(B)$, i.e. μ is monotone.

Definition (5.3)

Given two family $\mathcal{G} \subset \mathcal{F}$ of subsets of a set Ω and set functions $\lambda: \mathcal{G} \rightarrow \mathbb{R}^*, \mu: \mathcal{F} \rightarrow \mathbb{R}^*$, we say that μ is an extension of λ if, for all $A \in \mathcal{G}$, we have $\mu(A) = \lambda(A)$. Under the same condition we say that λ is the restriction of μ to $\mathcal{G}$.



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Remark

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For all $A \in \mathcal{F}$ define $\mu_A : \mathcal{F}_A \rightarrow \mathbb{R}^*$ by $\mu_A(B) = \mu(B)$ for all $B \in \mathcal{F}_A$, where $\mathcal{F}_A = \{B \in \mathcal{F} : B \subseteq A\}$. Then μ_A is the restriction of μ on A , and hence $(A, \mathcal{F}_A, \mu_A)$ is a measure space.

Definition (5.4)

Let $\mathcal{F}$ be a family of all subsets of a set Ω . The set function $\mu^* : \mathcal{F} \rightarrow \mathbb{R}^*$ is called an outer measure on Ω if,

1. $\mu^*(A) \geq 0$ for all $A \in \mathcal{F}$
2. $\mu^*(\phi) = 0$
3. If $A, B \in \mathcal{F}$ with $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$
4. $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$, whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$.

It is clear to show that $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$ for all $A, B \subset \Omega$. (we can construct a sequence $\{A_n\}$ of elements of $\mathcal{F}$ by setting $A_1 = A, A_2 = B$ and $A_k = \phi$ for all $k \geq 3$)

Remark

Every measure on the family of all subsets of a set Ω is an outer measure on Ω , but the converse is not true.

Example(5.5)

Let Ω be any set with more than one point, and $\mathcal{F}$ be a family of subsets of Ω . Define $\mu^* : \mathcal{F} \rightarrow \mathbb{R}^*$ by $\mu^*(\phi) = 0$, $\mu^*(A) = 1$ for all $A \neq \phi$. Then μ^* is an outer measure on Ω , but not measure.

Example(5.6)

Let μ^* be an outer measure on a set Ω , if $\mu^*(A) = 0$, show that $\mu^*(A \cup B) = \mu^*(B)$ for all $B \subseteq \Omega$.

Proof :

Since $B \subseteq A \cup B$, then $\mu^*(B) \leq \mu^*(A \cup B)$ (1)

Since $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$ for all $B \subseteq \Omega$

Since $\mu^*(A) = 0$, then $\mu^*(A \cup B) \leq 0 + \mu^*(B) = \mu^*(B)$ (2)

From (1) and (2), we have $\mu^*(A \cup B) = \mu^*(B)$

Theorem(5.7)

Let $\mathcal{F}$ be a ring on a set Ω , and μ be a measure on $\mathcal{F}$. For all $A \subseteq \Omega$, define

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{F}, n=1,2,\dots, A \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

Then μ^* is an outer measure on Ω and that $\mu^*(A) = \mu(A)$, for all $A \in \mathcal{F}$.

Proof :

Since $\mu^*(A)$ is the infimum over a set of a set of non-negative numbers, we have $\mu^*(A) \geq 0$ for all $A \subseteq \Omega$

Since $\mathcal{F}$ is a ring $\Rightarrow \phi \in \mathcal{F}$

Put $A_n = \phi$ for all $n=1,2,\dots \Rightarrow A_n \in \mathcal{F}$ for all $n=1,2,\dots$

$$\Rightarrow \mu(A_n) = \mu(\phi) = 0 \Rightarrow \mu^*(\phi) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0. \text{ But, } \mu^*(\phi) \geq 0 \Rightarrow \mu^*(\phi) = 0$$

Let $A \subseteq B \subseteq \Omega$, and $\{B_n\}$ be a sequence in $\mathcal{F}$ with $B \subseteq \bigcup_{n=1}^{\infty} B_n \Rightarrow A \subseteq \bigcup_{n=1}^{\infty} B_n$.

It follows that



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$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(B_n) \quad *$$

Hence $\mu^*(A)$ is a lower bounded of all sums involved in (*). But $\mu^*(B)$ is the infimum of the those sums $\Rightarrow \mu^*(A) \leq \mu^*(B)$

Let $\{A_n\}$ be a sequence of subsets of Ω . If $\mu^*(A_n) = \infty$ for some $n \geq 1$, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \text{ is obviously. If } \mu^*(A_n) < \infty \text{ for all } n \geq 1.$$

Given $\varepsilon > 0$: Since $\mu^*(A_n) < \infty$, we have $\mu^*(A_n) < \mu^*(A_n) + \varepsilon 2^{-n}$

It follows that $\mu^*(A_n) + \varepsilon 2^{-n}$ cannot be a lower bounded of all sums $\sum_{m=1}^{\infty} \mu(B_m)$, as $\{B_m\}$.

Hence, there exists $A_{n_k} \in \mathcal{F}$ with $A_n \subset \bigcup_k A_{n_k}$ and $\sum_{k=1}^{\infty} \mu(A_{n_k}) \leq \mu^*(A_n) + \varepsilon 2^{-n}$

It is important to assume $\mu^*(A_n) < \infty$, since otherwise the inequality $\mu^*(A_n) < \mu^*(A_n) + \varepsilon 2^{-n}$ may not be a strict inequality.

$$\Rightarrow \bigcup_n A_n \subset \bigcup_{n,k} A_{n_k} \Rightarrow \mu^*\left(\bigcup_n A_n\right) \leq \mu^*\left(\bigcup_{n,k} A_{n_k}\right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{n_k})$$

$$\Rightarrow \mu^*\left(\bigcup_n A_n\right) \leq \sum_{n=1}^{\infty} (\mu^*(A_n) + \varepsilon 2^{-n}) = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon, \quad \varepsilon \text{ arbitrary}$$

Now to prove $\mu^* = \mu$ on $\mathcal{F}$.

If $A \in \mathcal{F}$, then $\mu^*(A) \leq \mu(A)$ by the definition of μ^* , and if $A \subset \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{F}$, then

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Take the infimum over all such coverings of A to obtain $\mu(A) \leq \mu^*(A)$. Hence $\mu^* = \mu$.



Definition (5.8)

Let μ^* be an outer measure on the set Ω . We say that the set A is μ^* -measurable (or measurable with respect to μ^*) if $\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c)$ for all $T \subseteq \Omega$.

Note that

Since $T = (T \cap A) \cup (T \cap A^c)$ for all $A, T \subseteq \Omega$, by the subadditivity condition on outer measure $\mu^*(T) = \mu^*((T \cap A) \cup (T \cap A^c)) \leq \mu^*(T \cap A) + \mu^*(T \cap A^c)$

Hence A is μ^* -measurable iff $\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \cap A^c)$ for all $T \subseteq \Omega$

Theorem(5.9)

Let μ^* be an outer measure on the set Ω .

1. If $\mu^*(A) = 0$, then A is μ^* -measurable, and hence $\emptyset$ is μ^* -measurable.
2. If A is μ^* -measurable, so is A^c , and hence Ω is μ^* -measurable.

Proof :

1. Let $T \subseteq \Omega$

Since $T \cap A \subset A \Rightarrow \mu^*(T \cap A) \leq \mu^*(A) = 0$. Also $T \cap A^c \subset T \Rightarrow \mu^*(T \cap A^c) \leq \mu^*(T)$

$\Rightarrow \mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \cap A^c) \Rightarrow A$ is μ^* -measurable

Since $\mu^*(\emptyset) = 0$, then $\emptyset$ is μ^* -measurable.

2. Let $T \subseteq \Omega$, Since A is μ^* -measurable

$\Rightarrow \mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \cap A^c) = \mu^*(T \cap A^c) + \mu^*(T \cap A) = \mu^*(T \cap A^c) + \mu^*(T \cap (A^c)^c)$

$\Rightarrow A^c$ is μ^* -measurable, Since $\emptyset$ is μ^* -measurable and $\Omega = \emptyset^c$, then Ω is μ^* -measurable.

Theorem(5.10)

Let μ^* be an outer measure on the set Ω . If A and B are μ^* -measurable, then $A \cap B$,

$A \cup B$ and $A \setminus B$ are μ^* -measurable

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Proof :

(1) Let $T \subseteq \Omega \Rightarrow T \cap A \subset \Omega$, Since B is μ^* -measurable, then

$$\mu^*(T \cap A) = \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c) \quad \dots (1)$$

Since $A \cap B \subset A$, we have $A^c \subseteq (A \cap B)^c \Rightarrow T \cap A^c \subseteq T \cap (A \cap B)^c$

It follows

$$T \cap A^c = (T \cap (A \cap B)^c) \cap (T \cap A^c) = T \cap (A \cap B)^c \cap A^c \quad \dots (2)$$

Since $(A \cap B)^c \cap A = (A^c \cup B^c) \cap A = A \cap B^c$, we obtain

$$T \cap (A \cap B)^c \cap A = T \cap A \cap B^c \quad \dots (3)$$

Since $T \cap (A \cap B)^c \subset \Omega$ and A is μ^* -measurable

$$\mu^*(T \cap (A \cap B)^c) = \mu^*(T \cap (A \cap B)^c \cap A) + \mu^*(T \cap (A \cap B)^c \cap A^c)$$

$$\mu^*(T \cap (A \cap B)^c) = \mu^*(T \cap A \cap B^c) + \mu^*(T \cap A^c) \quad \dots (4)$$

$$\begin{aligned} \mu^*(T \cap (A \cap B)) + \mu^*(T \cap (A \cap B)^c) &= \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c) + \mu^*(T \cap A^c) \\ &= \mu^*(T \cap A) + \mu^*(T \cap A^c) \end{aligned}$$

Since $\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c)$, we have

$$\mu^*(T) = \mu^*(T \cap A \cap B) + \mu^*(T \cap (A \cap B)^c)$$

$\Rightarrow A \cap B$ is μ^* -measurable

(2) Since A and B are μ^* -measurable, then A^c and B^c are μ^* -measurable

$\Rightarrow A^c \cap B^c$ is μ^* -measurable $\Rightarrow A \cup B = (A^c \cap B^c)^c$ is μ^* -measurable

(3) $A \setminus B = A \cap B^c$, so $A \setminus B$ is μ^* -measurable

Theorem(5.11)

Let μ^* be an outer measure on the set Ω . If A is μ^* -measurable and B any subset of Ω such that $A \cap B = \emptyset$, then $\mu^*(T \cap (A \cup B)) = \mu^*(T \cap A) + \mu^*(T \cap B)$ for all $T \subseteq \Omega$.

Proof:

Since $T \cap (A \cup B) \subseteq \Omega$ and A is μ^* -measurable, then

$$\mu^*(T \cap (A \cup B)) = \mu^*((T \cap (A \cup B)) \cap A) + \mu^*((T \cap (A \cup B)) \cap A^c)$$

Since $A \cap B = \emptyset \Rightarrow B \subset A^c$, we have $\mu^*(T \cap (A \cup B)) = \mu^*(T \cap A) + \mu^*(T \cap B)$

Theorem(5.12)

Let μ^* be an outer measure on the set Ω , and denoted by $\mathcal{F}$ the family of μ^* -measurable sets. Then $\mathcal{F}$ is a σ -field on Ω and μ^* is a measure on $\mathcal{F}$.

Proof:

Since Ω is μ^* -measurable, then $\Omega \in \mathcal{F}$

Let $A \in \mathcal{F} \Rightarrow A$ is μ^* -measurable, then A^c is μ^* -measurable. So that $A^c \in \mathcal{F}$

Let $A_n \in \mathcal{F}$ for $n = 1, 2, \dots$. Define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (\bigcup_{k=1}^n A_k)$ for $n = 1, 2, \dots$ $\{B_n\}$ is a

sequence of Pairwise disjoint sets in $\mathcal{F}$ with $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

Take $\mathcal{G}_m = \bigcup_{n=1}^m B_n \Rightarrow \mathcal{G}_m \in \mathcal{F}$ for all m

$$\mu^*(T \cap \mathcal{G}_m) = \sum_{n=1}^m \mu^*(T \cap B_n) \quad \dots (1)$$

Put $B = \bigcup_{n=1}^{\infty} B_n \Rightarrow \mathcal{G}_m \subset B \Rightarrow T \cap B^c \subset T \cap \mathcal{G}_m^c$

$$\mu^*(T \cap B^c) \leq \mu^*(T \cap \mathcal{G}_m^c) \quad \dots (2)$$

From (1) and (2), we have $\mu^*(T \cap B^c) + \sum_{n=1}^m \mu^*(T \cap B_n) \leq \mu^*(T \cap \mathcal{G}_m^c) + \mu^*(T \cap \mathcal{G}_m)$

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Since $G_m \in \mathcal{F} \Rightarrow \mu^*(T \cap G_m^c) + \mu^*(T \cap G_m) = \mu^*(T) \Rightarrow \mu^*(T \cap B^c) + \sum_{n=1}^m \mu^*(T \cap B_n) \leq \mu^*(T)$

Taking the limit as $m \rightarrow \infty$, we conclude $\mu^*(T \cap B^c) + \sum_{n=1}^{\infty} \mu^*(T \cap B_n) \leq \mu^*(T)$

Since $\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B) \leq \mu^*(T \cap B^c) + \sum_{n=1}^{\infty} \mu^*(T \cap B_n) \leq \mu^*(T) \dots (3)$

$\mu^*(T) = \mu^*(T \cap B^c) + \mu^*(T \cap B)$ for all $T \subset \Omega$

$\Rightarrow B \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \Rightarrow \mathcal{F}$ is a σ -filed.

Now we shall prove μ^* is a measure on $\mathcal{F}$.

Let $\{B_n\}$ be a sequence of Pairwise disjoint sets in $\mathcal{F}$ and $B = \bigcup_{n=1}^{\infty} B_n$

from (3), we have $\mu^*(T) = \mu^*(T \cap B^c) + \sum_{n=1}^{\infty} \mu^*(T \cap B_n)$ for all $T \subset \Omega$

Take $T = B$, we have $\mu^*(B) = \mu^*(B \cap B^c) + \sum_{n=1}^{\infty} \mu^*(B \cap B_n) = \mu^*(\phi) + \sum_{n=1}^{\infty} \mu^*(B_n)$

Since $\mu^*(\phi) = 0$, then $\mu^*(B) = \sum_{n=1}^{\infty} \mu^*(B_n)$

It follows that μ^* is σ -additive on $\mathcal{F}$.

Since μ^* is an outer measure on the set Ω , then $\mu^*(A) \geq 0$ for all $A \in \mathcal{F}$,

so that μ^* is a measure on $\mathcal{F}$.

Definition(5.13)

The measure space $(\Omega, \mathcal{F}, \mu)$ is said to be complete if $\mathcal{F}$ contains all subsets of sets measure zero, i.e. if $A \in \mathcal{F}$ and $\mu(A) = 0$, then $B \in \mathcal{F}$ for all $B \subseteq A$.

Theorem(5.14)

Every measure space has completion

Proof :

The completion of a measure space $(\Omega, \mathcal{F}, \mu)$ is defined as follows. Let $\mathcal{F}_\mu$ be the family of sets $M \cup N$, where M ranges over $\mathcal{F}$ and N over all subsets of sets of measure zero in $\mathcal{F}$, i.e.

$$\mathcal{F}_\mu = \{M \cup N : M \in \mathcal{F}, N \subset K \in \mathcal{F}, \mu(K) = 0\}$$

Define $\mu^* : \mathcal{F}_\mu \rightarrow \mathbb{R}$ by $\mu^*(A) = \mu(M)$ for all $A = M \cup N \in \mathcal{F}_\mu$. We must prove $(\Omega, \mathcal{F}_\mu, \mu^*)$ is a complete measure space.

$$\text{Let } A \in \mathcal{F} \Rightarrow A = A \cup \emptyset \Rightarrow A \in \mathcal{F}_\mu \Rightarrow \mathcal{F} \subset \mathcal{F}_\mu$$

(1) $\mathcal{F}_\mu$ is σ -field on Ω

Since $\Omega \in \mathcal{F}$ and $\mathcal{F} \subset \mathcal{F}_\mu$, then $\Omega \in \mathcal{F}_\mu$

Let $A \in \mathcal{F}_\mu \Rightarrow A = M \cup N$ such that $M \in \mathcal{F}, N \subset K \in \mathcal{F}, \mu(K) = 0$

$$\Rightarrow A^c = (M \cup N)^c = M^c \cap N^c = (M^c \cap K^c) \cup (M^c \cap (N^c \setminus K^c))$$

$$\text{and } M^c \cap (N^c \setminus K^c) = M^c \cap (K \setminus N) \subseteq K \Rightarrow A^c \in \mathcal{F}_\mu$$

Let $A_n \in \mathcal{F}_\mu, n = 1, 2, \dots \Rightarrow A_n = M_n \cup N_n$ such that $M_n \in \mathcal{F}, N_n \subset K_n \in \mathcal{F}, \mu(K_n) = 0$

$$\Rightarrow \bigcup_{n=1}^{\infty} M_n \in \mathcal{F} \text{ and } \bigcup_{n=1}^{\infty} N_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (M_n \cup N_n) = (\bigcup_{n=1}^{\infty} M_n) \cup (\bigcup_{n=1}^{\infty} N_n) \in \mathcal{F}_\mu$$

$\Rightarrow \mathcal{F}_\mu$ is σ -field on Ω .

(2) μ^* is a measure on $\mathcal{F}_\mu$

$$\text{Let } A_i \in \mathcal{F}_\mu, i = 1, 2 \Rightarrow A_i = M_i \cup N_i \text{ such that } M_i \in \mathcal{F}, N_i \subset K_i \in \mathcal{F}, \mu(K_i) = 0$$

$$\text{If } A_1 = A_2 \Rightarrow M_1 \cup N_1 = M_2 \cup N_2$$

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$$\text{Since } M_1 = (M_1 \cap M_2) \cup (M_1 \cap M_2^c) \Rightarrow \mu(M_1) = \mu(M_1 \cap M_2) + \mu(M_1 \cap M_2^c)$$

Since $M_1 \cap M_2 \subset N_2 \Rightarrow \mu(M_1 \cap M_2) \leq \mu(N_2)$. This $\mu(M_1) \leq \mu(M_2)$, and by symmetry ,
 $\mu(M_1) = \mu(M_2)$, $\mu^*(A_1) = \mu^*(A_2) \Rightarrow \mu^*$ is well defined.

Let $A \in \mathcal{F}_\mu \Rightarrow A = M \cup N$ such that $M \in \mathcal{F}$, $N \subset K \in \mathcal{F}$, $\mu(K) = 0$

Since $\mu^*(A) = \mu(M)$ and μ is a measure on $\mathcal{F}$, then $\mu(M) \geq 0 \Rightarrow \mu^*(A) \geq 0$

Let $\{A_n\}$ be a sequence of disjoint sets in $\mathcal{F}_\mu$

$$\Rightarrow A_n = M_n \cup N_n \text{ such that } M_n \in \mathcal{F}, N_n \subset K_n \in \mathcal{F}, \mu(K_n) = 0$$

$$\text{Since } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (M_n \cup N_n) = (\bigcup_{n=1}^{\infty} M_n) \cup (\bigcup_{n=1}^{\infty} N_n) \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} M_n)$$

Since $A_n = M_n \cup N_n$ and $A_n \cap A_m = \emptyset$ for all $n \neq m$, then $M_n \cap M_m = \emptyset$ for all $n \neq m$

$$\text{Since } \mu \text{ is a measure on } \mathcal{F}, \text{ then } \mu(\bigcup_{n=1}^{\infty} M_n) = \sum_{n=1}^{\infty} \mu(M_n) \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$$

(3) The measure space $(\Omega, \mathcal{F}_\mu, \mu^*)$ is complete

Let $B \subseteq A$ such that $A \in \mathcal{F}_\mu$ and $\mu(A) = 0$, we want to show that $B \in \mathcal{F}_\mu$

Since $A \in \mathcal{F}_\mu \Rightarrow A = M \cup N$ such that $M \in \mathcal{F}$, $N \subset K \in \mathcal{F}$, $\mu(K) = 0$

$$\text{Since } \mu(A) = 0 \Rightarrow \mu(M \cup N) = 0 \Rightarrow \mu(M) = 0 \Rightarrow \mu(M \cup K) = 0$$

$$\text{Since } A = M \cup N \text{ and } N \subset K \Rightarrow A = M \cup N \subseteq M \cup K$$

$$\text{Since } B \subseteq A \Rightarrow B \subseteq M \cup K \Rightarrow B \in \mathcal{F}_\mu$$

Theorem(5.15) Caratheodory's Extension Theorem

Let μ be σ -finite measure on a field $\mathcal{F}$, then μ has a unique extension to a measure on $\sigma(\mathcal{F})$

Proof :

For all $A \subseteq \Omega$, define

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$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{F}, n = 1, 2, \dots, A \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

Then μ^* is an outer measure on Ω and that $\mu^*(A) = \mu(A)$, for all $A \in \mathcal{F}$.

Theorem (5.16)

The space $(\Omega, \mathcal{F}^*, \mu^*)$ is completion of $(\Omega, \sigma(\mathcal{F}_0), \mu^*)$.

Definition(5.17)

Let $\mathcal{F}$ be a ring on a set Ω , μ be a measure on Ω , and μ^* be an outer measure on Ω . The inner measure on Ω , denoted by μ_* and defined by

$$\mu_*(A) = \sup \{ \mu^*(B) - \mu^*(B \setminus A) : B \in \mathcal{F}, \mu^*(B \setminus A) < \infty \}$$

Then $\mathcal{F}$ is a σ -field on Ω and μ^* is a measure on $\mathcal{F}$.

Theorem(5.18)

Let $\mathcal{F}$ be a ring on a set Ω , μ be a measure on Ω , μ^* be an outer measure on Ω and μ_* be an outer measure on Ω , then

- (1) $\mu_*(A) \leq \mu^*(A)$ for all $A \subset \Omega$
- (2) If $A \in \mathcal{F}$, then $\mu_*(A) = \mu^*(A)$
- (3) If $A \subseteq B$, then $\mu_*(A) \leq \mu_*(B)$

Exercises(5)

1. Show that if μ^* is an outer measure on Ω , A_0 any fixed subset of Ω , then

$$\mu_0^*(A) = \mu^*(A \cap A_0) \text{ defines another outer measure on } \Omega.$$

2. Show that if μ_1^*, μ_2^* are outer measure on Ω , so is μ^* defined by

$$\mu^*(A) = \max \{ \mu_1^*(A), \mu_2^*(A) \} \text{ for all } A \subseteq \Omega$$

3. Let μ^* be an outer measure on a set Ω , and let $\mathcal{F}$ the family of μ^* -measurable sets.

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Show that

(1) If $A, B \in \mathcal{F}$, then $\mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B)$

(2) If $A, B \subseteq \Omega$ such that $\mu^*(A) = \mu^*(B)$ and $A \in \mathcal{F}$, then $\mu^*(B|A) = 0$

4. Let $\mathcal{G}$ the family of subsets of a set Ω , μ a nonnegative real valued set function on $\mathcal{G}$ such that $\mathcal{G}$ and μ satisfy the following conditions

(1) $\phi \in \mathcal{G}$ and $\mu(\phi) = 0$ (2) $\Omega \in \mathcal{G}$ and $\mu(\Omega) = 1$ (3) $0 \leq \mu(A) \leq 1$ for all $A \in \mathcal{G}$

(4) If $A, B \in \mathcal{G}$, then $A \cup B, A \cap B \in \mathcal{G}$ and $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$

(5) If $A, B \in \mathcal{G}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$

(6) If $A_n \in \mathcal{G}$, $n = 1, 2, \dots$ and $A_n \uparrow A$ then $A \in \mathcal{G}$ and $\mu(A_n) \rightarrow \mu(A)$

Define $\mu^*(A) = \inf\{\mu(G) : G \in \mathcal{G}, G \supset A\}$ for each $A \subseteq \Omega$. Show that

(1) $\mu^* = \mu$ on $\mathcal{G}$

(2) $0 \leq \mu^*(A) \leq 1$ for all $A \subseteq \Omega$

(3) $\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*(A) + \mu^*(B)$ In particular

$$\mu^*(A) + \mu^*(A^c) \geq \mu^*(A \cup A^c) + \mu^*(A \cap A^c) = \mu^*(\Omega) + \mu^*(\phi) = \mu(\Omega) + \mu(\phi) = 1 + 0 = 1$$

(4) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$

(5) If $A_n \uparrow A$ then $\mu^*(A_n) \rightarrow \mu^*(A)$

5. Under the hypothesis of the exercise 4. Show that

(1) μ^* is an outer measure on Ω

(2) $\mathcal{F} = \{A \subseteq \Omega : \mu^*(A) + \mu^*(A^c) = 1\}$, then $\mathcal{F}$ is a σ -field and μ^* is a probability measure on $\mathcal{F}$.

5.6 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{F}_\mu = \{A : N \subset A \subset M, N, M \in \mathcal{F}, \mu(M|N) = 0\}$.

For any $A \in \mathcal{F}_\mu$, set $\mu^*(A) = \mu(N) = \mu(M)$ for any pair of sets $N, M \in \mathcal{F}$ with $N \subset A \subset M$ and

$\mu(M|N) = 0$. Show that $(\Omega, \mathcal{F}_\mu, \mu^*)$ is a complete measure space.

6. Lebesgue Measure

Definition (6.1)

Let D be a subset of a set of real numbers $\mathbb{R}$ and let $f : D \rightarrow \mathbb{R}$ be a function . A real number L is said to be

- Limit of f at point $a \in D$, if every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in D$ with $|x - a| < \delta$ we have $|f(x) - L| < \varepsilon$. We can write " the limit of f as x approaches a is L " as

$$\lim_{x \rightarrow a} f(x) = L$$

- Right-hand limit of f at point $a \in D$, if every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in D$ with $a < x < a + \delta$ we have $|f(x) - L| < \varepsilon$. We can write " the right-hand limit of f as x approaches a is L " as

$$\lim_{x \rightarrow a^+} f(x) = L$$

In other word $\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a + h)$, $h > 0$, and denoted by $f(a^+)$, i.e.

$$f(a^+) = \lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a + h), \text{ where } h > 0$$

- Left hand limit of f at point $a \in D$, if every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in D$ with $a - \delta < x < a$ we have $|f(x) - L| < \varepsilon$. We can write " the left-hand limit of f as x approaches a is L " as

$$\lim_{x \rightarrow a^-} f(x) = L$$

In other word $\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a + h)$, where $h < 0$, and denoted by $f(a^-)$, i.e.

$$f(a^-) = \lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a + h), \text{ where } h < 0$$

$$\lim_{x \rightarrow a} f(x) = L \text{ iff } \lim_{x \rightarrow a^+} f(x) = L \text{ and } \lim_{x \rightarrow a^-} f(x) = L$$



Definition (6.2)

Let D be a subset of a set of real numbers $\mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ is said to be a

- continuous at a point $a \in D$, if $\lim_{x \rightarrow a} f(x) = f(a)$. The function f is said to be a continuous if its continuous at every points in D .
- Right continuous at a point $a \in D$, if $f(a^+) = f(a)$.
- Left continuous at a point $a \in D$, if $f(a^-) = f(a)$.

f continuous at a point $a \in D$, iff it is right continuous and left continuous at a .

Definition(6.3)

Lebesgue -Stieltjes Measure on $\mathbb{R}$ is a measure μ on $\beta(\mathbb{R})$ such that $\mu(I) < \infty$ for each bounded interval

Definition(6.4)

A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is said to be distribution function on $\mathbb{R}$ if

1. f is an increasing, i.e. $a < b$ implies $F(a) \leq F(b)$
2. f is right continuous, i.e. $F(a^+) = \lim_{x \rightarrow a^+} F(x) = F(a)$ for all $a \in \mathbb{R}$

We will show that the formula $\mu(a, b] = F(b) - F(a)$ sets a 1-1 correspondence between the Lebesgue–Stieltjes measures and distributions where two distributions that differ by a constant are identified. of course, probability measures correspond to distributions.

Theorem (6.5)

Let μ be a Lebesgue–Stieltjes measure on $\mathbb{R}$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$, up to additive constants, by $F(b) - F(a) = \mu((a, b])$. For example, fix $F(0)$ arbitrary and set $F(x) - F(0) = \mu((0, x])$, $x \geq 0$, $F(0) - F(x) = \mu((x, 0])$, $x < 0$. Then F is distribution function.



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Proof :

(1) Let $a, b \in \mathbb{R}$ such that $a < b$

$$\Rightarrow F(b) - F(a) = \mu((a, b]) \geq 0 \Rightarrow F(b) - F(a) \geq 0 \Rightarrow F(a) \leq F(b)$$

$\Rightarrow F$ is an increasing function

(2) Let $\{x_n\}$ be a sequence of point such that $x_1 > x_2 > \dots \rightarrow x$, then

$$F(x_n) - F(x) = \mu((x, x_n]) \rightarrow 0 \text{ implying that } F \text{ is right continuous.}$$

• We should notice also that

$$\mu(\{a\}) = \lim_{n \rightarrow \infty} \mu((a - \frac{1}{n}, a]) = \lim_{n \rightarrow \infty} F(a) - F(a - \frac{1}{n}) = F(a) - F(a^-) \text{ for all } a \in \mathbb{R}. \text{ Hence in fact } F$$

is continued at $\{a\}$ iff $\mu(\{a\}) = 0$.

Remark

$$\begin{aligned} \mu((a, b)) &= F(b^-) - F(a), & \mu([a, b]) &= F(b) - F(a^-) \\ \mu([a, b)) &= F(b^-) - F(a^-), & \mu(\mathbb{R}) &= F(\infty) - F(-\infty) \end{aligned}$$

Theorem (6.6)

Suppose F is a distribution function on $\mathbb{R}$. There is a unique measure μ on $\beta(\mathbb{R})$ such that

$$\mu(a, b] = F(b) - F(a).$$

Proof :

Let $A = \{(a, b] : -\infty < a < b < \infty\}$. Set $F(\infty) = \lim_{x \uparrow \infty} F(x)$ and $F(-\infty) = \lim_{x \downarrow -\infty} F(x)$.

These quantities exist since F is increasing. Define for any $\mu(a, b] = F(b) - F(a)$, for any

$-\infty \leq a < b \leq \infty$, where $F(\infty) > -\infty, F(-\infty) < \infty$. Suppose $(a, b] = \bigcup_{i=1}^n (a_i, b_i]$, where the union is

disjoint. By relabeling we may assume that $a_1 = a, b_n = b, a_i = b_{i-1}$



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Then $\mu((a_i, b_i]) = F(b_i) - F(a_i)$ and $\sum_{i=1}^n \mu((a_i, b_i]) = \sum_{i=1}^n (F(b_i) - F(a_i)) = F(b) - F(a) = \mu((a, b])$

which proves that condition (1) holds.

For (2), let $-\infty < a < b < \infty$ and $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i)$ where the union is disjoint. (We can also order them if we want.) By right continuity of F , given $\varepsilon > 0$ there is a $\delta > 0$ such that $F(a + \delta) - F(a) < \varepsilon$, or equivalently, $F(a + \delta) < F(a) + \varepsilon$.

Similarly, there is a $\delta_i > 0$ such that $F(b_i + \delta_i) < F(b_i) + \varepsilon 2^{-i}$, for all i . Now, $\{(a_i, b_i + \delta_i)\}$ forms a open cover for $[a + \delta, b]$. By compactness, there is a finite subcover. Thus,

$$[a + \delta, b] \subset \bigcup_{i=1}^N (a_i, b_i + \delta_i) \text{ and } (a + \delta, b] \subset \bigcup_{i=1}^N (a_i, b_i + \delta_i).$$

Therefore

$$\begin{aligned} F(b) - F(a + \delta) &= \mu((a + \delta, b]) \leq \sum_{i=1}^N \mu((a_i, b_i + \delta_i)) = \sum_{i=1}^N (F(b_i + \delta_i) - F(a_i)) \\ &= \sum_{i=1}^N (F(b_i + \delta_i) - F(b_i) + F(b_i) - F(a_i)) \leq \sum_{i=1}^N \varepsilon 2^{-i} + \sum_{i=1}^N (F(b_i) - F(a_i)) \leq \varepsilon + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \end{aligned}$$

$$\text{Therefore, } \mu((a, b]) = F(b) - F(a) \leq 2\varepsilon + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) = 2\varepsilon + \sum_{i=1}^{\infty} \mu((a_i, b_i]),$$

proving (ii) provided $-\infty < a < b < \infty$.

- If $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i)$, a and b arbitrary, and $(c, d] \subset (a, b]$ for any $-\infty < c < d < \infty$, we have

by above $F(d) - F(c) = \sum_{i=1}^{\infty} (F(b_i) - F(a_i))$ and the result follows by taking limits.



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Example (6.7)

Let F be the distribution function defined by

$$F(x) = \begin{cases} 0, & x < -1 \\ 1+x, & -1 < x < 0 \\ 2+x^2, & 0 \leq x < 2 \\ 9, & x \geq 2 \end{cases}$$

and let μ be the Lebesgue–Stieltjes measure corresponding to F . Find $\mu(A)$ for

- (1) $A = \{2\}$, (2) $A = \{-\frac{1}{2}, 3\}$, (3) $A = (-1, 0] \cup (1, 2)$, (4) $A = \{x : |x| + 2x^2 > 1\}$.

Remark

Lebesgue measure on $\mathbb{R}$ is special case of Lebesgue–Stieltjes Measure on $\mathbb{R}$ when

$F(x) = x$ for all $x \in \mathbb{R}$

$$\text{If } F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1 \\ 1, & x > 1 \end{cases}$$

the measure we obtain is called the Lebesgue measure on $\Omega = (0, 1]$. Notice that $\mu(\Omega) = 1$.

- If μ is a probability measure then $F(x) = \mu((-\infty, x])$ and $\lim_{x \rightarrow \infty} F(x) = 1$. $\lim_{x \rightarrow -\infty} F(x) = 0$

Theorem (6.8)

Let μ be Lebesgue measure on $\mathbb{R}$

- (1) If A is a signal set in $\mathbb{R}$, then $\mu(A) = 0$, in general if A is countable set in $\mathbb{R}$, then $\mu(A) = 0$
- (2) If $A_1 = (a, b)$, $A_2 = (a, b]$, $A_3 = [a, b)$, $A_4 = [a, b]$, then $\mu(A_i) = b - a$ for all $i = 1, 2, 3, 4$
- (3) If $A_1 = (a, \infty)$, $A_2 = [a, \infty)$, $A_3 = (-\infty, b)$, $A_4 = (-\infty, b]$, then $\mu(A_i) = \infty$ for all $i = 1, 2, 3, 4$
- (4) $\mu(\mathbb{R}) = \mu(-\infty, \infty) = \infty$.

Proof :



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Since μ is a Lebesgue measure on $\mathbb{R} \Rightarrow F(x) = x \Rightarrow F$ is continuous function

(1) Let $A = \{a\}$ (A is a signal set in $\mathbb{R}$)

Since F is continuous function, then $\mu(\{a\}) = 0$

Let $A = \{a_1, a_2, a_3, \dots\}$, then $\mu(A) = \mu(\bigcup_{n=1}^{\infty} \{a_n\}) = \sum_{n=1}^{\infty} \mu(\{a_n\}) = \sum_{n=1}^{\infty} (0) = 0$

(2) From theorem (6), we have $\mu(A_2) = \mu(a, b] = F(b) - F(a)$

Also, since F is continuous function, then $f(a^+) = f(a^-) = f(a)$ for all $a \in \mathbb{R}$

$$\mu(A_1) = \mu((a, b)) = F(b^-) - F(a) = F(b) - F(a)$$

$$\mu(A_3) = \mu([a, b)) = F(b^-) - F(a^-) = F(b) - F(a)$$

$$\mu(A_4) = \mu([a, b]) = F(b) - F(a^-) = F(b) - F(a)$$

Exercise(6)

1. Given an example of a function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that F is right continuous and is increasing in each coordinate separately, but F is not a distribution function on $\mathbb{R}^2$.
2. If μ is σ -finite measure on $\beta(\mathbb{R}^n)$ Show that for each $A \in \beta(\mathbb{R}^n)$

$$\mu(A) = \sup\{\mu(B) : B \subset A, B \text{ compact}\}$$

3. If μ is in fact a Lebesgue – Stieltjes measure. Show that

$$\mu(A) = \inf\{\mu(B) : B \supset A, B \text{ open}\}$$

4. Given an example of a σ -finite measure on $\beta(\mathbb{R}^n)$ that is not a Lebesgue – Stieltjes Measure and for which (2) fails.



7. General Set Functions

Definition(7.1)

Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\lambda: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ is called a **signed measure** on $\mathcal{F}$, if λ is a Countably additive set function, i.e.

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Whenever $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$.

Remark

Every measure is signed measure and the converse not true.

Definition(7.2)

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$. A set $A \in \mathcal{F}$ is said to be a positive set (with respect to λ) if $\lambda(B) \geq 0$ for every measurable subset B of A . Similarly, a set A is called a negative set (with respect to λ) if $\lambda(B) \leq 0$ for every measurable subset B of A .

- A set that is both positive and negative (with respect to λ) is called **null set**, i.e. a measurable set is called a null set iff every measurable subset of it has λ measure zero.

Remark

The distinction between a null set and a set of measure zero, while every null set must have measure zero, a set of measure zero may well be a union of two sets whose measure are not zero but are negative of each other.

Theorem(7.3)

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$, and let A be a measurable set.

1. A is positive iff for every measurable set B , $A \cap B$ is measurable and $\lambda(A \cap B) \geq 0$



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2. A is negative iff for every measurable set B , $A \cap B$ is measurable and $\lambda(A \cap B) \leq 0$

Proof :

(1) Assume A is positive and let B is a measurable set is measurable set

Since A is measurable set $\Rightarrow A \cap B$ is measurable set.

Since A is positive set, $A \cap B \subseteq A$ and $A \cap B$ measurable $\Rightarrow \lambda(A \cap B) \geq 0$

Conversely, let $A \cap B$ is measurable and $\lambda(A \cap B) \geq 0$ for every measurable set B

Let C be a measurable and $C \subseteq A \Rightarrow C = A \cap C \Rightarrow \lambda(C) = \lambda(A \cap C) \geq 0$

Theorem(7.4)

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$

1. Every measurable subset of a positive (rsp. negative) set is positive (rsp. negative)
2. The union of a countable of positive (rsp. negative) sets is positive (rsp. negative)

Proof :

(1) Let A be a measurable subset of a positive set B , and let C be a measurable subset of $A \Rightarrow C \subseteq B$

Since B is positive $\Rightarrow \lambda(C) \geq 0 \Rightarrow A$ is positive

(2) Let $\{A_n\}$ be a sequence of positive sets and let $A = \bigcup_{n=1}^{\infty} A_n$

Let B be a measurable subset of A . Put $B_n = B \cap A_n \cap A_{n-1}^c \cap \dots \cap A_1^c$

$\Rightarrow B_n$ is measurable subset of A_n and so $\lambda(B_n) \geq 0$.

Since the B_n are disjoint and $B = \bigcup_{n=1}^{\infty} B_n$, we have $\lambda(B) = \sum_{n=1}^{\infty} \lambda(B_n) \geq 0 \Rightarrow A$ is positive.

Theorem(7.5)

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$

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1. There is a positive set B contained in A with $\lambda(B) > 0$
2. There is a negative set C contained in A with $\lambda(C) < 0$

Proof :

(1) If A is positive we are done

Let n_1 be smallest positive number such that there is an $A_1 \subseteq A$ with $\lambda(A_1) \leq -\frac{1}{n_1}$

Since $A \setminus A_1 \subseteq A$. If $A \setminus A_1$ is a positive with $\lambda(A \setminus A_1) > 0$ we are done. If not n_2 smallest integer such that there is an $A_2 \subseteq A \setminus A_1$ with $\lambda(A_2) \leq -\frac{1}{n_2}$.

Continue : Let n_k be smallest positive such that there is $A_k \subseteq A \setminus \bigcup_{j=1}^{k-1} A_j$ with $\lambda(A_k) \leq -\frac{1}{n_k}$

$$\text{Let } B = A \setminus \bigcup_{k=1}^{\infty} A_k \Rightarrow A = B \cup \left(\bigcup_{k=1}^{\infty} A_k \right)$$

$$\Rightarrow \lambda(A) = \lambda(B \cup \left(\bigcup_{k=1}^{\infty} A_k \right)) = \lambda(B) + \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda(B) + \sum_{k=1}^{\infty} \lambda(A_k)$$

Infinite series $\sum_{k=1}^{\infty} \lambda(A_k)$ is absolutely converge

Since $0 < \lambda(A) < \infty \Rightarrow \sum_{k=1}^{\infty} \lambda(A_k)$ is converges

But $\sum_{k=1}^{\infty} \frac{1}{n_k} < -\sum_{k=1}^{\infty} \lambda(A_k) < \infty \Rightarrow \sum_{k=1}^{\infty} \frac{1}{n_k}$ is converges , $n_k \rightarrow \infty$

Since $\lambda(A_k) \leq 0$ and $\lambda(A_k) \leq 0 \Rightarrow \lambda(B) > 0$

To prove B is positive : let $\varepsilon > 0$, since $n_k \rightarrow \infty$, may be choose a large number k such that $(n_k - 1)^{-1} < \varepsilon$



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Since $B = A \setminus \bigcup_{k=1}^{\infty} A_k$, then B can contain any measurable set with measuring less than $-(n_k - 1)^{-1}$ that of the largest $-\varepsilon$, so B does not contain measurable set measured less than $-\varepsilon$.

Since ε any positive number and not on the appointment and consequent can contain any measure sets a negative and this is not possible, B is positive s.

Theorem(7.6) Hahn- Decomposition

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$. There is a positive set A and a negative set B with $A \cap B = \emptyset$, $A \cup B = \Omega$

Proof :

Assume λ dose not take ∞

$v = \sup\{\lambda(A) : A \text{ is positive set with respect } \lambda\}$

Since ϕ is positive, then $v \geq 0$

Let $\{A_n\}$ be a sequence of positive sets such that $v = \lim_{n \rightarrow \infty} \lambda(A_n)$

Set $A = \bigcup_{n=1}^{\infty} A_n$, by using part(2) of theorem (7.4), we have A is positive, also $\lambda(A) \leq v$

Since $A \setminus A_n \subset A \Rightarrow \lambda(A \setminus A_n) \geq 0$ and $\lambda(A) = \lambda(A_n) + \lambda(A \setminus A_n) \geq \lambda(A_n)$, so $\lambda(A) \geq v$
 $\Rightarrow 0 \leq \lambda(A) = v < \infty \Rightarrow \lambda(A) \geq 0$

Let $B = A^c$. To prove B is negative

Let C be positive set and $C \subseteq B$, then $A \cap C = \emptyset$ and $A \cup C$ positive set

$\Rightarrow v \geq \lambda(A \cup C) = \lambda(A) + \lambda(C) = v + \lambda(C) \Rightarrow \lambda(C) = 0$

Since $0 \leq v < \infty$, and therefore B does not contain a positive subsets with a positive measurements. And therefore does not positively measurements subsets, so B is negative set.



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Remarks

1. The Hahn decomposition is not unique.
2. The Hahn decomposition A, B give two measures λ^+ and λ^- defined by

$$\lambda^+(C) = \lambda(A \cap C), \quad \lambda^-(C) = -\lambda(B \cap C)$$

Notice that $\lambda^+(B) = 0$ and $\lambda^-(A) = 0$. Clearly $\lambda = \lambda^+ - \lambda^-$

Exercises(7)

1. Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$. Then λ assumes a maximum and a minimum values, i.e., there are sets $C, D \in \mathcal{F}$ such that

$$\lambda(C) = \sup\{\lambda(A) : A \in \mathcal{F}\} \text{ and } \lambda(D) = \inf\{\lambda(A) : A \in \mathcal{F}\}$$

- If λ is a measure on the measurable space $(\Omega, \mathcal{F})$, the result is trivial :
take $C = \Omega$ and $D = \emptyset$.

2. Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$. Define

$$\lambda^+(A) = \sup\{\lambda(B) : B \in \mathcal{F}, B \subset A\} \text{ and } \lambda^-(A) = \inf\{\lambda(B) : B \in \mathcal{F}, B \subset A\}$$

for all $A \in \mathcal{F}$. Then

(1) λ^+ and λ^- are measures on $(\Omega, \mathcal{F})$, and

(2) $\lambda = \lambda^+ - \lambda^-$, $|\lambda| = \lambda^+ + \lambda^-$, $(-\lambda)^+ = \lambda^-$, $(-\lambda)^- = \lambda^+$

λ^+ is called the upper variation (or positive part) of λ , and λ^- is called the lower variation (or negative part) of λ , and $|\lambda|$ is called the total variation (or the absolute value) of λ .

3. In exercise 7.2 show that

(1) If $\lambda > -\infty$, then λ^- is finite; if $\lambda < +\infty$, λ^+ is finite

(2) $|\lambda(A)| \leq |\lambda|(A)$, $A \in \mathcal{F}$

(3) If $A \in \mathcal{F}$, then $|\lambda|(A) = 0$ iff $\lambda(B) = 0$ for all $B \in \mathcal{F}$, $B \subset A$.



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- (4) If λ is the difference of two measures λ_1 and λ_2 (i.e. $\lambda = \lambda_1 - \lambda_2$), then $\lambda_1 \geq \lambda^+$ and $\lambda_2 \geq \lambda^-$
4. Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$.
- (1) The set function λ is the difference of two measures, at least one of which is finite.
- (2) If λ is finite (i.e. $\lambda(A)$ is never $\pm\infty$ for any $A \in \mathcal{F}$), then λ is bounded.
- (3) There is a set $D \in \mathcal{F}$ such that $\lambda(A \cap D) \leq 0$ and $\lambda(A \cap D^c) \geq 0$ for all $A \in \mathcal{F}$
- (4) If D is any set in $\mathcal{F}$, such that $\lambda(A \cap D) \leq 0$ and $\lambda(A \cap D^c) \geq 0$ for all $A \in \mathcal{F}$, then
- $$\lambda^+(A) = \lambda(A \cap D^c) \text{ and } \lambda^-(A) = \lambda(A \cap D) \text{ for all } A \in \mathcal{F}$$
- (5) If C is another set in $\mathcal{F}$, such that $\lambda(A \cap C) \leq 0$ and $\lambda(A \cap C^c) \geq 0$ for all $A \in \mathcal{F}$, then
- $$|\lambda|(C \Delta D) = 0.$$
- (6) $|\lambda(A)| = \sup \left\{ \sum_{i=1}^n |\lambda(A_i)| : A_1, A_2, \dots, A_n \text{ disjoint measurable subsets of } A, i = 1, 2, \dots, n \right\}$
5. Let λ_1 and λ_2 be two signed measures on the measurable space $(\Omega, \mathcal{F})$. Show that
- $$|\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|$$



8. Measurable Functions

In mathematics, particularly in measure theory, measurable functions are structure-preserving functions between measurable spaces; as such, they form a natural context for the theory of integration. Specifically, a function between measurable spaces is said to be measurable if the pre image of each measurable set is measurable, analogous to the situation of continuous functions between topological spaces.

Definition(8.1)

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. A function $f: \Omega \rightarrow \Omega'$ is said to be measurable with respect to $\mathcal{F}$ and $\mathcal{F}'$, if $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$

The notation $f: (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ will mean that $f: \Omega \rightarrow \Omega'$, measurable with respect to $\mathcal{F}$ and $\mathcal{F}'$. In special case, if $\Omega' = \mathbb{R}^n$ and $\mathcal{F}' = \beta(\mathbb{R}^n)$, we say that $f: \Omega \rightarrow \mathbb{R}^n$

Definition(8.2)

Let $(\Omega, \mathcal{F})$ be a measurable space. The function $f: \Omega \rightarrow \bar{\mathbb{R}}$ is said to be Borel measurable on $(\Omega, \mathcal{F})$ if f is measurable with respect to $\mathcal{F}$ and $\beta(\bar{\mathbb{R}})$, and we write $f \in \mathcal{F}$.

Example(8.3)

1. If $\mathcal{F}' = \{\emptyset, \Omega'\}$, then any function $f: \Omega \rightarrow \Omega'$ is measurable
2. If $\mathcal{F} = \{A : A \subseteq \Omega\}$, i.e. $\mathcal{F}$ is the family of all subsets of Ω , then any function $f: \Omega \rightarrow \Omega'$ is measurable.

Theorem(8.4)

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces, $D \subseteq \Omega'$, $f: \Omega \rightarrow \Omega'$ be a function $f: \Omega \rightarrow \Omega'$ such that $f(\Omega) \subseteq D$.

1. $f^{-1}(A) = f^{-1}(D \cap A)$ for all $A \in \mathcal{F}'$



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2. If $\mathcal{G}$ be the trace of $\mathcal{F}'$ on D , (i.e. $\mathcal{G} = \mathcal{F}'_D$), then $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable iff $f : (\Omega, \mathcal{F}) \rightarrow (D, \mathcal{G})$ is itself measurable .

Proof :

$$(1) \text{ since } D \cap A \subseteq A \Rightarrow f^{-1}(D \cap A) \subseteq f^{-1}(A)$$

$$\text{Let } x \in f^{-1}(A) \Rightarrow f(x) \in A$$

$$\text{Since } f^{-1}(A) \subseteq \Omega \Rightarrow x \in \Omega \Rightarrow f(x) \in f(\Omega)$$

$$\text{Since } f(\Omega) \subseteq D \Rightarrow f(x) \in D \Rightarrow f(x) \in D \cap A \Rightarrow x \in f^{-1}(D \cap A)$$

$$\Rightarrow f^{-1}(A) \subseteq f^{-1}(D \cap A) \Rightarrow f^{-1}(A) = f^{-1}(D \cap A)$$

$$(2) \mathcal{G} = \mathcal{F}'_D = \{D \cap B : B \in \mathcal{F}'\}$$

Since $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable iff $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$

From(1) , this equivalent to $f^{-1}(D \cap A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$, or in other words, $f^{-1}(B) \in \mathcal{F}$

for all $B \in \mathcal{G} \Rightarrow f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable iff $f : (\Omega, \mathcal{F}) \rightarrow (D, \mathcal{G})$ is measurable.

Theorem(8.5)

Let $(\Omega, \mathcal{F})$ be a measurable space, and let $f : \Omega \rightarrow \bar{\mathbb{R}}$ be a function. Then the following statements are equivalent.

1. $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^+, \beta(\mathbb{R}^+))$ is measurable

2. $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \beta(\mathbb{R}))$ is measurable

3. $f : (\Omega, \mathcal{F}) \rightarrow (\bar{\mathbb{R}}, \beta(\bar{\mathbb{R}}))$ is measurable

Proof:

Since $\beta(\mathbb{R}^+) = \beta(\mathbb{R}) \cap \mathbb{R}^+$ and $f(\Omega) \subseteq \mathbb{R}^+ \subseteq \mathbb{R}$, by using theorem (8.4), we have

$f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^+, \beta(\mathbb{R}^+))$ is measurable iff $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \beta(\mathbb{R}))$ is measurable. Also,

since $\beta(\mathbb{R}) = \beta(\bar{\mathbb{R}}) \cap \mathbb{R}$ and $f(\Omega) \subseteq \mathbb{R} \subseteq \bar{\mathbb{R}}$, by using theorem (3), we have



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$f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \beta(\mathbb{R}))$ is measurable iff $f : (\Omega, \mathcal{F}) \rightarrow (\overline{\mathbb{R}}, \beta(\overline{\mathbb{R}}))$ is measurable.

Theorem(8.6)

Let $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$, and $(\Omega'', \mathcal{F}'')$ be three measurable spaces, and let $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ and $g : (\Omega', \mathcal{F}') \rightarrow (\Omega'', \mathcal{F}'')$ be two measurable functions, then $g \circ f : (\Omega, \mathcal{F}) \rightarrow (\Omega'', \mathcal{F}'')$ is measurable function. Hence composition of measurable functions is measurable.

Proof :

Let $A \in \mathcal{F}''$

Since g is measurable $\Rightarrow g^{-1}(A) \in \mathcal{F}'$, and since f is measurable $\Rightarrow f^{-1}(g^{-1}(A)) \in \mathcal{F}$

But $f^{-1} \circ g^{-1} = (g \circ f)^{-1} \Rightarrow (g \circ f)^{-1}(A) \in \mathcal{F} \Rightarrow g \circ f$ is measurable.

Theorem(8.7)

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces, and let $\mathcal{G}$ be the family of subsets of Ω' such that $\sigma(\mathcal{G}) = \mathcal{F}'$, then $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable iff $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$.

Proof:

Suppose Let $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable $\Rightarrow f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$

Since $\mathcal{G} \subseteq \sigma(\mathcal{G}) = \mathcal{F}' \Rightarrow f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$

Conversely, suppose that $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$

Put $\mathcal{H} = \{A \in \mathcal{F}' : f^{-1}(A) \in \mathcal{F}\} \Rightarrow \mathcal{H}$ is a σ -field on Ω' and $\mathcal{G} \subseteq \mathcal{H}$

Since $\sigma(\mathcal{G})$ is the smallest σ -field on Ω' containing $\mathcal{G}$, we obtain that $\sigma(\mathcal{G}) \subseteq \mathcal{H}$.

However $\sigma(\mathcal{G}) = \mathcal{F}'$. It follows that $\mathcal{F}' \subseteq \mathcal{H}$, and in particular, $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{F}'$.

So that $f : (\Omega, \mathcal{F}) \rightarrow (\Omega', \mathcal{F}')$ is measurable.

Theorem(8.8)

Let (Ω, τ) and (Ω', τ') be two topological spaces. If the function $f : (\Omega, \tau) \rightarrow (\Omega', \tau')$ is continuous, then $f : (\Omega, \beta(\Omega)) \rightarrow (\Omega', \beta(\Omega'))$ is measurable.

Proof:

Let $A \in \beta(\Omega')$

Since $\beta(\Omega') = \sigma(\tau') \Rightarrow A \in \sigma(\tau')$

Since f is continuous $\Rightarrow f^{-1}(A) \in \tau$

In particular, for all $\tau \subseteq \sigma(\tau) \Rightarrow f^{-1}(A) \in \sigma(\tau)$. Using theorem(8.7), we conclude that $f : (\Omega, \beta(\Omega)) \rightarrow (\Omega', \beta(\Omega'))$ is measurable.

Corollary (8.9)

Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \rightarrow \mathbb{R}$ be a measurable function, $g : \mathbb{R} \rightarrow \mathbb{R}$ continuous function, then $g \circ f : \Omega \rightarrow \mathbb{R}$ is a measurable function.

Theorem(8.10)

Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, and $f, g : \Omega \rightarrow \mathbb{R}^*$ be two functions such that $f = g$ a.e., then f is measurable iff g is measurable.

Proof :

For any $a \in \mathbb{R}$ the set $\{f < a\} \Delta \{g < a\} \subseteq \{f \neq g\}$, so that $\{f < a\} \in \mathcal{F}$

Since $(\Omega, \mathcal{F}, \mu)$ be a complete measure space, then all such sets are in $\mathcal{F}$, so that

$$\{f < a\} \in \mathcal{F} \Leftrightarrow \{g < a\} \in \mathcal{F}$$

Remark

First : If $a, b \in \mathbb{R}$ with $a < b$, then



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$$1. (a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

$$2. [a, b] = \bigcap_{n=1}^{\infty} [a, b + \frac{1}{n}] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

Second : Let $a, b \in \mathbb{R}$ and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a function

$$1. f^{-1}([-\infty, b]) = \{x \in \Omega : f(x) \leq b\} = \{f \leq b\}$$

$$2. f^{-1}([a, \infty]) = \{x \in \Omega : f(x) \geq a\} = \{f \geq a\}$$

$$3. f^{-1}([-\infty, b)) = \{x \in \Omega : f(x) < b\} = \{f < b\}$$

$$4. f^{-1}((a, \infty]) = \{x \in \Omega : f(x) > a\} = \{f > a\}$$

$$5. \{f \leq b\}^c = \{f > b\}$$

$$6. \{f < b\}^c = \{f \geq b\}$$

$$7. \{f \geq a\} = \bigcap_{n=1}^{\infty} \{f > a - \frac{1}{n}\}$$

$$8. \{f > a\} = \bigcup_{n=1}^{\infty} \{f \geq a + \frac{1}{n}\}$$

$$9. \{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\}$$

$$10. \{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\}$$

$$11. \{f = a\} = \{f \geq a\} \cap \{f \leq a\}$$

$$12. \{a < f < b\} = \{f > a\} \cap \{f < b\}$$

$$13. \{f = \infty\} = \bigcap_{n=1}^{\infty} \{f \geq n\}$$

$$14. \{f = -\infty\} = \bigcap_{n=1}^{\infty} \{f \leq -n\}$$



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Third : Let $a, b \in \mathbb{R}$ and let $f : \Omega \rightarrow \overline{\mathbb{R}}, g : \Omega \rightarrow \overline{\mathbb{R}}$ be functions

$$1. (f \wedge g)(x) = f(x) \wedge g(x) = \min\{f(x), g(x)\}$$

$$2. (f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\}$$

$$3. \{f = g\} = \{f \leq g\} \cap \{f \geq g\}$$

$$4. \{f > g\} = \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} \{f > \frac{m}{n}\} \cap \{g < \frac{m}{n}\} \right)$$

$$5. \{\min\{f, g\} < a\} = \{f < a\} \cup \{g < a\}$$

$$6. \{\min\{f, g\} > a\} = \{f > a\} \cap \{g > a\}$$

$$7. \{\max\{f, g\} < a\} = \{f < a\} \cap \{g < a\}$$

$$8. \{\max\{f, g\} > a\} = \{f > a\} \cup \{g > a\}$$

Fourth : Let $a \in \mathbb{R}$ and $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ be a function for all n

$$1. \{\sup f_n \leq a\} = \bigcap_{n=1}^{\infty} \{f_n \leq a\}$$

$$2. \{\sup f_n > a\} = \bigcup_{n=1}^{\infty} \{f_n > a\}$$

$$3. \{\sup f_n < a\} = \bigcap_{n=1}^{\infty} \{f_n < a\}$$

$$4. \{\inf f_n > a\} = \bigcap_{n=1}^{\infty} \{f_n > a\}$$

$$5. \{\inf f_n < a\} = \bigcup_{n=1}^{\infty} \{f_n < a\}$$

Now we define the following classes of subsets of $\overline{\mathbb{R}}$

$$1. \mathcal{G}_1 = \{[-\infty, b] : b \in \mathbb{R}\}$$

$$2. \mathcal{G}_2 = \{[-\infty, b) : b \in \mathbb{R}\}$$



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$$3. \mathcal{G}_3 = \{[a, \infty] : a \in \mathbb{R}\}$$

$$4. \mathcal{G}_4 = \{(a, \infty] : a \in \mathbb{R}\}$$

It is clear to show that $\sigma(\mathcal{G}_i) = \beta(\overline{\mathbb{R}})$ for all $i = 1, 2, 3, 4$

Theorem(8.11)

Let $(\Omega, \mathcal{F})$ be a measurable space, and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a function. The following statements are equivalent.

1. f is a Borel measurable on $(\Omega, \mathcal{F})$
2. $\{f \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$
3. $\{f < a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$
4. $\{f \geq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$
5. $\{f > a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$

Proof :

$$(1) \Rightarrow (2) \quad \{f \leq a\} = f^{-1}([-\infty, a])$$

Since f is measurable and $[-\infty, a] \in \beta(\mathbb{R}^*) \Rightarrow f^{-1}([-\infty, a]) \in \mathcal{F} \Rightarrow \{f \leq a\} \in \mathcal{F}$

$$(2) \Rightarrow (3) \quad \{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \in \mathcal{F}$$

$$(3) \Rightarrow (4) \quad \{f \geq a\} = \{f < a\}^c \in \mathcal{F}$$

$$(4) \Rightarrow (5) \quad \{f > a\} = \bigcup_{n=1}^{\infty} \{f \geq a + \frac{1}{n}\} \in \mathcal{F}$$

(5) $\Rightarrow$ (1) Since $\sigma(\mathcal{G}_4) = \beta(\overline{\mathbb{R}})$. By using theorem(8.7), we have f is a Borel measurable on $(\Omega, \mathcal{F})$.



Corollary(8.12)

Let $(\Omega, \mathcal{F})$ be a measurable space, and $f : \Omega \rightarrow \overline{\mathbb{R}}$, $g : \Omega \rightarrow \overline{\mathbb{R}}$ be Borel measurable functions. Then

1. $\{f = a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$
2. $\{a < f < b\} \in \mathcal{F}$ for all $a, b \in \mathbb{R}$
3. $-f$ and $|f|$ are measurable with respect to $\mathcal{F}$ and $\beta(\overline{\mathbb{R}})$
4. The functions $\max\{f, g\}$ and $\min\{f, g\}$ are measurable iff the functions g, f are measurable.

Theorem(8.13)

Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g : \Omega \rightarrow \overline{\mathbb{R}}$ be two measurable functions, then

1. $\{f < g\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{r < g\}) = \bigcup_{r \in \mathbb{Q}} \{f < r < g\}$
2. The sets $\{f < g\}, \{f > g\}, \{f = g\}, \{f \leq g\}, \{f \geq g\}$ belong to the σ -field $\mathcal{F}$.
3. $\{f + g < a\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g < a - r\})$

Proof :

(1) Let $x \in \{f < g\} \Rightarrow f(x) < g(x)$ there exists a rational number $r \in \mathbb{Q}$ such that $f(x) < r < g(x)$. It follows that $x \in \{f < r\} \cap \{r < g\}$. So $\{f < g\} \subset \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{r < g\})$

The reverse inclusion is clear.

(2) Since f and g are measurable, $\{f < r\} = f^{-1}([-\infty, r))$ and $\{r < g\} = g^{-1}((r, \infty])$ are both elements of $\mathcal{F}$, for all $r \in \mathbb{Q}$ using (1), and the fact $\mathbb{Q}$ is countable set, it follows that $\{f < g\} \in \mathcal{F}$. Similarly $\{g < f\} \in \mathcal{F}$

Moreover, we $\{f \leq g\} = \{g < f\}^c \in \mathcal{F}$ and $\{g \leq f\} = \{f < g\}^c \in \mathcal{F}$.



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Finally, $\{f = g\} = \{f \leq g\} \cap \{g \leq f\} \in \mathcal{F}$

Theorem(8.14)

Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g: \Omega \rightarrow \mathbb{R}^*$ be two measurable functions, and $\alpha \in \mathbb{R}$.

Then each of the functions $\alpha + f, \alpha f, f \pm g, f^2, fg, \frac{1}{f}$ (where $\frac{1}{f}(x) = \infty$ if $f(x) = 0$), which is defined, is measurable.

Proof :

The measurability of the first two functions $\alpha + f, \alpha f$ follows immediately from any Part of theorem (8.10). Consider now the function $(f + g)$

$$\{f + g < a\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g < a - r\})$$

By theorem (8.10) each of the sets on the right-hand side is in $\mathcal{F}$ so that, because $\mathcal{F}$ is a σ -field, the set on the left-hand is also in $\mathcal{F}$, and $(f + g)$ must be measurable.

Now

$$\{f^2 \leq a\} = \begin{cases} \emptyset, & a < 0 \\ \{f = 0\}, & a = 0 \\ \{-a \leq f \leq a\}, & a > 0 \end{cases}$$

and each of these sets in $\mathcal{F}$, so that f^2 is measurable. $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$. Further

$$\left\{ \frac{1}{f} < a \right\} = \begin{cases} \left\{ \frac{1}{a} < f < 0 \right\}, & a < 0 \\ \{f = 0\}, & a = 0 \\ \{-\infty \leq f < 0\} \cup \left\{ f > \frac{1}{a} \right\}, & a > 0 \end{cases}$$

and each of these sets in $\mathcal{F}$, so that $\frac{1}{f}$ is measurable.

Theorem(8.15)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a measurable function . Define $\mu_f : \beta(\mathbb{R}) \rightarrow \mathbb{R}$ by $\mu_f(A) = \mu\{f^{-1}(A)\}$ for all $A \in \beta(\mathbb{R})$. Then μ_f is a measure on $\beta(\mathbb{R})$. This measure is called the measure generated (or induced) by f .

Proof :

(1) Let $A \in \beta(\mathbb{R})$

Since f is a measurable function $\Rightarrow f^{-1}(A) \in \mathcal{F}$ for all .

$\Rightarrow \mu\{f^{-1}(A)\} \geq 0$ for all $A \in \beta(\mathbb{R})$, then $\mu_f(A) \geq 0$ for all $A \in \beta(\mathbb{R})$,

(2) let $\{A_n\}$ be disjoint sequence of sets in $\beta(\mathbb{R})$

$\Rightarrow f^{-1}(A_n) \in \mathcal{F}$ for all n , and $f^{-1}(A_n) \cap f^{-1}(A_m) = f^{-1}(A_n \cap A_m) = f^{-1}(\emptyset) = \emptyset$ for all $n \neq m$

$\Rightarrow \{f^{-1}(A_n)\}$ is disjoint sequence of set in $\mathcal{F}$.

$$\mu_f\left\{\bigcup_{n=1}^{\infty} A_n\right\} = \mu\{f^{-1}(\bigcup_{n=1}^{\infty} A_n)\} = \mu\left\{\bigcup_{n=1}^{\infty} f^{-1}(A_n)\right\} = \sum_{n=1}^{\infty} \mu\{f^{-1}(A_n)\} = \sum_{n=1}^{\infty} \mu_f(A_n)$$

Therefore μ_f is a measure on $\beta(\mathbb{R})$.

Definition(8.16)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a measurable function .The σ -field generated by f is given by $\sigma(f) = f^{-1}(\beta(\mathbb{R}))$. Thus a set in $\sigma(f)$ is of the form $\{f \in A\}$ for some $A \in \beta(\mathbb{R})$.

Remark

• If f is a measurable function on a measure space $(\Omega, \mathcal{F}, \mu)$, then

1.) $\sigma(f)$ is the smallest σ -field that consists of all the sets of the form $\{f \leq a\}$, $a \in \mathbb{R}$

2. $f \in \sigma(f)$

Definition(8.17)

Let $f : \Omega \rightarrow \overline{\mathbb{R}}$ be an extended real valued function defined on a set Ω .

(1) A number $a \in \overline{\mathbb{R}}$ is called a lower bound for f if $a \leq f(x)$ for all $x \in \Omega$, i.e.

$$f^{-1}(-\infty, a) = \{x \in \Omega : f(x) < a\} = \phi$$

The largest lower bound for f is called the infimum of f and denoted by $\inf f$, i.e.

$$\inf f = \sup L_f \quad \text{where } L_f = \{a \in \mathbb{R} : f^{-1}(-\infty, a) = \phi\}$$

If $L_f = \phi$, then $\sup f = \infty$ and if $L_f = \mathbb{R}$, then $\sup f = -\infty$

(b) A number $b \in \overline{\mathbb{R}}$ is called an upper bound for f if $f(x) \leq b$ for all $x \in \Omega$, i.e.

$$f^{-1}(b, \infty) = \{x \in \Omega : f(x) > b\} = \phi$$

The smallest upper bound for f is called the supremum of f and denoted by $\sup f$, i.e.

$$\sup f = \inf U_f \quad \text{where } U_f = \{b \in \mathbb{R} : f^{-1}(b, \infty) = \phi\}$$

If $U_f = \phi$, then $\sup f = \infty$ and if $U_f = \mathbb{R}$, then $\sup f = -\infty$

Definition(8.18)

Let $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a measurable function on measure space $(\Omega, \mathcal{F}, \mu)$.

1. A number $a \in \overline{\mathbb{R}}$ is called an essential lower bound for f if $a \leq f(x)$ for almost all $x \in \Omega$,

$$\text{i.e. } \mu(f^{-1}(-\infty, a)) = 0$$

The largest essential lower bound for f is called the essential infimum of f and denoted by $\text{ess inf } f$, i.e.

$$\text{ess inf } f = \sup L_f^{\text{ess}} \quad \text{where } L_f^{\text{ess}} = \{a \in \mathbb{R} : \mu(f^{-1}(-\infty, a)) = 0\}$$

$$\text{ess inf } f = \sup \{a \in \overline{\mathbb{R}} : f \geq a \text{ a.e.}\} = \sup \{a \in \overline{\mathbb{R}} : \mu\{f < a\} = 0\}$$

If $L_f^{\text{ess}} = \phi$, then $\text{ess inf } f = \infty$ and if $L_f^{\text{ess}} = \mathbb{R}$, then $\text{ess inf } f = -\infty$

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2. A number $b \in \overline{\mathbb{R}}$ is called an essential upper bound for f if $f(x) \leq b$ for almost all $x \in \Omega$, i.e. $\mu(f^{-1}(b, \infty)) = 0$

The smallest essential upper bound for f is called the essential supremum of f and denoted by $\text{ess sup } f$, i.e.

$$\text{sup } f = \inf U_f^{\text{ess}} \quad \text{where } U_f^{\text{ess}} = \{b \in \mathbb{R} : \mu(f^{-1}(b, \infty)) = 0\}$$

$$\text{ess sup } f = \inf \{b \in \overline{\mathbb{R}} : f \leq b \text{ a.e.}\} = \inf \{b \in \overline{\mathbb{R}} : \mu\{f > b\} = 0\}$$

If $U_f^{\text{ess}} = \emptyset$, then $\text{ess sup } f = \infty$ and if $U_f^{\text{ess}} = \mathbb{R}$, then $\text{ess sup } f = -\infty$

Examples(8.19)

Let $\Omega = \mathbb{R}$, $\mathcal{F} = \beta(\mathbb{R})$, μ is Lebesgue measure.

1. If the function $f; \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \begin{cases} 5, & x = 1 \\ -4, & x = -1 \\ 2, & \text{o.w} \end{cases}$, then $\text{sup } f = 5$, $\text{inf } f = -4$

Since $\{x \in \mathbb{R} : f(x) = -4\} = \{-1\}$ and $\{x \in \mathbb{R} : f(x) = 5\} = \{1\}$

Also $\mu(\{x \in \mathbb{R} : f(x) = -4\}) = \mu(\{-1\}) = 0$ and $\mu(\{x \in \mathbb{R} : f(x) = 5\}) = \mu(\{1\}) = 0$

$\{-1\} = \{x \in \mathbb{R} : f(x) = -4\} = \{x \in \mathbb{R} : f(x) < 2\}$ and $\{1\} = \{x \in \mathbb{R} : f(x) = 5\} = \{x \in \mathbb{R} : f(x) > 2\}$

$\text{ess inf } f = \sup \{a \in \mathbb{R} : \mu\{f < a\} = 0\} = \sup \{2\} = 2$ and $\text{ess sup } f = \inf \{b \in \mathbb{R} : \mu\{f > b\} = 0\} = \inf \{2\} = 2$

2. If the function $f; \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \begin{cases} x^3, & x \in \mathbb{Q} \\ \tan^{-1} x & x \notin \mathbb{Q} \end{cases}$, then $\text{sup } f = \infty$, $\text{inf } f = -\infty$

$$\text{ess inf } f = -\frac{\pi}{2} \text{ and } \text{ess sup } f = \frac{\pi}{2}$$

3. If the function $f; \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, then for any $b \in \mathbb{R}$, we have



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$$\mu(\{x \in \mathbb{R} : \frac{1}{x} > b\}) \geq \frac{1}{|b|} \text{ and } \text{ess sup } f = \infty$$

Remarks

1. If $\mu(\Omega) > 0$, then $\inf f \leq \text{es } \inf f \leq \text{ess sup } f \leq \sup f$
2. If $\mu(\Omega) = 0$, then $\text{es } \inf f = \infty$ and $\text{ess sup } f = -\infty$
3. If $\text{ess sup } f = -\infty$, then $f = -\infty$ a.e.

Since by definition of $\text{ess sup } f \leq -n$ a.e. for all $n \geq 1$.

Now if $\text{ess sup } f$ is finite, and $A = \{x \in \Omega : \text{ess sup } f < f(x)\}$, define A_n for $n \geq 1$ by

$$A_n = \{x \in \Omega : \text{ess sup } f < \frac{1}{n}\}$$

These are null sets, hence so is $A = \bigcup_{n=1}^{\infty} A_n$, and thus we have verified : $f \leq \text{ess sup } f$ a.e.

The following is now straightforward to prove

Theorem (8.20)

If $f, g : \Omega \rightarrow \mathbb{R}$ are measurable functions on measure space $(\Omega, \mathcal{F}, \mu)$, then

1. $\text{ess sup}(f + g) \leq \text{ess sup } f + \text{ess sup } g$
2. $\text{ess sup}(fg) \leq (\text{ess sup } f)(\text{ess sup } g)$ whenever both the right are nonnegative.

Proof :

(1) Since $f \leq \text{ess sup } f$ a.e. and $g \leq \text{ess sup } g$ a.e.

By adding we have $f + g \leq \text{ess sup } f + \text{ess sup } g$ a.e.

So the number $f + g \leq \text{ess sup } f + \text{ess sup } g$ belongs to the set $\{\lambda \in \overline{\mathbb{R}} : f + g \leq \lambda \text{ a.e.}\}$

Hence the infimum of this set is smaller than this number.



Definition(8.21)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (S, β) an arbitrary probabilizable. A function $f : \Omega \rightarrow S$ is called random object (or random element) in S , if f is a measurable function.

• A random object is called a random variable, if $S = \mathbb{R}$ and $\beta = \beta(\mathbb{R})$, we the random variables by the captor letters $X, Y, Z, \dots$, and the values of the random variables denoted by the small letters $x, y, z, \dots$, i.e. a random variable X on a probability space $(\Omega, \mathcal{F}, P)$ is a Borel measurable function from Ω to $\mathbb{R}$.

A function $X : \Omega \rightarrow \mathbb{R}$ is a random variable on $(\Omega, \mathcal{F}, P)$ if $\{X \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$

• There are two types of random variables on $(\Omega, \mathcal{F}, P)$,

1. Discrete random variable : A random variable X is called discrete, if the set of values of X is countable (i.e. $X(\Omega)$ is countable)

2. Continuous random variable : A random variable X is called continuous, if the set of values of X is uncountable

• A random variable X on $(\Omega, \mathcal{F}, P)$ is called bounded, if there is $k > 0$ such that $P\{|X| > k\} = 0$

• Two random variables X and Y on $(\Omega, \mathcal{F}, P)$ are said to be equivalent, if $P\{X \neq Y\} = 0$

• Let X and Y be two random variables on $(\Omega, \mathcal{F}, P)$, and $\alpha \in \mathbb{R}$. Then each of the functions :

$\alpha + X, \alpha X, X \pm Y, Y^2, XY, \frac{1}{X}$ (where $\frac{1}{X}(x) = \infty$ if $X(x) = 0$), which is defined, is random.

The Distribution Function

Theorem(8.22)

Let X be a random variable on $(\Omega, \mathcal{F}, P)$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(x) = P\{X \leq x\}$ for all $x \in \mathbb{R}$.

Then



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1. $0 \leq F(x) \leq 1$
2. If $a \leq b$, then $P\{a < X \leq b\} = F(b) - F(a)$
3. The function F is an increasing, i.e. if $a < b$, then $F(a) \leq F(b)$

Proof :

$$(2) \text{ Let } A = \{X \leq b\} \text{ and } B = \{X \leq a\} \Rightarrow \{a < X \leq b\} = A \cap B^c$$

Since $a < b \Rightarrow B \subseteq A$, and since X is a random variable $\Rightarrow A, B \in \mathcal{F}$

$$P(A) = P(A \cap B^c) + P(A \cap B) = P(A \cap B^c) + P(B) \Rightarrow P(A \cap B^c) = P(A) - P(B)$$

$$P\{a < X \leq b\} = P\{X \leq b\} - P\{X \leq a\} = F(b) - F(a)$$

$$(3) \text{ Let } a, b \in \mathbb{R} \text{ such that } a < b$$

$$F(b) - F(a) = P\{a < X \leq b\} \geq 0 \Rightarrow F(a) \leq F(b)$$

- It is not hard to show that if f is bounded and either non-decreasing or non-increasing, then $f(a^+)$ and $f(a^-)$ exist for all a . Under the same conditions, f has limits $f(-\infty)$ as $x \rightarrow -\infty$ and $f(\infty)$ as $x \rightarrow \infty$

$$f(-\infty) = \lim_{x \rightarrow -\infty} f(x) \text{ and } f(\infty) = \lim_{x \rightarrow \infty} f(x)$$

Now, since the distribution function F of a random variable X is bounded (because $0 \leq F(x) \leq 1$) and increasing, then F has limits $f(a^+)$ and $f(a^-)$ for all a as well as limits $f(-\infty)$ and $f(\infty)$.

Theorem(8.23)

Let F be a distribution function of a random variable X on $(\Omega, \mathcal{F}, P)$. Then

1. $F(\infty) = 1$, and $F(-\infty) = 0$
2. The function F is a right continuous, i.e. $f(a^+) = f(a)$ for all $a \in \mathbb{R}$

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3. $F(a^+) = P\{X \leq a\}$ and $F(a^-) = P\{X < a\}$ for all $a \in \mathbb{R}$

4. $P\{X = a\} = F(a^+) - F(a^-) = F(a) - F(a^-)$ for all $a \in \mathbb{R}$

5. The function F is continuous at a iff $P\{X = a\} = 0$

Thus a random variable X is continuous iff F is continuous at every points in $\mathbb{R}$

Proof :

(1) Let $A_n = \{X \leq n\}$, i.e. $A_n = \{\omega \in \Omega : X(\omega) \leq n\}$, then $\dots \subseteq A_{-2} \subseteq A_{-1} \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$

and $\bigcup_{n=0}^{\infty} A_n = \Omega$, $\bigcap_{n=0}^{\infty} A_n = \phi$

since $F(n) = P\{X \leq n\} = P(A_n)$, we have $F(\infty) = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} P(A_n) = P\{\bigcup_{n=0}^{\infty} A_n\} = P\{\Omega\} = 1$

and $F(-\infty) = \lim_{n \rightarrow -\infty} F(n) = \lim_{n \rightarrow -\infty} P(A_n) = P\{\bigcap_{n=0}^{\infty} A_n\} = P\{\phi\} = 0$

(2) Let $a \in \mathbb{R}$: Assume $h > 0$

$$\lim_{h \rightarrow 0} P\{a < X \leq a + h\} = \lim_{h \rightarrow 0} (F(a + h) - F(a))$$

But $\lim_{h \rightarrow 0} P\{a < X \leq a + h\} = P\{\phi\} = 0$ and $\lim_{h \rightarrow 0} (F(a + h) - F(a)) = \lim_{h \rightarrow 0} F(a + h) - F(a) = F(a^+) - F(a)$

$$F(a^+) - F(a) = 0 \Rightarrow F(a^+) = F(a)$$

The function F is a right continuous at a , so that F is a right continuous

(3) Since F is a right continuous, then $f(a^+) = f(a)$ for all $a \in \mathbb{R}$

And since $F(a) = P\{X \leq a\}$ for all $a \in \mathbb{R}$, then $F(a^+) = P\{X \leq a\}$ for all $a \in \mathbb{R}$

(4) Let $a \in \mathbb{R}$: Assume $h > 0$

$$P\{X = a\} = \lim_{h \rightarrow 0} P\{a - h < X \leq a\} = \lim_{h \rightarrow 0} (F(a) - F(a - h)) = F(a) - F(a^-)$$

(5) Assume F is continuous at $a \Rightarrow F$ is a left continuous at a

$$\Rightarrow F(a^-) = F(a), \text{ but } P\{X = a\} = F(a) - F(a^-) \Rightarrow P\{X = a\} = 0$$

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Conversely : assume $P\{X = a\} = 0$

Since $P\{X = a\} = F(a) - F(a^-) \Rightarrow F(a^-) = F(a)$

$\Rightarrow$ The function F is a left continuous at a , but F is a right continuous at a , then F is continuous at a , so that F is a right continuous.

Remarks

1. Let X be a random variable on $(\Omega, \mathcal{F}, P)$. The function $F: \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = P\{X \leq x\}$ for all $x \in \mathbb{R}$ is distribution function and its called the distribution function of X .

2. If X is a continuous random variable on $(\Omega, \mathcal{F}, P)$, then

$$P\{a < X < b\} = P\{a < X \leq b\} = P\{a \leq X < b\} = P\{a \leq X \leq b\} = F(b) - F(a)$$

3. If $F: \mathbb{R} \rightarrow [0, 1]$ is an increasing and right continuous function with $F(\infty) = 1, F(-\infty) = 0$, then F is a distribution function.

Example(8.24)

Let X be a of a random variable on $(\Omega, \mathcal{F}, P)$ with distribution function

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}(x+2), & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Compute : $P\{-\frac{1}{2} < X \leq \frac{1}{2}\}, P\{X = 0\}, P\{X = 1\}, P\{2 < X \leq 3\}$

Ans :

$$P\{-\frac{1}{2} < X \leq \frac{1}{2}\} = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{1}{4}(\frac{1}{2} + 2) - \frac{1}{4}(-\frac{1}{2} + 2) = \frac{5}{8} - \frac{3}{8} = \frac{1}{4}$$

$$P\{X = 0\} = F(0) - F(0^-) = \frac{1}{4}(0 + 2) - \frac{1}{4}(0 + 2) = 0 \Rightarrow F \text{ is continuous at } 0$$

$$P\{X = 1\} = F(1) - F(1^-) = 1 - \frac{1}{4}(1 + 2) = 1 - \frac{3}{4} = \frac{1}{4}, \quad P\{2 < X \leq 3\} = F(3) - F(2) = 1 - 1 = 0$$

The Probability Density Function

Definition(8.25)

Let X be a discrete random variable on $(\Omega, \mathcal{F}, P)$, (i.e. $X(\Omega)$ is countable) and the values $\{x_n\}$ of X can be arranged so that $x_n \leq x_{n+1}$ for all n , i.e. $X(\Omega) = \{x_1, x_2, \dots\}$. The probability mass function of X is the function $f: X(\Omega) \rightarrow \mathbb{R}$, given by $f(x_i) = P\{X = x_i\}$, $i = 1, 2, \dots$

It follows that

$$1. \quad 0 \leq f(x_i) \leq 1 \text{ for all } i = 1, 2, \dots$$

$$2.) \sum_{i=1}^{\infty} f(x_i) = 1$$

- If $A \subseteq X(\Omega)$ then, $P(A) = \sum_{x \in A} p(x)$

Definition(8.26)

A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be absolute continuous if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all positive integers n and all families $(a_1, b_1), \dots, (a_n, b_n)$ of disjoint open subintervals of $[a, b]$ of total length at most δ , we have

$$\sum_{i=1}^n |f(b_i) - f(a_i)| \leq \varepsilon$$

- Every absolutely continuous function is continuous and the converse not true.

Definition(8.27)

The random variable X on $(\Omega, \mathcal{F}, P)$ is said to be absolutely continuous if there is a non-negative real-valued measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$F(x) = \int_{-\infty}^x f(t) dt$$



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For all $x \in \mathbb{R}$. We call f the density function of X .so $f(x) = \frac{dF(x)}{dx}$

Since $f(\infty) = 1 \Rightarrow \int_{-\infty}^{\infty} f(t)dt = 1$

Theorem (8.28)

If X be absolute continuous with density f , it follows that

1. $P(a < x < b) = \int_a^b f(x)dx$ for all $a, b \in \mathbb{R}$ such that $a \leq b$

2. $P_x(A) = \int_A f(x)dx$ for all $A \in \beta(\mathbb{R})$

Proof :

(1) $P(a < x < b) = F(b) - F(a) = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx = \int_a^b f(x)dx$

(2) Define the measure $\mu: \beta(\mathbb{R}) \rightarrow \mathbb{R}$ by $\mu(B) = \int_B f(x)dx$ for all $B \in \beta(\mathbb{R})$

Satisfies $\mu(a, b] = F(b) - F(a)$ for all $a < b$

Thus μ is the Lebesgue-Stieltjes measure corresponding to F ; hence $\mu = P_x$

Exercises (8)

1. Let $\Omega = \{a, b, c, d\}$, $\mathcal{F}_1 = \{\emptyset, \{a\}, \{b, c, d\}, \Omega\}$, $\mathcal{F}_2$ is the set of all subsets of Ω

(1) Define $f: (\Omega, \mathcal{F}_1) \rightarrow (\Omega, \mathcal{F}_2)$ by $f(\omega) = a$ for all $\omega \in \Omega$. Show that f is measurable.

(2) Define $f: (\Omega, \mathcal{F}_1) \rightarrow (\Omega, \mathcal{F}_2)$ by $f(\omega) = \begin{cases} a, & \omega = a, b \\ c, & \omega = c, d \end{cases}$. Is f is measurable? why.

2. Define $f: (\mathbb{R}, \beta(\mathbb{R})) \rightarrow (\mathbb{R}, \beta(\mathbb{R}))$ by $f(x) = \begin{cases} \sin 2x, & x > 0 \\ 1 + \cos x, & x \leq 0 \end{cases}$. Is f is measurable? why.

3. Is every measurable function $f: (\mathbb{R}, \beta(\mathbb{R})) \rightarrow (\mathbb{R}, \beta(\mathbb{R}))$ is continuous ? why.

4. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces, such that $\mathcal{F}_2 = \sigma(\mathcal{G})$ for some family



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of subsets $\mathcal{G}$ of Ω_2 . If $f : \Omega_1 \rightarrow \Omega_2$ is a function such that $f^{-1}(A) \in \mathcal{F}_1$ for all $A \in \mathcal{G}$.

Show that f is a measurable.

5. Prove that every monotone function is measurable.

6. Let $f, g : \Omega \rightarrow \mathbb{R}$ be measurable functions on the measurable space $(\Omega, \mathcal{F})$. Show that

every $A \in \mathcal{F}$ the function $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \notin A \end{cases}$ is measurable.

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Find $\sigma(f)$ for the following function

$$(1) f(x) = x \quad (2) f(x) = x^2 \quad (3) f(x) = |x|$$

8. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. Find $\sigma(f)$ for the following function

$$(1) f(x, y) = x + y \quad (2) f(x) = x^2 + y^2$$

9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Explain why f and f' are measurable?

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. $\{x\} \in \sigma(f)$ for all $x \in \mathbb{R}$ iff f is injective

11. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. Which of the following are measurable

$$(1) f(x-2) \quad (2) e^{f(x)} \quad (3) \sin(f(x)+8) \quad (4) f'' \quad (5) \operatorname{sgn} f(x-7)$$

12. Suppose that $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. If f and g are measurable functions.

Show that $h(x) = F(f(x), g(x))$ is also measurable.

13. Show that if f is measurable, then $f_\lambda(x) = \begin{cases} \lambda, & f(x) > \lambda \\ f(x), & f(x) \leq \lambda \end{cases}$ is also measurable.

14. Find a non measurable f such that f^2 is measurable.

15. Show that for measurable f , $\operatorname{ess\,sup} f \leq \operatorname{ess\,sup} g$. Show that these quantities coincide when f is continuous.

16. Let X be a random variable on $(\Omega, \mathcal{F}, P)$. Define $P_x : \beta(\mathbb{R}) \rightarrow \mathbb{R}$ by $P_x(A) = P\{X^{-1}(A)\}$

for all $A \in \beta(\mathbb{R})$. Then P_x is a probability measure on $\beta(\mathbb{R})$.

9. Sequences of Measurable Functions

Recall that a sequence $\{x_n\}$ of real numbers is said to be Converge to the point $x \in \mathbb{R}$, if for each $\varepsilon > 0$, there is $k \in \mathbb{Z}^+$ such that $|x_n - x| < \varepsilon$ for all $n \geq k$ and we write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$. Given a sequence $\{f_n\}$ of real valued functions defined on Ω , for $x \in \Omega$, we have real sequence $\{f_n(x)\}$. If $\{f_n(x)\}$ is converge for all point of Ω . we can define the function $f : \Omega \rightarrow \mathbb{R}$ by, for any $x \in X$, then $f(x)$ is Limit point of $\{f_n(x)\}$; that is $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ or $f_n(x) \rightarrow f(x)$.

Definition (9.1)

Let $\{f_n\}$ be a sequence of real valued functions defined on Ω and a function $f : \Omega \rightarrow \mathbb{R}$, $A \subseteq \Omega$. We sat that

1. $\{f_n\}$ converges to f (pointwise) on A , if for every $x \in A$, then $f_n(x) \rightarrow f(x)$, i.e. if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$.

We write $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ or $f_n \rightarrow f$ on A

2. $\{f_n\}$ is a Cauchy sequence (pointwise) on A , if for every $x \in A$, then $\{f_n(x)\}$ is a Cauchy sequence, i.e., for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$.

Note that, in the above definition, when $A = \Omega$, we can omit " on A " from the statements. i.e. $f_n \rightarrow f$, if for every $x \in \Omega$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$.

- This has meaning only if $f_n : \Omega \rightarrow \mathbb{R}$ is finite valued. Because $\mathbb{R}$ is complete it is clear that if $\{f_n\}$ is a Cauchy sequence pointwise on Ω , there must be an $f : \Omega \rightarrow \mathbb{R}$ such that $f_n \rightarrow f$ on Ω .

Definition (9.2)

Let $\{f_n\}$ be a sequence of real valued functions defined on Ω and a function $f : \Omega \rightarrow \mathbb{R}$, $A \subseteq \Omega$. We sat that

1. $\{f_n\}$ converges uniformly to f on A , if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and all $x \in A$. We write $f_n \xrightarrow{u} f$ on A
2. $\{f_n\}$ is a Cauchy sequence uniformly on A , if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$ and all $x \in A$.

• It is clear that every converges uniformly sequence is convergent pointwise, but the converse is not true.

Definition(9.3)

Let f and g be real valued measurable functions defined on a measure space , we say that

1. f is a constant a.e. if there is $b \in \mathbb{R}$ such that $\mu(\{f \neq b\}) = 0$
2. f is bounded if there is $k > 0$ such that $\mu(\{|f| > k\}) = 0$
3. f is a essentially bounded if it bounded a.e., i.e. if there exists a positive , finite constant c such that $\mu(\{f > c\}) = 0$.
4. f and g are equivalent (i.e. $f = g$ a.e.), if $\mu(\{f \neq g\}) = 0$.

Theorem (9.4)

Let $\{f_n\}$ be a sequence of measurable functions on a measurable space $(\Omega, \mathcal{F})$, then the following are measurable function also:

$$\max_{n \leq k} f_n, \min_{n \leq k} f_n, \sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$$

Proof :

It is sufficient to note the following are measurable sets

$$\begin{aligned} \{x \in \Omega : \max_{n \leq k} f_n(x) > a\} &= \bigcup_{n=1}^k \{x \in \Omega : f_n(x) > a\} & \{x \in \Omega : \min_{n \leq k} f_n(x) > a\} &= \bigcap_{n=1}^k \{x \in \Omega : f_n(x) > a\} \\ \{x \in \Omega : \sup_{n \in \mathbb{N}} f_n(x) > a\} &= \bigcup_{n \in \mathbb{N}} \{x \in \Omega : f_n(x) > a\} & \{x \in \Omega : \inf_{n \in \mathbb{N}} f_n(x) \geq a\} &= \bigcap_{n \in \mathbb{N}} \{x \in \Omega : f_n(x) \geq a\} \end{aligned}$$

For the upper limit, by definition $\limsup_{n \rightarrow \infty} f_n = \inf_{n \geq 1} \{\sup_{m \geq n} f_m\}$ and the above relations show

that $g_n = \sup_{m \geq n} f_m$ is measurable , hence $\inf_{n \geq 1} g_n$ is measurable. The lower is done similarly.

Corollary(9.5)

Let $\{f_n\}$ be a sequence of measurable functions on a measurable space $(\Omega, \mathcal{F})$. If $f_n \rightarrow f$ on Ω , then f is measurable on $(\Omega, \mathcal{F})$.

Proof :

This is immediate since $\lim_{n \rightarrow \infty} f_n = \limsup_{n \rightarrow \infty} f_n$ which is measurable.

Convergence almost everywhere

Definition(9.6)

Let $f, f_n \in X$, $n \in \mathbb{N}$ and let $A \in \mathcal{F}$, we say that

1. $\{f_n\}$ converges almost everywhere to f on A , denoted by $f_n \xrightarrow{a.e.} f$ on A , if there is

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a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \rightarrow f$ on $A \setminus B$.

2. $\{f_n\}$ Cauchy almost everywhere on A , denoted by f_n Cauchy a.e. on A , if there is $B \subseteq A$ such that $\mu(B) = 0$ and f_n Cauchy on $A \setminus B$.

Note that, in the above definition,

- when $A = \Omega$, we can omit " on A " from the statements.
i.e. $f_n \xrightarrow{a.e.} f$, if there is a subset $B \subseteq \Omega$ such that $\mu(B) = 0$ and $f_n \rightarrow f$ on B^c .
- When μ is a probability measurer, the convergence is called converges almost surely (a.s.) or converges almost certainly, $f_n \xrightarrow{a.s.} f$ or $f_n \xrightarrow{a.c.} f$, therefore $f_n \xrightarrow{a.s.} f$ if $\mu\{\lim_{n \rightarrow \infty} f_n = f\} = 1$,
i.e., $f_n \xrightarrow{a.c.} f$, if there exists $A \subseteq \Omega$ with $\mu(A) = 1$ such that $\lim_{n \rightarrow \infty} |f_n(x) - f(x)| = 0$ for all $x \in A$

Theorem(9.7)

Let $f_n \in X$, $n \in \mathbb{N}$, if $f_n \xrightarrow{a.e.} f$, then $f \in X$.

Proof:

Let $B_n = \{x \in \Omega : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}$, since $f_n \xrightarrow{a.e.} f \Rightarrow \mu(B) = 0$. Define

$$g_n(x) = \begin{cases} f_n(x), & x \notin B \\ 0, & x \in B \end{cases}$$

$\Rightarrow \lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} f_n(x)$ if $x \notin B \Rightarrow g(x) = f(x)$ if $x \notin B \Rightarrow \{g_n\}$ converges pointwise to g on Ω . Also $g_n \in \mathbb{M}(\Omega)$ for all n . Hence $g \in \mathbb{M}(\Omega)$. Consequently $f \in X$.

Theorem (9.8)

Let $f, f_n \in X$, $n \in \mathbb{N}$, if $f_n \xrightarrow{a.e.} f$ on A then

- f_n is a Cauchy a.e.
- If $g \in X$ and $f_n \xrightarrow{a.e.} g$ then $f = g$ a.e.
- If g is real-valued measurable function such that $f = g$ a.e. then $f_n \xrightarrow{a.e.} g$.
- If $\{g_n\}$ is a sequence of real-valued measurable functions such that $f_n = g_n$ a.e. for each n , then $g_n \xrightarrow{a.e.} f$.

Proof:

(1) Since $f_n \xrightarrow{a.e.} f$ on $A \Rightarrow$ there exist $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$, $\forall x \in A \setminus B \Rightarrow f_n(x)$ is Cauchy sequence for all $x \in A \setminus B \Rightarrow f_n$ Cauchy a.e.

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(2) Since $f_n \xrightarrow{a.e.} f$, there exist $B \subset A$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A \setminus B$ since $f_n \xrightarrow{a.e.} g$, there exist $C \subset A$ such that $\mu(C) = 0$ and $f_n(x) \rightarrow g(x)$ for all $x \in A \setminus C$.

Let $D = B \cup C$, since μ is a measure $\Rightarrow \mu(D) = 0$ and for any $x \in A \setminus D$,
 $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\lim_{n \rightarrow \infty} f_n(x) = g(x)$. So $f(x) = g(x)$, $\forall x \notin D \Rightarrow f = g$ a.e.

(3) Since $f_n \xrightarrow{a.e.} f$

$\Rightarrow$ There exist $B \subset A$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A \setminus B$.
since $f = g$ a.e., there exists $C \subset A$ such that $\mu(C) = 0$ and $f(x) = g(x)$, $\forall x \notin C$

Let $D = B \cup C$ since μ a measure $\Rightarrow \mu(D) = 0$ and $\forall x \notin D$

$\lim_{n \rightarrow \infty} f_n(x) = f(x) = g(x)$ So $\lim_{n \rightarrow \infty} f_n(x) = g(x)$, $\forall x \notin D$ Therefore $f_n \xrightarrow{a.e.} g$

(4) Since $f_n \xrightarrow{a.e.} f \Rightarrow$ there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \rightarrow f$ on $A \setminus B$.

Let B_n be a sequence in A such that $f_n(x) = g_n(x)$ for all $x \notin B_n$ and $\mu(B_n) = 0$.

Let $D = B \cup (\bigcup_n B_n)$, since μ is a measure $\Rightarrow \mu(D) = 0$, and for all $x \notin D$,

$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$. So $g_n(x) \rightarrow f(x)$ for all $x \notin B$. Therefore $g_n \xrightarrow{a.e.} f$

Theorem (9.9)

Let $f, f_n \in X$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$. If $f_n \xrightarrow{a.e.} f$, and $g_n \xrightarrow{a.e.} g$, on A then

1. $\lambda f_n \xrightarrow{a.e.} \lambda f$.
2. $f_n + g_n \xrightarrow{a.e.} f + g$.
3. $|f_n| \xrightarrow{a.e.} |f|$.
4. $f_n \vee g_n \xrightarrow{a.e.} f \vee g$ and $f_n \wedge g_n \xrightarrow{a.e.} f \wedge g$
5. $f_n^+ \xrightarrow{a.e.} f^+$ and $f_n^- \xrightarrow{a.e.} f^-$
6. $I_A f_n \xrightarrow{a.e.} I_A f$ for all $A \subseteq \Omega$
7. $f_n g_n \xrightarrow{a.e.} fg$

Proof :

(1) Since $f_n \xrightarrow{a.e.} f \Rightarrow$ There exist $B \subset A$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A \setminus B$, then $\lambda f_n(x) \rightarrow \lambda f(x)$ for all $x \in A \setminus B$, therefore $\lambda f_n \xrightarrow{a.e.} \lambda f$

(2) Since $f_n \xrightarrow{a.e.} f$, and $g_n \xrightarrow{a.e.} g \Rightarrow$ There exist $B \subset A$ and $C \subset A$ such that $\mu(B) = 0$, $\mu(C) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A \setminus B$, $g_n(x) \rightarrow g(x)$ for all $x \in A \setminus C$



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Let $D = B \cup C$, Since μ is a measure, this implies that $\mu(D) = 0 \Rightarrow f_n(x) \rightarrow f(x)$ and $g_n(x) \rightarrow g(x)$ for all $x \in A \setminus D$, so that $f_n(x) + g_n(x) \rightarrow f(x) + g(x)$ for all $x \in A \setminus D$,

$$\Rightarrow f_n + g_n \xrightarrow{a.e.} f + g$$

(3) Since $f_n \xrightarrow{a.e.} f \Rightarrow$ There exist $B \subset A$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A \setminus B$, $\Rightarrow |f_n(x)| \rightarrow |f(x)|$ for all $x \in A \setminus B$, $\Rightarrow |f_n| \xrightarrow{a.e.} |f|$

Theorem (9.10)

Let $f_n, g_n \in X$, $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $A \subseteq \Omega$, If $\{f_n\}$ and $\{g_n\}$ are Cauchy a.e. then all of the following sequence of functions are Cauchy a.e. : λf_n , $f_n + g_n$, $|f_n|$, $f_n \wedge g_n$, $f_n \vee g_n$, f_n^+ , f_n^- , $I_A f_n$ and $f_n g_n$

Theorem (9.11)

Let $f_n \in X$, $n \in \mathbb{N}$. If f_n is a Cauchy a.e. then there exists a real-value measurable function f such that $f_n \xrightarrow{a.e.} f$

Proof :

Since f_n is a Cauchy a.e. $\Rightarrow$ There exist $B \subset \Omega$ such that $\mu(B) = 0$ and f_n is a pointwise Cauchy on B^c , i.e. for every $x \in B^c$ and for every $\varepsilon > 0$, there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$. Define $f: \Omega \rightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & x \in B^c \\ 0 & x \in B \end{cases}$$

f is measurable function and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \notin B$, therefore $f_n \xrightarrow{a.e.} f$

Theorem (9.12)

Let $f, g, f_n \in X$, $n \in \mathbb{N}$ such that $f_n \xrightarrow{a.e.} f$ then

1. If and $f_n \geq 0$ a.e. then $f \geq 0$ a.e.
2. If and $f_n \leq g$ a.e. for each n then $f \leq g$ a.e.
3. If $|f_n| \leq |g|$ a.e. then $|f| \leq |g|$ a.e.
4. If $f_n \leq f_{n+1}$ a.e. for each n , then $f_n \uparrow f$ a.e.

Proof :

since $f_n \xrightarrow{a.e.} f$, this implies that there exist $B \subset \Omega$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \notin B$

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(1) since $f_n \geq 0$ a.e. for each n , so there exist $B_n \subset \Omega$ such that $\mu(B_n) = 0$ and $f_n(x) \geq 0$ for all $x \notin B_n$.

Let $A = B \cup \left(\bigcup_{n=1}^{\infty} B_n \right) \Rightarrow \mu(A) = 0$ and for any $x \notin A$ implies $f(x) = \lim_{n \rightarrow \infty} f_n(x) \geq 0$ therefore $f \geq 0$ a.e.

(2) since $f_n \leq g$ a.e. $\Rightarrow g - f_n \geq 0$ a.e., Since $f_n \xrightarrow{a.e.} f \Rightarrow g - f_n \xrightarrow{a.e.} g - f$ by (1) $g - f \geq 0$ a.e., so $f \leq g$ a.e.

(3) Since $f_n \xrightarrow{a.e.} f$, this implies that $|f_n| \xrightarrow{a.e.} |f|$, Since $|f_n| \leq |g|$ a.e. by (2) $|f| \leq |g|$ a.e.

(4) Since $f_n \leq f_{n+1}$ a.e. for each $n \Rightarrow$ there exist $A_n \subset \Omega$ such that $\mu(A_n) = 0$ and $f_n(x) \leq f_{n+1}(x)$ for all $x \notin A_n$

Let $A = B \cup \left(\bigcup_{n=1}^{\infty} A_n \right) \Rightarrow \mu(A) = 0$ and for any $x \notin A$ implies $f_n(x) \uparrow f(x)$ and $f_n(x) \rightarrow f(x)$, hence $f_n \uparrow f$ a.e.

Theorem (9.13)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$, then

1. If $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$ and $f_n = g_n$ a.e. for all n , then $f = g$ a.e.
2. If $f_n \xrightarrow{a.e.} f$, $f_n = g_n$ a.e. for all n , and $f = g$ a.e. then $g_n \xrightarrow{a.e.} g$ a.e.

Proof:

(1) since $f_n \xrightarrow{a.e.} f$, this implies that there exist $B \subset \Omega$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \notin B$ and $g_n \xrightarrow{a.e.} g \Rightarrow$ there exist $C \subset \Omega$ such that $\mu(C) = 0$ and $g_n(x) \rightarrow g(x)$ for all $x \notin C$ also $f_n = g_n$ a.e. $\Rightarrow$ there exist $B_n \subset \Omega$ such that $\mu(B_n) = 0$ for all n

Let $D = (B \cup C) \cup \left(\bigcup_{n=1}^{\infty} B_n \right)$ implies that $\mu(D) = 0$

$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} g_n(x) = g(x)$ for all $x \notin D$, so that $f(x) = g(x)$ for all $x \notin D$ therefore $f = g$ a.e.

(2) since $f_n \xrightarrow{a.e.} f$, this implies that there exist $B \subset \Omega$ such that $\mu(B) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \notin B$ and $f_n = g_n$ a.e. for all $n \Rightarrow$ there exist $B_n \subset \Omega$ such that $\mu(B_n) = 0$ for all n and $g_n(x) = f_n(x)$ for all $x \notin B_n$ also $f = g$ a.e. $\Rightarrow$ there exist $C \subset \Omega$ such that $\mu(C) = 0$ and $f(x) = g(x)$ for all $x \notin C$

Let $D = (B \cup C) \cup \left(\bigcup_{n=1}^{\infty} B_n \right)$ implies that $\mu(D) = 0$

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$g(x) = f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} g_n(x)$ for all $x \notin D$, so that $g_n(x) \rightarrow g(x)$ for all $x \notin D$

Therefore $g_n \xrightarrow{a.e.} g$

Almost Uniformly Convergence

Definition (9.14)

Let $f, f_n \in X$, $n \in \mathbb{N}$ and let $A \in \mathcal{F}$, we say that

1. $\{f_n\}$ converges almost uniformly to f on A , denoted by $f_n \xrightarrow{a.u.} f$ on A , if for each $\varepsilon > 0$, there is a subset $B \subseteq A$ such that $\mu(B) < \varepsilon$ and $f_n \xrightarrow{u} f$ on $A \setminus B$.
2. $\{f_n\}$ is a Cauchy almost uniformly, denoted by f_n Cauchy a.u., if for each $\varepsilon > 0$, there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and f_n Cauchy uniformly on A^c .

Note that, in the above definition,

1. when $A = \Omega$, we can omit " on A " from the statements. i.e. $f_n \xrightarrow{a.u.} f$, if for each $\varepsilon > 0$, there is a subset $B \subseteq \Omega$ such that $\mu(B) < \varepsilon$ and $f_n \xrightarrow{u} f$ on B^c .
2. When μ is a probability measurer, the convergence is called converges almost certainly uniformly.

Theorem (9.15)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$, such that $f_n \xrightarrow{a.u.} f$, then

1. f_n Cauchy a.u.
2. If $f_n \xrightarrow{a.u.} g$, then $f = g$ a.e.
3. If $f = g$ a.e., then $f_n \xrightarrow{a.u.} g$.
4. If $f_n = g_n$ a.e. for all n , then $g_n \xrightarrow{a.u.} f$.
5. If $f_n = g_n$ a.e. for all n and $f = g$ a.e., then $g_n \xrightarrow{a.u.} g$.

Proof :

(1) Let $\varepsilon > 0$

Since $f_n \xrightarrow{a.u.} f$, then there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c , thus f_n is uniformly Cauchy on A^c , therefore f_n Cauchy a.u.

(2) Let $\varepsilon > 0$, since $f_n \xrightarrow{a.u.} f$ and $f_n \xrightarrow{a.u.} g$, there is a subset $A, B \subset \Omega$ such that $\mu(A) < \varepsilon$, $\mu(B) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c and $f_n \xrightarrow{u} g$ on B^c , $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin A$, $f_n(x) \rightarrow g(x)$ uniformly for any $x \notin B$.

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$\Rightarrow \mu(A \cup B) < \varepsilon$, $\varepsilon > 0$ and $f_n(x) \rightarrow f(x)$, $f_n(x) \rightarrow g(x)$ uniformly for any $x \notin A \cup B$.

Since ε arbitrary so $A \cup B$ is a null set and $f(x) = g(x)$ for any $x \notin A \cup B$,

Therefore $f = g$ a.e.

(3) Since $f_n \xrightarrow{a.u.} f \Rightarrow$ for any $\varepsilon > 0$, there exists $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and

$f_n \xrightarrow{u} f$ on A^c , i.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin A$.

Since $f = g$ a.e. $\Rightarrow$ there exist $B \subset \Omega$ such that $\mu(B) = 0$ and $f(x) = g(x)$ for any $x \notin B$.

$\Rightarrow \mu(A \cup B) < \varepsilon$ and for $x \notin A \cup B$, $f_n(x) \rightarrow f(x) = g(x)$ Uniformly, therefore

$f_n \xrightarrow{a.u.} g$.

(4) Since $f_n \xrightarrow{a.u.} f \Rightarrow$ for any $\varepsilon > 0$, there exists $B \subset \Omega$ such that $\mu(B) < \varepsilon$ and $f_n \xrightarrow{u} f$ on B^c , i.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin B$,

Since $f_n = g_n$ a.e., then there exist a sequence $\{A_n\}$ of subsets of Ω such that $\mu(A_n) = 0$ and $f_n(x) = g_n(x)$ for all $x \notin A_n$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, then $\mu(A) = 0 \Rightarrow \mu(A \cup B) < \varepsilon$

and for any $x \notin A \cup B$, $g_n(x) = f_n(x) \rightarrow f(x)$ uniformly

Therefore $g_n(x) \rightarrow f(x)$ uniformly for any $x \notin A \cup B$, thus $g_n \xrightarrow{a.u.} f$

(5) Since $f_n \xrightarrow{a.u.} f \Rightarrow$ for any $\varepsilon > 0$, there exists $B \subset \Omega$ such that $\mu(B) < \varepsilon$ and

$f_n \xrightarrow{u} f$ on B^c , i.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin B$,

Since $f_n = g_n$ a.e., then there exist a sequence $\{A_n\}$ of subsets of Ω such that $\mu(A_n) = 0$ and $f_n(x) = g_n(x)$ for all $x \notin A_n$.

Since $f = g$ a.e. $\Rightarrow$ there exist $C \subset \Omega$ such that $\mu(C) = 0$ and $f(x) = g(x)$ for any $x \notin C$.

Let $A = C \cup \left(\bigcup_{n=1}^{\infty} A_n \right)$, then $\mu(A) = 0 \Rightarrow \mu(A \cup B) < \varepsilon$

and for any $x \notin A \cup B$, $g_n(x) = f_n(x) \rightarrow f(x) = g(x)$ uniformly, thus for any $x \notin A \cup B$,

$g_n(x) \rightarrow g(x)$ uniformly. Therefore $g_n \xrightarrow{a.u.} g$.

Theorem (9.16)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, such that $f_n \xrightarrow{a.u.} f$ and $g_n \xrightarrow{a.u.} g$, then

1. $\lambda \cdot f_n \xrightarrow{a.u.} \lambda \cdot f$

2. $f_n + g_n \xrightarrow{a.u.} f + g$

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3. $|f_n| \xrightarrow{a.u.} |f|$

Proof

(1) Since $f_n \xrightarrow{a.u.} f$, then for any $\varepsilon > 0$, there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c , i.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin A$, therefore $c.f_n \xrightarrow{a.u.} c.f$.

(2)

since $f_n \xrightarrow{a.u.} f$ and $g_n \xrightarrow{a.u.} g$, then for any $\varepsilon > 0$, there is a subset $A, B \subset \Omega$ such that $\mu(A) < \varepsilon, \mu(B) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c and $g_n \xrightarrow{u} g$ on B^c , $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin A$, $g_n(x) \rightarrow g(x)$ uniformly for any $x \notin B$.

$\Rightarrow \mu(A \cup B) < \varepsilon, \varepsilon > 0$ and $f_n(x) \rightarrow f(x), g_n(x) \rightarrow g(x)$ uniformly for any $x \notin A \cup B$.

So $f_n(x) + g_n(x) \rightarrow f(x) + g(x)$ uniformly for any $x \notin A \cup B$. Therefore $f_n + g_n \xrightarrow{a.u.} f + g$

(3) Since $f_n \xrightarrow{a.u.} f$, then for any $\varepsilon > 0$, there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c , i.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \notin A$, so $|f_n(x)| \rightarrow |f(x)|$ uniformly for any $x \notin A$, therefore $|f_n| \xrightarrow{a.u.} |f|$.

Theorem (9.16)

Let $f_n, g_n \in X, n \in \mathbb{N}, \lambda \in \mathbb{R}$ and $A \subseteq \Omega$, If $\{f_n\}$ and $\{g_n\}$ are Cauchy a.u. then all of the following sequence of functions are Cauchy a.u.: $\lambda f_n, f_n + g_n, |f_n|$ and

Convergence in Measure

Definition (9.17)

Let $f, f_n \in X, n \in \mathbb{N}$ and let $A \in \mathcal{F}$, we say that

1. $\{f_n\}$ converges in measure to f on A , denoted by $f_n \xrightarrow{\mu} f$ on A , if

$$\lim_{n \rightarrow \infty} \mu(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

for each $\varepsilon > 0$.

2. $\{f_n\}$ is a Cauchy sequence in measure, denoted by f_n Cauchy in measure, if for each $\varepsilon > 0$,

$$\mu(\{x \in \Omega : |f_n(x) - f_m(x)| \geq \varepsilon\}) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Note that, in the above definition,

1. when $A = \Omega$, we can omit " on A " from the statements.

i.e. $f_n \xrightarrow{\mu} f$, if $\lim_{n \rightarrow \infty} \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$ for each $\varepsilon > 0$.

2. When μ is a probability measurer, the convergence is called convergence in probability.



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$f_n \xrightarrow{\mu} f$ if $\lim_{n \rightarrow \infty} (\mu\{|f_n - f| \geq \varepsilon\}) = 0$ for any $\varepsilon > 0$, or equivalent $\lim_{n \rightarrow \infty} (\mu\{|f_n - f| < \varepsilon\}) = 1$ for any $\varepsilon > 0$, i.e., $f_n \xrightarrow{\mu} f$, for any $\varepsilon, \delta > 0$, there is $k \in \mathbb{Z}^+$ such that $\mu\{|f_n - f| > \varepsilon\} < \delta$ for all $n \geq k$

3. In general if (V, d) is a metric space and $f, f_n: V \rightarrow V$ -valued measurable functions, then $f_n \xrightarrow{\mu} f$, if $d(f_n, f) \xrightarrow{\mu} 0$, that is, if $\lim_{n \rightarrow \infty} \mu(\{x \in \Omega : d(f_n(x), f(x)) \geq \varepsilon\}) = 0$ for each $\varepsilon > 0$.

For $V = \mathbb{R}$, taking $d(f, g) = |f - g|$ we obtain above definition.

Theorem (9.18)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mu} f$, then

1. f_n Cauchy in measure
2. If $f_n \xrightarrow{\mu} g$, then $f = g$ a.e.
3. If $f = g$ a.e., then $f_n \xrightarrow{\mu} g$
4. If $f_n = g_n$ a.e. for all n , then $g_n \xrightarrow{\mu} f$.

Proof:

(1) Let $\varepsilon > 0$ and define

$$A_{nm} = \{x \in \Omega : |f_n(x) - f_m(x)| \geq \varepsilon\} \text{ and } A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\}$$

Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$ this implies that $A_{nm} \subseteq A_n \cup A_m$
 $\Rightarrow \mu(A_{nm}) \leq \mu(A_n \cup A_m) \leq \mu(A_n) + \mu(A_m)$

Since $f_n \xrightarrow{\mu} f \Rightarrow \mu(A_n) \rightarrow 0$ and $\mu(A_m) \rightarrow 0 \Rightarrow \mu(A_{nm}) \rightarrow 0$ as $n, m \rightarrow \infty$

Therefore f_n Cauchy in measure.

(2) Given any $\varepsilon > 0$, define

$$A = \{x \in \Omega : |f(x) - g(x)| \geq \varepsilon\}, \quad A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\}, \quad B_n = \{x \in \Omega : |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\}$$

Since $|f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|$ this implies that $A \subseteq A_n \cup B_n$
 $\Rightarrow \mu(A) \leq \mu(A_n \cup B_n) \leq \mu(A_n) + \mu(B_n) \Rightarrow \mu(A) = 0$

$$N(f - g) = \{x \in \Omega : (f - g)(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in \Omega : |f(x) - g(x)| \geq \frac{1}{n}\}$$

$$\Rightarrow \mu(N(f - g)) = 0 \Rightarrow f = g \text{ a.e.}$$

(3) Since $f = g$ a.e $\Rightarrow$ there exists $A \in F$ with $\mu(A) = 0$ and $f(x) \neq g(x)$ for all $x \in A$.

For any $\varepsilon > 0$, we have $\{x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon\} \subset A \cup \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}$

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$$\mu(\{x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon\}) \leq \mu(A) + \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow f_n \xrightarrow{\mu} g$$

(4) Since $f_n = g_n$ a.e. for all $n \Rightarrow$ there exists $A_n \in F$ with $\mu(A_n) = 0$ and $f_n(x) \neq g_n(x)$ for all $x \in A_n$.

Put $A = \bigcup_{n=1}^{\infty} A_n \Rightarrow \mu(A) = 0$. For any $\varepsilon > 0$, we have

$$\{x \in \Omega : |g_n(x) - f(x)| \geq \varepsilon\} \subset A \cup \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}$$

$$\mu(\{x \in \Omega : |g_n(x) - f(x)| \geq \varepsilon\}) \leq \mu(A) + \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow g_n \xrightarrow{\mu} f$$

Theorem (9.19)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mu} f$, $g_n \xrightarrow{\mu} g$ and $\lambda \in \mathbb{R}$, then

1. $\lambda f_n \xrightarrow{\mu} \lambda f$.
2. $f_n + g_n \xrightarrow{\mu} f + g$.
3. $|f_n| \xrightarrow{\mu} |f|$.
4. $f_n \vee g_n \xrightarrow{\mu} f \vee g$ and $f_n \wedge g_n \xrightarrow{\mu} f \wedge g$.
5. $f_n^+ \xrightarrow{\mu} f^+$ and $f_n^- \xrightarrow{\mu} f^-$.
6. $I_A f_n \xrightarrow{\mu} I_A f$ for all locally measurable set A .

Proof :

(1) This clear if $\lambda = 0$, if $\lambda \neq 0$, let $\varepsilon > 0$, Since $f_n \xrightarrow{\mu} f$ and

$$\{x \in \Omega : |\lambda f_n(x) - \lambda f(x)| \geq \varepsilon\} = \{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{|\lambda|}\}$$

This implies that

$$\mu(\{x \in \Omega : |\lambda f_n(x) - \lambda f(x)| \geq \varepsilon\}) = \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{|\lambda|}\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

So that $\lambda f_n \xrightarrow{\mu} \lambda f$

(2) Let $h_n = f_n + g_n$, $h = f + g$, since $|h_n(x) - h(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$

This implies that

$$\{x \in \Omega : |h_n(x) - h(x)| \geq \varepsilon\} \subseteq \left\{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x \in \Omega : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}$$

Since $f_n \xrightarrow{\mu} f$, $g_n \xrightarrow{\mu} g$. Thus

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$$\mu(\{x \in \Omega : |h_n(x) - h(x)| \geq \varepsilon\}) \leq \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\}) + \mu(\{x \in \Omega : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\}) \rightarrow 0$$

As $n \rightarrow \infty$ So that $f_n + g_n \xrightarrow{\mu} f + g$

(3) Since $\|f_n(x) - f(x)\| \leq |f_n(x) - f(x)|$ this implies that

$$\{x \in \Omega : \|f_n(x) - f(x)\| \geq \varepsilon\} \subseteq \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}$$

Since $f_n \xrightarrow{\mu} f$ so $\mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}) \rightarrow 0$. Therefore $|f_n| \xrightarrow{\mu} |f|$

Theorem (9.20)

Let $f_n, g_n \in X$, $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $A \subseteq \Omega$, If $\{f_n\}$ and $\{g_n\}$ are Cauchy in measure then all of the following sequence of functions are Cauchy in measure. : λf_n , $f_n + g_n$, $|f_n|$, $f_n \wedge g_n$, $f_n \vee g_n$, f_n^+ , f_n^- and $I_A f_n$ for all locally measurable set A

Theorem (9.21)

Let $f, f_n, g, g_n \in X$, $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mu} f$ then

1. If $f_n \geq 0$ a.e. then $f \geq 0$ a.e.
2. If $f_n \leq g$ a.e. for each n then $f \leq g$ a.e.
3. If $|f_n| \leq |g|$ a.e. for all n then $|f| \leq |g|$ a.e.
4. If $g_n \xrightarrow{\mu} g$ and $f_n = g_n$ a.e. for all n , then $f = g$ a.e.

Proof :

(1) Let $\varepsilon > 0$, $A = \{x : f(x) \leq -\varepsilon\}$ and $A_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$,

Since $f_n \xrightarrow{\mu} f \Rightarrow \mu(A_n) \rightarrow 0$,

Let $x \in A \Rightarrow f(x) \leq -\varepsilon$, since $f(x) = f(x) - f_n(x) + f_n(x) \geq f(x) - f_n(x)$

$\Rightarrow f(x) - f_n(x) \leq -\varepsilon \Rightarrow |f(x) - f_n(x)| \geq \varepsilon \Rightarrow x \in A_n$ so $A \subseteq A_n$.

This implies that $\mu(A) \rightarrow 0$

Since $\{x \in \Omega : f(x) < 0\} = \bigcup_{k=1}^{\infty} \{x : f(x) \leq -\frac{1}{k}\}$. Therefore $\mu(\{x : f(x) < 0\}) = 0$, thus $f \geq 0$ a.e.

(2) since $f_n \leq g$ a.e. $\Rightarrow g - f_n \geq 0$ a.e.

since $f_n \xrightarrow{\mu} f \Rightarrow g - f_n \rightarrow g - f$, by (1) $g - f \geq 0$ a.e., so $f \leq g$ a.e.

(3) Since $f_n \xrightarrow{\mu} f$, so $|f_n| \xrightarrow{\mu} |f|$ (by theorem (9.18), therefore $|f| \leq |g|$ by (3)

(4) Since $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ this implies that

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \rightarrow 0 \text{ and } \mu(\{x : |g_n(x) - g(x)| \geq \varepsilon\}) \rightarrow 0$$

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Let $A = \{x \in \Omega : f_n(x) \neq g_n(x)\}$

Since $|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g_n(x)| + |g_n(x) - g(x)|$

So $\{x \in \Omega : |f(x) - g(x)| \geq \varepsilon\} \subseteq \{x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon\} \cup A \cup \{x \in \Omega : |g_n(x) - g(x)| \geq \varepsilon\}$,

thus $\mu(\{x \in \Omega : |f(x) - g(x)| \geq \varepsilon\}) \rightarrow 0$

Since $\{x \in \Omega : |f(x) \neq g(x)|\} = \bigcup_{k=1}^{\infty} \{x : |f(x) - g(x)| \geq \frac{1}{k}\}$.

Therefore $\mu(\{x : f(x) \neq g(x)\}) = 0$, thus $f = g$ a.e.

Relation Between Types of Convergence

Almost uniform convergence is stronger than both almost everywhere and convergence in measure, as we now prove

Theorem(9.22)

Let $f, f_n \in X$, $n \in \mathbb{N}$ and 1

1. If $f_n \xrightarrow{a.u.} f$, then $f_n \xrightarrow{\mu} f$.
2. If $f_n \xrightarrow{a.u.} f$, then $f_n \xrightarrow{a.e.} f$.
- 3.. If f_n Cauchy sequence almost uniformly, then f_n is a Cauchy sequence in measure.
4. If f_n Cauchy sequence almost uniformly, then f_n is a Cauchy sequence in measure.

Proof :

1. Let $\varepsilon > 0$

Since $f_n \xrightarrow{a.u.} f$, then there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and $f_n \xrightarrow{u} f$ on A^c .

If $\delta > 0$, then eventually $|f_n(x) - f(x)| < \delta$ for any $x \notin A$, so $\{x \in \Omega : |f_n(x) - f(x)| \geq \delta\} \subset A$

Thus $\mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \delta\}) \leq \mu(A) < \varepsilon$, therefore $f_n \xrightarrow{\mu} f$

2. Let k positive integer

Since $f_n \xrightarrow{a.u.} f$, then there is a subsets $A_k \subset \Omega$ such that $\mu(A_k) < \frac{1}{k}$ and $f_n \xrightarrow{u} f$ on A_k^c .

If $B = \bigcup_{k=1}^{\infty} A_k^c$, then $f_n \xrightarrow{u} f$ on B and $\mu(B^c) = \mu((\bigcup_{k=1}^{\infty} A_k^c)^c) = \mu(\bigcap_{k=1}^{\infty} A_k) \leq \mu(A_k) \rightarrow 0$ as

$k \rightarrow \infty$. Thus $\mu(B^c) = 0$, therefore $f_n \xrightarrow{a.e.} f$

Theorem(9. 23)

Let $f_n \in X$, $n \in \mathbb{N}$ and let $A \in \mathcal{F}$,

1. If f_n Cauchy in measure, then there is a subsequence f_{n_k} is Cauchy almost uniformly.
2. If f_n Cauchy in measure, then there is $f \in X$ such that $f_n \xrightarrow{\mu} f$.

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3. If $f_n \xrightarrow{\mu} f$, then there is a subsequence f_{n_k} such that $f_{n_k} \xrightarrow{\mu} f$.

4. 3. If $f_n \xrightarrow{\mu} f$, then there is a subsequence f_{n_k} such that $f_{n_k} \xrightarrow{a.u.} f$.

Proof :

1. For each positive integer k , choose a positive integer N_k such that $N_{k+1} > N_k$ for all k and $\mu(\{x \in \Omega : |f_n(x) - f_m(x)| \geq 2^{-k}\}) \leq 2^{-k}$ for $n, m \geq N_k$.

Put $A_k = \{x \in \Omega : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq 2^{-k}\}$, $B_k = \bigcup_{i=k}^{\infty} A_i$. Then

$$\mu(B_k) = \mu\left(\bigcup_{i=k}^{\infty} A_i\right) \leq \sum_{i=k}^{\infty} \mu(A_i) \leq 2^{1-k}$$

Let $\varepsilon > 0$, we can choose k so that $\varepsilon > 2^{1-k}$; then $\mu(B_k) < \varepsilon$ and for all $x \in B_k^c$ we have

$$|f_{n_i}(x) - f_{n_{i+1}}(x)| \leq 2^{-i} \text{ for all } i \geq k$$

Thus $j \geq i \geq k \Rightarrow |f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{t=i}^{j-1} |f_{n_t}(x) - f_{n_{t+1}}(x)| \leq 2^{-i}$ so that the sequence

f_{n_i} Cauchy uniformly on B_k^c , that is $\mu(B_k) \rightarrow 0$ as $k \rightarrow \infty$, it is a Cauchy sequence almost uniformly.

2. Since f_n Cauchy in measure, then there is a subsequence f_{n_k} is Cauchy almost uniformly and therefore Cauchy almost everywhere and therefore.

We write $f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$ for every x for which the limit exists.

We observe that, for every $\varepsilon > 0$,

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \{x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\} \cup \{x : |f_{n_k}(x) - f(x)| \geq \frac{\varepsilon}{2}\}$$

Given $\delta > 0$ we can find a set $A_\delta \in \mathcal{F}$ and integers k_0, N such that $\mu(A_\delta) < \frac{\delta}{2}$,

$$|f_{n_k}(x) - f(x)| \geq \frac{\varepsilon}{2} \text{ for } k \geq k_0, x \in A_\delta^c \text{ and } \mu(\{x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\}) < \frac{\delta}{2} \text{ for } n, n_k \geq N$$

It follows that $n \geq \max\{N, n_k\} \Rightarrow \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \delta$.

Remark

Let $f, f_n \in X$, $n \in N$. For every $\delta > 0$, define

$$\beta_n(\delta) = \{x \in \Omega : |f_n(x) - f(x)| \geq \delta\}, \quad n \in N, \text{ and } \beta_\delta = \limsup_{n \rightarrow \infty} \beta_n(\delta)$$

Since each $f_n - f$ is measurable function, the sets $\beta_n(\delta)$ are measurable.

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Theorem (9.24)

Let $f, f_n \in X$, $n \in N$. Then $f_n \xrightarrow{a.e.} f$ iff $\lim_{n \rightarrow \infty} \mu(\bigcup_{k=n}^{\infty} \beta_k(\delta)) = 0$ for every $\delta > 0$

Proof :

It is clear to show that $\bigcup_{k=n}^{\infty} \beta_k(\delta) \downarrow \beta_\delta$. Hence $\mu(\bigcup_{k=n}^{\infty} \beta_k(\delta)) \rightarrow \mu(\beta_\delta)$ as $n \rightarrow \infty$

Now

$$\{x \in \Omega : f_n(x) \not\rightarrow f(x)\} = \bigcup_{\delta > 0} \beta_\delta$$

Since $\beta_{\delta_1} \subset \beta_{\delta_2}$ for $\delta_1 > \delta_2$, we have $\{x \in \Omega : f_n(x) \not\rightarrow f(x)\} = \bigcup_{m=1}^{\infty} \beta_{\frac{1}{m}}$

Therefore $f_n \xrightarrow{a.e.} f$ on A iff $\mu(\beta_\delta) = 0$ for every $\delta > 0$

$f_n \xrightarrow{a.e.} f$ on A iff $\lim_{n \rightarrow \infty} \mu(\bigcup_{k=n}^{\infty} \beta_k(\delta)) = 0$ for every $\delta > 0$

Theorem (9.25)

Let $f, f_n \in X$, $n \in N$.

1. If μ is finite and $f_n \xrightarrow{a.e.} f$, then $f_n \xrightarrow{a.u.} f$ (Egoroff's theorem)

2. If μ is finite and $f_n \xrightarrow{a.e.} f$, then $f_n \xrightarrow{\mu} f$

Proof :

2. Let $\varepsilon > 0$

Take $A_j = \bigcup_{k=n(j)}^{\infty} \{x : |f_k(x) - f(x)| \geq \frac{1}{j}\}$, for any positive integer j , then $\mu(A_j) = \frac{\varepsilon}{2^j}$ for all j

Let $A = \bigcup_{j=1}^{\infty} A_j$, then $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j) < \varepsilon$.

Also if $\delta > 0$, there exists positive integer j such that $\frac{1}{j} < \delta$, we have, for any $k \geq n(j)$

and $x \in A^c$ (hence $x \notin A_j$), $|f_n(x) - f(x)| < \frac{1}{j} < \delta$. thus $f_n \xrightarrow{u} f$ on A^c , so $f_n \xrightarrow{a.u.} f$

2. Since μ is finite and $f_n \xrightarrow{a.e.} f$, by (2), we have $f_n \xrightarrow{a.u.} f$ also by theorem (9.22), we have $f_n \xrightarrow{\mu} f$.

Example(9.26)

1. Let $\Omega = [0, \infty)$ with Lebesgue measure and define $f_n(x) = \begin{cases} 1, & n \leq x \leq n + \frac{1}{n} \\ 0, & \text{elsewhere} \end{cases}$, then



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$f_n \xrightarrow{a.e.} 0$, and $f_n \xrightarrow{\mu} 0$ but does not converge almost uniformly.

For if $f_n \xrightarrow{a.u.} 0$ on A and $\mu(A^c) < \varepsilon$, then eventually $f_n < 1$ on A ; hence if $A_n = [n, n + \frac{1}{n}]$ we have $A \cap \bigcup_{k \geq n} A_k = \emptyset$ for sufficiently large n . Therefore, $\bigcup_{k \geq n} A_k \subseteq A^c$, and consequently

$$\mu(A^c) \geq \sum_{k=n}^{\infty} \mu(A_k)$$

2. Let $\Omega = [0, \infty)$ with Lebesgue measure and define $f_n(x) = \begin{cases} 1, & n \leq x \leq n+1 \\ 0, & \text{elsewhere} \end{cases}$,

then $f_n \xrightarrow{a.e.} 0$, but not in measure, hence not almost uniformly.

3. Let $\Omega = (0, 1]$, $\mathcal{F} = \beta(\Omega)$, and μ is the Lebesgue measure. Let

$$A_1 = (0, 1], \quad A_2 = (0, \frac{1}{2}], \quad A_3 = (\frac{1}{2}, 1], \quad A_4 = (0, \frac{1}{3}], \quad A_5 = (\frac{1}{3}, \frac{2}{3}], \quad A_6 = (\frac{2}{3}, 1], \dots$$

If $f_n = I_{A_n}$, then, for arbitrary $\varepsilon \in (0, 1)$, the $\mu(\{f_n > \varepsilon\})$ form the sequence $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\}$, whose limit is obviously 0. So $f_n \xrightarrow{\mu} 0$

4. Let $\Omega = [0, 1]$, $\mathcal{F} = \beta(\Omega)$, and μ is the Lebesgue measure, if for $n = 1, 2, \dots$, $A_n^i = [\frac{i-1}{n}, \frac{i}{n}]$,

$i = 1, 2, \dots, n$. If $f_n^i = I_{A_n^i}$, then $\{f_n^i\}$ converges in measure to 0, but fails to converge at any point of X .

5. Let $\Omega = [0, 1]$, $\mathcal{F} = \beta(\Omega)$, and μ is the Lebesgue measure. Let

$$A_{r,k} = [\frac{r-1}{2^k}, \frac{r}{2^k}] \quad , \quad r = 1, 2, \dots, 2^k \quad , \quad k = 1, 2, \dots$$

and arrange these intervals as a single sequence of sets $\{B_n\}$ by taking first those for which $k = 1$, then those with $k = 2$, etc. If $f_n = I_{A_n}$, then, for arbitrary $\varepsilon \in (0, 1)$, $\{x : |f_n(x)| \geq \varepsilon\} = B_n$ so that, for any $\varepsilon > 0$, $\mu\{x : |f_n(x)| \geq \varepsilon\} \leq \mu(B_n) \rightarrow 0$. This means that $f_n \rightarrow 0$ in measure in $[0, 1]$.

However, at no point $x \in [0, 1]$ does $f_n(x) \rightarrow 0$, in fact, since every x is in infinitely many of the sets B_n and infinitely many of the sets B_n^c we have $\liminf_{n \rightarrow \infty} f_n(x) = 0$, $\limsup_{n \rightarrow \infty} f_n(x) = 1$ for all $x \in [0, 1]$.

6. Let $\Omega = \mathbb{Z}^+$, $\mathcal{F}$ the family of all subsets of Ω , and $\mu(A)$ the number of integers in $A \subseteq \Omega$. If $f_n = I_{A_n}$ where $A_n = \{1, 2, \dots, n\}$, then $f_n(x) \rightarrow 1$ for all x . However, f_n does not converge almost uniformly to 1, (this example showing that Egoroff theorem is false without $\mu(A) < \infty$)



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Exercises(9)

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions. Show that $A = \{x \in \Omega : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$ is a measurable set.
2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions converging pointwise to the function f . Show that f is a measurable function.
3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on this space. Show that if $f_n \xrightarrow{\mu} f$, then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \xrightarrow{a.e.} f$.
4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on this space. Show that $f_n \xrightarrow{\mu} f$ iff every subsequence $\{f_{n_k}\}$ contains a further subsequence converging a.e. to f .
5. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Show that
 - (1) The function $\mu\{|f| > \lambda\}$, for $\lambda > 0$, is right continuous and nonincreasing.
 - (2) If f, g, h are nonnegative measurable and λ_1, λ_2 are positive numbers with the property that $f \leq \lambda_1 g + \lambda_2 h$, then for all $\lambda > 0$, $\mu\{f > (\lambda_1 + \lambda_2)\lambda\} \leq \mu\{g > \lambda\} + \mu\{h > \lambda\}$.
6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on this space satisfying $\sum_{n=1}^{\infty} \mu\{|f_n| > \lambda_n\} < \infty$ for some of real numbers λ_n . Show that $\limsup_{n \rightarrow \infty} \frac{|f_n|}{\lambda_n} \leq 1$.
7. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of measurable functions on this space. Show that
 - (1) $f_n \xrightarrow{a.e.} f$ iff for any $\varepsilon > 0$, $\lim_{m \rightarrow \infty} \mu\{|f_n - f_m| > \varepsilon\} = 0$, for some $k > n \geq m$
 - (2) $f_n \xrightarrow{a.e.} 0$ iff for any $\varepsilon > 0$, $\mu\{|f_n| > \varepsilon\} = 0$
8. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of nonnegative measurable functions on this space. Show that $f_n \xrightarrow{a.e.} \infty$ iff for any $M > 0$, $\mu\{f_n < M\} = 0$.
9. Let $\{A_n\}$ be a sequence of measurable sets and let $f_n = I_{A_n}$. Then $\{f_n\}$ is Cauchy in measure iff $\mu(A_n \Delta A_m) \rightarrow 0$ as $n, m \rightarrow \infty$.



10. The Integration

The notion of integral of a real valued function f defined over a closed interval $[a, b]$, originally came from geometry " the area under the curve $f(x)$ in between a and b . In the classical theory of integration on $\mathbb{R}$, $\int_a^b f(x)dx$ is defined as a limit of Riemann sums, which are integrals of functions that approximate f and are constant on subintervals of $[a, b]$. Therefore, the theory developed for the measure space $(\Omega, \mathcal{F}, \mu)$ in the following sections can be extended virtually without change to an abstractly given measure space $(\Omega, \mathcal{F}, \mu)$.

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a measurable function. The goal of this section is to define the integral of f with respect to the measure μ .

Definition (10.1)

Let A be a subset of a set Ω . A function $I_A : \Omega \rightarrow \mathbb{R}$ defined by

$$I_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

is called the indicator (or characteristic) of A . Note that : $I_\phi = 0$, $I_\Omega = 1$

Theorem (10.2)

Let A, B be subsets of a set Ω

1. $A = B$ iff $I_A = I_B$
2. $A \subseteq B$ iff $I_A \leq I_B$
3. $I_{A \cap B} = I_A \cdot I_B$
4. $I_{A \cup B} = I_A + I_B - I_{A \cap B}$

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5. $I_{A^c} = 1 - I_A$

6. $I_{A \Delta B} = |I_A - I_B|$

Proof :

(1) Suppose $A = B$

$$I_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases} = \begin{cases} 1 & , x \in B \\ 0 & , x \notin B \end{cases} = I_B(x) \Rightarrow I_A(x) = I_B(x) \text{ for all } x \in \Omega \Rightarrow I_A = I_B$$

Conversely, suppose that $I_A = I_B$

Let $x \in A \Rightarrow I_A(x) = 1$, but $I_A = I_B \Rightarrow I_B(x) = 1 \Rightarrow x \in B \Rightarrow A \subseteq B$

Similarly to prove $B \subseteq A \Rightarrow A = B$

(3) Let $x \in \Omega$

If $x \in A \cap B \Rightarrow I_{A \cap B}(x) = 1$ and $x \in A \cap B \Rightarrow x \in A, x \in B$

$$\Rightarrow I_A(x) = 1, I_B(x) = 1 \Rightarrow I_A(x) \times I_B(x) = 1 \Rightarrow I_{A \cap B}(x) = I_A(x) \times I_B(x)$$

or if $x \notin A \cap B \Rightarrow I_{A \cap B}(x) = 0$ and $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$

$$\Rightarrow I_A(x) = 0 \text{ or } I_B(x) = 0 \Rightarrow I_A(x) \times I_B(x) = 0 \Rightarrow I_{A \cap B}(x) = I_A(x) \times I_B(x)$$

So $I_{A \cap B}(x) = I_A(x) \times I_B(x)$ for all $x \in \Omega \Rightarrow I_{A \cap B} = I_A \times I_B$

(4) Let $x \in \Omega$

If $x \in A \cup B \Rightarrow I_{A \cup B}(x) = 1$ and $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

There are three cases

(a) $x \in A$ and $x \in B \Rightarrow x \in A \cap B \Rightarrow I_A(x) = 1, I_B(x) = 1, I_{A \cap B}(x) = 1$

$$\Rightarrow I_A(x) + I_B(x) - I_{A \cap B}(x) = 1 + 1 - 1 = 1$$

(b) $x \in A$ and $x \notin B \Rightarrow x \notin A \cap B \Rightarrow I_A(x) = 1, I_B(x) = 0, I_{A \cap B}(x) = 0$

$$\Rightarrow I_A(x) + I_B(x) - I_{A \cap B}(x) = 1 + 0 - 0 = 1$$

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$$(c) \ x \notin A \text{ and } x \in B \Rightarrow x \notin A \cap B \Rightarrow I_A(x) = 0, \ I_B(x) = 1, \ I_{A \cap B}(x) = 0$$

$$\Rightarrow I_A(x) + I_B(x) - I_{A \cap B}(x) = 0 + 1 - 0 = 1 \Rightarrow I_{A \cup B}(x) = I_A(x) + I_B(x) - I_{A \cap B}(x)$$

$$\text{Now if } x \notin A \cup B \Rightarrow I_{A \cup B}(x) = 0 \text{ and } x \notin A \cup B \Rightarrow x \notin A, \ x \notin B \Rightarrow x \notin A \cap B$$

$$I_A(x) = 0, \ I_B(x) = 0, \ I_{A \cap B}(x) = 0 \Rightarrow I_A(x) + I_B(x) - I_{A \cap B}(x) = 0 + 0 - 0 = 0$$

$$\text{So } I_{A \cup B}(x) = I_A(x) + I_B(x) - I_{A \cap B}(x) \text{ for all } x \in \Omega \Rightarrow I_{A \cup B} = I_A + I_B - I_{A \cap B}$$

(5) Let $x \in \Omega$

$$\text{If } x \in A^c \Rightarrow I_{A^c}(x) = 1 \text{ and } x \in A^c \Rightarrow x \notin A \Rightarrow I_A(x) = 0 \Rightarrow 1 - I_A(x) = 1 - 0 = 1$$

$$\text{If } x \notin A^c \Rightarrow I_{A^c}(x) = 0 \text{ and } x \in A^c \Rightarrow x \notin A \Rightarrow I_A(x) = 0 \Rightarrow 1 - I_A(x) = 1 - 0 = 1$$

$$\text{So } I_{A^c}(x) = 1 - I_A(x) \text{ for } x \in \Omega \Rightarrow I_{A^c} = 1 - I_A$$

Theorem(10.3)

Let $(\Omega, \mathcal{F})$ be a measurable space and $A \subset \Omega$, then I_A is a measurable function iff $A \in \mathcal{F}$

Proof :

Suppose I_A is a measurable function, Since $\{1\} \in \beta(\mathbb{R}) \Rightarrow I_A^{-1}(\{1\}) \in \mathcal{F}$

$$\text{But } I_A^{-1}(\{1\}) = A \Rightarrow A \in \mathcal{F}$$

Conversely, suppose $A \in \mathcal{F}$. Let $a \in \mathbb{R}$

$$\{I_A > a\} = \begin{cases} \Omega, & a > 0 \\ A, & 0 \leq a < 1 \\ \phi, & a \geq 1 \end{cases} \quad \text{Thus } \{I_A > a\} \in \mathcal{F} \Rightarrow I_A \text{ Is measurable function.}$$

Example(10.4)

Let $(\Omega, \mathcal{F})$ be a measurable space, $A_1, A_2, \dots, A_n$ be mutually disjoint sets and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$,

then the function $g : \Omega \rightarrow \mathbb{R}$ defined by $g(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$ is measurable

Proof :

Since $\{g > a\} = (\bigcup_{i: \alpha_i > a} A_i) \in \mathcal{F} \Rightarrow g$ is measurable .

Definition(10.5)

Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \rightarrow \overline{\mathbb{R}}$ is said to be a simple if it takes only finitely many distinct values, i.e. the range of f is a finite set of distinct values

$$\{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

Remark

Let $A_i = \{x \in \Omega : f(x) = \alpha_i\}$ for $i = 1, 2, \dots, n \Rightarrow A_i = f^{-1}\{\alpha_i\}$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^n A_i = \Omega$

Since $I_{A_i}(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases}$, we can write $f(x) = \alpha_1 I_{A_1}(x) + \alpha_2 I_{A_2}(x) + \dots + \alpha_n I_{A_n}(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$

Thus , a function $f : \Omega \rightarrow \overline{\mathbb{R}}$ is a simple function if it can be expressed as

$$f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$$

Where $n \geq 1$, $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$, $A_i = \{x \in \Omega : f(x) = \alpha_i\}$. Thus a simple function is one which takes a constant value α_i on the set A_i where the set A_i are disjoint sets .

Theorem(10.6)

Let $(\Omega, \mathcal{F})$ be a measurable space. A simple function $f : \Omega \rightarrow \overline{\mathbb{R}}$ is measurable iff $A_i \in \mathcal{F}$ for all $i = 1, 2, \dots, n$

Proof:

Since $f : \Omega \rightarrow \overline{\mathbb{R}}$ is a simple function if it can be expressed as $f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$

Where $n \geq 1$, $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$, $A_i = \{x \in \Omega : f(x) = \alpha_i\}$



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Suppose $A_i \in \mathcal{F} \Rightarrow I_{A_i}$ is measurable function for all $i = 1, 2, \dots, n$

By using theorem(8.14), we have $\alpha_i I_{A_i}$ is measurable function for all $i = 1, 2, \dots, n$

Also by using theorem (8.14), we have $\alpha_1 I_{A_1} + \alpha_2 I_{A_2} + \dots + \alpha_n I_{A_n}$ is measurable function

Thus f is measurable function. Conversely, suppose that f is measurable function

Since $\{\alpha_i\} \in \beta(\mathbb{R})$ and f is measurable function, then $f^{-1}(\{\alpha_i\}) \in \mathcal{F}$

But $f^{-1}(\{\alpha_i\}) = A_i \Rightarrow A_i \in \mathcal{F}$ for all $i = 1, 2, \dots, n$

• It is clear to show that

1. Every constant function is simple
2. If Ω is a finite, then any function on Ω is simple.

Theorem(10.7)

The sum, difference and product of two simple functions is a simple function.

Proof :

Let f, g be two simple functions on a measurable space $(\Omega, \mathcal{F})$

$$f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x) \text{ and } g(x) = \sum_{j=1}^m \beta_j I_{B_j}(x)$$

Then the sets $C_{ij} = A_i \cap B_j$, $i = 1, 2, \dots, n$ $j = 1, 2, \dots, m$ are in $\mathcal{F}$ form a partition of Ω .

That is $(f + g)(x) = \sum_{i=1}^n \sum_{j=1}^m (\alpha_i + \beta_j) I_{A_i \cap B_j}(x)$ and $(fg)(x) = \sum_{i=1}^n \sum_{j=1}^m (\alpha_i \beta_j) I_{A_i \cap B_j}(x)$

Theorem(10.8)

Let f be a real-valued function defined on a set Ω . Then there exists a sequence $\{f_n\}$ of simple functions on Ω such that $f_n \rightarrow f$.

Proof :

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For all $n \geq 1$, we define $f_n : \Omega \rightarrow \mathbb{R}$ by $f_n = \sum_{k=0}^{n-1} \frac{k}{2^n} I_{A_{nk}} + n I_{B_n}$

Where $A_{nk} = \{x \in \Omega : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}$ and $B_n = \{x \in \Omega : n \leq f(x)\}$

It is clear to show that $\{f_n\}$ is a sequence of simple functions and $f_n \rightarrow f$

- If f is a measurable function on a measurable space $(\Omega, \mathcal{F})$, then $\{f_n\}$ may be chosen to be a sequence of measurable functions.

$$A_{nk} = f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n})) \quad \text{and} \quad B_n = f^{-1}([n, \infty))$$

- If $f \geq 0$, then $\{f_n\}$ may be chosen to be an increasing sequence, i.e.

$$0 \leq f_1 \leq \dots \leq f, \quad f_n \uparrow f.$$

Note that : The fundamental idea of integration is to measure the area between the graph of a function and the abscissa. For simple function f , if $f = \sum_{i=1}^n \alpha_i I_{A_i}$, then $\sum_{i=1}^n \alpha_i \mu(A_i)$ should be the μ -area enclosed by the graph and the abscissa.

Theorem(10.9)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and f be a measurable simple function on $(\Omega, \mathcal{F})$.

If f has a different representation $\sum_{i=1}^n \alpha_i I_{A_i}$ and $\sum_{j=1}^m \beta_j I_{B_j}$. Then

$$1. f = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} I_{A_i \cap B_j} \quad \text{where} \quad \lambda_{ij} = \alpha_i = \beta_j$$

$$2. \sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{j=1}^m \beta_j \mu(B_j)$$

Proof :



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$$(1) \text{ Let } f = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} I_{A_i \cap B_j}$$

Since $A_i \in \mathcal{F}$ for all $i = 1, 2, \dots, n$ and $B_j \in \mathcal{F}$ for all $j = 1, 2, \dots, m$

$\Rightarrow A_i \cap B_j \in \mathcal{F}$ for all $i = 1, 2, \dots, n$ and for all $j = 1, 2, \dots, m$

Since $A_i \cap A_r = \emptyset$ for all $i \neq r$ and $B_j \cap B_s = \emptyset$ for all $j \neq s$ $(A_i \cap B_j) \cap (A_r \cap B_s) = \emptyset$. It

follows that the $A_i \cap B_j$'s are Pairwise disjoint, and $\bigcup_{i=1}^n \bigcup_{j=1}^m (A_i \cap B_j) = \Omega$.

Let $x \in \Omega$. There exists a unique (i, j) such that $x \in A_i \cap B_j$. We have $g(x) = \alpha_i = f(x)$.

It follows that $g = f$.

(2) Since $A_i \cap A_j = \emptyset$, $\bigcup_{i=1}^n A_i = \Omega$ and $B_i \cap B_j = \emptyset$ and $\bigcup_{j=1}^m B_j = \Omega$, we have

$$A_i = \bigcup_{j=1}^m (A_i \cap B_j) \text{ for all } i = 1, 2, \dots, n \text{ and } B_j = \bigcup_{i=1}^n (A_i \cap B_j) \text{ for all } j = 1, 2, \dots, m$$

$$\Rightarrow \mu(A_i) = \mu\left(\bigcup_{j=1}^m (A_i \cap B_j)\right) = \sum_{j=1}^m \mu(A_i \cap B_j) \text{ for all } i = 1, 2, \dots, n$$

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^n \alpha_i \sum_{j=1}^m \mu(A_i \cap B_j) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \mu(A_i \cap B_j) \quad (1)$$

$$\text{Similarly, we have } \sum_{j=1}^m \beta_j \mu(B_j) = \sum_{i=1}^n \sum_{j=1}^m \beta_j \mu(A_i \cap B_j) \quad (2)$$

If $A_i \cap B_j = \emptyset$, then $\mu(A_i \cap B_j) = \mu(\emptyset) = 0 \Rightarrow \alpha_i \mu(A_i \cap B_j) = \beta_j \mu(A_i \cap B_j)$

Or $A_i \cap B_j \neq \emptyset$, there exists $x \in A_i \cap B_j$ $x \in A_i$, $x \in B_j \Rightarrow f(x) = \alpha_i, f(x) = \beta_j$

$\Rightarrow \alpha_i \mu(A_i \cap B_j) = \beta_j \mu(A_i \cap B_j)$, and we conclude from (1) and (2) that

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{j=1}^m \beta_j \mu(B_j)$$



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Definition(10.10)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and f be a measurable simple function on $(\Omega, \mathcal{F})$. We define the integral of f with respect to μ , as the sum, denoted $\int_{\Omega} f d\mu$, defined by

$$\int_{\Omega} f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i)$$

As long as $+\infty$ and $-\infty$ do not both appear in the sum, if they do, we say that the integral does not exist.

- If $A \in \mathcal{F}$, we define $\int_A f d\mu = \int_{\Omega} f I_A d\mu$. Hence $\int_A f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap A)$
- If $\mu(A) = 0$, then $\int_A f d\mu = 0$

Since f is a measurable simple functions on a measurable space $(\Omega, \mathcal{F})$

If $f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$, where $n \geq 1$, $\alpha_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ for $i = 1, 2, \dots, n$, $A_i = \{x \in \Omega : f(x) = \alpha_i\}$

Then we have $(f I_A)(x) = \sum_{i=1}^n \alpha_i I_{A_i \cap A}(x)$, where $A_i \cap A \in \mathcal{F}$, from theorem(10.7), we have

$f I_A$ is a measurable simple function on $(\Omega, \mathcal{F})$. Since $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^n A_i = \Omega$.

It follows that $\bigcup_{i=1}^n (A_i \cap A) \cup A^c = \Omega$. Hence $f I_A = \sum_{i=1}^n \alpha_i I_{A_i \cap A} + 0 \times I_{A^c}$

By definition(10.10), we have $\int_A f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap A) + 0 \times \mu(A^c) = \sum_{i=1}^n \alpha_i \mu(A_i \cap A)$

Theorem(10.11)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space. Then $\int_A d\mu = \mu(A)$ for all $A \in \mathcal{F}$.

Proof :

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Define $f : \Omega \rightarrow \mathbb{R}$ by $f(x) = 1$ for all $x \in \Omega$, then f is simple function and

$$n = 1, A_1 = \Omega, \alpha_1 = 1$$

$$\int_A d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap A) = \alpha_1 \mu(A_1 \cap A) = \mu(\Omega \cap A) = \mu(A)$$

Example(10.12)

Let $\Omega = \mathbb{R}$, $\mathcal{F} = \beta(\Omega)$, and μ is the Lebesgue measure

1. If $f : [a, b] \rightarrow \mathbb{R}$ be a function, defined by $f(x) = \begin{cases} 2, & x \in \mathbb{Q} \cap [a, b] \\ 5, & x \in \mathbb{Q}^c \cap [a, b] \end{cases}$, then

$$\begin{aligned} \alpha_1 &= 2, & A_1 &= \mathbb{Q} \cap [a, b], & \mu(A_1) &= 0 \\ \alpha_2 &= 5, & A_2 &= \mathbb{Q}^c \cap [a, b], & \mu(A_2) &= b - a \end{aligned}$$

$$\int_{\Omega} f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i) = \alpha_1 \mu(A_1) + \alpha_2 \mu(A_2) = 2 \times 0 + 5 \times (b - a) = 5(b - a)$$

$$2. \int_{\mathbb{Q}} d\mu = \mu(\mathbb{Q}) = 0$$

$$3. \int_1^3 d\mu = \int_{[1,3]} d\mu = \mu([1,3]) = 3 - 1 = 2$$

Theorem(10.13)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f, g be two measurable simple functions on $(\Omega, \mathcal{F})$.

$$1. \int_{\Omega} (af + bg) d\mu = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu \text{ for all } a, b \in \mathbb{R}$$

In special cases

$$(i) \text{ If } a = 1, b = 1, \text{ then } \int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$$

$$(ii) \text{ If } b = 0, \text{ then } \int_{\Omega} (af) d\mu = a \int_{\Omega} f d\mu$$

$$2. \int_A (af + bg) d\mu = a \int_A f d\mu + b \int_A g d\mu \text{ for all } A \in \mathcal{F} \text{ and for all } a, b \in \mathbb{R}$$

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3. If $A, B \in \mathcal{F}$ such that $A \cap B = \phi$, then $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$

4. If $f \geq 0$, then $\int_A f d\mu \geq 0$ for all $A \in \mathcal{F}$

5. If $f \leq g$, then $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in \mathcal{F}$

6. If $a \leq f \leq b$, where $a, b \in \mathbb{R}$, then $a\mu(A) \leq \int_A f d\mu \leq b\mu(A)$

Proof :

(1) Since f, g be two simple functions on a measurable space $(\Omega, \mathcal{F})$

$$f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x) \text{ and } g(x) = \sum_{j=1}^m \beta_j I_{B_j}(x)$$

Where $A_i = \{x \in \Omega : f(x) = \alpha_i\} = f^{-1}(\{\alpha_i\})$, $i = 1, 2, \dots, n$ and $B_j = \{x \in \Omega : g(x) = \beta_j\} = g^{-1}(\{\beta_j\})$, $j = 1, 2, \dots, m$

$$A_i \cap A_j = \phi, \bigcup_{i=1}^n A_i = \Omega \text{ and } B_i \cap B_j = \phi, \bigcup_{j=1}^m B_j = \Omega$$

Take $C_{ij} = A_i \cap B_j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$

$$C_{ij} \cap C_{st} = (A_i \cap B_j) \cap (A_s \cap B_t) = (A_i \cap A_s) \cap (B_j \cap B_t) = \phi \cap \phi = \phi$$

$$\bigcup_{i,j} C_{ij} = \bigcup_{i=1}^n \bigcup_{j=1}^m C_{ij} = \bigcup_{i=1}^n \bigcup_{j=1}^m (A_i \cap B_j) = (\bigcup_{i=1}^n A_i) \cap (\bigcup_{j=1}^m B_j) = \Omega \cap \Omega = \Omega$$

Then the sets $C_{ij} = A_i \cap B_j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ are form a partition of Ω .

$$\text{That is } (af + bg)(x) = \sum_{i=1}^n \sum_{j=1}^m (a\alpha_i + b\beta_j) I_{A_i \cap B_j}(x)$$

$$\begin{aligned} \int_{\Omega} (af + bg) d\mu &= \sum_{i,j} (a\alpha_i + b\beta_j) \mu(C_{ij}) = \sum_{i=1}^n \sum_{j=1}^m (a\alpha_i + b\beta_j) \mu(A_i \cap B_j) \\ &= a \sum_{i=1}^n \sum_{j=1}^m \alpha_i \mu(A_i \cap B_j) + b \sum_{i=1}^n \sum_{j=1}^m \beta_j \mu(A_i \cap B_j) = a \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^m \mu(A_i \cap B_j) \right) + b \sum_{j=1}^m \beta_j \left(\sum_{i=1}^n \mu(A_i \cap B_j) \right) \end{aligned}$$

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Since $\sum_{j=1}^m \mu(A_i \cap B_j) = \mu(A_i \cap B_1) + \mu(A_i \cap B_2) + \dots + \mu(A_i \cap B_m) = \mu(A_i \cap \bigcup_{j=1}^m B_j) = \mu(A_i \cap \Omega) = \mu(A_i)$

and $\sum_{i=1}^n \mu(A_i \cap B_j) = \mu(B_j)$

$$\int_{\Omega} (af + bg) d\mu = a \sum_{i=1}^n \alpha_i \mu(A_i) + b \sum_{j=1}^m \beta_j \mu(B_j) = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu$$

(2) since f and g are simple and measurable functions on $(\Omega, \mathcal{F})$, then $f I_A$ and $g I_A$ are simple and measurable functions on $(\Omega, \mathcal{F})$.

$$\int_A (af + bg) d\mu = \int_{\Omega} (af + bg) I_A d\mu = \int_{\Omega} (af I_A + bg I_A) d\mu = a \int_{\Omega} f I_A d\mu + b \int_{\Omega} g I_A d\mu = a \int_A f d\mu + b \int_A g d\mu$$

(3) Since $A_i \cap (A \cup B) = (A_i \cap A) \cup (A_i \cap B)$

Since $A \cap B = \emptyset$, then $(A_i \cap A) \cap (A_i \cap B) = \emptyset$

Therefore $\mu(A_i \cap (A \cup B)) = \mu((A_i \cap A) \cup (A_i \cap B)) = \mu(A_i \cap A) + \mu(A_i \cap B)$

$$\begin{aligned} \int_{A \cup B} f d\mu &= \sum_{i=1}^n \alpha_i \mu(A_i \cap (A \cup B)) = \sum_{i=1}^n \alpha_i (\mu(A_i \cap A) + \mu(A_i \cap B)) = \sum_{i=1}^n \alpha_i \mu(A_i \cap A) + \sum_{i=1}^n \alpha_i \mu(A_i \cap B) \\ &= \int_A f d\mu + \int_B f d\mu \end{aligned}$$

(4) since f is a simple function, then $f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$

Since $f \geq 0 \Rightarrow \alpha_i \geq 0$ for $i = 1, 2, \dots, n$

Since $\mu(A_i \cap A) \geq 0$ for $i = 1, 2, \dots, n \Rightarrow \alpha_i \mu(A_i \cap A) \geq 0$ for $i = 1, 2, \dots, n$

$$\Rightarrow \sum_{i=1}^n \alpha_i \mu(A_i \cap A) \geq 0 \Rightarrow \int_A f d\mu \geq 0$$

(5) Let $h = g - f$, Since $f \leq g \Rightarrow h \geq 0$

By using (4), we have $\int_A h d\mu \geq 0 \Rightarrow \int_A (g - f) d\mu \geq 0$

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$$\int_A g d\mu - \int_A f d\mu \geq 0 \Rightarrow \int_A g d\mu \geq \int_A f d\mu \Rightarrow \int_A f d\mu \leq \int_A g d\mu$$

(6) since $a \leq f \leq b$, by using (5), we have $\int_A a d\mu \leq \int_A f d\mu \leq \int_A b d\mu$

$$a \int_A d\mu \leq \int_A f d\mu \leq b \int_A d\mu \Rightarrow a\mu(A) \leq \int_A f d\mu \leq b\mu(A)$$

Theorem(10.14)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space, $A \in \mathcal{F}$ and let f be a measurable simple function on $(\Omega, \mathcal{F})$.

Define $\lambda: \mathcal{F} \rightarrow \mathbb{R}$ by $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Then

1. λ is a signed measure
2. If $f \geq 0$, then λ is a measure
3. If $A_n \uparrow A$, where $A, A_n \in \mathcal{F}$, $n \geq 1$, and $f \geq 0$, then $\int_{A_n} f d\mu \uparrow \int_A f d\mu$.

Proof :

(1) Let $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$.

$$\Rightarrow \lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \int_{\bigcup_{n=1}^{\infty} A_n} f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu = \sum_{n=1}^{\infty} \lambda(A_n) \Rightarrow \lambda \text{ is signed measure}$$

$$(2) \text{ since } f \geq 0 \Rightarrow I_A f \geq 0 \Rightarrow \int_{\Omega} I_A f d\mu \geq 0 \Rightarrow \int_A f d\mu \geq 0$$

$$\Rightarrow \lambda(A) \geq 0 \Rightarrow \lambda \text{ is a measure}$$

(3) Since $A_n \uparrow A$ and λ is a measure on $\mathcal{F}$, then $\lambda(A_n) \uparrow \lambda(A)$

$$\text{Since } \lambda(A) = \int_A f d\mu \text{ for all } A \in \mathcal{F}, \text{ then } \int_{A_n} f d\mu \uparrow \int_A f d\mu.$$

Definition(10.15)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and $f: \Omega \rightarrow \mathbb{R}$ be a non-negative measurable function. The



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(abstract) Lebesgue integral of f with respect to μ , denoted by $\int_{\Omega} f d\mu$, or $\int_{\Omega} f(x) d\mu(x)$, or $\int_{\Omega} f(x) \mu(dx)$ and defined as :

$$\int_{\Omega} f d\mu = \sup \left\{ \int_{\Omega} h d\mu : 0 \leq h \leq f, \quad h, \text{ is measurable simple function on } (\Omega, \mathcal{F}) \right\}$$

Note that according to the definition, the integral of a non-negative measurable function always exists; it may be $+\infty$.

Remark

If $A \in \mathcal{F}$, we define $\int_A f d\mu = \int_{\Omega} f I_A d\mu$. Hence

$$\int_A f d\mu = \sup \left\{ \int_A h d\mu : 0 \leq h \leq f, \quad h, \text{ is measurable simple function on } (\Omega, \mathcal{F}) \right\}$$

Theorem(10.16)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f, g be two non-negative measurable functions on $(\Omega, \mathcal{F})$.

1. $\int_A f d\mu \geq 0$ for all $A \in \mathcal{F}$
2. If $f \leq g$ on $A \in \mathcal{F}$, then $\int_A f d\mu \leq \int_A g d\mu$
3. If $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\int_A f d\mu \leq \int_B f d\mu$
4. If $\lambda \geq 0$, then $\int_A \lambda f d\mu = \lambda \int_A f d\mu$ for all $A \in \mathcal{F}$
5. If $\mu(A) = 0$, then $\int_A f d\mu = 0$
6. If $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, then $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$
7. If $a \leq f \leq b$, where $a, b \in \mathbb{R}$, then $a\mu(A) \leq \int_A f d\mu \leq b\mu(A)$ for all $A \in \mathcal{F}$
8. $f = 0$ a.e. on $A \in \mathcal{F}$ iff $\int_A f d\mu = 0$

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9. If $f \leq g$ a.e. on $A \in \mathcal{F}$, then $\int_A f d\mu \leq \int_A g d\mu$

Proof :

(1) since $\int_A f d\mu = \sup\{\int_A h d\mu : 0 \leq h \leq f, h, \text{ is measurable simple function on } (\Omega, \mathcal{F})\}$

$\Rightarrow \int_A f d\mu \geq \int_A h d\mu$ for every measurable simple function h such that $0 \leq h \leq f$

Since 0 is measurable simple function and $0 \leq f$, then $\int_A f d\mu \geq \int_A 0 d\mu$

Since $\int_A 0 d\mu = 0 \Rightarrow \int_A f d\mu \geq 0$

(2) Let h be a simple function on $(\Omega, \mathcal{F})$ such that $0 \leq h \leq f$

Since $f \leq g \Rightarrow 0 \leq h \leq g$

Since $\int_A g d\mu = \sup\{\int_A h d\mu : 0 \leq h \leq g, h, \text{ is measurable simple function on } (\Omega, \mathcal{F})\}$

$\Rightarrow \int_A h d\mu \leq \int_A g d\mu$.

Hence $\int_A g d\mu$ is an upper bound of all $\int_A h d\mu$, for simple function on $(\Omega, \mathcal{F})$ with

$0 \leq h \leq f$. The integral $\int_A f d\mu$ being the smallest of such upper bound, we have

$$\int_A f d\mu \leq \int_A g d\mu.$$

(3) Let h be a simple function on $(\Omega, \mathcal{F})$ such that $0 \leq h \leq f$

$$h(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x) \text{ Where } A_i = \{x \in \Omega : f(x) = \alpha_i\}, \alpha_i \geq 0, i = 1, 2, \dots, n$$

since $A \subseteq B \Rightarrow A_i \cap A \subseteq A_i \cap B \Rightarrow \mu(A_i \cap A) \leq \mu(A_i \cap B)$

$$\Rightarrow \sum_{i=1}^n \alpha_i \mu(A_i \cap A) \leq \sum_{i=1}^n \alpha_i \mu(A_i \cap B) \Rightarrow \int_A h d\mu \leq \int_B h d\mu$$

$\Rightarrow \sup\{\int_A h d\mu : 0 \leq h \leq f, h, \text{ is measurable simple function on } (\Omega, \mathcal{F})\}$



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$$\leq \sup \left\{ \int_B h d\mu : 0 \leq h \leq f, \quad h, \text{ is measurable simple function on } (\Omega, \mathcal{F}) \right\}$$

$$\Rightarrow \int_A f d\mu \leq \int_B f d\mu$$

(4) (i) If $\lambda > 0$

Let h be a simple function on $(\Omega, \mathcal{F})$ such that $0 \leq h \leq f$

Since $\lambda > 0$, then $0 \leq \lambda h \leq \lambda f$

Since h is a simple function on $(\Omega, \mathcal{F})$, so $\lambda h \Rightarrow \int_A (\lambda h) d\mu \leq \int_A (\lambda f) d\mu$

Since $\int_A (\lambda h) d\mu = \lambda \int_A h d\mu \Rightarrow \lambda \int_A h d\mu \leq \int_A (\lambda f) d\mu$

Since $\lambda > 0$, we have $\int_A h d\mu \leq \frac{1}{\lambda} \int_A (\lambda f) d\mu$

Hence $\frac{1}{\lambda} \int_A (\lambda f) d\mu$ is an upper bound of all $\int_A h d\mu$, for h simple function on $(\Omega, \mathcal{F})$

With $0 \leq h \leq f$. The integral $\int_A f d\mu$ being the smallest of such upper bound, we have

$\int_A f d\mu \leq \frac{1}{\lambda} \int_A (\lambda f) d\mu$. Multiplying both sides by λ , we obtain that

$$\lambda \int_A f d\mu \leq \int_A (\lambda f) d\mu.$$

Similarly, since $\frac{1}{\lambda} > 0$, we have $\frac{1}{\lambda} \int_A (\lambda f) d\mu \leq \int_A \frac{1}{\lambda} (\lambda f) d\mu$, i.e. $\int_A (\lambda f) d\mu \leq \lambda \int_A f d\mu$.

We conclude that $\int_A (\lambda f) d\mu = \lambda \int_A f d\mu$.

(ii) If $\lambda = 0$, whether or not $\int_A f d\mu = \infty$, we have $\lambda \int_A f d\mu = 0$.

Since 0 is measurable simple function on $(\Omega, \mathcal{F})$, we have $\int_A 0 d\mu = 0$.

It follows that the equality $\int_A (\lambda f) d\mu = \lambda \int_A f d\mu$ is still true in the case when $\lambda = 0$.



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Theorem(10.17) Monotone Convergence Theorem

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let $\{f_n\}$ be a sequence of non-negative measurable functions on $(\Omega, \mathcal{F})$ such that $f_n \uparrow f$. Then $\int_{\Omega} f_n d\mu \uparrow \int_{\Omega} f d\mu$

Proof :

- Since $f_n \uparrow f \Rightarrow f_n \rightarrow f$ and $f_n \leq f_{n+1}$ for all n (in other word $\sup_{n \geq 1} f_n = f$)

Since $f_n \leq f_{n+1}$ for all n , then $\int_{\Omega} f_n d\mu \leq \int_{\Omega} f_{n+1} d\mu$ for all n

Let $\alpha_n = \int_{\Omega} f_n d\mu$. Then $\{\alpha_n\}$ is non decreasing sequence and it converges to $\alpha \in [0, \infty]$,

$\alpha_n \uparrow \alpha$, i.e. $\sup_{n \geq 1} \alpha_n = \alpha$,

- Since $\sup_{n \geq 1} f_n = f$, for all $n \geq 1$, $f_n \leq f$, we have $\int_{\Omega} f_n d\mu \leq \int_{\Omega} f d\mu$ for all n .

It follows that $\int_{\Omega} f d\mu$ is an upper bound of all $\int_{\Omega} f_n d\mu$ for all $n \geq 1$.

Since α is the smallest of such upper bound, we have $\alpha \leq \int_{\Omega} f d\mu$.

- Let $0 < \lambda < 1$, and let h be any simple function on $(\Omega, \mathcal{F})$ such that $0 \leq h < f$. Define

$$A_n = \{x \in \Omega : f_n(x) \geq \lambda h(x)\}$$

Since h is measurable simple function on $(\Omega, \mathcal{F})$, then λh is measurable simple function on $(\Omega, \mathcal{F})$. It follows that $A_n \in \mathcal{F}$.

Let $n \geq 1$. Suppose $x \in A_n$. Then $\lambda h(x) \leq f_n(x) \leq f_{n+1}(x)$. So $x \in A_{n+1}$ and $A_n \subseteq A_{n+1}$.

Let $x \in \Omega$. If $h(x) = 0$, then $x \in A_n$ for all $n \geq 1$. Suppose $h(x) > 0$. Then, we have

$$0 < h(x) < \infty.$$

Since $0 < \lambda < 1$, we have $\lambda h(x) \leq h(x)$. It follows that $\lambda h(x) \leq f(x) = \sup_{n \geq 1} f_n(x)$.

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Since $f(x)$ is the smallest upper bound of all $f_n(x)$ for $n \geq 1$, we see that $\lambda h(x)$ cannot be such upper bound. There exist $n \geq 1$ such that $\lambda h(x) \leq f_n(x)$. In particular, there

exists $n \geq 1$, such that $x \in A_n$. Hence, $\Omega = \bigcup_{n=1}^{\infty} A_n$, with $A_n \subseteq A_{n+1}$, i.e. $A_n \uparrow \Omega$.

- For $n \geq 1$, we have $\lambda h I_{A_n} \leq f_n$. Hence $\int_{\Omega} \lambda h I_{A_n} d\mu \leq \int_{\Omega} f_n d\mu$. But $\int_{\Omega} \lambda h I_{A_n} d\mu = \lambda \int_{\Omega} h I_{A_n} d\mu$

We conclude that $\lambda \int_{\Omega} h I_{A_n} d\mu \leq \int_{\Omega} f_n d\mu$ for all $n \geq 1$

- since $A_n \uparrow \Omega$, $\int_{\Omega} h I_{A_n} d\mu \uparrow \int_{\Omega} h d\mu$. In particular, $\lambda \int_{\Omega} h I_{A_n} d\mu \uparrow \lambda \int_{\Omega} h d\mu$

Since $\int_{\Omega} f_n d\mu \uparrow \alpha$ and $\lambda \int_{\Omega} h I_{A_n} d\mu \leq \int_{\Omega} f_n d\mu$ for all $n \geq 1$. Taking the limit as $n \rightarrow \infty$, we conclude that $\lambda \int_{\Omega} h d\mu \leq \alpha$ for all $0 < \lambda < 1$.

- Since $\lambda \int_{\Omega} h d\mu \leq \alpha$ for all $0 < \lambda < 1$, we have $\int_{\Omega} h d\mu \leq \alpha$.
- α is an upper bound of all $\int_{\Omega} h d\mu$ for h simple function on $(\Omega, \mathcal{F})$ such that $0 \leq h \leq f$.

The integral $\int_{\Omega} f d\mu$ being the smallest of such upper bound, we have $\int_{\Omega} f d\mu \leq \alpha$.

- Since $\alpha \leq \int_{\Omega} f d\mu$, we have $\alpha = \int_{\Omega} f d\mu$, we conclude $\int_{\Omega} f_n d\mu \uparrow \int_{\Omega} f d\mu$

Theorem(10.18)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f, g be two non negative measurable functions on $(\Omega, \mathcal{F})$.

1. If $\{f_n\}$ and $\{g_n\}$ are two sequences of non-negative measurable functions on $(\Omega, \mathcal{F})$

such that $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n + g_n \uparrow f + g$

2. $\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$

3. $\int_{\Omega} (af + bg) d\mu = a \int_{\Omega} f d\mu + b \int_{\Omega} g d\mu$ for all $a, b \in \mathbb{R}$

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Proof :

(2) since f and g be non-negative measurable functions, then there exist two sequences $\{f_n\}$ and $\{g_n\}$ of simple functions on $(\Omega, \mathcal{F})$ such that $f_n \uparrow f$ and $g_n \uparrow g$. Thus $f_n + g_n \uparrow f + g$, $f_n + g_n$ non-negative, simple, measurable. From monotone convergence theorem, we have $\int_{\Omega} (f_n + g_n) d\mu \uparrow \int_{\Omega} (f + g) d\mu$. Hence

$$\int_{\Omega} (f + g) d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} (f_n + g_n) d\mu = \lim_{n \rightarrow \infty} \left(\int_{\Omega} f_n d\mu + \int_{\Omega} g_n d\mu \right) = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$$

(3) this is an immediate application of (2), and part(4) of theorem(10.16).

Theorem(10.19)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of non-negative measurable

functions. Then $\int_{\Omega} \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_{\Omega} f_n d\mu \right)$

Proof :

$$\text{Take } f = \sum_{n=1}^{\infty} f_n, \quad g_n = \sum_{k=1}^n f_k$$

Since $g_n \uparrow f$, from the monotone convergence theorem, we have $\int_{\Omega} g_n d\mu \uparrow \int_{\Omega} f d\mu$.

However, from theorem(10.18), $\int_{\Omega} g_n d\mu = \sum_{k=1}^n \left(\int_{\Omega} f_k d\mu \right)$. Hence, we see that the sequence

$$\sum_{k=1}^n \left(\int_{\Omega} f_k d\mu \right) \text{ converges to } \int_{\Omega} f d\mu. \text{ In other words, we have } \int_{\Omega} \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \left(\int_{\Omega} f_n d\mu \right)$$

Theorem(10.20) Fatou's Lemma

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let $\{f_n\}$ be a sequence of non-negative measurable functions on $(\Omega, \mathcal{F})$. Then



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$$\int_{\Omega} (\liminf_{n \rightarrow \infty} f_n) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$$

Proof :

We define $g_n(x) = \inf\{f_k(x) : k \geq n\}$, $n \geq 1$

- Since g_n is a countable infimum of measurable functions. It is therefore measurable, and is obviously nonnegative.
- Let $x \in \Omega$ and $n \geq 1$. For all $k \geq n$, we have $g_n(x) \leq f_k(x)$. In particular, $g_n(x)$ is a lower bound of all $f_k(x)$ for $k \geq n+1$.

Since $g_{n+1}(x)$ is the greatest of such lower bound, we have $g_n(x) \leq g_{n+1}(x)$. It follows that $\{g_n(x)\}$ is non decreasing sequence of real numbers, which therefore converges to its supremum. Hence $g_n \uparrow \sup_{n \geq 1} g = \liminf_{n \rightarrow \infty} f_n$.

- For all $n \geq 1$, we have $g_n \leq f_n$. From theorem(10.16), we have $\int_{\Omega} g_n d\mu \leq \int_{\Omega} f_n d\mu$.

It follows that $\liminf_{n \rightarrow \infty} \int_{\Omega} g_n d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$ (1)

Since $g_n \uparrow \liminf_{n \rightarrow \infty} f_n$, by the monotone convergence theorem, we have

$$\int_{\Omega} g_n d\mu \uparrow \int_{\Omega} (\liminf_{n \rightarrow \infty} f_n) d\mu$$

In particular, the sequence $\{\int_{\Omega} g_n d\mu\}$ converges to $\int_{\Omega} (\liminf_{n \rightarrow \infty} f_n) d\mu$. It follows that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} g_n d\mu = \int_{\Omega} (\liminf_{n \rightarrow \infty} f_n) d\mu \quad \dots(2)$$

Comparing (1) with(2), we conclude that $\int_{\Omega} (\liminf_{n \rightarrow \infty} f_n) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$.



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Exercises (10)

- Let $\{A_n\}$ be a sequence of subsets of Ω . If $A = \limsup_{n \rightarrow \infty} A_n$, $B = \liminf_{n \rightarrow \infty} A_n$. Show that

$$(1) I_A(x) = \limsup_{n \rightarrow \infty} I_{A_n}(x) \quad (2) I_B(x) = \liminf_{n \rightarrow \infty} I_{A_n}(x) \text{ for all } x \in \Omega$$
- Let $\Omega = \mathbb{R}$, $\mathcal{F} = \beta(\Omega)$, and μ is the Lebesgue measure. Find $\int_A f d\mu$ if

$$(1) f(x) = [x], \quad A = [0, 10] \quad (2) f(x) = [x^2], \quad A = [0, 2] \quad (3) f(x) = [\sin x], \quad A = [0, 2\pi]$$
- Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f be non-negative measurable function on $(\Omega, \mathcal{F})$. Show that $\int_{\Omega} f d\mu = 0$ iff $\mu\{f > 0\} = 0$
- Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f be non-negative measurable function on $(\Omega, \mathcal{F})$.
For $n \geq 1$, put $A_n = \{f > \frac{1}{n}\}$, and $h_n = \frac{1}{n} I_{A_n}$. Show that

$$(1) h_n \text{ is a simple function on } (\Omega, \mathcal{F}) \text{ with } h_n \leq f \quad (2) A_n \uparrow \mu\{f > 0\}$$
- Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f, g be two non-negative measurable function on $(\Omega, \mathcal{F})$ with $f = g$ a.e. . show that $\int_{\Omega} f d\mu = \int_{\Omega} g d\mu$
- Let $(\Omega, \mathcal{F}, \mu)$ be measure space, let f, g be two non-negative measurable function on $(\Omega, \mathcal{F})$. Define $\lambda: \mathcal{F} \rightarrow \overline{\mathbb{R}}$ by $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Show that

$$(1) \lambda \text{ is a measure on } \mathcal{F} \quad (2) \int_{\Omega} g d\lambda = \int_{\Omega} gf d\mu$$
- Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let $\{f_n\}$ be a sequence of non-negative measurable functions on $(\Omega, \mathcal{F})$. Show that $\limsup_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \leq \int_{\Omega} (\limsup_{n \rightarrow \infty} f_n) d\mu$.



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11. Integrable Functions

We can extend the integral very easily to general real valued functions, using the positive part f^+ and the negative part f^- of any measurable function $f : \Omega \rightarrow \mathbb{R}$. We will use the nonnegative measurable function $|f|$.

Definition (11.1)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and $f : \Omega \rightarrow \mathbb{R}$ be a measurable function. We say that f is μ -integrable (or simply integrable if μ is understood) if $\int_{\Omega} |f| d\mu < \infty$.

- The set of all μ -integrable functions denoted by $L^1(\Omega, \mathcal{F}, \mu)$,

Note that : $L^1(\Omega, \mathcal{F}, \mu)$ is subspace of the space of all measurable functions on $(\Omega, \mathcal{F})$, i.e.

If $f, g \in L^1(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in L^1(\Omega, \mathcal{F}, \mu)$

Indeed :

Since $|\alpha f + \beta g| \leq |\alpha| |f| + |\beta| |g|$, we have $\int_{\Omega} |\alpha f + \beta g| d\mu \leq |\alpha| \int_{\Omega} |f| d\mu + |\beta| \int_{\Omega} |g| d\mu < \infty$

We conclude that $\alpha f + \beta g \in L^1(\Omega, \mathcal{F}, \mu)$.

Definition (11.2)

Let Ω be any set and $f : \Omega \rightarrow \mathbb{R}$ a function, we define the positive and negative parts f^+ and f^- by

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = -\min\{f, 0\} = \max\{-f, 0\}$$

i.e.

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

It follows that f^+ and f^- are non-negative



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Theorem (11.3)

Let Ω be any set and $f : \Omega \rightarrow \mathbb{R}$ be a function

1. $f = f^+ - f^-$ and $|f| = f^+ + f^-$
2. $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$
3. $(-f)^+ = f^-$ and $(-f)^- = f^+$
4. If $\lambda > 0$, then $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- = \lambda f^-$

Proof :

$$(1) \quad f^+ - f^- = \max\{f, 0\} - \max\{-f, 0\} = \max\{f, 0\} + \min\{f, 0\} = f + 0 = f$$

Let $x \in \Omega$. If $f(x) \geq 0$, then $f^+(x) = f(x)$ and $f^-(x) = 0$

If $f(x) \leq 0$, then $f^+(x) = 0$ and $f^-(x) = -f(x)$

In any case, $f^+(x) + f^-(x) = |f|(x)$. So $|f| = f^+ + f^-$

Theorem (11.4)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and $f : \Omega \rightarrow \mathbb{R}$ be a function. Then

1. f is measurable function iff f^+ and f^- are measurable functions
2. If $f \in L^1(\Omega, \mathcal{F}, \mu)$, then $f^+, f^-, |f| \in L^1(\Omega, \mathcal{F}, \mu)$
3. If $f \in L^1(\Omega, \mathcal{F}, \mu)$, then the integrals $\int_{\Omega} f^+ d\mu$ and $\int_{\Omega} f^- d\mu$ are all well-defined

Proof :

(1) suppose f is measurable, then $|f|$ is measurable .

It follows that $\frac{1}{2}(|f| + f)$ and $\frac{1}{2}(|f| - f)$ are measurable functions, but $f^+ = \frac{1}{2}(|f| + f)$

and $f^- = \frac{1}{2}(|f| - f)$, then f^+ and f^- are measurable functions.

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Conversely suppose f^+ and f^- are measurable functions, then $f^+ - f^-$ is measurable function, but $f = f^+ - f^-$ then f is measurable function.

(2) since $|f| = f^+ + f^-$, then $f^+ \leq |f|$ and $f^- \leq |f|$, we have

$$\int_{\Omega} f^+ d\mu \leq \int_{\Omega} |f| d\mu < \infty \text{ and } \int_{\Omega} f^- d\mu \leq \int_{\Omega} |f| d\mu < \infty \Rightarrow f^+, f^-, |f| \in L^1(\Omega, \mathcal{F}, \mu)$$

(3) since f^+ and f^- are non-negative and measurable functions. Their integrals $\int_{\Omega} f^+ d\mu$, $\int_{\Omega} f^- d\mu$ are therefore well-defined.

Definition (11.5)

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let $f \in L^1(\Omega, \mathcal{F}, \mu)$. We define the integral of f with respect to μ , denoted $\int_{\Omega} f d\mu$, as

$$\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$$

If this is not of the form $\infty - \infty$, if it is, we say that the integral does not exists.

- The function f is μ -integrable if $\int_{\Omega} f d\mu$ is finite, i.e. if $\int_{\Omega} f^+ d\mu$ and $\int_{\Omega} f^- d\mu$ are both finite.

Remarks

1. If $A \in \mathcal{F}$, we define $\int_A f d\mu = \int_{\Omega} f I_A d\mu$. Hence $\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu$. The function f is said to be integrable over A if $\int_A f d\mu$ is finite, that is if $\int_A f^+ d\mu$ and $\int_A f^- d\mu$ are both finite.
2. The set of all integrable functions over A denoted by $L_A^1(\Omega, \mathcal{F}, \mu)$.
3. $f \in L^1(\Omega, \mathcal{F}, \mu)$ iff $f \in L_A^1(\Omega, \mathcal{F}, \mu)$ for all $A \in \mathcal{F}$.

Indeed $(f I_A)^+ = f^+ I_A \leq f^+$, $(f I_A)^- = f^- I_A \leq f^-$

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$$4. f \in L_A^1(\Omega, \mathcal{F}, \mu) \text{ iff, and } \int_A |f| d\mu = \int_A f^+ d\mu + \int_A f^- d\mu$$

The properties of the integral of non-negative functions extend to any, not necessarily non-negative, integrable functions.

Theorem(11.6)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, g \in L^1(\Omega, \mathcal{F}, \mu)$. Then

1. $\int_{\Omega} (f + g)^+ d\mu + \int_{\Omega} f^- d\mu + \int_{\Omega} g^- d\mu = \int_{\Omega} (f + g)^- d\mu + \int_{\Omega} f^+ d\mu + \int_{\Omega} g^+ d\mu$
2. $\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$
3. $\int_{\Omega} (-f) d\mu = -\int_{\Omega} f d\mu$
4. $\int_{\Omega} (\lambda f) d\mu = \lambda \int_{\Omega} f d\mu$ for all $\lambda \in \mathbb{R}$
5. If $f \leq g$, then $\int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu$

Proof :

(1) Take $h = f + g$

From $h = h^+ - h^-$, $f = f^+ - f^-$ and $g = g^+ - g^-$, we obtain that $h^+ + f^- + g^- = h^- + f^+ + g^+$,

we have $\int_{\Omega} (h^+ + f^- + g^-) d\mu = \int_{\Omega} (h^- + f^+ + g^+) d\mu$, by theorem(10.18), we have

$\int_{\Omega} h^+ d\mu + \int_{\Omega} f^- d\mu + \int_{\Omega} g^- d\mu = \int_{\Omega} h^- d\mu + \int_{\Omega} f^+ d\mu + \int_{\Omega} g^+ d\mu$, we conclude that

$$\int_{\Omega} (f + g)^+ d\mu + \int_{\Omega} f^- d\mu + \int_{\Omega} g^- d\mu = \int_{\Omega} (f + g)^- d\mu + \int_{\Omega} f^+ d\mu + \int_{\Omega} g^+ d\mu$$

(2) since $f, g, f + g \in L^1(\Omega, \mathcal{F}, \mu)$, then all six integrals in part(1) of this theorem are finite.

It follows that $\int_{\Omega} (f + g)^+ d\mu - \int_{\Omega} (f + g)^- d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu + \int_{\Omega} g^+ d\mu - \int_{\Omega} g^- d\mu$

We conclude that $\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$.

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(3) since $(-f)^+ = f^-$ and $(-f)^- = f^+$

$$\int_{\Omega} (-f) d\mu = \int_{\Omega} (-f)^+ d\mu - \int_{\Omega} (-f)^- d\mu = \int_{\Omega} (f)^- d\mu - \int_{\Omega} (f)^+ d\mu = -\int_{\Omega} f d\mu$$

(4)) If $\lambda \geq 0$, then $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- = \lambda f^-$

$$\int_{\Omega} (\lambda f) d\mu = \int_{\Omega} (\lambda f)^+ d\mu - \int_{\Omega} (\lambda f)^- d\mu = \int_{\Omega} \lambda (f)^+ d\mu - \int_{\Omega} \lambda (f)^- d\mu = \lambda \left(\int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu \right) = \lambda \int_{\Omega} f d\mu$$

If $\lambda \leq 0$, then $-\alpha \geq 0$, we have $\int_{\Omega} (-\lambda f) d\mu = -\lambda \int_{\Omega} f d\mu$,

But by using (3), we have $\int_{\Omega} (-\lambda f) d\mu = -\int_{\Omega} (\lambda f) d\mu \Rightarrow -\int_{\Omega} (\lambda f) d\mu = -\lambda \int_{\Omega} f d\mu$

We conclude that $\int_{\Omega} (\lambda f) d\mu = \lambda \int_{\Omega} f d\mu$

(5) since $f \leq g$, then $f^+ \leq g^+$ and $f^- \geq g^- \Rightarrow f^+ + g^- \leq f^- + g^+$. By linearity for nonnegative function, we obtain $\int_{\Omega} f^+ d\mu + \int_{\Omega} g^- d\mu \leq \int_{\Omega} f^- d\mu + \int_{\Omega} g^+ d\mu$. all integrals being finite,

$\Rightarrow \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu \leq \int_{\Omega} g^+ d\mu - \int_{\Omega} g^- d\mu$, we conclude that $\int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu$.

Corollary (11.7)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, g \in L^1(\Omega, \mathcal{F}, \mu)$, $\alpha, \beta \in \mathbb{R}$. Then

$$\int_{\Omega} (\alpha f + \beta g) d\mu = \alpha \int_{\Omega} f d\mu + \beta \int_{\Omega} g d\mu$$

Theorem(11.8)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f \in L^1(\Omega, \mathcal{F}, \mu)$. Then $\left| \int_{\Omega} f d\mu \right| \leq \int_{\Omega} |f| d\mu$

Proof :

Since $-|f| \leq f \leq |f|$ so by theorems (11.6), we have $-\int_{\Omega} |f| d\mu \leq \int_{\Omega} f d\mu \leq \int_{\Omega} |f| d\mu$.

Hence $\left| \int_{\Omega} f d\mu \right| \leq \int_{\Omega} |f| d\mu$.

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Theorem(11.9)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f be a measurable function. If $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, then $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$

Proof :

$$\text{Since } A \cap B = \emptyset \Rightarrow I_{A \cup B} = I_A + I_B$$

$$\int_{A \cup B} f d\mu = \int_{\Omega} I_{A \cup B} f d\mu = \int_{\Omega} (I_A + I_B) f d\mu = \int_{\Omega} (I_A f + I_B f) d\mu = \int_{\Omega} I_A f d\mu + \int_{\Omega} I_B f d\mu = \int_A f d\mu + \int_B f d\mu$$

Corollary(11.10)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f be a measurable function. If $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$, then $\int_{\bigcup_{n=1}^{\infty} A_n} f d\mu = \sum_{n=1}^{\infty} \left(\int_{A_n} f d\mu \right)$

Theorem(11.11) Dominated Convergence Theorem

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions such that $f_n \rightarrow f$. Suppose that there exists some $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that $|f_n| \leq g$ for all $n \geq 1$.

Then $f, f_n \in L^1(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$, and $\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0$. In particular,

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

Proof :

$$\text{Since } |f_n(x)| \leq g(x) \text{ for all } n \geq 1 \text{ and } g \in L^1(\Omega, \mathcal{F}, \mu), \text{ then } \int_{\Omega} |f_n| d\mu \leq \int_{\Omega} g d\mu < \infty$$

It follows that $f_n \in L^1(\Omega, \mathcal{F}, \mu)$.

Again since $|f_n(x)| \leq g(x)$ for all $n \geq 1$ and $x \in \Omega$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$

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$$\Rightarrow |f(x)| = \lim_{n \rightarrow \infty} |f_n(x)| \leq g(x), \text{ we see that } \int_{\Omega} |f| d\mu \leq \int_{\Omega} g d\mu < \infty.$$

We conclude that $f \in L^1(\Omega, \mathcal{F}, \mu)$.

For $n \geq 1$, define $h_n = 2g - |f_n - f|$

Since $f, f_n \in L^1(\Omega, \mathcal{F}, \mu) \Rightarrow f_n - f \in L^1(\Omega, \mathcal{F}, \mu) \Rightarrow |f_n - f| \in L^1(\Omega, \mathcal{F}, \mu)$

Since $g \in L^1(\Omega, \mathcal{F}, \mu)$, we have $h_n \in L^1(\Omega, \mathcal{F}, \mu)$. In particular, h_n is a measurable function.

Since $|f_n - f| \leq |f_n| + |f| \leq g + g = 2g$ $2g - |f_n - f| \geq 0 \Rightarrow h_n \geq 0$ for $n \geq 1$.

By Fatou's Lemma, we have $\int_{\Omega} (\liminf_{n \rightarrow \infty} h_n) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} h_n d\mu$

Since $f_n \rightarrow f$, we have $h_n \rightarrow 2g$. In particular,

$$\liminf_{n \rightarrow \infty} h_n = 2g \Rightarrow \int_{\Omega} (2g) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} h_n d\mu$$

$$\Rightarrow \int_{\Omega} (2g) d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} (2g - |f_n - f|) d\mu \Rightarrow \int_{\Omega} (2g) d\mu \leq \liminf_{n \rightarrow \infty} (\int_{\Omega} (2g) d\mu - \int_{\Omega} |f_n - f| d\mu)$$

$$\text{Since } \liminf_{n \rightarrow \infty} (\int_{\Omega} (2g) d\mu - \int_{\Omega} |f_n - f| d\mu) = \int_{\Omega} (2g) d\mu - \liminf_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu$$

$$\Rightarrow \int_{\Omega} (2g) d\mu \leq \int_{\Omega} (2g) d\mu - \liminf_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu$$

$$\text{Since } g \in L^1(\Omega, \mathcal{F}, \mu), \int_{\Omega} (2g) d\mu \in \mathbb{R}, \text{ we obtain } 0 \leq -\liminf_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu$$

$$\text{We conclude that } \liminf_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0$$

$$\text{Since } x_n \rightarrow 0 \text{ iff } \liminf_{n \rightarrow \infty} |x_n| = 0. \text{ It follows that } \lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0$$

Theorem(11.12) Chebyshev's Inequality

Let $(\Omega, \mathcal{F}, \mu)$ be measure space and let f be a non-negative measurable function on $(\Omega, \mathcal{F})$.

If $0 < p < \infty$ and $0 < \varepsilon < \infty$. Then $\mu\{x \in \Omega : f(x) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \int_{\Omega} f^p d\mu$.

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In particular, if $A \in \mathcal{F}$, then $\mu\{x \in A : f(x) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \int_A f^p d\mu$

Proof :

$$\text{Since } \mu\{x \in \Omega : f(x) \geq \varepsilon\} = \int_{\{x \in \Omega : f(x) \geq \varepsilon\}} d\mu$$

$$\Rightarrow \varepsilon^p \mu\{x \in \Omega : f(x) \geq \varepsilon\} = \int_{\{x \in \Omega : f(x) \geq \varepsilon\}} \varepsilon^p d\mu \leq \int_{\{x \in \Omega : f(x) \geq \varepsilon\}} f^p d\mu \leq \int_{\Omega} f^p d\mu$$

Theorem(11.13)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f, g be measurable functions

1. If $f = 0$ a.e., then $\int_{\Omega} f d\mu = 0$.

2. If $f \in L^1(\Omega, \mathcal{F}, \mu)$ and $f = g$ a.e., then $g \in L^1(\Omega, \mathcal{F}, \mu)$ and $\int_{\Omega} f d\mu = \int_{\Omega} g d\mu$

3. If $f \in L^1(\Omega, \mathcal{F}, \mu)$, then f is finite a.e.

4. If $f \geq 0$ and $\int_{\Omega} f d\mu = 0$, then $f = 0$ a.e.

Proof :

(1) If f is a simple function, $f(x) = \sum_{i=1}^n \alpha_i I_{A_i}(x)$ Where $n \geq 1$, $\alpha_i \in \mathbb{R}$, and $A_i \in \mathcal{F}$ for

$i = 1, 2, \dots, n$ $A_i = \{x \in \Omega : f(x) = \alpha_i\}$, then $\alpha_i \neq 0$ implies $\mu(A_i) = 0$ by hypothesis, hence

$$\int_{\Omega} f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i) = 0.$$

If $f \geq 0$ and $0 \leq S \leq f$, S is simple, then $S = 0$ a.e., hence $\int_{\Omega} S d\mu = 0$

For general f , Since $f = f^+ - f^-$ and if $f = 0$ a.e., then $f^+ = 0$ a.e. and $f^- = 0$ a.e.

Hence $\int_{\Omega} f^+ d\mu = 0$ and $\int_{\Omega} f^- d\mu = 0$, so $\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu = 0$

(2) Let $A = \{x \in \Omega : f(x) = g(x)\}$, $B = A^c$, then $f = f I_A + f I_B$ and $g = g I_A + g I_B = f I_A + g I_B$

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Since $f I_B = g I_B = 0$ a.e. (because $I_B = 0$, this $\mu(B) = 0$), by part (1), we have

$$\int_{\Omega} f I_B = \int_{\Omega} g I_B d\mu = 0, \text{ then } \int_{\Omega} f d\mu = \int_{\Omega} f I_A d\mu + \int_{\Omega} f I_B d\mu = \int_{\Omega} f I_A d\mu \text{ and}$$

$$\int_{\Omega} g d\mu = \int_{\Omega} f I_A d\mu + \int_{\Omega} g I_B d\mu = \int_{\Omega} f I_A d\mu. \text{ Hence } \int_{\Omega} f d\mu = \int_{\Omega} g d\mu.$$

(3) Let $A = \{x \in \Omega : |f(x)| = \infty\}$.

If $\mu(A) > 0$, then $\int_{\Omega} |f| d\mu \geq \int_A |f| d\mu = \infty \mu(A) = \infty$, a contradiction

(4) Let $B = \{x \in \Omega : f(x) > 0\}$, $B_n = \{x \in \Omega : f(x) \geq \frac{1}{n}\}$

$$\Rightarrow \bigcup_{n=1}^{\infty} B_n = B \text{ and } B_n \subseteq B_{n+1} \text{ for all } n. \Rightarrow B_n \uparrow B$$

$$\text{Since } \mu(B_n) = \mu(\{x \in \Omega : f(x) \geq \frac{1}{n}\}) \leq n \int_{\Omega} f d\mu = n \times 0 = 0$$

$$\text{Since } B_n \uparrow B \Rightarrow \mu(B_n) \rightarrow \mu(B) \Rightarrow \mu(B) = 0$$

Definition(11.14)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f, f_n \in \mathbb{M}(\Omega)$, $n \in \mathbb{N}$ and $A \in \mathcal{F}$, we say that $\{f_n\}$ mean converges to f on A , denoted by $f_n \xrightarrow{f-m} f$ on A , if $\lim_{n \rightarrow \infty} \int_A |f_n - f| d\mu = 0$.

Note that, in the above definition, when $A = \Omega$, we can omit "on A " from the statements.

Theorem(11.15)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f, f_n \in \mathbb{M}(\Omega)$, $n \in \mathbb{N}$ and $A \in \mathcal{F}$. If $f_n \xrightarrow{f-m} f$ on A , then $f_n \xrightarrow{\mu} f$ on A .

Proof:

Chebyshev's inequality $\mu\{|f_n - f| \geq \varepsilon\} \leq \int_{\Omega} |f_n - f| d\mu$ and the result follows.

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Definition(11.16)

A function $u : (a,b) \rightarrow \mathbb{R}$ (the interval " $(a,b) = \mathbb{R}$ " is permitted) is said to be convex if for all $x, y \in (a,b)$ and $0 \leq \lambda \leq 1$, we have $u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y)$

Remark

Let $u : (a,b) \rightarrow \mathbb{R}$ be a function

1. u is convex iff for all $x_1, x_2, \dots, x_n \in (a,b)$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i = 1$, $n \geq 1$,

$$\text{we have } u\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i u(x_i)$$

2. u is convex iff for all x, y, z with $a < x < y < z < b$ we have $u(y) \leq \frac{z-y}{z-x}u(x) + \frac{y-x}{z-x}u(z)$

3. u is convex iff for all x, y, z with $a < x < y < z < b$ we have $\frac{u(y)-u(x)}{y-x} \leq \frac{u(z)-u(y)}{z-y}$

4. If u is convex, and $x_0, r, s, r', s' \in (a,b)$ such that $r < r' < x_0 < s < s'$.

Show that for all $x \in (x_0, s)$

$$\frac{u(r')-u(r)}{r'-r} \leq \frac{u(x)-u(x_0)}{x-x_0} \leq \frac{u(s')-u(s)}{s'-s} \text{ and deduce that } \lim_{x \downarrow x_0} u(x) = u(x_0)$$

5. If u is convex, then u is continuous

Theorem(11.18) Jensen's Inequality

Let $(\Omega, \mathcal{F}, \mu)$ be a Probability space and $f \in L^1(\Omega, \mathcal{F}, \mu)$ such that $f(\Omega) \subseteq (a,b)$. Suppose

$u : (a,b) \rightarrow \mathbb{R}$ is convex function. Then $u(f)$ is measurable and $u\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} u(f) d\mu$

Proof:

- Since u is convex, then u is continuous

Since f is measurable, then $u(f)$ is measurable.

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- (1) If $\int_{\Omega} f d\mu = a$, then $\int_{\Omega} (f - a) d\mu = 0$

Since $f \in L^1(\Omega, \mathcal{F}, \mu)$, then $\int_{\Omega} f d\mu \in \mathbb{R} \Rightarrow a \in \mathbb{R}$

Since $a \leq f \Rightarrow f - a \geq 0$, so $f - a \geq a$ and $\int_{\Omega} (f - a) d\mu = 0$, we conclude that $f = a$

a.e., which contradicts $f(\Omega) \subseteq (a, b)$

- (2) If $\int_{\Omega} f d\mu = b$, then $\int_{\Omega} (b - f) d\mu = 0$

since $f \in L^1(\Omega, \mathcal{F}, \mu)$, then $\int_{\Omega} f d\mu \in \mathbb{R} \Rightarrow b \in \mathbb{R}$

Since $f(\Omega) \subseteq (a, b) \Rightarrow f \leq b \Rightarrow b - f \geq 0$, so $b - f \geq a$ and $\int_{\Omega} (b - f) d\mu = 0$,

we conclude that $f = b$ a.e., which contradicts $f(\Omega) \subseteq (a, b)$

- Let $\int_{\Omega} f d\mu = t$

Since $f(\Omega) \subseteq (a, b) \Rightarrow a < f < b$. It follows that $a \leq t \leq b$. From (1) and (2), $t = a$ or $t = b$ leads to a contradiction. We conclude that $a < t < b$.

- We define $\beta = \sup_{a < x < t} \frac{u(t) - u(x)}{t - x}$

Since $a < t$, $(a, t) \neq \emptyset$ and $\beta \neq -\infty$. Let $y \in (t, b)$

Since u is convex, then for all $x \in (a, t)$, we have $\frac{u(t) - u(x)}{t - x} \leq \frac{u(y) - u(t)}{y - t}$

It follows that $\beta \leq \frac{u(y) - u(t)}{y - t}$. In particular, $\beta < \infty$ and finally $\beta \in \mathbb{R}$.

- Let $x \in (a, b)$.

If $x \in (a, t)$, then by definition of β , we have $\frac{u(t) - u(x)}{t - x} \leq \beta$ and consequently

$$u(t) + \beta(x - t) \leq u(x)$$

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If $x \in (t, b)$, then $\beta \leq \frac{u(x) - u(t)}{x - t}$ and consequently $u(t) + \beta(x - t) \leq u(x)$

i.e. $u(t) + \beta(x - t) \leq u(x)$ for all $x \in (a, b)$

- Let $\omega \in \Omega \Rightarrow f(\omega) \in f(\Omega)$,

Since $f(\Omega) \subseteq (a, b) \Rightarrow f(\omega) \in (a, b)$, we obtain $u(t) + \beta(f(\omega) - t) \leq u(f(\omega))$

Integrating this inequality, we have $\int_{\Omega} (u(t) + \beta(f(\omega) - t)) d\mu \leq \int_{\Omega} u(f(\omega)) d\mu$.

$$\Rightarrow \int_{\Omega} u(t) d\mu + \beta \left(\int_{\Omega} f(\omega) d\mu - \int_{\Omega} t d\mu \right) \leq \int_{\Omega} u(f(\omega)) d\mu$$

$$\Rightarrow u(t) \int_{\Omega} d\mu + \beta \left(\int_{\Omega} f(\omega) d\mu - t \int_{\Omega} d\mu \right) \leq \int_{\Omega} u(f(\omega)) d\mu$$

$$\Rightarrow u(t) \mu(\Omega) + \beta \left(\int_{\Omega} f(\omega) d\mu - t \mu(\Omega) \right) \leq \int_{\Omega} u(f(\omega)) d\mu$$

Since $\mu(\Omega) = 1$, we have $u(t) + \beta \left(\int_{\Omega} f(\omega) d\mu - t \right) \leq \int_{\Omega} u(f(\omega)) d\mu$

Since $\int_{\Omega} f d\mu = t$, we have $u \left(\int_{\Omega} f d\mu \right) + \beta(t - t) \leq \int_{\Omega} u(f) d\mu \Rightarrow u \left(\int_{\Omega} f d\mu \right) \leq \int_{\Omega} u(f) d\mu$

Example(11.19)

1. If $u(x) = e^x$, then u is convex function, so $\exp \left(\int_{\Omega} f d\mu \right) \leq \int_{\Omega} e^f d\mu$

- If $\Omega = \{1, 2, \dots, n\}$ with the measure μ defined by $\mu\{i\} = \frac{1}{n}$ and the function f given

by $f(i) = x_i$, we obtain $\exp \left\{ \frac{1}{n} (x_1 + x_2 + \dots + x_n) \right\} \leq \frac{1}{n} (e^{x_1} + e^{x_2} + \dots + e^{x_n})$

Setting $y_i = e^{x_i}$ we obtain the Geometric mean inequality. That is

$$(y_1 y_2 \cdots y_n)^{\frac{1}{n}} \leq \frac{1}{n} (y_1 + y_2 + \dots + y_n)$$

More generally, extend this example in the following way

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Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be a sequence of positive numbers with $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ and let $y_1, y_2, \dots, y_n$ positive numbers, then $y_1^{\alpha_1} \times y_2^{\alpha_2} \times \dots \times y_n^{\alpha_n} \leq \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n$

2. If $u(x) = \log(x)$ then u is convex function, so $\log\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} \log(f) d\mu$

3. If $u(x) = x^3$ then u is convex function if $x > 0$ and concave if $x < 0$, so

$$\left(\int_{\Omega} f d\mu\right)^3 \leq \int_{\Omega} f^3 d\mu \text{ if } f > 0 \text{ and } \left(\int_{\Omega} f d\mu\right)^3 \geq \int_{\Omega} f^3 d\mu \text{ if } f < 0$$

Comparison of Lebesgue and Riemann Integrals

In this section we show that integration with respect to Lebesgue measure is more general Riemann integration, we obtain a precise criterion for Riemann integrability.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function, fixed throughout the discussion.

Definition(11.20)

Let $[a, b]$ be a given interval. A partition P of $[a, b]$ is a finite set of points, say

$$P = \{x_0, x_1, \dots, x_n\} \text{ or } P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\}$$

such that $a = x_0 < x_1 < \dots < x_n = b$. $J_i = [x_{i-1}, x_i]$

- The symbol Δx_i denotes the difference $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \dots, n$, so that $\sum_{i=1}^n \Delta x_i = b - a$
- The norm of a partition P is the length of the largest subinterval of P and is denoted by $\|P\|$.

$$\|P\| = \max\{\Delta x_i : i = 1, 2, \dots, n\}$$

- A partition Q of $[a, b]$ is said to be finer than P (or a refinement of P) if $P \subseteq Q$.

Note that if $P \subseteq Q$, then $\|Q\| \leq \|P\|$



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That is, refinement of a partition decreases its norm, but the converse does not necessarily hold.

- The set of all possible partitions of $[a, b]$ is denoted by $P([a, b])$

Definition(11.21)

Let f is a bounded real valued function on $J = [a, b]$. i.e. $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function.

- We define

$$M = \sup \{f(x) : x \in J\} = \sup \{f(x) : a \leq x \leq b\} \quad m = \inf \{f(x) : x \in J\} = \inf \{f(x) : a \leq x \leq b\}$$

- If $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ is a partition of interval $J = [a, b]$, we may construct upper and lower sums of f relative to P as follows:

$$M_i = \sup \{f(x) : x \in J_i\} = \sup \{f(x) : x_{i-1} \leq x \leq x_i\} \quad m_i = \inf \{f(x) : x \in J_i\} = \inf \{f(x) : x_{i-1} \leq x \leq x_i\}$$

The upper and lower sums are given by

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i, \quad L(f, P) = \sum_{i=1}^n m_i \Delta x_i$$

Finally , the upper and lower Riemann integrals over $[a, b]$ are given by

$$\int_a^b f dx = \inf \{U(f, P) : P \text{ is partitions of } [a, b]\}, \quad \int_{-a}^b f dx = \sup \{U(f, P) : P \text{ is partitions of } [a, b]\}$$

Respectively .

Definition(11.22)

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. We say that f is Riemann integrable on $[a, b]$ if the upper and lower Riemann integrals over $[a, b]$ are equal, i.e.

$$\int_a^b f dx = \int_{-a}^b f dx .$$



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We write $f \in R[a,b]$, and we denoted the common value of $\int_a^b f dx$ and $\int_{-a}^b f dx$ by $\int_a^b f$ or $\int_a^b f dx$ or $\int_a^b f(x) dx$. This is the Riemann integral of f over $[a,b]$.

Remark

Since f is bounded, there exist two numbers, α and β such that $\alpha \leq f(x) \leq \beta$ for all $x \in [a,b]$. Hence, for every P ,

$$\alpha(b-a) \leq L(f, P) \leq U(f, P) \leq \beta(b-a)$$

So that the numbers $L(f, P)$ and $U(f, P)$ from a bounded set. This shows that the upper and lower Riemann integrals are defined for every bounded function f .

Definition(11.23)

Let $f : [a,b] \rightarrow \mathbb{R}$ be a bounded function. Define step functions α and β , called the upper and lower functions corresponding to P , by

$$\alpha(x) = M_i, \quad \beta(x) = m_i, \quad x \in J_i, \quad i = 1, 2, \dots, n$$

[$\alpha(a)$ and $\beta(a)$ may be chosen arbitrarily].

Now we take as a measure space $\Omega = [a,b]$, $\mathcal{F} = \beta([a,b])$, the Lebesgue measurable subsets of $[a,b]$, μ be a Lebesgue measure. Since α and β are simple functions, we have

$$U(f, P) = \int_a^b \alpha d\mu, \quad L(f, P) = \int_a^b \beta d\mu$$

Now let $\{P_n\}$ be a sequence of partitions of $[a,b]$ such that P_{k+1} is a refinement of P_k for each k , and such that $\|P_k\| \rightarrow 0$ as $k \rightarrow \infty$. If α_k and β_k are the upper and lower functions corresponding to P_k , then $\alpha_1 \geq \alpha_2 \geq \dots \geq f \geq \dots \geq \beta_2 \geq \beta_1$. Thus α_k and β_k approach limit functions α and β . If $|f|$ is bounded by M , then all $|\alpha_k|$ and $|\beta_k|$ are bounded by M as



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well, and the function that is constant at M is integrable on $[a, b]$ with respect to μ , since $\mu([a, b]) = b - a < \infty$

By the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} U(f, P_k) = \lim_{k \rightarrow \infty} \int_a^b \alpha_k d\mu = \int_a^b \alpha d\mu, \quad \lim_{k \rightarrow \infty} L(f, P_k) = \lim_{k \rightarrow \infty} \int_a^b \beta_k d\mu = \int_a^b \beta d\mu$$

We shall need one other fact, namely that if x is not an endpoint of any of the subintervals of the P_k , f is continuous at x iff $\alpha(x) = f(x) = \beta(x)$

If $\lim_{k \rightarrow \infty} U(f, P_k) = \lim_{k \rightarrow \infty} L(f, P_k) = r$, where r is a finite number, independent of the particular sequence of partitions, f is said to be Riemann integrable on $[a, b]$, and $r = r_{ab}(f)$ is said to be the (value of the) Riemann integral of f on $[a, b]$. The above argument shows that

$$f \text{ is Riemann integrable on } [a, b] \text{ iff } \int_a^b \alpha d\mu = \int_a^b \beta d\mu = r$$

independent of the particular sequence of partitions. If f is Riemann integrable,

$$r_{ab}(f) = \int_a^b \alpha d\mu = \int_a^b \beta d\mu$$

We are now ready for the main results,

Theorem (11.24)

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function,

1. The function f is Riemann integrable on $[a, b]$ iff f is continuous a.e. on $[a, b]$.
(with respect to Lebesgue measure)
2. If f is Riemann integrable on $[a, b]$, then f is integrable with respect to Lebesgue measure on $[a, b]$ and the two integrals are equal.

Proof :



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(1) suppose that f is Riemann integrable on $[a, b]$

Exercises(11)

1. Given an example of a sequence of nonnegative measurable functions f_n for which we have strict inequality in Fatou's Lemma.
2. Compute $\lim_{n \rightarrow \infty} \int_0^n (1 - \frac{x}{n})^n e^{\frac{x}{2}} dx$
3. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $f, f_n \in C(\Omega)$
 - (1) If $f_n \xrightarrow{\mu} f$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous function, then $g(f_n) \xrightarrow{\mu} g(f)$
 - (2) If $f_n \xrightarrow{\mu} f$ and $f_n \geq 0$, then $\int_{\Omega} f d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$
4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{f_n\}$ be a sequence of measurable functions converging to zero in measure. Let g be abounded uniformly continuous on $\mathbb{R}$. Prove that $\lim_{n \rightarrow \infty} \int_{\Omega} g(f_n) dP = g(0)$
5. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Suppose f and g are positive measurable functions Such that $fg \geq 1$ a.e. on Ω . Prove that $\int_{\Omega} fg d\mu \geq 1$.
6. Let $\Omega = [0, 1]$ and let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and f a measurable function on this space. Let A be the set of all $x \in \Omega$ such that $f(x)$ is an integer. Prove that the set A is measurable and that $\lim_{n \rightarrow \infty} \int_{\Omega} (\cos(\pi f(x)))^{2n} d\mu = \mu(A)$
7. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space such that and f a nonnegative real valued measurable function on Ω . Prove that $\lim_{n \rightarrow \infty} \int_{\Omega} f^n d\mu$ exists, as a finite number, iff $\mu\{f > 1\} = 0$.
8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f and g be measurable functions.



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If $|f| \leq g$ and g is integrable. Show that f is integrable .

9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions, f, g , measurable functions. $|f_n| \leq g$ for all n , where $|g|^p$ is integrable ($p > 0$), and $f_n \xrightarrow{a.e.} f$.

Show that $|f|^p$ is integrable, and $\int_{\Omega} |f_n - f|^p d\mu \rightarrow 0$.

10. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f and g are measurable, . If μ is σ -finite on $\mathcal{F}$, $\int_{\Omega} f d\mu$ and $\int_{\Omega} g d\mu$ exists, and $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in \mathcal{F}$. Show that $f \leq g$ a.e.

11. Let $T: (\Omega, \mathcal{F}) \rightarrow (\Omega_0, \mathcal{F}_0)$ be a measurable function, and let μ be a measure on $\mathcal{F}$.

Define a measure $\mu_0 = \mu T^{-1}$ on $\mathcal{F}_0$ by $\mu_0(A) = \mu(T^{-1}(A))$, $A \in \mathcal{F}_0$. If $\Omega_0 = \Omega$, $\mathcal{F}_0 = \mathcal{F}$, and $\mu_0 = \mu$, T is said to preserve the measure μ . If $f: (\Omega_0, \mathcal{F}_0) \rightarrow (R^*, \beta(R^*))$ and $A \in \mathcal{F}_0$, then

$$\int_{T^{-1}(A)} f(T(\omega)) d\mu(\omega) = \int_A f(\omega) d\mu_0(\omega).$$

12. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions, f, g , measurable functions. $|f_n| \leq g$ for all n , where $g \in L^1(\Omega, \mathcal{F}, \mu)$. Show that

(1) If $f_n \xrightarrow{a.e.} f$, then $f \in L^1(\Omega, \mathcal{F}, \mu)$, and $\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$.

(2) (1) If $f_n \xrightarrow{\mu} f$, then $f \in L^1(\Omega, \mathcal{F}, \mu)$, and $\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$.

13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f \in L^1(\Omega, \mathcal{F}, \mu)$. Define $\lambda: \mathcal{F} \rightarrow \mathbb{R}$ by

$$\lambda(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{F}. \text{ Show that}$$

(1) λ is a signed measure (2) If $f \geq 0$, then λ is a measure

$$(3) \lambda^+(A) = \int_A f^+ d\mu, \lambda^-(A) = \int_A f^- d\mu, |\lambda|(A) = \int_A |f| d\mu.$$

14. Show that f is Riemann-Stieltjes integrable iff f is continuous a.e. on $[a, b]$.



12. Radon-Nikodym Theorem and Related Results

Definition(12.1)

Let μ_1, μ_2 be two measures on measurable space $(\Omega, \mathcal{F})$. We say that μ_1 is singular with respect to μ_2 (written $\mu_1 \perp \mu_2$) if there are $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, $A \cup B = \Omega$ and $\mu_1(A) = 0$, $\mu_2(B) = 0$

Remarks

1. If μ_1, μ_2 are two measures on measurable space $(\Omega, \mathcal{F})$, then $\mu_1 \perp \mu_2$ if there is a set $A \in \mathcal{F}$ such that $\mu_1(A) = 0$, $\mu_2(A^c) = 0$
- 2.) μ_1 is singular with respect to μ_2 iff μ_2 is singular with respect to μ_1 , so we may say that μ_1 and μ_2 are mutually singular
3. Let λ_1, λ_2 be two signed measures on the measurable space $(\Omega, \mathcal{F})$, we say that λ_1 and λ_2 are mutually singular iff $|\lambda_1| \perp |\lambda_2|$
4. If λ is a signed measure on the measurable space $(\Omega, \mathcal{F})$, then $\lambda^+ \perp \lambda^-$

Example(12.2)

Let $\Omega = \mathbb{R}$, $\mathcal{F} = \beta(\mathbb{R})$, μ is Lebesgue measure, $\lambda = \sum_{n \in \mathbb{Z}} \delta_n \Rightarrow \mu \perp \lambda$

Theorem(12.3) Jordan - Decomposition

Let λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$. There are two mutually singular measures λ^+ and λ^- such that $\lambda = \lambda^+ - \lambda^-$. This decomposition is unique.

Proof:

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Since λ be a signed measure on the measurable space $(\Omega, \mathcal{F})$, by using Hahn-decomposition There is a positive set A and a negative set B with

$$A \cap B = \emptyset, \quad A \cup B = \Omega$$

Defined λ^+ and λ^- by $\lambda^+(C) = \lambda(A \cap C)$, $\lambda^-(C) = -\lambda(B \cap C)$ for all $C \in \mathcal{F}$

$$\lambda^+(B) = \lambda(A \cap B) = \lambda(\emptyset) = 0, \quad \lambda^-(A) = -\lambda(B \cap A) = -\lambda(\emptyset) = 0 \Rightarrow \lambda^+ \perp \lambda^-$$

Clearly $\lambda = \lambda^+ - \lambda^-$

Definition(12.4)

Let μ_1, μ_2 be two measures on measurable space $(\Omega, \mathcal{F})$. We say that μ_1 is absolute continuous with respect to μ_2 (written $\mu_1 \ll \mu_2$) if $\mu_2(A) = 0$ implies $\mu_1(A) = 0$ for every $A \in \mathcal{F}$

Remarks

1. Let μ and λ be two signed measures on the measurable space $(\Omega, \mathcal{F})$. We say that λ is absolute continuous with respect to μ (written $\lambda \ll \mu$) if for every $A \in \mathcal{F}$ with $|\mu|(A) = 0$, we have $\lambda(A) = 0$.
2. Two measures μ_1 and μ_2 on the measurable space $(\Omega, \mathcal{F})$, for which $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$, are called equivalent, in symbols $\mu_1 \sim \mu_2$, i.e.

$$\mu_1 \sim \mu_2 \text{ iff } (\mu_1(A) = 0 \Leftrightarrow \mu_2(A) = 0 \text{ for all } A \in \mathcal{F})$$

Example(12.5)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f \geq 0$ be measurable function such that $\int_{\Omega} f d\mu$

exists. Define $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Then $\lambda \ll \mu$

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Example(12.6)

1. Let $\Omega = \mathbb{R}$, $\mathcal{F} = \beta(\mathbb{R})$, μ is Lebesgue measure, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$

$\lambda(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$, a probability measure, we know as the standard normal.

2. Let $\Omega = \mathbb{R}^+$, $\mathcal{F} = \beta(\mathbb{R}^+)$, μ is Lebesgue measure, $f(x) = e^{-x}$, $x > 0$

$\lambda(A) = \int_A e^{-x} dx$, another probability measure called standard exponential distribution.

3. Let $\Omega = \mathbb{N}$, $\mathcal{F} = P(\mathbb{N})$, $\mu = \#$, $f(n) = n^{-\alpha}$, $\lambda(A) = \sum_{n \in A} n^{-\alpha}$. Then $\lambda \ll \mu$, also $\mu \ll \lambda$

4. μ_1, μ_2 any measures. Then $\mu_1 \ll \mu_1 + \mu_2$ and $\mu_2 \ll \mu_1 + \mu_2$

Remark

If $f(x) > 0$ a.e. $[\mu]$, then also $\mu \ll \lambda$. This is because $\mu(A) > 0 \Rightarrow \int_A f d\mu > 0$

Thus example(1), $\mu \ll \lambda$, but in example(2), $\mu \not\ll \lambda$

Lemma(12.7)

Let $\{A_\alpha\}_{\alpha \in D}$ where D countable set of real numbers. Suppose $A_\alpha \subset A_\beta$ whenever $\alpha < \beta$.

Then there is a measurable function f such that $f(x) \leq \alpha$ on A_α and $f(x) \geq \alpha$ on A_α^c

Proof :

For $x \in \Omega$, set $f(x) = \text{first } \alpha \text{ such that } x \in A_\alpha \Rightarrow f(x) = \inf\{\alpha \in D : x \in A_\alpha\}$. $\inf\{\emptyset\} = \infty$

- If $x \notin A_\alpha$, $x \notin A_\beta$ for any $\beta < \alpha$ and so, $f(x) \geq \alpha$
- If $x \in A_\alpha$, then $f(x) \leq \alpha$ provided we show that f is measurable

Claim: $\forall \gamma$ real, $\{x : f(x) < \gamma\} = \bigcup_{\beta < \gamma, \beta \in D} A_\beta$

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If $f(x) < \gamma$, then $x \in D_\beta$ save $\beta < \gamma$. If $x \in A_\beta$, $\beta < \gamma \Rightarrow f(x) < \gamma$

Remarks

1. Suppose $\{A_\alpha\}_{\alpha \in D}$ as in lemma (12.7) but this time $\alpha < \beta$ implies only $\mu\{D_\alpha \mid A_\alpha\} = 0$.

Then there exists a measurable function f on Ω such that $f(x) \leq \alpha$ a.e. on A_α and $f(x) \geq \alpha$ a.e. on A_α^c

2. Suppose D is dense. Then the function in lemma (12.7) is unique and the function in (1) is unique μ a.e.

Theorem(12.8) Radon-Nikodym theorem

Let μ be a σ -finite measure on the measurable space $(\Omega, \mathcal{F})$, and λ be a measure on $(\Omega, \mathcal{F})$.

Assume $\lambda \ll \mu$, then there is a nonnegative measurable function $f: \Omega \rightarrow \mathbb{R}^*$ such that

$$\lambda(A) = \int_A f d\mu$$

for all $A \in \mathcal{F}$. The function f is unique a.e. $[\mu]$. We call f is called the Radon-

Nikodym derivative of λ with respect to μ and write $f = \frac{d\lambda}{d\mu}$

Proof :

Assume $\mu(\Omega) = 1$. Let $\lambda_\alpha = \lambda - \alpha\mu$, $\alpha \in \mathbb{Q} \Rightarrow \lambda_\alpha$ is a signed measure.

By using Hahn- Decomposition, There is a positive set A_α and a negative set B_α with

$$A_\alpha \cap B_\alpha = \emptyset, A_\alpha \cup B_\alpha = \Omega$$

$$\Omega = B_\alpha, B_\alpha = \emptyset, \text{ if } \alpha \leq 0$$

$$B_\alpha \cap B_\beta = B_\alpha \cap B_\beta^c = B_\alpha \cap A_\beta$$

Since A_α is positive and B_α is negative $\Rightarrow \lambda_\beta(B_\alpha \mid B_\beta) \geq 0$ and $\lambda_\alpha(B_\alpha \mid B_\beta) \geq 0$

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Since $\lambda_\alpha = \lambda - \alpha\mu$, where $\alpha \in \mathbb{Q} \Rightarrow \lambda(B_\alpha | B_\beta) - \beta\mu(B_\alpha | B_\beta) \geq 0$ and

$$\lambda(B_\alpha | B_\beta) - \alpha\mu(B_\alpha | B_\beta) \leq 0$$

Thus $\beta\mu(B_\alpha | B_\beta) \leq \lambda(B_\alpha | B_\beta) \leq \alpha\mu(B_\alpha | B_\beta)$

Thus, if $\alpha < \beta$, we have $\mu(B_\alpha | B_\beta) = 0$

Thus, there exists a measurable function such that for all $\alpha \in \mathbb{Q}$, $f(x) \geq \alpha$ a.e. on A_α and $f(x) \leq \alpha$ a.e. on B_α . Since $B_0 = 0$, $f \geq 0$ a.e.

Let N be very large. Put $E_k = A \cap (\frac{B_{k+1}}{N} | \frac{B_k}{N})$, $k = 0, 1, 2, \dots$, $E_\infty = \Omega \setminus \bigcup_{k=0}^{\infty} \frac{B_k}{N}$

Then $E_0, E_1, \dots, E_\infty$ are disjoint and $A = \bigcup_{k=0}^{\infty} E_k \cup E_\infty$.

So $\lambda(A) = \lambda(E_\infty) + \sum_{k=0}^{\infty} \lambda(E_k)$ on $E_k \subset (\frac{B_{k+1}}{N} | \frac{B_k}{N}) = \frac{B_{k+1}}{N} \cap A_{\frac{k}{N}}$

We have, $\frac{k}{N} \leq f(x) \leq \frac{k+1}{N}$ a.e. and so, $\frac{k}{N} \mu(E_k) \leq \int_{E_k} f(x) d\mu \leq \frac{k+1}{N} \mu(E_k)$ (1)

Also $E_k \subset A_{\frac{k}{N}} \Rightarrow \frac{k}{N} \mu(E_k) \leq \lambda(E_k)$ (2) and $E_k \subset \frac{B_{k+1}}{N} \Rightarrow \lambda(E_k) \leq \frac{k+1}{N} \mu(E_k)$ (3)

Thus

$$\lambda(E_k) - \frac{1}{N} \mu(E_k) \leq \frac{k}{N} \mu(E_k) \leq \int_{E_k} f(x) dx \leq \frac{k}{N} \mu(E_k) + \frac{1}{N} \mu(E_k) \leq \lambda(E_k) + \frac{1}{N} \mu(E_k) \text{ on } E_\infty,$$

$f \equiv \infty$ a.e.

If $\mu(E_\infty) > 0$, then $\lambda(E_\infty) = 0$ since $(\lambda - \alpha\mu)(E_\infty) \geq 0$ for all α

If $\mu(E_\infty) = 0$, then $\lambda(E_\infty) = 0$, so either way: $\lambda(E_\infty) = \int_{E_\infty} f d\mu$

$$\text{Add } \lambda(A) - \frac{1}{N} \mu(A) \leq \int_A f d\mu \leq \lambda(A) + \frac{1}{N} \mu(A)$$

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Since N is arbitrary, we are done

Uniqueness : If $\lambda(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$

$$\Rightarrow \lambda(A) - \alpha\mu(A) = \int_A (g - \alpha) d\mu \text{ for all } \alpha \text{ for all } A \subset A_\alpha$$

Since $0 \leq \lambda(A) - \alpha\mu(A) = \int_A (g - \alpha) d\mu$, we have $g - \alpha \geq 0$ a.e. $[\mu]$ on A_α

$$\Rightarrow g \geq \alpha \text{ a.e. } [\mu] \text{ on } A_\alpha. \text{ similarly } \Rightarrow g \leq \alpha \text{ a.e. } [\mu] \text{ on } B_\alpha \Rightarrow f = g \text{ a.e.}$$

Suppose μ is σ -finite : $\lambda \ll \mu$. Let Ω_i be such that $X_i \cap X_j = \emptyset$, $\bigcup_i X_i = \Omega$, $\mu(X_i) < \infty$

Put $\mu_i(A) = \mu(A \cap X_i)$ and $\lambda_i(A) = \lambda(A \cap X_i)$. Then $\lambda_i \ll \mu_i$

$$\Rightarrow \text{there is } f_i \geq 0 \text{ such that } \lambda_i(A) = \int_A f_i d\mu_i \Rightarrow \lambda(A \cap X_i) = \int_{A \cap X_i} f_i d\mu = \int_A f_i d\mu$$

Remark

The space needs to be σ -finite

For example : let $\Omega = [0,1]$, $\mathcal{F} = \beta([0,1])$, μ is counting measure

$\Rightarrow \mu$ is a measure on $\mathcal{F}$, but not σ -finite

$\Rightarrow \lambda \ll \mu$, where λ is Lebesgue measure

If $\lambda(A) = \int_A f d\mu \Rightarrow f(x) = 0$ for all $x \in \Omega \Rightarrow \lambda = 0$. This contradiction

Theorem(12.9) Lebesgue Decomposition Theorem

Let μ and λ be σ -finite measures on the measurable space $(\Omega, \mathcal{F})$. Then λ has a unique decomposition as $\lambda_1 + \lambda_2$, where λ_1 and λ_2 are measures on $(\Omega, \mathcal{F})$ such that $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$.

Proof :



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Let $\nu = \mu + \lambda \Rightarrow \nu$ is σ -finite measure

Since $\nu(A) = 0 \Rightarrow \mu(A) = \lambda(A) = 0$

$\Rightarrow \mu \ll \nu$ and $\lambda \ll \nu$, by using radon-Nikodym theorem there are a nonnegative measurable functions f, g such that $\mu(A) = \int_A f d\nu$ and $\lambda(A) = \int_A g d\nu$ for all $A \in \mathcal{F}$.

Let $A = \{f > 0\}$, $B = \{f = 0\} \Rightarrow A, B \in \mathcal{F}$ and $A \cap B = \emptyset$, $A \cup B = \Omega$ and $\mu(B) = 0$

Let $\lambda_1(C) = \lambda(B \cap C)$, then $\lambda_1(A) = \nu(B \cap A) = \nu(\emptyset) = 0$, so $\lambda_1 \perp \mu$

Set $\lambda_2(C) = \lambda(A \cap C) \Rightarrow \lambda_2(C) = \int_{A \cap C} g d\nu$

Clearly $\lambda_1 + \lambda_2 = \lambda$ and it only remains to show that $\lambda_1 \ll \mu$. Assume $\mu(A) = 0$

Then $\mu(A) = \int_A f d\nu = 0 \Rightarrow f = 0$ a.e. $[\lambda]$. ($f \geq 0$) on A

Since $f > 0$ on $A \cap C \Rightarrow \nu(A \cap C) = 0$. Thus $\lambda_2(C) = \int_{A \cap C} g d\nu = 0 \Rightarrow \lambda_2 \perp \mu$

To prove uniqueness, if $\lambda = \lambda_1 + \lambda_2 = \eta_1 + \eta_2$, where $\lambda_1, \eta_1 \ll \mu$ and $\lambda_2, \eta_2 \perp \mu$, then

$\lambda_1 - \eta_1 = \eta_2 - \lambda_2$ is both absolutely continuous and singular with respect to μ , hence it is identically 0.



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Exercises(12)

1. Let μ be a σ -finite measure on the measurable space $(\Omega, \mathcal{F})$, and λ be a σ -finite signed measure on $(\Omega, \mathcal{F})$ (i.e. $|\lambda|$ is σ -finite measure). Then λ has a unique decomposition as $\lambda_1 + \lambda_2$ where λ_1 and λ_2 are signed measures on $(\Omega, \mathcal{F})$ such that $\lambda_1 \ll \lambda$ and $\lambda_2 \perp \mu$.
 - If λ be a σ -finite measure on $(\Omega, \mathcal{F})$, then λ_1 and λ_2 are measures on $(\Omega, \mathcal{F})$.
2. Let μ and λ be two signed measures on the measurable space $(\Omega, \mathcal{F})$. Then the following conditions are equivalent :
 - (1) $\lambda \ll \mu$
 - (2) $\lambda^+ \ll \mu$
 - (3) $|\lambda| \ll |\mu|$
3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \rightarrow \mathbb{R}$ be a measurable function such that $\int_{\Omega} f d\mu$ exists. Define $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$. Then
 - (1) λ is a signed measure
 - (2) If $f \geq 0$, then λ is a measure
 - (3) λ is finite iff f is integrable.
 - (4) $\lambda \ll \mu$
4. Let μ be a σ -finite measure on the measurable space $(\Omega, \mathcal{F})$, and λ be a signed measure on $(\Omega, \mathcal{F})$. Assume $\lambda \ll \mu$, then there is a Borel measurable function f such that $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$.



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5. In exercise (4) show that

(1) If g is any measurable function with this property, then $g = f$ a.e. $[\mu]$.

(2) If $|\lambda|$ is finite, then f is μ -integrable, hence finite a.e. $[\mu]$.

(3) If $|\lambda|$ is a σ -finite, then f is finite a.e. $[\mu]$.

(4) If λ is a measure, $f \geq 0$ a.e. $[\mu]$.

6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, f a nonnegative measurable function on Ω .

Define a measure λ on $\mathcal{F}$ by $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$.

Show that if g is measurable function on Ω , $\int_{\Omega} g d\lambda = \int_{\Omega} fg d\mu$.

7. Let μ , λ_1 and λ_2 be σ -finite measures on the measurable space $(\Omega, \mathcal{F})$.

Show that the Radon-Nikodym derivative $[\frac{d\lambda}{d\mu}]$ has the following proper.

(1) $[\frac{d(\lambda_1 + \lambda_2)}{d\mu}] = [\frac{d\lambda_1}{d\mu}] + [\frac{d\lambda_2}{d\mu}]$

(2) If $\lambda_1 \ll \mu$ and f is a non-negative measurable function, then $\int_{\Omega} fd\lambda_1 = \int_{\Omega} \frac{d\lambda_1}{d\mu} d\mu$

(3) If $\lambda_1 \ll \mu \ll \lambda_2$, then $[\frac{d\lambda_1}{d\lambda_2}] = [\frac{d\lambda_1}{d\mu}] \cdot [\frac{d\mu}{d\lambda_2}]$

(4) If $\lambda_1 \ll \mu$ and $\mu \ll \lambda$, then $[\frac{d\lambda_1}{d\mu}] = [\frac{d\mu}{d\lambda}]^{-1}$



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13. Spaces of Measurable Functions

Let Ω be a non empty set and let $L(\Omega)$ denote the set of all real valued functions defined on Ω . Then $L(\Omega)$ is a linear space under the following addition and scalar multiplication

1. $(f + g)(x) = f(x) + g(x)$ for all $f, g \in L(\Omega)$
2. $(\lambda f)(x) = \lambda f(x)$ for all $f \in L(\Omega)$ and for all $\lambda \in \mathbb{R}$

Recall that $L(\Omega, \mathcal{F}, \mu)$, denote the set of all real valued measurable functions defined on $(\Omega, \mathcal{F})$ and $L_+(\Omega, \mathcal{F}, \mu)$, denote the set of all nonnegative real valued measurable functions defined on $(\Omega, \mathcal{F})$, i.e. $L_+(\Omega, \mathcal{F}, \mu) = \{f \in L(\Omega, \mathcal{F}, \mu) : f \geq 0\}$

Definition(13.1)

Let P and q be positive real numbers, we say that P and q is a pair of conjugate exponents

if $\frac{1}{p} + \frac{1}{q} = 1$.

Remarks

(1) If P and q are conjugate exponents, then

- (i) $1 < p < \infty$ and $1 < q < \infty$
- (ii) If $p = 2$, then $q = 2$
- (iii) As $p \rightarrow 1$, then $q \rightarrow \infty$

(2) For $\alpha \in (0, \infty)$, we define $u^\alpha : [0, \infty] \rightarrow [0, \infty]$ by $u^\alpha(x) = \begin{cases} x^\alpha, & x \in \mathbb{R}^+ \\ \infty, & x = \infty \end{cases}$, then u^α is continuous.

Theorem(13.2)

If P and q are conjugate exponents and $a, b \in [0, \infty]$, then



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$$1. ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

$$2. ab = \frac{1}{p}a^p + \frac{1}{q}b^q \text{ iff } a^p = b^q$$

$$3. \text{ If } 0 < p \leq 1, \text{ then } (a+b)^p \leq a^p + b^p$$

$$4. \text{ If } p \geq 1, \text{ then } (a+b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof :

(1) Let $u(t) = -\ln(t)$, then u is convex function on $(0, \infty)$

If $a, b \in (0, \infty)$, since $\frac{1}{p} + \frac{1}{q} = 1$, we have $-\ln(\frac{1}{p}a^p + \frac{1}{q}b^q) \leq -\frac{1}{p}\ln(a^p) - \frac{1}{q}\ln(b^q) = -\ln(ab)$

and consequently $\ln(ab) \leq \ln(\frac{1}{p}a^p + \frac{1}{q}b^q)$

Since the function $f(x) = e^x$ non decreasing, we conclude that $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$

It is easy to check that inequality is fact true for all $a, b \in [0, \infty]$.

$$(3) \text{ Since } (\frac{a}{a+b})^p + (\frac{b}{a+b})^p \geq \frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1.$$

$$\Rightarrow \frac{a^p}{(a+b)^p} + \frac{b^p}{(a+b)^p} \geq 1 \Rightarrow \frac{a^p + b^p}{(a+b)^p} \geq 1 \Rightarrow (a+b)^p \leq a^p + b^p$$

(4) Let $u(t) = t^p$, then u is convex function on $(0, \infty)$

$$\text{If } a, b \in (0, \infty), \text{ since } p > 1, \text{ we have } (\frac{1}{2}a + \frac{1}{2}b)^p \leq \frac{1}{2}a^p + \frac{1}{2}b^p \Rightarrow (\frac{a+b}{2})^p \leq \frac{a^p + b^p}{2}$$

we conclude that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$

In fact, it is easy to check that inequality is fact true for all $a, b \in [0, \infty]$.



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Theorem (13.3)

Let $f, g \in L_+(\Omega, \mathcal{F}, \mu)$ and p, q be conjugate exponents. Define

$$A = \left(\int_{\Omega} f^p d\mu \right)^{\frac{1}{p}}, \quad B = \left(\int_{\Omega} g^q d\mu \right)^{\frac{1}{q}} \quad \text{and} \quad C = \int_{\Omega} fg d\mu$$

1. A, B and C are well defined elements of $[0, \infty]$
2. If $A = 0$ or $B = 0$, then $C \leq AB$
3. If $A = \infty$ or $B = \infty$, then $C \leq AB$
4. If $0 < A < \infty$ and $0 < B < \infty$. Define $F = \frac{f}{A}$ and $G = \frac{g}{B}$, then $\int_{\Omega} F^p d\mu = \int_{\Omega} G^q d\mu = 1$
5. For all $x \in \Omega$, we have $F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^q$

Proof :

$$(1) \quad f^p \text{ can be viewed as } f^p = u^p \circ f, \text{ where } u^p \text{ by } u^p(x) = \begin{cases} x^p, & x \in \mathbb{R}^+ \\ \infty, & x = \infty \end{cases}$$

We proved that u^p is a continuous function. It is therefore measurable with respect to the σ -field $\beta([0, \infty])$ on $[0, \infty]$. It follows that f^p is a measure function which is also nonnegative. Hence the integral $\int_{\Omega} f^p d\mu$ is a well defined elements of $[0, \infty]$ and

$A = \left(\int_{\Omega} f^p d\mu \right)^{\frac{1}{p}}$ is also well defined, being understood that $(\infty)^{\frac{1}{p}} = \infty$ elements of $[0, \infty]$

Similarly $B = \left(\int_{\Omega} g^q d\mu \right)^{\frac{1}{q}}$ is a well defined elements of $[0, \infty]$.

Finally, the function $fg \in L_+(\Omega, \mathcal{F}, \mu)$ and $C = \int_{\Omega} fg d\mu$ is a well defined elements of $[0, \infty]$.

(2) If $A = 0$. Then $\int_{\Omega} f^p d\mu = 0$, and since f^p is nonnegative, we see that $f^p = 0$ a.e., and



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Consequently $f = 0$ a.e. So $fg = 0$ a.e. and finally $C = \int_{\Omega} fg d\mu = 0$. So $C \leq AB$

Similarly, $B = 0$ implies $C = 0$, and therefore $C \leq AB$.

(3) since $A = \infty$. Then either $B = 0$ or $B > 0$. If $B = 0$, then $C \leq AB$ is true from (2). If

$B > 0$, then $AB = \infty$, and consequently $C \leq AB$. In any case, we see that $C \leq AB$.

Similarly, $B = \infty$ implies $C \leq AB$.

(4) Since $0 < A < \infty$, $\int_{\Omega} F^p d\mu = \int_{\Omega} \left(\frac{f}{A}\right)^p d\mu = \frac{1}{A^p} \int_{\Omega} f^p d\mu = \frac{A^p}{A^p} = 1$ and similarly $\int_{\Omega} G^q d\mu = 1$

(5) For all $x \in \Omega$, $F(x)$ and $G(x)$ are elements of $[0, \infty]$. From inequality $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$,

we have $F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^q$

Theorem (13.4) Holder's inequality

Let $f, g \in L_+(\Omega, \mathcal{F}, \mu)$ and P, q be conjugate exponents. Then

$$\int_{\Omega} fg d\mu \leq \left(\int_{\Omega} f^p d\mu\right)^{\frac{1}{p}} \left(\int_{\Omega} g^q d\mu\right)^{\frac{1}{q}}$$

Proof :

Define $A = \left(\int_{\Omega} f^p d\mu\right)^{\frac{1}{p}}$, $B = \left(\int_{\Omega} g^q d\mu\right)^{\frac{1}{q}}$ and $C = \int_{\Omega} fg d\mu$

Integration on both side of $F(x)G(x) \leq \frac{1}{p}(F(x))^p + \frac{1}{q}(G(x))^q$, we obtain

$$\int_{\Omega} FG d\mu \leq \frac{1}{p} \int_{\Omega} F^p d\mu + \frac{1}{q} \int_{\Omega} G^q d\mu$$

Since $\int_{\Omega} F^p d\mu = \int_{\Omega} G^q d\mu = 1$, then $\int_{\Omega} FG d\mu \leq \frac{1}{p} + \frac{1}{q} = 1$

Since $\int_{\Omega} FG d\mu = \int_{\Omega} \frac{f}{A} \times \frac{g}{B} d\mu = \frac{1}{AB} \int_{\Omega} fg d\mu = \frac{C}{AB} \Rightarrow \frac{C}{AB} \leq 1 \Rightarrow C \leq AB$



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We conclude that $\int_{\Omega} fg d\mu \leq (\int_{\Omega} f^p d\mu)^{\frac{1}{p}} (\int_{\Omega} g^q d\mu)^{\frac{1}{q}}$

Corollary (13.5) Cauchy-Schwarz Inequality

Let $f, g \in L_+(\Omega, \mathcal{F}, \mu)$ Then

$$\int_{\Omega} fg d\mu \leq (\int_{\Omega} f^2 d\mu)^{\frac{1}{2}} (\int_{\Omega} g^2 d\mu)^{\frac{1}{2}}$$

Proof :

Take $p=2 \Rightarrow q=2$. By using Holder's inequality, we have

$$\int_{\Omega} fg d\mu \leq (\int_{\Omega} f^2 d\mu)^{\frac{1}{2}} (\int_{\Omega} g^2 d\mu)^{\frac{1}{2}}$$

Theorem (13.6)

Let $f, g \in L_+(\Omega, \mathcal{F}, \mu)$ and $1 < P < \infty$. Define

$$A = (\int_{\Omega} f^p d\mu)^{\frac{1}{p}}, B = (\int_{\Omega} g^p d\mu)^{\frac{1}{p}} \text{ and } C = (\int_{\Omega} (f+g)^p d\mu)^{\frac{1}{p}}$$

1. A, B and C are well defined elements of $[0, \infty]$
2. If $A = \infty$ or $B = \infty$ or $C = 0$, then $C \leq A + B$
3. If $A < \infty$ and $B < \infty$, then $C < \infty$
4. If $0 \leq A < \infty$, $0 \leq B < \infty$ and $0 < C < \infty$, then there exists $q \in \mathbb{R}^+$ such that P, q are conjugate exponents

$$5. \int_{\Omega} f(f+g)^{p-1} d\mu \leq AC^{\frac{p}{q}} \text{ and } \int_{\Omega} g(f+g)^{p-1} d\mu \leq BC^{\frac{p}{q}}$$

$$6. \int_{\Omega} (f+g)^p d\mu \leq C^{\frac{p}{q}} (A+B)$$

Proof :

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(1) f^p , g^p and $(f + g)^p$ are all nonnegative measurable functions.

All three integrals $\int_{\Omega} f^p d\mu$, $\int_{\Omega} g^p d\mu$ and $\int_{\Omega} (f + g)^p d\mu$ are therefore well defined.

It follows that A, B and C are well defined elements of $[0, \infty]$.

(2) If $A = \infty$ or $B = \infty$, then $A + B = \infty$, and $C \leq A + B$. If $C = 0$, then clearly $C \leq A + B$.

(3) using $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, for all $x \in \Omega$, we have

$$(f(x) + g(x))^p \leq 2^{p-1}((f(x))^p + (g(x))^p)$$

Integrating on both side of the inequality, we obtain

$$\int_{\Omega} (f + g)^p d\mu \leq 2^{p-1}(\int_{\Omega} f^p d\mu + \int_{\Omega} g^p d\mu)$$

Since $A < \infty$ and $B < \infty$, then both integrals $\int_{\Omega} f^p d\mu$ and $\int_{\Omega} g^p d\mu$ are finite, then see that $\int_{\Omega} (f + g)^p d\mu$ is itself finite. So $C < \infty$.

(4) Take $q = \frac{p}{p-1}$. Since $1 < p < \infty$, q is a well defined element of $\mathbb{R}^+$, and $\frac{1}{p} + \frac{1}{q} = 1$

(5) using Holder's inequality, since $q(p-1) = p$, we have

$$\int_{\Omega} f (f + g)^{p-1} d\mu \leq (\int_{\Omega} f^p d\mu)^{\frac{1}{p}} (\int_{\Omega} (f + g)^{q(p-1)} d\mu)^{\frac{1}{q}} = AC^{\frac{p}{q}}$$

and

$$\int_{\Omega} g (f + g)^{p-1} d\mu \leq (\int_{\Omega} g^p d\mu)^{\frac{1}{p}} (\int_{\Omega} (f + g)^{q(p-1)} d\mu)^{\frac{1}{q}} = AC^{\frac{p}{q}}$$

(6) Since $\int_{\Omega} (f + g)^p d\mu = \int_{\Omega} f (f + g)^{p-1} d\mu + \int_{\Omega} g (f + g)^{p-1} d\mu$

And using (5), we obtain $\int_{\Omega} (f + g)^p d\mu \leq C^{\frac{p}{q}} (A + B)$

Theorem (13.7) Minkokowsk's Inequality

Let $f, g \in L_+(\Omega, \mathcal{F}, \mu)$ and $1 \leq p < \infty$. Then $\int_{\Omega} (f + g)^{\frac{1}{p}} d\mu \leq (\int_{\Omega} f^p d\mu)^{\frac{1}{p}} + (\int_{\Omega} g^p d\mu)^{\frac{1}{p}}$

Proof :

Define $A = (\int_{\Omega} f^p d\mu)^{\frac{1}{p}}$, $B = (\int_{\Omega} g^p d\mu)^{\frac{1}{p}}$ and $C = (\int_{\Omega} (f + g)^p d\mu)^{\frac{1}{p}}$

Since $\int_{\Omega} (f + g)^p d\mu \leq C^{\frac{p}{q}} (A + B)$, we have $C^p \leq C^{\frac{p}{q}} (A + B)$

Having assumed in (4) of theorem(13.7), that $C \in (0, \infty)$, we can divide both of this

inequality by $C^{\frac{p}{q}}$, to obtain $C^{p-\frac{p}{q}} \leq A + B$

Since $p - \frac{p}{q} = 1$, we conclude that $C \leq A + B$. Hence

$$\int_{\Omega} (f + g)^{\frac{1}{p}} d\mu \leq (\int_{\Omega} f^p d\mu)^{\frac{1}{p}} + (\int_{\Omega} g^p d\mu)^{\frac{1}{p}}$$

Definition(13.8)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $1 \leq p < \infty$. We define the space

$L^p = L^p(\Omega, \mathcal{F}, \mu)$ as the family of all real valued measurable functions f such

that $\int_{\Omega} |f|^p d\mu < \infty$. That is $L^p = L^p(\Omega, \mathcal{F}, \mu) = \{f \in L(\Omega, \mathcal{F}, \mu) : \int_{\Omega} |f|^p d\mu < \infty\}$

For all $f \in L^p(\Omega, \mathcal{F}, \mu)$, we put $\|f\|_p = (\int_{\Omega} |f|^p d\mu)^{\frac{1}{p}}$

Theorem(13.9)

$L^p(\Omega, \mathcal{F}, \mu)$ is a sublinear space of $L(\Omega, \mathcal{F}, \mu)$ be a measure space, where $1 \leq p < \infty$, .i.e.

If $f, g \in L^p(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in L^p(\Omega, \mathcal{F}, \mu)$



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Proof :

Let $f, g \in L^p(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$,

$\Rightarrow f, g$ are measurable functions $\Rightarrow \alpha f + \beta g$ is measurable function

Moreover, we have $|\alpha f + \beta g| \leq |\alpha| |f| + |\beta| |g|$

Since $p \geq 1$, the function $u(x) = x^p$ is non decreasing on $\mathbb{R}^+$, so

$$|\alpha f + \beta g|^p \leq (|\alpha| |f| + |\beta| |g|)^p$$

Hence, we see $\int_{\Omega} |\alpha f + \beta g|^p d\mu \leq \int_{\Omega} (|\alpha| |f| + |\beta| |g|)^p d\mu$

However, using Minkokowsk's Inequality, we have

$$\left(\int_{\Omega} (|\alpha| |f| + |\beta| |g|)^p d\mu \right)^{\frac{1}{p}} \leq |\alpha| \left(\int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} + |\beta| \left(\int_{\Omega} |g|^p d\mu \right)^{\frac{1}{p}}$$

We conclude that $\int_{\Omega} |\alpha f + \beta g|^p d\mu < \infty$. So $\alpha f + \beta g \in L^p(\Omega, \mathcal{F}, \mu)$

Theorem(13.10)

$\| \cdot \|_p$ is a seminorm on $L^p(\Omega, \mathcal{F}, \mu)$ where $1 \leq p < \infty$. i.e.

1. $\|f\|_p \geq 0$ for all $f \in L^p(\Omega, \mathcal{F}, \mu)$
2. If $f = 0$, then $\|f\|_p = 0$
3. $\|\lambda f\|_p = |\lambda| \|f\|_p$ for all $f \in L^p(\Omega, \mathcal{F}, \mu)$ and all $\lambda \in \mathbb{R}$
4. $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ for all $f, g \in L^p(\Omega, \mathcal{F}, \mu)$

Proof :

(1) It follows from the definition of $\| \cdot \|_p$

(2) Suppose $f = 0$, then $|f|^p = 0$, so $\int_{\Omega} |f|^p d\mu = 0$ and finally $\|f\|_p = 0$.

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$$(3) \|\lambda f\|_p = \left(\int_{\Omega} |\lambda f|^p d\mu \right)^{\frac{1}{p}} = |\lambda| \left(\int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} = |\lambda| \|f\|_p$$

(4) Using $|f + g|^p \leq (|f| + |g|)^p$ and Minkokowsk's Inequality, we have

$$\left(\int_{\Omega} (|f| + |g|)^p d\mu \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} + \left(\int_{\Omega} |g|^p d\mu \right)^{\frac{1}{p}}$$

We see that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

Remarks

1. If $\|f\|_p = 0$, then $f = 0$ a.e.

$$\text{Indeed } \|f\|_p = 0 \Rightarrow \int_{\Omega} |f|^p d\mu = 0$$

Since $|f|^p$ is nonnegative, $|f|^p = 0$ a.e., and consequently $f = 0$ a.e.

2. The function $d : L^p(\Omega, \mathcal{F}, \mu) \times L^p(\Omega, \mathcal{F}, \mu) \rightarrow \mathbb{R}$ by $d(f, g) = \|f - g\|_p$ for all $f, g \in L^p(\Omega, \mathcal{F}, \mu)$

is not a metric function

indeed $\|f - g\|_p = 0$ implies that $f = g$ a.e., and not necessarily $f = g$.

We can change $\|\cdot\|_p$ into a norm by passing to equivalence as follows.

If $f, g \in L^p(\Omega, \mathcal{F}, \mu)$, define $f \sim g$ iff $f = g$ a.e. $\Rightarrow \|f\|_p = \|g\|_p$. Then $\|f\|_p$ is the same for all f in given equivalence class. Thus $L^p(\Omega, \mathcal{F}, \mu)$ is the family of equivalence class, $L^p(\Omega, \mathcal{F}, \mu)$ becomes a linear space, and $\|\cdot\|_p$ is a seminorm on $L^p(\Omega, \mathcal{F}, \mu)$. In fact $\|\cdot\|_p$ is a norm, since $\|f\|_p = 0$ implies $f = 0$ a.e.

Since every normed space is a metric space, so $L^p(\Omega, \mathcal{F}, \mu)$ is a metric space.

Remark

Recall that, if $D = \{\alpha \in \mathbb{R}^+ : \mu\{x \in \Omega : |f(x)| > \alpha\} = 0\} = \{\alpha \in \mathbb{R}^+ : |f| \leq \alpha \text{ a.e. } [\mu]\}$. Define

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$$\|f\|_{\infty} = \begin{cases} \infty, & D = \emptyset \\ \inf D, & D \neq \emptyset \end{cases}$$

Suppose $\|f\|_{\infty} < \infty$.

Since $f^{-1}((\|f\|_{\infty}, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}((\|f\|_{\infty} + \frac{1}{n}, \infty])$ and $\mu(f^{-1}((\|f\|_{\infty} + \frac{1}{n}, \infty]) = 0$, we see $\|f\|_{\infty} \in D$.

The quantity $\|f\|_{\infty}$ is called the essential supremum of $|f|$. The essential supremum of f is denoted by $\text{ess sup } f$ and defined as:

$$\text{ess sup } f = \inf\{\lambda \in \mathbb{R} : f \leq \lambda \text{ a.e.}\} = \inf\{\lambda \in \mathbb{R} : \mu\{x \in \Omega : f(x) > \lambda\} = 0\}, \quad \|f\|_{\infty} = \text{ess sup } |f|.$$

For example, if the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$, then

$$\sup |f| = 1, \text{ but } \text{ess sup } |f| = 0$$

Definition(13.11)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f \in L(\Omega, \mathcal{F}, \mu)$. We define the space $L^{\infty} = L^{\infty}(\Omega, \mathcal{F}, \mu)$ as the family of all real valued measurable functions f such that $\|f\|_{\infty} < \infty$. That is $L^{\infty} = L^{\infty}(\Omega, \mathcal{F}, \mu) = \{f \in L(\Omega, \mathcal{F}, \mu) : \|f\|_{\infty} < \infty\}$

Theorem(13.12)

1. If $f \in L^{\infty}(\Omega, \mathcal{F}, \mu)$, then $|f| \leq \|f\|_{\infty}$ a.e.
2. $L^{\infty}(\Omega, \mathcal{F}, \mu)$ is a sublinear space of $L(\Omega, \mathcal{F}, \mu)$ be a measure space, i.e.

If $f, g \in L^{\infty}(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in L^{\infty}(\Omega, \mathcal{F}, \mu)$

Proof:

(1) Since $\|f\|_{\infty} < \infty$, for all $n \geq 1$, we have $\|f\|_{\infty} < \|f\|_{\infty} + \frac{1}{n}$

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$\|f\|_\infty$ being the greatest lower bound of all almost everywhere upper bounds of $|f|$,

$\|f\|_\infty + \frac{1}{n}$ cannot be such lower bound.

There exists $\lambda \in \mathbb{R}^+$, such that $|f| \leq \lambda$ a.e., and $\lambda < \|f\|_\infty + \frac{1}{n}$

In particular, $|f| < \|f\|_\infty + \frac{1}{n}$ a.e. .

Let A_n be the set defined by $A_n = \{\|f\|_\infty + \frac{1}{n} \leq |f|\}$. Then $A_n \in \mathcal{F}$ and $\mu(A_n) = 0$.

Moreover, $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = \{|f| < \|f\|_\infty\}$

It follows that $A_n \uparrow \{|f| < \|f\|_\infty\}$ and we see that $\mu\{|f| < \|f\|_\infty\} = \lim_{n \rightarrow \infty} \mu(A_n) = 0$

We conclude that $|f| \leq \|f\|_\infty$ a.e.

(2) Let $f, g \in L^\infty(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbb{R}$,

$\Rightarrow f, g$ are measurable functions $\Rightarrow \alpha f + \beta g$ is measurable function

Moreover, we have $|\alpha f + \beta g| \leq |\alpha| |f| + |\beta| |g| \leq |\alpha| \|f\|_\infty + |\beta| \|g\|_\infty$ a.e.

It follows that $\|\alpha f + \beta g\|_\infty \leq |\alpha| \|f\|_\infty + |\beta| \|g\|_\infty < \infty$. We conclude that $\alpha f + \beta g \in L^\infty(\Omega, \mathcal{F}, \mu)$

Theorem(13.13)

$\|\cdot\|_\infty$ is a seminorm on $L^\infty(\Omega, \mathcal{F}, \mu)$, i.e.

1. $\|f\|_\infty \geq 0$ for all $f \in L^\infty(\Omega, \mathcal{F}, \mu)$
2. If $f = 0$, then $\|f\|_\infty = 0$
3. $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$ for all $f \in L^\infty(\Omega, \mathcal{F}, \mu)$ and all $\lambda \in \mathbb{R}$
4. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in L^\infty(\Omega, \mathcal{F}, \mu)$

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Proof :

(1) Since $\|f\|_\infty$ is an infimum of subset of $\mathbb{R}^+$, it is either ∞ (when such subset is empty), or lies in $\mathbb{R}^+$. So $\|f\|_\infty \geq 0$

(2) Suppose $f = 0$, then $|f| \leq 0$, and 0 is therefore almost everywhere bound of $|f|$, so $\|f\|_\infty \leq 0$

Since $\|f\|_\infty \geq 0$, we have $\|f\|_\infty = 0$.

(3) We have $|\lambda f| \leq |\lambda| \|f\|_\infty$ a.e. and hence $\|\lambda f\|_\infty \leq |\lambda| \|f\|_\infty$.

If $\lambda \neq 0$, we have $\|f\|_\infty = \left\| \frac{1}{\lambda} (\lambda f) \right\|_\infty \leq \frac{1}{|\lambda|} \|\lambda f\|_\infty$

It follows that $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$ (also true if $\lambda = 0$)

(4) since $|f| \leq \|f\|_\infty$ a.e. and $|g| \leq \|g\|_\infty$ a.e., then $|f| + |g| \leq \|f\|_\infty + \|g\|_\infty$ a.e.

Since $|f + g| \leq |f| + |g|$, we have $|f + g| \leq \|f\|_\infty + \|g\|_\infty$ a.e.

Hence $\|f\|_\infty + \|g\|_\infty$ is almost everywhere upper bound of $|f + g|$.

$\|f + g\|_\infty$ being a lower bound of all such upper bounds, we have $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Remarks

1. If $\|f\|_\infty = 0$, then $f = 0$ a.e.

Indeed $\|f\|_\infty = 0 \Rightarrow |f| \leq 0$ a.e. Consequently $f = 0$ a.e.

2. The function $d : L^\infty(\Omega, \mathcal{F}, \mu) \times L^\infty(\Omega, \mathcal{F}, \mu) \rightarrow \mathbb{R}$ by $d(f, g) = \|f - g\|_\infty$ for all $f, g \in L^\infty(\Omega, \mathcal{F}, \mu)$ is not a metric function

indeed $\|f - g\|_\infty = 0$ implies that $f = g$ a.e., and not necessarily $f = g$.

We can change $\|\cdot\|_\infty$ into a norm by passing to equivalence as follows.



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If $f, g \in L^\infty(\Omega, \mathcal{F}, \mu)$, define $f \sim g$ iff $f = g$ a.e. $\Rightarrow \|f\|_\infty = \|g\|_\infty$. Then $\|f\|_\infty$ is the same for all f in given equivalence class. Thus $L^\infty(\Omega, \mathcal{F}, \mu)$ is the family of equivalence class, $L^\infty(\Omega, \mathcal{F}, \mu)$ becomes a linear space, and $\|\cdot\|_\infty$ is a seminorm on $L^\infty(\Omega, \mathcal{F}, \mu)$. In fact $\|\cdot\|_\infty$ is a norm, since $\|f\|_\infty = 0$ implies $f = 0$ a.e.

Since every normed space is a metric space, so $L^\infty(\Omega, \mathcal{F}, \mu)$ is a metric space.

Definition(13.14)

Let $1 \leq p \leq \infty$. For all $r > 0$ and $f_0 \in L^p(\Omega, \mathcal{F}, \mu)$, we define the so-called open ball in $L^p(\Omega, \mathcal{F}, \mu)$:

$$\beta_r(f_0) = \{f \in L^p(\Omega, \mathcal{F}, \mu) : \|f - f_0\|_p < r\}$$

We call usual topology in $L^p(\Omega, \mathcal{F}, \mu)$, the set τ defined by

$$\tau = \{G \subseteq L^p(\Omega, \mathcal{F}, \mu) : \forall f \in G, \exists r > 0, \beta_r(f) \subseteq G\}$$

Note that if $d(f, g) = \|f - g\|_p$, was a metric, the usual topology in $L^p(\Omega, \mathcal{F}, \mu)$, would be noting but the metric topology.

Example(13.15)

Let $1 \leq p \leq \infty$. Suppose there exists $A \in \mathcal{F}$ with $\mu(A) = 0$ and $A \neq \emptyset$. Put $f = I_A$ and $g = 0$. Then

1. $f, g \in L^p(\Omega, \mathcal{F}, \mu)$ and $f \neq g$
2. Any open set containing f also contains g
3. $L^p(\Omega, \mathcal{F}, \mu)$ is not Hausdorff

Proof :

(1) Since $A \neq \emptyset$, $I_A \neq 0$, so $f \neq g$

Since $A \in \mathcal{F}$, the function $f = I_A$ is measurable. Moreover, since $\mu(A) = 0$, $\|f\|_p = 0 < \infty$,

and we see that $f \in L^p(\Omega, \mathcal{F}, \mu)$

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since $g = 0$, it is measurable function and $\|g\|_p = 0 < \infty$, so $g \in L^p(\Omega, \mathcal{F}, \mu)$

(2) Let G be open in $L^p(\Omega, \mathcal{F}, \mu)$ such that $f \in G$. By definition of open set, there exists

$r > 0$ such that $\beta_r(f) \subseteq G$. However, $\|f - g\|_p = \|f\|_p = 0$. So in particular $\|f - g\|_p < r$.

So $g \in \beta_r(f)$ and finally $g \in G$

(3) Since $f \neq g$, there would exist G_1, G_2 open sets in $L^p(\Omega, \mathcal{F}, \mu)$ such that $f \in G_1$, $g \in G_2$

and $G_1 \cap G_2 = \emptyset$. However from (2), this is impossible, as g would always be an

element of G_1 as well as G_2 . We conclude that $L^p(\Omega, \mathcal{F}, \mu)$ is not Hausdorff

Definition(13.16)

A sequence $\{f_n\}$ in $L^p(\Omega, \mathcal{F}, \mu)$ is said to be

1. convergent in $L^p(\Omega, \mathcal{F}, \mu)$ if there is $f \in L^p(\Omega, \mathcal{F}, \mu)$ such that $\|f_n - f\|_p \rightarrow 0$ as $n \rightarrow \infty$.

and we write $f_n \xrightarrow{L^p} f$

2. Cauchy in $L^p(\Omega, \mathcal{F}, \mu)$, if $\|f_n - f_m\|_p \rightarrow 0$ as $n, m \rightarrow \infty$.

Theorem(13.17)

Let $f, g, f_n \in L^p(\Omega, \mathcal{F}, \mu)$, and $1 \leq p \leq \infty$

1. If $f_n \xrightarrow{L^p} f$, then $\{f_n\}$ is a Cauchy in $L^p(\Omega, \mathcal{F}, \mu)$

2. If $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$, then $f = g$ a.e.

Proof :

(1) since $f_n \xrightarrow{L^p} f$ then $\|f_n - f\|_p \rightarrow 0$

Since $\|f_n - f_m\|_p \leq \|f_n - f\|_p + \|f_m - f\|_p$

It follows that we have $\|f_n - f_m\|_p \rightarrow 0$. We conclude that $\{f_n\}$ is a Cauchy in $L^p(\Omega, \mathcal{F}, \mu)$.

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(2) Since $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$, then $\|f_n - f\|_p \rightarrow 0$ and $\|f_n - g\|_p \rightarrow 0$

Since $\|f - g\|_p \leq \|f_n - f\|_p + \|f_n - g\|_p$ for all $n \geq 1$. It follows that we have $\|f - g\|_p = 0$. We conclude that $f = g$ a.e..

Example(13.18)

Let $1 \leq p \leq \infty$. Suppose there exists $A \in \mathcal{F}$ with $\mu(A) = 0$ and $A \neq \emptyset$. Find $f, g, f_n \in L^p(\Omega, \mathcal{F}, \mu)$ such that $f \neq g$ and $f_n \xrightarrow{L^p} f$, $f_n \xrightarrow{L^p} g$.

Proof :

Take $f_n = I_A = f$ for all $n \geq 1$ and $g = 0$

Then f_n, f and g are all elements of $L^p(\Omega, \mathcal{F}, \mu)$, and $f \neq g$.

Moreover, for all $n \geq 1$, we have $\|f_n - f\|_p = \|f_n - g\|_p = 0$. So $f_n \xrightarrow{L^p} f$ and $f_n \xrightarrow{L^p} g$.

Theorem(13.19)

Let $f, f_n \in L^p(\Omega, \mathcal{F}, \mu)$, and $0 < p < \infty$. If $f_n \xrightarrow{f-m} f$, then $f_n \xrightarrow{\mu} f$.

Proof :

Chebyshev's inequality $\mu\{|f_n - f| \geq \varepsilon\} \leq \int_{\Omega} |f_n - f| d\mu$ and the result follows.

Example(13.20)

1. Let $\Omega = [0, 1]$, with the Lebesgue measure. Define $f : \Omega \rightarrow \mathbb{R}$, by

$$f_n(x) = \begin{cases} e^x, & 0 \leq x \leq \frac{1}{n} \\ 0, & \text{else} \end{cases}$$

Then $f_n \xrightarrow{a.e.} 0$, hence $f_n \xrightarrow{\mu} 0$. But for each $0 < p \leq \infty$, f_n fails to converge in L^p .

To see this .



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For $p < \infty$, we have $\|f_n\|_p^p = \int_{\Omega} |f_n|^p d\mu = \int_0^1 |f_n(x)|^p dx = \frac{1}{n} e^{np} \rightarrow \infty$

And $\|f\|_{\infty} = e^n \rightarrow \infty$

2. Let $\Omega = \mathbb{R}$, with the Lebesgue measure. Define $f : \Omega \rightarrow \mathbb{R}$, by

$$f_n(x) = \begin{cases} \frac{1}{n}, & 0 \leq x \leq e^x \\ 0, & \text{else} \end{cases}$$

Then $f_n \xrightarrow{u} 0$, so that $f_n \xrightarrow{L^{\infty}} 0$. It follows quickly that $f_n \xrightarrow{a.e.} 0$ and $f_n \xrightarrow{\mu} 0$

But for each $0 < p < \infty$, f_n fails to converge in L^p .

To see this. $\|f_n\|_p^p = \int_{\Omega} |f_n|^p d\mu = \int_0^1 |f_n(x)|^p dx = n^{-p} e^n \rightarrow \infty$

Theorem(13.21)

Let $\{f_n\}$ be a sequence in $L^p(\Omega, \mathcal{F}, \mu)$, $0 < p < \infty$ satisfying $\|f_n - f_{n+1}\|_p < (\frac{1}{4})^n$, $n = 1, 2, \dots$.

Then $\{f_n\}$ converges a.e.

Proof :

Set $A_n = \{x \in \Omega : |f_n(x) - f_{n+1}(x)| > 2^{-n}\}$. By Chebyshev's inequality, we have

$$\mu(A_n) \leq 2^{n_p} \int_{\Omega} |f_n - f_{n+1}|^p d\mu \leq \left(\frac{1}{4}\right)^{n_p} \times 2^{n_p} = 2^{-n_p} \quad . \text{ This show that } \sum_{n=1}^{\infty} \mu(A_n) < \infty$$

By Borel Cantelli lemma, we have $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$

But if $x \notin \limsup_{n \rightarrow \infty} A_n$, then $|f_n(x) - f_{n+1}(x)| < 2^{-n}$ for large n , so $\{f_n(x)\}$ is Cauchy sequence in $\mathbb{R}$ and hence $\{f_n(x)\}$ converges.

Theorem(13.22)

$L^p(\Omega, \mathcal{F}, \mu)$, where $1 \leq p < \infty$ is a Banach space

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Proof :

Since $L^p(\Omega, \mathcal{F}, \mu)$ is a normed space. We need to show that $L^p(\Omega, \mathcal{F}, \mu)$ is complete

Let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega, \mathcal{F}, \mu)$, where $1 \leq p < \infty$

Given any $\varepsilon > 0$, there is $N \in \mathbb{Z}^+$ such that $\|f_n - g_m\|_p < \varepsilon$ for all $n, m > N$

The for each $k = 1, 2, \dots$ there is a n_k such that $\|f_n - f_m\|_p < (\frac{1}{4})^n$ for all $n, m > n_k$

By theorem(13.21), we have $f_{n_k}(x) \rightarrow f$ a.e. . We need to show that $f \in L^p(\Omega, \mathcal{F}, \mu)$ and that it is the $L^p(\Omega, \mathcal{F}, \mu)$ limit of $\{f_n\}$. Let $\varepsilon > 0$. Take N so large that $\|f_n - g_m\|_p < \varepsilon$ for all $n, m > N$

Fix such an m . Then by the pointwise convergence of the subsequence and by Fatou's lemma we have

$$\int_{\Omega} |f - f_m|^p d\mu = \int_{\Omega} \lim_{k \rightarrow \infty} |f_{n_k} - f_m|^p d\mu \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |f_{n_k} - f_m|^p d\mu < \varepsilon^p$$

Therefore $f_n \rightarrow f$ in $L^p(\Omega, \mathcal{F}, \mu)$ and $\int_{\Omega} |f - f_m| d\mu < \infty$ for m sufficiently large. But then,

$\|f\|_p = \|f_m - f_m - f\|_p \leq \|f_m\|_p + \|f_m - f\|_p$, which shows that $f \in L^p(\Omega, \mathcal{F}, \mu)$.

Remark

Let Ω is an arbitrary set, $\mathcal{F}$ consists of all subsets of Ω , μ is counting measure, (i.e. if A has n numbers, $n = 0, 1, 2, \dots$, then $\mu(A) = n$; if A is an infinite set, then $\mu(A) = \infty$).

If $f = (f(\alpha): \alpha \in \Omega)$ is a nonnegative real-valued function on Ω , then

$$\int_{\Omega} f d\mu = \sum_{\alpha} f(\alpha) \quad (1)$$



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where the series is defined as $\sup \left\{ \sum_{\alpha} f(\alpha) : A \subset \Omega, A \text{ is finite} \right\}$. If $f(\alpha) > 0$ for

uncountably many α , then for some $\delta > 0$ we have $f(\alpha) \geq \delta$ for infinitely many α , so that $\sum_{\alpha} f(\alpha) = \infty$.

- If the nonnegative hypothesis is dropped, we apply the above results to f^+ and f^- to again obtain equation (1), where the series is interpreted as

$$\sum_{\alpha} f^+(\alpha) - \sum_{\alpha} f^-(\alpha)$$

- The space $L^p(\Omega, \mathcal{F}, \mu)$ will be denoted by $\ell^p(\Omega)$, it consists of all real-valued functions $f = (f(\alpha) : \alpha \in \Omega)$ such that $f(\alpha) = 0$ for all but countably many α and

$$\|f\|_p = \left(\sum_{n=1}^{\infty} |f(\alpha)|^p \right)^{\frac{1}{p}} < \infty$$

- If Ω is the set of positive integers, the space $\ell^p(\Omega)$ will be denoted simply by ℓ^p , it consists of all sequences $f = \{x_n\}$ such that

$$\|f\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty$$

It will be useful to state the Holder and Minkowski inequalities for sums

- If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where p and q are conjugate exponent, $1 \leq p < \infty$, then

$$fg \in L^1(\Omega) \quad \text{and} \quad \sum_{\alpha} |f(\alpha)g(\alpha)| \leq \left(\sum_{\alpha} |f(\alpha)|^p \right)^{\frac{1}{p}} \left(\sum_{\alpha} |g(\alpha)|^q \right)^{\frac{1}{q}}$$

- If $f, g \in L^p(\Omega)$, $1 \leq p < \infty$, then $f + g \in L^p(\Omega)$ and

$$\left(\sum_{\alpha} |f(\alpha) + g(\alpha)|^p \right)^{\frac{1}{p}} \leq \left(\sum_{\alpha} |f(\alpha)|^p \right)^{\frac{1}{p}} + \left(\sum_{\alpha} |g(\alpha)|^p \right)^{\frac{1}{p}}$$



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Theorem(13.23)

1. A sequence $\{f_n\}$ in $L^\infty(\Omega, \mathcal{F}, \mu)$ converges to $f \in L^\infty(\Omega, \mathcal{F}, \mu)$ iff there is a set $A \in \mathcal{F}$ with $\mu(A) = 0$ such that $f_n \xrightarrow{u} f$ on A^c .
3. A sequence $\{f_n\}$ is Cauchy in $L^\infty(\Omega, \mathcal{F}, \mu)$ iff there is a set $A \in \mathcal{F}$ with $\mu(A) = 0$ such that $\{f_n\}$ is uniformly Cauchy on A^c .

Proof :

Suppose that $f_n \rightarrow f$ in $L^\infty(\Omega, \mathcal{F}, \mu) \Rightarrow \|f_n - f\|_\infty \rightarrow 0$

Then for each $k > 1$ there is an $n > n_k$ sufficiently large so that $\|f_n - f\|_\infty < \frac{1}{k}$. Thus, there

is a set A_k so such that $\mu(A_k) = 0$ and $|f_n(x) - f(x)| < \frac{1}{k}$ for every $x \in A_k^c$. Let $A = \bigcup_{k=1}^\infty A_k$.

Then $\mu(A) = 0$ and $f_n \rightarrow f$ uniformly on A^c .

Conversely : Suppose $f_n \rightarrow f$ uniformly on A^c and $\mu(A) = 0$.

Then given $\varepsilon > 0$ there is an N such that for all $n > N$ and $x \in A^c$, $|f_n(x) - f(x)| < \varepsilon$. This is the same as saying that $\|f_n - f\|_\infty < \varepsilon$ for all $n > N$.

Theorem(13.24)

$L^\infty(\Omega, \mathcal{F}, \mu)$ is a Banach space

Proof :

Since $L^\infty(\Omega, \mathcal{F}, \mu)$ is a normed space. We need to show that $L^\infty(\Omega, \mathcal{F}, \mu)$ is complete

Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega, \mathcal{F}, \mu)$, by theorem(12), there is a set $A \in \mathcal{F}$ with $\mu(A) = 0$ such that $\{f_n\}$ is uniformly Cauchy on A^c . That is, given any $\varepsilon > 0$, there is

$N \in \mathbb{Z}^+$ such that for all $n, m > N$ and all $x \in A^c$, $|f_n(x) - f_m(x)| < \varepsilon$.



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Therefore the sequence $\{f_n\}$ converges uniformly on A^c to a function f . Define $f(x) = 0$ for $x \in A$. Then $\{f_n\}$ converges to f in $L^\infty(\Omega, \mathcal{F}, \mu)$ and $f \in L^\infty(\Omega, \mathcal{F}, \mu)$

Remark

If Ω is an arbitrary set, $\mathcal{F}$ consists of all subsets of Ω , μ is counting measure, then $L^\infty(\Omega, \mathcal{F}, \mu)$ is the set of all bounded real-valued functions $f = (f(\alpha) : \alpha \in \Omega)$, denoted by $\ell^\infty(\Omega)$. The essential supremum is simply the supremum; in other words, $\|f\|_\infty = \sup\{|f(\alpha)| : \alpha \in \Omega\}$ is the set of positive integers, $\ell^\infty(\Omega)$ is the space of bounded sequences of real numbers denoted by ℓ^∞

Definition(13.25)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $0 < p < \infty$. We define the space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ as the family of all real valued measurable functions f such that $\int_\Omega |f|^p d\mu < \infty$. That is

$$L^p = L^p(\Omega, \mathcal{F}, \mu) = \{f \in L(\Omega, \mathcal{F}, \mu) : \int_\Omega |f|^p d\mu < \infty\}$$

For all $f \in L^p(\Omega, \mathcal{F}, \mu)$, we put $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$

Remark

If $0 < p < \infty$, $\|\cdot\|_p$ is not a seminorm on $L^p(\Omega, \mathcal{F}, \mu)$. For let A and B be disjoint sets with $\alpha = \mu(A)$, $\beta = \mu(B)$ assumed finite and positive. If $f = I_A$, $g = I_B$

$$\Rightarrow \|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega I_A^p d\mu\right)^{\frac{1}{p}} = (\mu(A))^{\frac{1}{p}} = \alpha^{\frac{1}{p}} \text{ similarly we have } \|g\|_p = \beta^{\frac{1}{p}}$$

Since $A \cap B = \emptyset$, then $I_{A \cup B} = I_A + I_B \Rightarrow f + g = I_A + I_B = I_{A \cup B}$

$$\Rightarrow \|f + g\|_p = \left(\int_\Omega |f + g|^p d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega (I_A + I_B)^p d\mu\right)^{\frac{1}{p}} = (\mu(A) + \mu(B))^{\frac{1}{p}} = (\alpha + \beta)^{\frac{1}{p}}$$



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Since $(\alpha + \beta)^{\frac{1}{p}} > \alpha^{\frac{1}{p}} + \beta^{\frac{1}{p}}$ when $0 < p < 1 \Rightarrow \|f + g\|_p > \|f\|_p + \|g\|_p$

Exercises(13)

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let f be a measurable function with the property that $\|f\|_2 = 1$ and $\|f\|_1 = \frac{1}{2}$. Prove that $\frac{1}{4}(1 - \eta)^2 \leq P\{x \in \Omega : |f(x)| \geq \frac{\eta}{2}\}$

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $f \in L^1(\Omega, \mathcal{F}, P)$ in nonnegative.

Prove that

$$\sqrt{1 + \|f\|_1^2} \leq \int_{\Omega} \sqrt{1 + f^2} dP \leq 1 + \|f\|_1$$

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f_1, f_2, \dots, f_n$ be measurable functions.

Suppose $1 < p < \infty$. Prove that

$$\int_{\Omega} \left| \frac{1}{n} \sum_{i=1}^n f_i(x) \right|^p d\mu(x) \leq \frac{1}{n} \int_{\Omega} \frac{1}{n} \sum_{i=1}^n |f_i(x)|^p d\mu(x) \text{ and } \int_{\Omega} \left| \frac{1}{n} \sum_{i=1}^n f_i(x) \right|^p d\mu(x) \leq \left(\frac{1}{n} \sum_{i=1}^n \|f_i\|_p \right)^p$$

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions.

Satisfying $\|f_n\|_p \leq n^{\frac{1}{p}}$ for $2 < p < \infty$. Prove that the sequence $\{\frac{1}{n}f_n\}$ converges to zero almost everywhere.

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $f \in L^{\infty}(\Omega, \mathcal{F}, P)$ and $\|f\|_{\infty} > 0$. Prove that

$$\lim_{n \rightarrow \infty} \left(\frac{\int_{\Omega} |f|^{n+1} dP}{\int_{\Omega} |f|^n dP} \right) = \|f\|_{\infty}$$

6. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose $f \in L^1(\Omega, \mathcal{F}, P)$. Prove that



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$$\lim_{p \rightarrow 0} (\|f\|_p) = \exp\left\{\int_{\Omega} \log |f| dP\right\}$$

Where $\exp(-\infty)$ is defined to be zero.

7. Suppose $f \in L^1(\Omega, \mathcal{F}, \mu)$. Prove that $\int_{\{|f|>n\}} d\mu = 0$

8. Let $\Omega = [0, 1]$ with its Lebesgue measure. Suppose $f \in L^1(\Omega, \mathcal{F}, P)$. Prove that

$$x^n f \in L^1(\Omega, \mathcal{F}, P) \text{ for every } n = 1, 2, \dots \text{ and compute } \lim_{n \rightarrow \infty} \int_{\Omega} x^n f(x) dx.$$

9. Let $\Omega = [0, \infty)$ with its Lebesgue measure and define $f_n(x) = I_{A_n}(x)$ where

$$A_n = \{x \in \Omega : n \leq x \leq n + \frac{1}{n}\}. \text{ Show that}$$

$$f_n \xrightarrow{a.e.} 0, f_n \xrightarrow{\mu} 0 \text{ and } f_n \xrightarrow{L^p} 0, \text{ but not } f_n \xrightarrow{a.u.} 0$$

10. If $0 < p < 1$. Define $d_p : L^p(\Omega, \mathcal{F}, \mu) \times L^p(\Omega, \mathcal{F}, \mu) \rightarrow \mathbb{R}$ by $d_p(f, g) = \|f - g\|_p$ for all

$f, g \in L^p(\Omega, \mathcal{F}, \mu)$. Show that $(L^p(\Omega, \mathcal{F}, \mu), d_p)$ is a complete metric space.



14. Stochastic Process

In this chapter, we introduce the concept of a stochastic process. The theory of stochastic process was first developed in connection with the study of fluctuations and noise in physical systems. A stochastic process is the mathematical model of an empirical process whose development is governed by probability laws. stochastic process provides useful model for the studies of such diverse fields as statistical physics. Communication and control, time series analysis, population growth, and management sciences. Let $\mathbb{T}$ be an index set, $(\Omega, \mathcal{F}, P)$ be a probability space, and let (S, β) be a measurable space.

Definition(14.1)

A function $X : \Omega \times T \rightarrow S$ is called S -valued stochastic process, if for each $t \in \mathbb{T}$, the function $\omega \rightarrow X(t, \omega)$ is an S -valued random variable, i.e. $\{\omega : X(t, \omega) \in A\} \in \mathcal{F}$ for every $A \in \beta$.

- The function $\omega \rightarrow X(t, \omega)$ from Ω into S is called the random function of the process, this function is often denoted by X_t (some times X_t is called the state at time t), and the process itself by $X = (X_t)_{t \in \mathbb{T}}$ or $X = \{X_t; t \in \mathbb{T}\}$
- The probability space $(\Omega, \mathcal{F}, P)$ is called the base of the S -valued stochastic process X , and the space Ω is called sample space of X , the point $\omega \in \Omega$ is called the sample point.
- The measurable space (S, β) is called the state space of the S -valued stochastic process X , (i.e. the state space of X is a set of values of X_t).
- We say that the S -valued stochastic process X has initial value $\alpha \in S$, if $0 \in T$ and $X_0(\cdot) = X(0, \cdot) = \alpha$



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- If $S = \mathbb{R}^d$ and $\beta = \beta(\mathbb{R}^d)$, then X is called $\mathbb{R}^d$ -valued stochastic process. In particular, if $d = 1$ (i.e. $S = \mathbb{R}$, and $\beta = \beta(\mathbb{R})$), then X is called $\mathbb{R}$ -valued stochastic process (or, more briefly, X is called stochastic process). i.e.

A stochastic process is a family of random variables $X = \{X_t; t \in \mathbb{T}\}$ on $(\Omega, \mathcal{F}, P)$

Remarks

- The set $\mathbb{T}$ will always be a subset of the extended of real numbers $\mathbb{R}^*$, and $\mathbb{T}$ is often called the parameter space or the index set of the process.

$$\mathbb{T} = \{1, 2, \dots\}, \quad \mathbb{T} = \{0, \pm 1, \pm 2, \dots\}, \quad \mathbb{T} = \{t : t \geq 0\} = [0, \infty),$$

$$\mathbb{T} = \{t : -\infty < t < \infty\} = (-\infty, \infty) = \mathbb{R}, \quad \mathbb{T} = [-\infty, \infty] = \overline{\mathbb{R}}$$

- For a fixed sample $\omega \in \Omega$, the function $t \rightarrow X(t, \omega)$ from T into S is the (the sample) path (or, trajectory or realization) of the S -valued stochastic process X associated with ω . The totality of all sample paths is called an ensemble.

- Let S be a topological space. An S -valued stochastic process X is said to be a
1. continuous (resp. a right continuous, resp. a left continuous) when its paths are continuous (resp. a right continuous, resp. a left continuous)

2. cad lag when its paths are right continuous with left limits.

3. cad lag when its paths are left continuous with right limits.

Description of a stochastic process

In a stochastic process $X = (X_t)_{t \in \mathbb{T}}$,

- The index set T is called the parameter set of the stochastic process.
- The values assumed by X_t are called states, and the set of all possible values forms the state space E of the stochastic process.
- The stochastic process is called a discrete parameter (or discrete time) process if



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the index set $\mathbb{T}$ is discrete. A discrete parameter process is also called stochastic sequence and is denoted by $\{X_n; n=1,2,\dots\}$ or $\{X_n\}$. If $\mathbb{T}$ is continuous, then we have a continuous parameter (or continuous time) process.

- The stochastic process is called a discrete state process if the state space E of a stochastic process is discrete, often referred to as a chain. In this case, the state space E is often assume to be $\{0,1,2,\dots\}$. If the state space E is continuous, then we have a continuous state process.
- A complex stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is defined by $X_t = Y_t + iZ_t$ where $Y = (Y_t)_{t \in \mathbb{T}}$ and $Z = (Z_t)_{t \in \mathbb{T}}$ are real stochastic processes and $i = \sqrt{-1}$. Throughout this book, all stochastic are real stochastic processes unless specified otherwise.

Definition(14.2)

Consider a stochastic process $\{X_t\}_{t \in \mathbb{T}}$. For a fixed time t_1 , $X_{t_1} = X_1$ is a random variable, and its distribution function $F_X(x_1, t_1)$ is defined as

$$F_X(x_1; t_1) = P\{X_{t_1} \leq x_1\}$$

$F_X(x_1; t_1)$ is known as the first order distribution of X_t . similarly, given t_1 and t_2 ,

$X_{t_1} = X_1$ and $X_{t_2} = X_2$ represent two random variables. Their joint distribution is known as the second order distribution of X_t and is given by

$$F_X(x_1, x_2; t_1, t_2) = P\{X_{t_1} \leq x_1, X_{t_2} \leq x_2\}$$

In general, we define the nth order distribution of X_t by

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P\{X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n\}$$

- If X_t is a discrete time process, then X_t is specified by a collection of p.d.f :

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$$f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P\{X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n\}$$

- If X_t is a continuous time process, then X_t is specified by a collection of p.d.f :

$$f_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \frac{\partial^n F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$$

The complete characterization of X_t requires knowledge of all the distributions as $n \rightarrow \infty$. Fortunately, often much less is sufficient.

Example(14.3)

Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = U \cos \lambda t$, $t \geq 0$ where λ is a constant and U is a uniform random variable over $(0,1)$.

Thus $(X_t)_{t \geq 0}$ is a continuous time, continuous state stochastic process and the state space is $E = \{x : -1 < x < 1\}$, and the index set is $\mathbb{T} = \{t : t \geq 0\}$

Since $U \sim u(0,1) \Rightarrow f(x) = 1, 0 < x < 1$

1. If $t = 0$, then $X_0 = X \cos 0 = X \Rightarrow f(x; 0) = 1, 0 < x < 1$

2. If $t = \frac{\pi}{4\lambda}$, then $X_{\frac{\pi}{4\lambda}} = X \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} X \Rightarrow f(x; \frac{\pi}{4\lambda}) = \sqrt{2}, 0 < x < \frac{1}{\sqrt{2}}$

3. If $t = \frac{\pi}{2\lambda}$, then $X_{\frac{\pi}{2\lambda}} = X \cos \frac{\pi}{2} = 0$ irrespective of the value of X . Thus the p.d.f of $X_{\frac{\pi}{2\lambda}}$ is

$$f(x; \frac{\pi}{2\lambda}) = P\{X = 0\} = 1$$

4. If $t = \frac{\pi}{\lambda}$, then $X_{\frac{\pi}{\lambda}} = X \cos \pi = -X \Rightarrow f(x; \frac{\pi}{\lambda}) = 1, -1 < x < 0$

Definition(14.4)

Let $(X_t)_{t \geq 0}$ be a stochastic process

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1. The mean of X_t is defined by $\mu_X(t) = E(X_t)$ where X_t is treated a random variable for a fixed value of t . In general $\mu_X(t)$ is a function of time, and it is often called the ensemble average of X_t .

2. The characteristic function $\phi_X(\lambda_1, \lambda_2, \dots, \lambda_n)$ of X_t is defined by

$$\phi_X(\lambda_1, \lambda_2, \dots, \lambda_n) = E(e^{i \sum_{k=1}^n \lambda_k X_k})$$

3. A measure of dependence among the random variables of X_t is provided by its autocorrelation function, defined by $R_X(t, s) = E(X_t X_s)$ and $R_X(\tau) = E(X_t X_{t+\tau})$

Note that $R_X(t, s) = R_X(s, t)$ and $R_X(t, t) = E(X_t^2)$

4. The autocovariance function of X_t is defined by

$$K_X(t, s) = \text{Cov}(X_t, X_s) = E((X_t - \mu_X(t))(X_s - \mu_X(s))) = R_X(t, s) - \mu_X(t)\mu_X(s)$$

It is clear that if the mean of X_t is zero, then $K_X(t, s) = R_X(t, s)$

Note that the variance of X_t is given by $\sigma_X^2(t) = \text{Var}(X_t) = E((X_t - \mu_X(t))^2) = K_X(t, s)$

• If $X = (X_t)_{t \in T}$ is a complex stochastic process, then its autocorrelation function

$R_X(t, s)$ and autocovariance function $K_X(t, s)$ are defined, respectively by

$$R_X(t, s) = E(X_t \overline{X_s}) \text{ and } K_X(t, s) = E((X_t - \mu_X(t))(\overline{X_s - \mu_X(s)}))$$

Where $\bar{}$ denote the complex conjugate.

Theorem (14.5)

Let $(X_t)_{t \geq 0}$ be a stochastic process

1. $R_X(-\tau) = \overline{R_X(\tau)}$

2. $|R_X(\tau)| \leq R_X(0)$

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$$3. R_X(0) = E(X_t^2) \geq 0$$

Proof :

$$\text{Since } R_X(\tau) = E(X_t X_{t+\tau})$$

$$(1) \text{ Setting } t + \tau = s, \text{ we get } R_X(\tau) = E(X_{s-\tau} X_s) = E(X_s X_{s-\tau}) = R_X(-\tau)$$

$$(2) \text{ since } E((X_t \pm X_{t+\tau})^2) \geq 0, \text{ we have}$$

$$E(X_t^2 \pm 2X_t X_{t+\tau} + X_{t+\tau}^2) \geq 0 \Rightarrow E(X_t^2) \pm 2E(X_t X_{t+\tau}) + E(X_{t+\tau}^2) \geq 0$$

$$\text{Since } R_X(0) = E(X_t X_t) = E(X_t^2)$$

$$\text{Thus } 2R_X(0) \pm 2R_X(\tau) \geq 0 \Rightarrow |R_X(\tau)| \leq R_X(0)$$

$$(3) \text{ since } |R_X(\tau)| \leq R_X(0) \Rightarrow R_X(0) \geq 0, \text{ also } R_X(0) = E(X_t^2) \Rightarrow R_X(0) = E(X_t^2) \geq 0$$

Example(14.6)

Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = U \cos \lambda t$, $t \geq 0$ where λ is a constant and U is a uniform random variable over $(0,1)$.

Solution :

$$\text{Since } U \sim u(0,1) \Rightarrow f(x) = 1, 0 < x < 1 \Rightarrow E(U) = \frac{1}{2}, E(U^2) = \frac{1}{3}$$

$$\text{Thus } E(X_t) = E(U \cos \lambda t) = E(U) \cos \lambda t = \frac{1}{2} \cos \lambda t$$

$$R_X(t, s) = E(X_t X_s) = E((U \cos \lambda t)(U \cos \lambda s)) = E(U^2 \cos \lambda t \cos \lambda s) = E(U^2) \cos \lambda t \cos \lambda s = \frac{1}{3} \cos \lambda t \cos \lambda s$$

$$K_X(t, s) = R_X(t, s) - E(X_t)E(X_s) = \frac{1}{3} \cos \lambda t \cos \lambda s - \left(\frac{1}{2} \cos \lambda t\right)\left(\frac{1}{2} \cos \lambda s\right) = \frac{1}{3} \cos \lambda t \cos \lambda s - \frac{1}{4} \cos \lambda t \cos \lambda s = \frac{1}{12} \cos \lambda t \cos \lambda s$$

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Example(14.7)

Consider a discrete parameter stochastic process $\{X_n : n \geq 1\}$ where X_n 's are independent identically distributed with common distribution function $F_X(x; t)$, mean μ , and variance σ^2 . Show that (1) $F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = (F_X(x; t))^n$ (2)

$$K_X(n, m) = \begin{cases} 0, & n \neq m \\ \sigma^2, & n = m \end{cases}$$

Since $F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P\{X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n\}$

Since $X_1, X_2, \dots, X_n$ are independent, then

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P\{X_{t_1} \leq x_1\} \times P\{X_{t_2} \leq x_2\} \times \dots \times P\{X_{t_n} \leq x_n\} = \prod_{i=1}^n P\{X_{t_i} \leq x_i\}$$

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \prod_{i=1}^n P\{X_{t_i} \leq x_i\} = \prod_{i=1}^n F_X(x_i; t_i) = \prod_{i=1}^n F_X(x; t) = (F_X(x; t))^n$$

$$\mu_X(n) = E(X_n) = \mu \text{ for all } n.$$

$$\text{If } n \neq m, \text{ then } R_X(n, m) = E(X_n X_m) = E(X_n)E(X_m) = \mu \times \mu = \mu^2$$

$$\text{If } n = m, \text{ then } R_X(n, n) = E(X_n X_n) = E(X_n^2) = \text{Var}(X_n) + (E(X_n))^2 = \sigma^2 + \mu^2$$

$$\text{Hence } R_X(n, m) = \begin{cases} \mu^2, & n \neq m \\ \sigma^2 + \mu^2, & n = m \end{cases}$$

$$K_X(n, m) = R_X(n, m) - \mu_X(n)\mu_X(m) = \begin{cases} \mu^2, & n \neq m \\ \sigma^2 + \mu^2, & n = m \end{cases} - \mu^2 = \begin{cases} 0, & n \neq m \\ \sigma^2, & n = m \end{cases}$$

Definition(14.8)

The cross correlation of two stochastic processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ is defined by

$$R_{XY}(\tau) = R_{XY}(t, t + \tau) = E(X_t Y_{t+\tau})$$

Two stochastic processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ is called (mutually) orthogonal if



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$$R_{XY}(\tau) = 0 \text{ for all } \tau$$

Theorem(14.9)

Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be two stochastic processes

1. $R_{XY}(-\tau) = R_{YX}(\tau)$
2. $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$
3. $|R_{XY}(\tau)| \leq \frac{1}{2}(R_X(0) + R_Y(0))$

Proof :

Since $R_{XY}(-\tau) = E(X_t Y_{t-\tau})$, $R_{YX}(\tau) = E(Y_t X_{t+\tau})$

(1) Setting $t - \tau = s$, we get $R_{XY}(-\tau) = E(X_{s+\tau} Y_s) = E(Y_s X_{s+\tau}) = R_{YX}(\tau)$

(2) By using Cauchy Schwarz inequality, it follows that

$$(E(X_t Y_{t+\tau}))^2 \leq E(X_t^2) E(Y_{t+\tau}^2) \Rightarrow (R_{XY}(\tau))^2 \leq R_X(0) R_Y(0)$$

$$\Rightarrow |R_{XY}(\tau)| \leq \sqrt{R_X(0) R_Y(0)}$$

(3) Since $E((X_t - Y_{t+\tau})^2) \geq 0$

$$\Rightarrow E(X_t^2 - 2X_t Y_{t+\tau} + Y_{t+\tau}^2) \geq 0 \Rightarrow E(X_t^2) - 2E(X_t Y_{t+\tau}) + E(Y_{t+\tau}^2) \geq 0$$

$$\Rightarrow R_X(0) - 2R_{XY}(\tau) + R_Y(0) \geq 0 \Rightarrow R_{XY}(\tau) \leq \frac{1}{2}(R_X(0) + R_Y(0))$$

Example(14.10)

Two stochastic processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are given by

$$X_t = \alpha \cos(\lambda t + U), \quad Y_t = \alpha \cos(\lambda t + U)$$

where α, λ are constants, and U is a uniform random variable over $(0, 2\pi)$.

Show that $R_{XY}(\tau) = \frac{\alpha^2}{2} \sin \lambda \tau$

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Solution :

$$\begin{aligned} R_{XY}(\tau) &= E(X_t X_{t+\tau}) = E((\alpha \cos(\lambda t + U)) \times (\alpha \sin(\lambda(t + \tau) + U))) \\ &= \frac{\alpha^2}{2} E(\sin(2\lambda t + \lambda \tau) + 2U) - \sin(-\lambda \tau) = \frac{\alpha^2}{2} \sin(\lambda \tau) \end{aligned}$$

Definition(14.11)

- A stochastic process $(X_t)_{t \in \mathbb{T}}$ is said to be stationary or strict sense stationary if , for all n and for every set of time of time instants $\{t_i \in T : i = 1, 2, \dots, n\}$,

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = F_X(x_1, x_2, \dots, x_n; t_1 + \tau, t_2 + \tau, \dots, t_n + \tau) \quad (1)$$

For any τ . Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and X_t and $X_{t+\tau}$ will have the same distribution for any τ . Thus, for the first order distribution.

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x) \text{ and } f_X(x; t) = f_X(x)$$

Then

$$\mu_X(t) = E(X_t) = \mu \text{ and } Var(X_t) = \sigma^2$$

Where μ and σ^2 are constants. Similarly, for the second order distribution,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1) \text{ and } f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$$

Nonstationary processes are characterized by distribution depending on the points $t_1, t_2, \dots, t_n$

- A stochastic process $(X_t)_{t \in \mathbb{T}}$ is said to be stationary to order k , if stationary condition (1) not hold for all n but holds for $n \leq k$.
- A stochastic process $(X_t)_{t \in \mathbb{T}}$ is said to be wide sense stationary(WSS) or weak stationary if $(X_t)_{t \in \mathbb{T}}$ is stationary to order 2



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Remark

If $(X_t)_{t \in \mathbb{T}}$ is WSS stochastic process, then we have

1. $E(X_t) = \mu$ (constant)
2. $R_X(t, s) = E(X_t X_s) = R_X(|s - t|)$

Note that a strict sense stationary process is also a wss process, but, in general, the converse is not true.

Example (14.12)

Show that a stochastic process $(X_t)_{t \in \mathbb{T}}$ which is stationary to order n is also stationary to all orders lower than n

solution:

Assume that the equation

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = F_X(x_1, x_2, \dots, x_n; t_1 + \tau, t_2 + \tau, \dots, t_n + \tau)$$

holds for some particular n ; i.e.

$$P(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n) = P\{X_{t_1 + \tau} \leq x_1, X_{t_2 + \tau} \leq x_2, \dots, X_{t_n + \tau} \leq x_n\}$$

For any τ . Letting $x_n \rightarrow \infty$, we have

$$P(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_{n-1}} \leq x_{n-1}) = P\{X_{t_1 + \tau} \leq x_1, X_{t_2 + \tau} \leq x_2, \dots, X_{t_{n-1} + \tau} \leq x_{n-1}\}$$

and the process is stationary to order $n-1$. Continuing the same procedure, we see that the process is stationary to all orders lower than n .

Example (14.13)

Show that if $(X_t)_{t \in \mathbb{T}}$ is a strict sense stationary stochastic process, then it is also WSS.

solution :

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Since X_t is a strict sense stationary, the first and second order distributions are invariant through time translation for all $t \in \mathbb{T}$. Then we have

$$\mu_X(t) = E(X_t) = E(X_{t+\tau}) = \mu_X(t + \tau)$$

and hence the mean function $\mu_X(t)$ must be constant; that is $E(X_t) = \mu$ (constant).

Similarly, we have $E(X_s X_t) = E(X_{s+\tau} X_{t+\tau})$

so that the autocorrelation function would depend on the time points s and t only through the difference $|t - s|$. Thus X_t is WSS.

Example (14.14)

Let $\{X_n, n \geq 0\}$ be a sequence of independent and identically distributed random variables with mean zero and variance 1. show that $\{X_n, n \geq 0\}$ is a WSS process.

solution :

Since $E(X_n) = 0$ (constant) for all n

$$R_X(n, n+k) = E(X_n X_{n+k}) = \begin{cases} E(X_n)E(X_{n+k}), & k \neq 0 \\ E(X_n^2), & k = 0 \end{cases} \text{ which depends on } k. \text{ Thus } \{X_n\} \text{ is}$$

WSS process

Example (14.15)

Show that if $(X_t)_{t \in \mathbb{T}}$ is WSS process, then it must also be covariance stationary.

solution :

If X_t is WSS, then $E(X_t) = \mu$ (constant) for all t and $R_X(t, t+\tau) = R_X(\tau)$ for all t

Now $K_X(t, t+\tau) = R_X(t, t+\tau) - E(X_t)E(X_{t+\tau}) = R_X(\tau) - \mu^2$ which indicates that $K_X(t, t+\tau)$ depends only on τ ; thus X_t is covariance stationary.



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Example(14.16)

Consider a stochastic process $(X_t)_{t \in \mathbb{R}}$ defined by $X_t = U \cos \lambda t + V \sin \lambda t$ where λ is a positive constant and U, V are random variables. Show that

1. The condition $E(U) = E(V) = 0$ is necessary for X_t to be stationary.
2. X_t is WSS iff U and V are uncorrelated with equal variance; i.e.

$$E(UV) = 0, E(U^2) = E(V^2) = \sigma^2$$

Solution :

(1) $\mu_X(t) = E(X_t) = E(U \cos \lambda t + V \sin \lambda t) = E(U) \cos \lambda t + E(V) \sin \lambda t$ must be independent of t for X_t to be stationary. Thus is possible only if $\mu_X(t) = 0$, that is $E(U) = E(V) = 0$

(2) If X_t is WSS, then $E(X_0^2) = E(X_{\frac{\pi}{2\lambda}}^2) = R_{XX}(0) = \sigma_X^2$

But $X_0^2 = U^2$ and $X_{\frac{\pi}{2\lambda}}^2 = V^2$, thus $E(U^2) = E(V^2) = \sigma_X^2 = \sigma^2$

Using the above result, we obtain

$$R_X(t, t + \tau) = E(X_t X_{t+\tau}) = E((U \cos \lambda t + V \sin \lambda t)(U \cos \lambda(t + \tau) + V \sin \lambda(t + \tau)))$$

$$\Rightarrow R_X(t, t + \tau) = \sigma^2 \cos \lambda \tau + E(UV) \sin(2\lambda t + \lambda \tau)$$

Which will be a function of τ only if $E(UV) = 0$.

Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part(1) and

$$R_X(t, t + \tau) = \sigma^2 \cos \lambda \tau + E(UV) \sin(2\lambda t + \lambda \tau), \text{ we have } \mu_X(t) = 0,$$

$$R_X(t, t + \tau) = \sigma^2 \cos \lambda \tau = R_X(\tau)$$

Hence X_t is WSS.



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Example(14.17)

Consider a stochastic process $(X_t)_{t \in \mathbb{R}}$ defined by $X_t = U \cos t + V \sin t$ where U and V are independent random variables each of which assume the values -2 and 1 with the probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Show that X_t is WSS but not strict sense stationary.

solution:

$$E(U) = E(V) = \frac{1}{3}(-2) + \frac{2}{3}(1) = 0, \quad E(U^2) = E(V^2) = \frac{1}{3}(-2)^2 + \frac{2}{3}(1)^2 = 2$$

Since U and V are independent, then $E(UV) = E(U)E(V) = 0$

Then, by above example X_t is WSS.

To see if X_t is strict sense stationary, we consider $E(X_t^3)$.

$$E(X_t^3) = E((U \cos t + V \sin t)^3) = E(U^3 \cos^3 t + 3U^2V \cos^2 t \sin t + 3UV^2 \cos t \sin^2 t + V^3 \sin^3 t)$$

$$E(X_t^3) = E(U^3) \cos^3 t + 3E(U^2V) \cos^2 t \sin t + 3E(UV^2) \cos t \sin^2 t + E(V^3) \sin^3 t$$

Now

$$E(U^2V) = E(U^2)E(V) = 0, \quad E(UV^2) = E(U)E(V^2) = 0, \quad E(U^3) = E(V^3) = \frac{1}{3}(-2)^3 + \frac{2}{3}(1)^3 = -2$$

Thus $E(X_t^3) = -2 \cos^3 t - 2 \sin^3 t = -2(\cos^3 t + \sin^3 t)$ Which is a function of t

Since all the moments of strict sense stationary process must be independent of time.

Thus X_t but not strict sense stationary.

Example(14.18)

Consider a stochastic process $(X_t)_{t \in \mathbb{R}}$ defined by $X_t = a \cos(\lambda t + U)$ where a and λ are constants and U is a uniform random variable over $(-\pi, \pi)$. Show that X_t is WSS

Solution :

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Since $U \sim u(-\pi, \lambda) \Rightarrow f(x) = \frac{1}{2\pi}, -\pi < x < \pi$

Thus $\mu_X(t) = E(X_t) = E(a \cos(\lambda t + U)) = \int_{-\pi}^{\pi} \frac{1}{2\pi} (a \cos(\lambda t + u)) du = 0$

$$R_X(t, t+\tau) = E(X_t X_{t+\tau}) = \int_{-\pi}^{\pi} \frac{1}{2\pi} (a \cos(\lambda t + u)) (a \cos(\lambda(t+\tau) + u)) du = \frac{a^2}{2\pi} \int_{-\pi}^{\pi} (\cos \lambda t + \cos(2\lambda t + 2u + \lambda \tau)) du = \frac{a^2}{2} \cos \lambda \tau$$

Since the mean of X_t is constant and the autocorrelation of X_t is a function of time difference only, we conclude that X_t is WSS.

Definition(14.19)

A stochastic process $(X_t)_{t \geq 0}$ is called independent stochastic process if $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ are independent random variables for $n = 2, 3, \dots$, i.e. if

$$F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \prod_{i=1}^n F_X(x_i; t_i)$$

Thus, a first order distribution is sufficient to characterize an independent stochastic process X_t .

Theorem(14.20)

If $\{X_n\}$ is a sequence of independent random variables and $S_n = \sum_{k=1}^n X_k$, then

$$\phi_{S_n}(\lambda_1, \lambda_2, \dots, \lambda_n) = \prod_{k=1}^n \phi_{X_k}(\lambda_k)$$

Proof :

$$\phi_{S_n}(\lambda_1, \lambda_2, \dots, \lambda_n) = E(e^{i \sum_{k=1}^n \lambda_k X_k}) = E(e^{i \lambda_1 X_1} \cdot e^{i \lambda_2 X_2} \dots e^{i \lambda_n X_n})$$

Since $X_1, X_2, \dots, X_n$ independent $\Rightarrow E(e^{i \lambda_1 X_1} \cdot e^{i \lambda_2 X_2} \dots e^{i \lambda_n X_n}) = E(e^{i \lambda_1 X_1}) \cdot E(e^{i \lambda_2 X_2}) \dots E(e^{i \lambda_n X_n})$

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$$\phi_{S_n}(\lambda_1, \lambda_2, \dots, \lambda_n) = E(e^{i\lambda_1 X_1}) \cdot E(e^{i\lambda_2 X_2}) \cdots E(e^{i\lambda_n X_n}) = \phi_{X_1}(\lambda_1) \cdot \phi_{X_2}(\lambda_2) \cdots \phi_{X_n}(\lambda_n) = \prod_{k=1}^n \phi_{X_k}(\lambda_k)$$

Definition(14.21)

A stochastic process $(X_t)_{t \geq 0}$ is said to have

1. Independent increments if for $n \geq 1$ and time points $0 \leq t_0 < t_1 < \cdots < t_n$, the increments :

$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent random variables.

2. Stationary independent increment if $\{X_t\}_{t \geq 0}$ independent increments and $X_t - X_s$

has the same distribution as $X_{t+h} - X_{s+h}$ for all $s, t, h \geq 0, s < t$

Theorem(14.22)

Let $(X_t)_{t \geq 0}$ be a stochastic process with stationary independent increment and assume that $X_0 = 0$. Show that

1. $E(X_t) = \mu_1 t$

2. $Var(X_t) = \sigma_1^2 t$

3. $Var(X_t - X_s) = \sigma_1^2 (t - s), \quad t > s$

4. $K_X(t, s) = \sigma_1^2 \min\{t, s\}$, where $\mu_1 = E(X_1)$ and $\sigma_1^2 = Var(X_1)$

Proof :

(1) Let $f(t) = E(X_t)$, Since $E(X_t - X_0) = E(X_t) \Rightarrow f(t) = E(X_t - X_0)$

For any t and s , by using the property of the stationary independent increment,

we have $f(t+s) = E(X_{t+s} - X_0) = E(X_{t+s} - X_s + X_s - X_0) = E(X_{t+s} - X_s) + E(X_s - X_0)$

$f(t+s) = E(X_t - X_0) + E(X_s - X_0) = f(t) + f(s)$

The only solution to the above functional equation is $f(t) = ct$, where c is a constant.

Since $c = f(1) = E(X_1)$, we obtain $E(X_t) = \mu_1 t$, where $\mu_1 = E(X_1)$

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(2) Let $g(t) = \text{Var}(X_t)$, Since $\text{Var}(X_t - X_0) = \text{Var}(X_t) \Rightarrow f(t) = \text{Var}(X_t - X_0)$

For any t and s , by using the property of the stationary independent increment,

we have $g(t+s) = \text{Var}(X_{t+s} - X_0) = \text{Var}(X_{t+s} - X_s + X_s - X_0) = \text{Var}(X_{t+s} - X_s) + \text{Var}(X_s - X_0)$
 $f(t+s) = \text{Var}(X_t - X_0) + \text{Var}(X_s - X_0) = g(t) + g(s)$

Which is the same functional equation as (1). Thus $g(t) = kt$, where k is a constant.

Since $k = g(1) = \text{Var}(X_1)$, we obtain $\text{Var}(X_t) = \sigma_1^2 t$, where $\sigma_1^2 = \text{Var}(X_1)$

(3) Let $t > s$. Then

$$\text{Var}(X_t) = \text{Var}(X_t - X_0) = \text{Var}(X_t - X_s + X_s - X_0) = \text{Var}(X_t - X_s) + \text{Var}(X_s - X_0)$$

$$\text{Var}(X_t) = \text{Var}(X_t - X_s) + \text{Var}(X_s)$$

$$\text{Since } \text{Var}(X_t) = \sigma_1^2 t \Rightarrow \sigma_1^2 t = \text{Var}(X_t - X_s) + \sigma_1^2 s \Rightarrow \text{Var}(X_t - X_s) = \sigma_1^2 (t - s)$$

$$(4) \text{Var}(X_t - X_s) = E((X_t - X_s - E(X_t - X_s))^2) = E(((X_t - E(X_t)) - (X_s - E(X_s))))^2)$$

$$\text{Var}(X_t - X_s) = E((X_t - E(X_t))^2) - 2E((X_t - E(X_t))(X_s - E(X_s))) + E((X_s - E(X_s))^2)$$

$$\Rightarrow \text{Var}(X_t - X_s) = \text{Var}(X_t) - 2\text{Cov}(X_t, X_s) + \text{Var}(X_s)$$

$$\Rightarrow K_X(t, s) = \text{Cov}(X_t, X_s) = \frac{1}{2}(\text{Var}(X_t) + \text{Var}(X_s) - \text{Var}(X_t - X_s))$$

We obtain

$$K_X(t, s) = \begin{cases} \frac{1}{2} \sigma_1^2 (t+s - (t-s)), & t > s \\ \frac{1}{2} \sigma_1^2 (t+s - (s-t)), & t < s \end{cases} = \begin{cases} \sigma_1^2 s, & t > s \\ \sigma_1^2 t, & t < s \end{cases} = \sigma_1^2 \min\{t, s\}$$

Some Special Stochastic Processes

In this section we introduce some special stochastic process, Markov process, Bernoulli process, random walk, counting processes, Poisson processes, normal process, and Winer process.



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Definition (14.23) Markov Process

A stochastic process $(X_t)_{t \in \mathbb{T}}$ is said to be a Markov process if

$$P\{X_{t_{n+1}} \leq x_{n+1} | X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n\} = P\{X_{t_{n+1}} \leq x_{n+1} | X_{t_n} = x_n\} \quad (1)$$

Whenever $t_1 < t_2 < \dots < t_n < t_{n+1}$.

A discrete state Markov Process is called a Markov chain. For a discrete parameter Markov chain $\{X_n, n \geq 0\}$, we have for every n

$$P\{X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n\} = P\{X_{n+1} = x_{n+1} | X_n = x_n\} \quad (2)$$

Equation (1) or equation (2) is referred to as the Markov property (which is also known as the memoryless property). This property of a Markov process states that the future state of the process depends only on the present state and not on the past history.

Clearly, any process with independent increments is a Markov process.

Example (14.24)

Using the Markov property, the n th-order distribution of a Markov process X_t can be

expressed as $F_X(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = F_X(x_1, t_1) \prod_{k=2}^n P\{X_{t_k} \leq x_k | X_{t_{k-1}} = x_{k-1}\}$

Thus, all finite order distribution of a Markov Process can be expressed in terms of the second order distributions.

Examples(14.25)

Show that for a Markov process $(X_t)_{t \in \mathbb{T}}$, the second order distribution is sufficient to characterize X_t .

solution :

Let $(X_t)_{t \in \mathbb{T}}$ be a Markov process with the n th order distribution

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$$F_X \{x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n\} = P\{X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n\}$$

Then, using the Marko property, we have

$$F_X \{x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n\} = P\{X_{t_n} \leq x_n | X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_{n-1}} \leq x_{n-1}\} \times P\{X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_{n-1}} \leq x_{n-1}\}$$

$$F_X \{x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n\} = P\{X_{t_n} \leq x_n | X_{t_{n-1}} \leq x_{n-1}\} \times F_X \{x_1, x_2, \dots, x_{n-1}; t_1, t_2, \dots, t_{n-1}\}$$

Applying the above relation for lower order distribution, we can write

$$F_X \{x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n\} = F_X(x_1, t_1) \prod_{k=2}^n P\{X_{t_k} \leq x_k | X_{t_{k-1}} \leq x_{k-1}\}$$

Hence, all finite order distribution of a Marko process can be completely determined by the second order distribution.

Remark

Recall that a random variable X is defined to have a Binomial distribution if the probability density function of X is given by

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$

Where $0 \leq p \leq 1$, $n = 1, 2, \dots$ and dented by $X \sim b(n, p)$

- If X has a Binomial distribution (i.e. $X \sim b(n, p)$), then

$$E(X) = np, \quad Var(X) = np(1-p)$$

- A Bernoulli distribution is a special case of a Binomial distribution when $n = 1$, i.e.

A random variable X is defined to have a Bernoulli distribution if the probability density function of X is given by

$$f(x) = p^x (1-p)^{1-x}, \quad x = 0, 1$$

Where $0 \leq p \leq 1$ and dented by $X \sim br(p)$

- If X has a Bernoulli distribution (i.e. $X \sim br(p)$), then $E(X) = p$, $Var(X) = p(1-p)$



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- If $X_1, X_2, \dots, X_n$ are random sample have Bernoulli distribution with parameter p , then

$$S_n = \sum_{k=1}^n X_k \sim b(n, p), \quad n > 0$$

Thus $E(S_n) = np$, $Var(S_n) = np(1-p)$, $E(S_n^2) = np(1-p) + np^2$

Definition (14.26) Bernoulli Process

A stochastic process $\{X_n, n \geq 1\}$ is called a Bernoulli process if $X_1, X_2, \dots$ are independent Bernoulli random variables with $p\{X_n = 1\} = p$ and $p\{X_n = 0\} = q$, where $q = 1 - p$. Thus The Bernoulli process $\{X_n, n \geq 1\}$ is a discrete time, discrete state process and the state space is $E = \{0, 1\}$, and the index set is $T = \{1, 2, \dots\}$

Definition (14.27) Random Walk

A stochastic process $\{X_n, n \geq 0\}$ is called the simple random walk process if $X_0 = 0$,

$X_n = \sum_{k=1}^n Y_k$, Where $Y_1, Y_2, \dots$ are independent identically distribution random variables with

$$P\{Y_n = 1\} = p, \quad P\{Y_n = -1\} = 1 - p \quad \text{for all } n.$$

Thus . The simple random walk process $\{X_n, n \geq 0\}$ is a discrete time, discrete state process and the state space is $E = \{\dots, -2, -1, 0, 1, 2, \dots\}$, and the index set is $T = \{0, 1, 2, \dots\}$

Example(14.28)

Let $\{X_n, n \geq 0\}$ be a simple random walk. Define $\{X_t\}_{t \geq 0}$ by $X_t = X_n$, $n \leq t < n+1$. Thus $\{X_t\}_{t \geq 0}$ is a continuous time, discrete state stochastic process and the state space is $E = \{\dots, -2, -1, 0, 1, 2, \dots\}$, and the index set is $T = \{t : t \geq 0\}$

Theorem(14.29)

Let $\{X_n, n \geq 0\}$ be a simple random walk. Then

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$$1. E(X_n) = n(p - q), \quad q = 1 - p$$

$$2. Var(X_n) = 4npq, \quad q = 1 - p$$

$$3.. R_X(n, m) = \begin{cases} m + (n - 1)m(p - q)^2, & m < n \\ n + (m - 1)n(p - q)^2 & n < m \end{cases}$$

Proof :

$$\text{Since } E(Y_k) = 1 \times p + (-1) \times q = p - q, \quad E(Y_k^2) = 1^2 \times p + (-1)^2 \times q = p + q = 1$$

$$Var(Y_k) = E(Y_k^2) - (E(Y_k))^2 = 1 - (p - q)^2$$

$$Var(Y_k) = 1 - p^2 + 2pq - q^2 = 1 + 4pq - p^2 - 2pq - q^2 = 1 + 4pq - (p + q)^2 = 1 + 4pq - 1 = 4pq$$

$$E(X_n) = E\left(\sum_{k=1}^n Y_k\right) = \sum_{k=1}^n E(Y_k) = \sum_{k=1}^n (p - q) = n(p - q), \quad q = 1 - p$$

$$Var(X_n) = Var\left(\sum_{k=1}^n Y_k\right) = \sum_{k=1}^n Var(Y_k) = \sum_{k=1}^n 4pq = 4npq, \quad q = 1 - p$$

Note that if $p = q = \frac{1}{2}$, then $E(X_n) = 0$, $Var(X_n) = n$

$$R_X(n, m) = E(X_n X_m) = \sum_{i=0}^n \sum_{j=0}^m E(Y_i Y_j) = \sum_{i=0}^{\min\{n, m\}} E(Y_i^2) + \sum_{i=0}^n \sum_{\substack{j=0 \\ i \neq j}}^m E(Y_i) E(Y_j)$$

$$R_X(n, m) = \min\{n, m\} + (nm - \min\{n, m\})(p - q)^2 = \begin{cases} m + (n - 1)m(p - q)^2, & m < n \\ n + (m - 1)n(p - q)^2 & n < m \end{cases}$$

Note that if $p = q = \frac{1}{2}$, then $R_X(n, m) = \min\{n, m\}$, $n, m > 0$

Theorem(14.30)

A simple random walk $\{X_n, n \geq 0\}$ is a Markov chain.

Proof :

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Since, $X_0 = 0$ and $X_n = \sum_{k=1}^n Y_k$, Where $Y_1, Y_2, \dots$ are independent identically distribution

random variables with $P\{Y_n = k\} = a_k$, $k = 1, -1$ and for all n .

Since $P\{X_{n+1} = i_{n+1} | X_0 = 0, X_1 = i_1, \dots, X_n = i_n\} = P\{Y_{n+1} + i_n = i_{n+1} | X_0 = 0, X_1 = i_1, \dots, X_n = i_n\}$

Since Y_{n+1} is independent of $X_0, X_1, \dots, X_n$, then

$P\{X_{n+1} = i_{n+1} | X_0 = 0, X_1 = i_1, \dots, X_n = i_n\} = P\{Y_{n+1} = i_{n+1} - i_n\} = a_{i_{n+1} - i_n} = P\{X_{n+1} = i_{n+1} | X_n = i_n\}$

Thus $\{X_n, n \geq 0\}$ is a Markov chain.

Definition(14.31) Counting Processes

Let t represent a time variable. Suppose an experiment begins at $t = 0$. Events of a particular kind occur randomly, the first at T_1 , the second at T_2 , and so on. The random variable T_i denotes the time at which i th event occurs, and the values t_i of T_i , $i = 1, 2, \dots$ are called points of occurrence.

Definition(14.32)

A stochastic process $(X_t)_{t \geq 0}$ is said to be a counting process if X_t represents the total number of "events" that have occurred in the interval $(0, t)$.

From its definition, we see that for a counting process, X_t must satisfy the following conditions

1. $X_t \geq 0$ and $X_0 = 0$
2. X_t is integer valued
3. $X_s \leq X_t$ if $s < t$
4. $X_t - X_s$ equals the number of events that occurred on the interval (s, t)



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Poisson Process

Recall that a random variable X is defined to have a Poisson distribution if the probability density function of X is given by

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots$$

Where the parameter λ satisfies $\lambda > 0$, and denoted by $X \sim p(\lambda)$

- If X has a Poisson distribution (i.e. $X \sim p(\lambda)$), then $E(X) = \lambda$, $Var(X) = \lambda$

One of the most important types of counting processes is the Poisson process (or Poisson counting process), which is defined as follows :

Definition (14.33) Counting Processes

A counting process $\{X_t\}_{t \geq 0}$ is said to be a Poisson process with rate(intensity) $\lambda(>0)$ if

1. $X_0 = 0$
2. X_t has independent increment
3. The number of events in any interval of length t is Poisson distribution with mean λt ;

$$\text{i.e. for all } s, t > 0, f(X_{t+s} - X_t = x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \dots$$

It follows from condition 3, of definition() that a Poisson process has stationary increment and that $E(X_t) = \lambda t$, $Var(X_t) = \lambda t$. Thus, the expected number of events in the interval $(0, 1)$, or any other interval of unit length, is just λ (hence the name of the rate or intensity).

An alternative definition of a Poisson process is given as follows :

Theorem (14.34)

A counting process $\{X_t\}_{t \geq 0}$ is a Poisson process with rate (intensity) $\lambda(>0)$ if

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1. $X_0 = 0$
2. X_t has independent and stationary increments
3. $f(X_{t+\Delta t} - X_t = 1) = \lambda \Delta t + o(\Delta t)$
4. $f(X_{t+\Delta t} - X_t \geq 2) = o(\Delta t)$

Where $o(\Delta t)$ is a function of Δt which goes to zero faster than does Δt , i.e. $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$.

Example (14.35)

The autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Poisson process X_t with rate λ are given by $R_X(t, s) = \lambda \min\{t, s\} + \lambda^2 ts$ and $K_X(t, s) = \lambda \min\{t, s\}$

Normal Process

Recall that a random variable X is defined to have a Normal distribution if the

probability density function of X is given by $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

Where $-\infty < \mu < \infty$, $\sigma > 0$, and denoted by $X \sim N(\mu, \sigma^2)$

- If X has a Normal distribution (i.e. $X \sim p(\lambda)$), then $E(X) = \mu$, $Var(X) = \sigma^2$
- We X has Standard Normal Distribution if $X \sim N(0,1)$, i.e. if $\mu = 0$, $\sigma^2 = 1$

Definition (14.36)

A stochastic process $\{X_t\}_{t \in T}$ is said to be a normal(or Gaussian) process if for any integer n and any subset $\{t_1, t_2, \dots, t_n\}$ of T , then the random variables $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ are jointly normally distributed in the sense that their joint characteristic function is given

by $\varphi_{X_{t_1}, X_{t_2}, \dots, X_{t_n}}(\lambda_1, \lambda_2, \dots, \lambda_n) = E(\exp(i \sum_{k=1}^n \lambda_k X_{t_k})) = \exp(i \sum_{k=1}^n \lambda_k E(X_{t_k}) - \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n \lambda_k \lambda_l \text{Cov}(X_{t_k}, X_{t_l}))$



Winer Processes

Another example of stochastic process with stationary independent increment is a Winer process.

Definition (14.37)

A stochastic process $\{X_t\}_{t \geq 0}$ is called a Winer process if

1. X_t has stationary independent increment
2. The increment $X_t - X_s$, $s < t$ is normally distributed
3. $E(X_t) = 0$
4. $X_0 = 0$

The Wiener process is also known as Brownian motion process, since it originates as model for Brownian motion, the motion of particles suspended in a fluid. From Def(*), we can verify that a Wiener process is normal process

Example (14.38)

Show that a Wiener process X_t is normal process and $E(X_t) = 0$, $Var(X_t) = \sigma^2 t$, where σ^2 is a parameter of the wiener process which must be determined from observations.

Remark

A Wiener process X_t is called a standard Wiener (or standard Brownian motion) process if $\sigma^2 = 1$.

Example (14.39)

The autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Wiener process X_t are given by $R_X(t, s) = K_X(t, s) = \sigma^2 \min\{t, s\}$, $s, t \geq 0$

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Definition(14.40)

A stochastic process $\{X_t\}_{t \geq 0}$ is called a Winer process with drift coefficient μ if

1. X_t has stationary independent increment
2. X_t is normally distributed with mean μ
3. $X_0 = 0$

From condtion2, the p.d.f of a standard Wiener process with drift coefficient μ is given

by
$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-\mu)^2}$$

Definition (14.41)

A stochastic process $\{X_t : t \in T\}$ is said to be continuous in probability if for every $\varepsilon > 0$ and

$$t \in T, \lim_{h \rightarrow 0} P\{|X_{t+h} - X_t| > \varepsilon\} = 0.$$

Example(14.42)

Show that a Winer process X_t is continuous in probability

Solution :

Let $\varepsilon > 0$. From Chebyshev's inequality , we have $P\{|X_{t+h} - X_t| > \varepsilon\} \leq \frac{Var(X_{t+h} - X_t)}{\varepsilon^2}$

Since X_t has stationary increments, we have $Var(X_{t+h} - X_t) = Var(X_h) = \sigma^2 h$

$$\Rightarrow P\{|X_{t+h} - X_t| > \varepsilon\} \leq \frac{\sigma^2 h}{\varepsilon^2} \Rightarrow \lim_{h \rightarrow 0} P\{|X_{t+h} - X_t| > \varepsilon\} \leq \lim_{h \rightarrow 0} \frac{\sigma^2 h}{\varepsilon^2} = 0$$

$\Rightarrow \lim_{h \rightarrow 0} P\{|X_{t+h} - X_t| > \varepsilon\} = 0$. Thus the Winer process X_t is continuous in probability.



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Exercises (14)

In exercises (1- 5) . Find

(1) $E(\max\{X_t : 0 \leq t \leq 1\})$ (2) $E(\max\{|X_t| : 0 \leq t \leq 1\})$

1. Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = Ut + V$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.
2. Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = t^2 + Ut + V$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.
3. Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = e^{Ut}$ where random variable U is and normal distributed $N(0, \sigma^2)$.
4. Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = U \cos \pi t$ where random variable U is normal distributed $N(0, \sigma^2)$.
5. Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = \begin{cases} U, & 0 < t \leq \frac{1}{2} \\ V, & t > \frac{1}{2} \end{cases}$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.
6. Consider a stochastic process $\{X_n, n \geq 1\}$, where $X_n = \sum_{k=1}^n Y_k$, and $Y_1, Y_2, \dots$ are random sample with mean zero and variance σ^2 . Is X_n stationary? Ans No.
7. Consider a stochastic process X_t defined by $X_t = U \cos(\lambda t + V)$, where U and V are independent and are uniformly distributed over $(-a, a)$ and $(-\pi, \pi)$, respectively.

Show that $E(X_t) = 0$ and $R_X(t, s) = \frac{1}{6} a^2 \cos \lambda(t - s)$



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8. Suppose that a stochastic process X_t is wide-sense stationary with

$$R_X(t, t + \tau) = \exp\left(-\frac{|\tau|}{2}\right). \text{ Show that } E(X_s^2) = 1 \text{ and } E((X_s - X_3)^2) = 2\left(\frac{e^{-1}}{e}\right)$$

9. Consider a stochastic process X_t defined by $X_t = U \cos t + (V + 1) \sin t$, $-\infty < t < \infty$, where U and V are independent random variables for which $E(U) = E(V) = 0$, $E(U^2) = E(V^2) = 1$. Show that (1) $K_X(t, s) = \cos(s - t)$ (2) X_t is not WSS.

10. Consider the stochastic processes $X_t = \alpha \cos(\lambda_1 t + U)$, $Y_t = \beta \cos(\lambda_2 t + V)$, where $\alpha, \beta, \lambda_1, \lambda_2$ are constants, and random variables U and V are independent and uniformly distributed over $(-\pi, \pi)$. Show that

$$(1) R_{XY}(t, t + \tau) = 0$$

$$(2) R_{XY}(t, t + \tau) = \frac{\alpha\beta}{2} \cos((\lambda_2 - \lambda_1)t + \lambda_1\tau), \text{ if } U = V$$

11. Let X_t be a Poisson process with rate λ . Show that $E((X_t - X_s)^2) = \lambda t + \lambda^2 t^2$ for $t > s$.

12. Let X_t be a Poisson process with rate λ . Show that

$$P\{X_{t-d} = k | X_t = j\} = \frac{j!}{k!(j-k)!} \left(\frac{t-d}{t}\right)^k \left(\frac{d}{t}\right)^{j-k}, \quad d > 0$$

13. Let T_n denote the time of the n th event of a Poisson with rate λ . Show that

$$\text{Var}(T_n) = \frac{n}{\lambda^2}$$

14. Let $X_1, X_2, \dots, X_n$ be jointly normal random variables. Let $Y_i = X_i + c_i$, $i = 1, 2, \dots, n$, where c_i are constant. Show that $Y_1, Y_2, \dots, Y_n$ are also jointly normal random variables.

15. Show that if a normal process is WSS, then it is also strict sense stationary.

15. Stochastic Calculus

In this section, we introduce the methods for analysis and processing of stochastic processes. First, we introduce the definitions of stochastic continuity, stochastic derivatives, and stochastic integrals of stochastic processes. Next, the notation of power spectral density is introduced. This concept enables us to study wide sense stationary process in the frequency domain and define a white noise process. The response of linear systems to stochastic processes is then studied. Finally, orthogonal and spectral representations or stochastic process are presented.

Definition(15.1)

A continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is said to be continuous in mean square or mean square (m.s.)continuous if $\lim_{\varepsilon \rightarrow 0} E((X_{t+\varepsilon} - X_t)^2) = 0$

Theorem(15.2)

A continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous iff $R_X(t, s)$ is continuous .

Proof :

Suppose that $R_X(t, s)$ is continuous, then

$$\text{Since } E((X_{t+\varepsilon} - X_t)^2) = E(X_{t+\varepsilon}^2 - 2X_{t+\varepsilon}X_t + X_t^2) = E(X_{t+\varepsilon}^2) - 2E(X_{t+\varepsilon}X_t) + E(X_t^2)$$

$$E((X_{t+\varepsilon} - X_t)^2) = R_X(t + \varepsilon, t + \varepsilon) - 2R_X(t + \varepsilon, t) + R_X(t, t)$$

Since $R_X(t, s)$ is continuous, we have

$$\lim_{\varepsilon \rightarrow 0} E((X_{t+\varepsilon} - X_t)^2) = \lim_{\varepsilon \rightarrow 0} (R_X(t + \varepsilon, t + \varepsilon) - 2R_X(t + \varepsilon, t) + R_X(t, t)) = 0$$

$$\Rightarrow X = (X_t)_{t \in \mathbb{T}} \text{ is m.s. continuous}$$

Conversely : suppose that $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous

Since

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$R_X(t + \varepsilon_1, t + \varepsilon_2) - R_X(t, t) = E((X_{t+\varepsilon_1} - X_t)(X_{t+\varepsilon_2} - X_t)) + E((X_{t+\varepsilon_1} - X_t)X_t) + E((X_{t+\varepsilon_2} - X_t)X_t)$ applying Cauchy-Schwarz inequality, we obtain

$$R_X(t + \varepsilon_1, t + \varepsilon_2) - R_X(t, t) \leq (E((X_{t+\varepsilon_1} - X_t)^2))^{\frac{1}{2}} (E((X_{t+\varepsilon_2} - X_t)^2))^{\frac{1}{2}} + (E((X_{t+\varepsilon_1} - X_t)^2))^{\frac{1}{2}} (E(X_t)^2)^{\frac{1}{2}} + (E((X_{t+\varepsilon_2} - X_t)^2))^{\frac{1}{2}} (E(X_t)^2)^{\frac{1}{2}}$$

Since $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous, we have $R_X(t + \varepsilon_1, t + \varepsilon_2) - R_X(t, t)$

Theorem(15.3)

A WSS stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous iff $R_X(\tau)$ is continuous at $\tau = 0$.

Proof :

Since $X = (X_t)_{t \in \mathbb{T}}$ is a WSS stochastic process, then $E((X_{t+\varepsilon} - X_t)^2) = 2(R_X(0) - R_X(t))$

Suppose that $R_X(\tau)$ is continuous at $\tau = 0$, i.e. $\lim_{\varepsilon \rightarrow 0} (R_X(\varepsilon) - R_X(0)) = 0$, then

$$\lim_{\varepsilon \rightarrow 0} E((X_{t+\varepsilon} - X_t)^2) = 0, \Rightarrow X = (X_t)_{t \in \mathbb{T}} \text{ is m.s. continuous.}$$

Similarly, we can show that if $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous, then $R_X(\tau)$ is continuous at $\tau = 0$.

Theorem(15.4)

If a continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous, then its mean continuous; that is $\lim_{\varepsilon \rightarrow 0} \mu_X(t + \varepsilon) = \mu_X(t)$.

Proof :

$$\text{We have } Var(X_{t+\varepsilon} - X_t) = E((X_{t+\varepsilon} - X_t)^2) - (E(X_{t+\varepsilon} - X_t))^2 \geq 0$$

$$\text{Thus } E((X_{t+\varepsilon} - X_t)^2) \geq (E(X_{t+\varepsilon} - X_t))^2 = (\mu_X(t + \varepsilon) - \mu_X(t))^2$$

Since $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous, then as $\varepsilon \rightarrow 0$, the left hand side of the above expression approaches zero. Thus $\lim_{\varepsilon \rightarrow 0} (\mu_X(t + \varepsilon) - \mu_X(t)) = 0$ or $\lim_{\varepsilon \rightarrow 0} \mu_X(t + \varepsilon) = \mu_X(t)$

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Example(15.5)

Show that every Wiener process is m.s. continuous .

Proof :

Let $X = (X_t)_{t \in \mathbb{T}}$ be the Wiener process $\Rightarrow R_X(t, s) = \sigma^2 \min\{t, s\}$

Thus , we have $|R_X(t + \varepsilon_1, t + \varepsilon_2) - R_X(t, t)| = \sigma^2 |\min(t + \varepsilon_1, t + \varepsilon_2) - t| \leq \sigma^2 \max(\varepsilon_1, \varepsilon_2)$

Since $\lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \max(\varepsilon_1, \varepsilon_2) = 0 \Rightarrow R_X(t, s)$ is continuous .

Hence the Wiener process is m.s. continuous.

Example(15.6)

Show that every m.s. continuous stochastic process is continuous in probability.

Proof :

suppose that $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous $\Rightarrow \lim_{\varepsilon \rightarrow 0} E((X_{t+\varepsilon} - X_t)^2) = 0$

since $P\{|X_{t+\varepsilon} - X_t| > a\} \leq \frac{E(|X_{t+\varepsilon} - X_t|^2)}{a^2} \Rightarrow \lim_{\varepsilon \rightarrow 0} P\{|X_{t+\varepsilon} - X_t| > a\} = 0$

Hence $X = (X_t)_{t \in \mathbb{T}}$ is continuous in probability.

Definition(15.7)

A continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is said to have a m.s. derivative

$X' = (X'_t)_{t \in \mathbb{T}}$ if $\text{l.i.m.}_{\varepsilon \rightarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = X'_t$, where l.i.m. denotes limit in the mean(square); that is

$$\lim_{\varepsilon \rightarrow 0} E\left(\left(\frac{X_{t+\varepsilon} - X_t}{\varepsilon} - X'_t\right)^2\right) = 0$$

Remarks

1. A continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ has a m.s. derivative $X' = (X'_t)_{t \in \mathbb{T}}$

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if $\frac{\partial^2 R_X(t, s)}{\partial t \partial s}$ exists at $s = t$.

2. If $X = (X_t)_{t \in \mathbb{T}}$ has the m.s. derivative $X' = (X'_t)_{t \in \mathbb{T}}$, then its mean and autocorrelation function are given by

$$E(X'_t) = \frac{d}{dt} E(X_t) = \mu'_X(t) \quad \text{and} \quad R_{X'}(t, s) = \frac{\partial^2 R_X(t, s)}{\partial t \partial s}$$

Example(15.8)

Suppose a stochastic process $X = (X_t)_{t \in \mathbb{T}}$ has a m.s. derivative $X' = (X'_t)_{t \in \mathbb{T}}$, then

1. $E(X'_t) = E\left(\lim_{\varepsilon \rightarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon}\right) = \lim_{\varepsilon \rightarrow 0} E\left(\frac{X_{t+\varepsilon} - X_t}{\varepsilon}\right) = \lim_{\varepsilon \rightarrow 0} E\left(\frac{\mu_X(t+\varepsilon) - \mu_X(t)}{\varepsilon}\right) = \mu'_X(t)$
2. $R_{XX}(t, s) = E(X_t X'_s) = E\left(X_t \lim_{\varepsilon \rightarrow 0} \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\right) = \lim_{\varepsilon \rightarrow 0} \frac{E(X_t X_{s+\varepsilon}) - E(X_t X_s)}{\varepsilon}$
 $= \lim_{\varepsilon \rightarrow 0} \frac{R_X(t, s+\varepsilon) - R_X(t, s)}{\varepsilon} = \frac{\partial R_X(t, s)}{\partial s}$
3. $R_X(t, s) = E(X'_t X'_s) = E\left(\lim_{\varepsilon \rightarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} \lim_{\varepsilon \rightarrow 0} \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\right) = \lim_{\varepsilon \rightarrow 0} \frac{E(X_{t+\varepsilon} X_{s+\varepsilon}) - E(X_t X_s)}{\varepsilon^2}$
 $= \lim_{\varepsilon \rightarrow 0} \frac{R_{XX}(t+\varepsilon, s+\varepsilon) - R_{XX}(t, s)}{\varepsilon^2} = \frac{\partial^2 R_{XX}(t, s)}{\partial t \partial s}$

Example(15.9)

If $X = (X_t)_{t \in \mathbb{T}}$ is WSS stochastic process and has a m.s. derivative. Show that

1. $R_{XX'}(\tau) = \frac{dR_X(\tau)}{d\tau}$
2. $R_{X'}(\tau) = \frac{d^2 R_X(\tau)}{d\tau^2}$

Proof:

(1) Since $R_{XX}(t, s) = \frac{\partial R_X(t, s)}{\partial s}$

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since $X = (X_t)_{t \in \mathbb{T}}$ is WSS stochastic process, then $R_X(t, s) = R_X(s - t)$

Thus, setting $s - t = \tau$, we obtain

$$\frac{\partial R_X(s - t)}{\partial s} = \frac{dR_X(\tau)}{d\tau} \text{ and } R_{XX}(t, t + \tau) = R_{XX}(\tau) = \frac{dR_X(\tau)}{d\tau}$$

$$(2) \text{ Since } \frac{\partial R_X(s - t)}{\partial t} = -\frac{dR_X(\tau)}{d\tau}. \text{ Thus } \frac{\partial^2 R_X(s - t)}{\partial t \partial s} = -\frac{d^2 R_X(\tau)}{d\tau^2}$$

$$\text{Since } R_X(t, s) = \frac{\partial^2 R_X(t, s)}{\partial t \partial s}, \text{ we have } R_X(t, t + \tau) = R_X(\tau) = \frac{d^2 R_X(\tau)}{d\tau^2}$$

Example(15.10)

Show that the Wiener process $X = (X_t)_{t \in \mathbb{T}}$ does not have a m.s. derivative.

Proof :

Since $X = (X_t)_{t \in \mathbb{T}}$ is the Wiener process, then $R_X(t, s) = \sigma^2 \min\{t, s\} = \begin{cases} \sigma^2 s, & t > s \\ \sigma^2 t, & t < s \end{cases}$

$$\frac{\partial R_X(t, s)}{\partial s} = \sigma^2 u(t - s) = \begin{cases} \sigma^2, & t > s \\ 0, & t < s \end{cases} \text{ where } u(t - s) \text{ is a unit step function defined by}$$

$$u(t - s) = \begin{cases} 1, & t > s \\ 0, & t < s \end{cases} \text{ and it is not continuous at } s = t. \text{ Thus } \frac{\partial^2 R_X(t, s)}{\partial t \partial s} \text{ does not exists at } s = t,$$

and the Wiener process $X = (X_t)_{t \in \mathbb{T}}$ does not have a m.s. derivative.

Definition(15.11)

A m.s. integral of a stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is defined by

$$Y_t = \int_{t_0}^t X_\alpha d\alpha = l.i.m. \sum_{\Delta t_i} X_{t_i} \Delta t_i$$

Where $t_0 < t_1 < \dots < t$ and $\Delta t_i = t_{i+1} - t_i$.

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This implies that if $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous, then its m.s. integral Y_t exists if the following integral exists :

$$\int_{t_0}^t \int_{t_0}^s R_X(\alpha, \beta) d\beta d\alpha$$

Theorem (15.12)

The m.s. integral of a stochastic process $X = (X_t)_{t \in \mathbb{T}}$ exists if the following integral exists :

$$\int_{t_0}^t \int_{t_0}^s R_X(\alpha, \beta) d\beta d\alpha$$

Proof :

A m.s. integral of a stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is defined by

$$Y_t = \int_{t_0}^t X_\alpha d\alpha = l.i.m. \sum_{\Delta t_i} X_{t_i} \Delta t_i$$

Again using the Cauchy criterion, the m.s. integral Y_t of X_t exists if

$$\lim_{\Delta t_i, \Delta t_k} E((\sum_i X_{t_i} \Delta t_i - \sum_k X_{t_k} \Delta t_k)^2) = 0$$

Since

$$E((\sum_i X_{t_i} \Delta t_i - \sum_k X_{t_k} \Delta t_k)^2) = E(\sum_i \sum_k X_{t_i} X_{t_k} \Delta t_i \Delta t_k + \sum_i \sum_k X_{t_i} X_{t_k} \Delta t_i \Delta t_k - 2 \sum_i \sum_k X_{t_i} X_{t_k} \Delta t_i \Delta t_k)$$

$$E((\sum_i X_{t_i} \Delta t_i - \sum_k X_{t_k} \Delta t_k)^2) = \sum_i \sum_k R_X(t_i, t_k) \Delta t_i \Delta t_k + \sum_i \sum_k R_X(t_i, t_k) \Delta t_i \Delta t_k - 2 \sum_i \sum_k R_X(t_i, t_k) \Delta t_i \Delta t_k$$

Then $\lim_{\Delta t_i, \Delta t_k} E((\sum_i X_{t_i} \Delta t_i - \sum_k X_{t_k} \Delta t_k)^2) = 0$ holds if $\lim_{\Delta t_i, \Delta t_k} \sum_i \sum_k R_X(t_i, t_k) \Delta t_i \Delta t_k$ exists, or

equivalently, $\int_{t_0}^t \int_{t_0}^s R_X(\alpha, \beta) d\beta d\alpha$ exists.

Example(15.13)

Consider a stochastic process $(X_t)_{t \geq 0}$ defined by $X_t = U \cos \lambda t + V \sin \lambda t$ where λ is a positive constant and U, V are independent random variables having normal distributions $N(0, \sigma^2)$.

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Find $P\{\int_0^{\frac{2\pi}{\lambda}} X_t^2 dt > b\}$ where b is constant.

Solution :

$$\text{Let } L = \frac{2\pi}{\lambda}$$

$$\begin{aligned} \int_0^{\frac{2\pi}{\lambda}} X_t^2 dt &= \int_0^L (U \cos \lambda t + V \sin \lambda t)^2 dt \\ &= U^2 \int_0^L \cos^2 \lambda t dt + 2UV \int_0^L \cos \lambda t \sin \lambda t dt + V^2 \int_0^L \sin^2 \lambda t dt = \frac{L}{2}(U^2 + V^2) \end{aligned}$$

$$\text{Since } Z = \frac{1}{\sigma^2}(U^2 + V^2) \sim \chi^2(2)$$

$$P\{\int_0^{\frac{2\pi}{\lambda}} X_t^2 dt > b\} = P\{\frac{L}{2}(U^2 + V^2) > b\} = P\{\frac{1}{\sigma^2}(U^2 + V^2) > \frac{2b}{\sigma^2 L}\} = P\{Z > \frac{\lambda b}{\sigma^2 \pi}\} = \int_{\frac{\lambda b}{\pi \sigma^2}}^{\infty} \frac{1}{2} e^{-\frac{z}{2}} dz = e^{-\frac{\lambda b}{2\pi \sigma^2}}$$

Example(15.14)

If $X = (X_t)_{t \in \mathbb{T}}$ is a normal stochastic process, then its integral Y_t is also a normal stochastic process. This follows from the fact that $\sum_i X_{t_i} \Delta t_i$ is a linear combination of the jointly normal random variables.

Theorem(15.15)

Let $X = (X_t)_{t \in \mathbb{T}}$ be a stochastic process

1. $E(\int_a^b X_t dt) = \int_a^b \mu_X(t) dt$ for all $a < b$
2. $E(|\int_a^b X_t dt|^2) = \int_a^b \int_a^b E(X_s X_t) ds dt$ for all $a < b$
3. $Var(\int_a^b X_t dt) = \int_a^b \int_a^b K_X(s, t) ds dt = 2 \int_a^b \int_a^t K_X(s, t) ds dt$ for all $a < b$
4. $E((\int_a^b X_s ds)(\int_c^d X_t dt)) = \int_a^b \int_c^d E(X_s X_t) dt ds$ for all $a, b, c, d \geq 0$

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$$5. \text{Cov}\left(\int_a^b X_s ds, \int_c^d X_t dt\right) = \int_a^b \int_c^d K_X(X_s, X_t) dt ds \text{ for all } a, b, c, d \geq 0$$

Proof :

$$(1) \text{ Let } a < b, \text{ then } E\left(\int_a^b X_t dt\right) = \int_a^b E(X_t) dt = \int_a^b \mu_X(t) dt$$

$$(2) \text{ Let } a < b, \text{ then } E\left(\left|\int_a^b X_t dt\right|^2\right) = E\left(\int_a^b \int_a^b X_s X_t ds dt\right) = \int_a^b \int_a^b E(X_s X_t) ds dt$$

$$(3) \text{ Let } a < b, \text{ then}$$

$$\begin{aligned} \text{Var}\left(\int_a^b X_t dt\right) &= E\left(\left|\int_a^b X_t dt\right|^2\right) - \left(E\left(\int_a^b X_t dt\right)\right)^2 = E\left(\int_a^b \int_a^b X_s X_t ds dt\right) - \int_a^b \int_a^b \mu_X(s) \mu_X(t) ds dt \\ &= \int_a^b \int_a^b (E(X_s X_t) - \mu_X(s) \mu_X(t)) ds dt = \int_a^b \int_a^b K_X(s, t) ds dt \end{aligned}$$

$$(5) \text{ Let } a, b, c, d \geq 0, \text{ then}$$

$$\begin{aligned} \text{Cov}\left(\int_a^b X_s ds, \int_c^d X_t dt\right) &= E\left(\int_a^b X_s ds\right) E\left(\int_c^d X_t dt\right) - E\left(\int_a^b X_s ds\right) E\left(\int_c^d X_t dt\right) \\ &= \int_a^b \int_c^d E(X_s X_t) ds dt - \int_a^b \int_c^d \mu_X(s) \mu_X(t) ds dt \\ &= \int_a^b \int_c^d (E(X_s X_t) - \mu_X(s) \mu_X(t)) ds dt = \int_a^b \int_c^d K_X(s, t) ds dt \end{aligned}$$

Example(15.16)

Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = U \cos \pi t$, $t \geq 0$ where X is a normal distribution $N(0, \sigma^2)$. find $E\left\{\int_0^1 X_t^2 dt\right\}$

Solution :

$$\int_0^1 X_t^2 dt = \int_0^1 U^2 \cos^2 \pi t dt = U^2 \int_0^1 \cos^2 \pi t dt = \frac{1}{2} U^2$$

$$\text{Since } U \sim N(0, \sigma^2) \Leftrightarrow \frac{1}{\sigma^2} U^2 \sim \chi^2(1) \Rightarrow E\left(\frac{1}{\sigma^2} U^2\right) = 1 \Rightarrow E(U^2) = \sigma^2$$

$$E\left\{\int_0^1 X_t^2 dt\right\} = \frac{1}{2} E(U^2) = \frac{1}{2} \sigma^2$$

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Example(15.17)

If $X = (X_t)_{t \in \mathbb{T}}$ is a Poisson stochastic process with parameter λ , and $Y_t = \int_0^t X_u du$, then

$$E(Y_t) = \int_0^t \mu_X(u) du = \int_0^t \lambda u du = \frac{1}{2} \lambda t^2 \text{ and } Var(Y_t) = 2 \int_0^t \int_0^r K_X(s, r) ds dr = 2 \int_0^t \int_0^r \lambda s ds dr = \lambda \int_0^t r^2 dr = \frac{\lambda t^3}{3}$$

Example(15.18)

If $X = (X_t)_{t \in \mathbb{T}}$ is a Wiener stochastic process with parameter σ^2 , and $Y_t = \int_0^t X_u du$.

Show that

1. $E(Y_t) = 0$
2. $Var(Y_t) = \frac{\sigma^2 t^3}{3}$
3. $R_Y(t, s) = \begin{cases} \frac{1}{6} \sigma^2 s^2 (3t - s), & t > s \geq 0 \\ \frac{1}{6} \sigma^2 t^2 (3s - t), & s > t \geq 0 \end{cases}$

Solution :

Since $X = (X_t)_{t \in \mathbb{T}}$ is a Wiener stochastic process with parameter σ^2 , then

$$\mu_X(t) = E(X_t) = 0 \text{ and } Var(X_t) = \sigma^2 t$$

$$(1) E(Y_t) = \int_0^t \mu_X(u) du = \int_0^t 0 du = 0$$

$$(2) Var(Z_t) = 2 \int_0^t \int_0^r K_X(s, r) ds dr = 2 \int_0^t \int_0^r \sigma^2 s ds dr = \sigma^2 \int_0^t r^2 dr = \frac{1}{3} \sigma^2 t^3$$

$$(3) \text{ Since } E\left(\left(\int_a^b X_s ds\right)\left(\int_c^d X_t dt\right)\right) = \int_a^b \int_c^d E(X_s X_t) dt ds \text{ for all } a, b, c, d \geq 0$$

$$\Rightarrow R_Y(t, s) = E(Y_t Y_s) = \int_0^t \int_0^s E(X_v X_u) du dv = \int_0^t \int_0^s R_X(v, u) du dv$$



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$$\text{Hence } R_Y(t, s) = \begin{cases} \frac{1}{6} \sigma^2 s^2 (3t - s), & t > s \geq 0 \\ \frac{1}{6} \sigma^2 t^2 (3s - t), & s > t \geq 0 \end{cases}$$

Exercises (15)

1. Show that a continuous time stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is m.s. continuous if it is WSS.

2. Show that if $X = (X_t)_{t \in \mathbb{T}}$ is a normal stochastic for which the m.s. derivative

$X' = (X'_t)_{t \in \mathbb{T}}$ exists, then $X' = (X'_t)_{t \in \mathbb{T}}$ is also a normal stochastic process.

3. Let $X = (X_t)_{t \in \mathbb{T}}$ be a WSS stochastic process and let X'_t be its m.s. derivative.

Show that $E(X_t X'_t) = 0$

4. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = U \cos \lambda t$ where λ is a constant and U is a uniform random variable over $(0, 1)$.

(1) Is X m.s. continuous ?

(2) Does X has a m.s. derivative ?

(3) If $Y_t = \frac{2}{T} \int_t^{t+\frac{T}{2}} X_u du$ and $\lambda = \frac{2\pi}{T}$, show that

$$E(Y_t) = -\frac{1}{\pi} \sin \lambda t \quad \text{and} \quad R_Y(t, s) = \frac{4}{3\pi^2} \sin \lambda t \sin \lambda s$$

In exercises (5- 9) . Find $E(\int_0^1 X_t dt)$ and $E(\int_0^1 X_t^2 dt)$

5. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = Ut + V$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.



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6. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = t^2 + Ut + V$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.
7. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = e^{Ut}$ where random variable U is and normal distributed $N(0, \sigma^2)$.
8. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = U \cos \pi t$ where random variable U is normal distributed $N(0, \sigma^2)$.
9. Consider a stochastic process $\{X_t\}_{t \geq 0}$ defined by $X_t = \begin{cases} U, & 0 < t \leq \frac{1}{2} \\ V, & t > \frac{1}{2} \end{cases}$ where random variables U and V are independent and normal distributed $N(0, \sigma^2)$.