History of Complex Analysis

Chong-Kyu Han

October 8, 2009

Numbers

$\mathbb{N}$
$\mathbb{Q}$
negative numbers
irrational numbers $\sqrt{2}$
$\mathbb{C}$

imaginary unit $\sqrt{-1} := i$

Kronecker (Leopold Kronecker, 1823-1891): God created the integers, all else is made by man.
Gauss: The true metaphysics of $\sqrt{-1}$ is not easy.

Atiyah’s viewpoint

Atiyah: Are theorems discovered or invented?

An example of a mathematical idea which, to my mind, represents an invention is $\sqrt{-1}$, the square root of minus one. Since the square of any number (positive or negative) is always positive, there is no number whose square is -1. However, over the centuries, mathematicians found themselves using the fictional number $\sqrt{-1}$ with great success, so much so that they eventually admitted such ’imaginary’ number into their world. A good claim can be made that this was the most inventive step taken in the history of mankind. It opened entirely new doors in mathematics and in the 20th century it was found to be essential in the formulation of quantum mechanics.

Cardano

Girolamo Cardano (1501 - 1576)

Ars magna (1545) Solution of cubic equations,
Nicolo Tartalia (1500-1557)

cubic equation

$x^3 = px + q, \quad p, q \geq 0.$

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

e.g. $\quad x^3 = 15x + 4 \quad \Rightarrow$

$$\begin{aligned} x &= \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}} \\ &= 4. \end{aligned}$$

Split 10 into two pieces so that their products is 40.

17th Century: beginning of modern mathematics

Issac Newton (1643-1727)

Gottfried Lebniz (1646-1716)

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$$

I did not understand how... a quantity could be real, when imaginary or impossible numbers were used to express it.

18th Century: beginning of function theory

Leonhard Euler (1707 - 1783)

Extension of functions to complex variables

Euler: $f(x) \to f(z)$
Functions = Elementary functions $\longrightarrow$ "function theory"
$a_0x^n + a_1x^{n-1} + \cdots$
$a_0z^n + a_1z^{n-1} + \cdots$

Exponential function $e^z = e^{x+iy} = e^x e^{iy}$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$\begin{aligned} e^{it} =& 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \\ =& 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \\ & + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) \\ =& \cos t + i \sin t \end{aligned}$$

symbols Euler used

$\cos t = \frac{e^{it}+e^{-it}}{2}$

$\cos z$
$\sin z$
$\tan z$ tang z

$e^{\pi i} + 1 = 0$

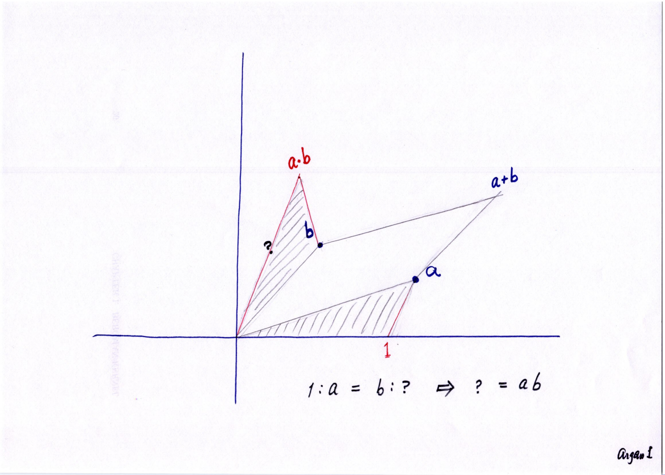
$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n = 2.714567$$

$e, \quad i, \quad \pi, \quad f(x), \quad \Sigma$

complex plane

Complex plane: Caspar Wessel (1745-1818) Norwegian

1813 Jean-Robert Argand, Swiss



de Moivre

$$|a \cdot b| = |a|\,|b|$$

$$\arg(a \cdot b) = \arg a + \arg b$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$n=2$ $\quad \cos^2\theta - \sin^2\theta + i\,2\cos\theta\sin\theta$

$$= \cos(2\theta) + i\sin(2\theta)$$

Abraham de Moivre (1667-1754)

Argan 2

1831 Gauss (1777-1855) sufficiently ripened state concerning "Complex numbers "
Gauss first used the word "complex"

Even in 1880's $\sqrt{-1}$ was suspected at Cambridge in trigonometric formula...

e.g., $\tan z$

Differential equations

2nd order ODE: $y'' = ky$

If $k > 0$, $e^{\sqrt{k}x}, e^{-\sqrt{k}x}$
If $k < 0$, let $k = -\omega^2$, $\cos \omega x$, $\sin \omega x$
In complex exponents, $e^{i\omega x}$, $e^{-i\omega x}$
Fourier analysis, Electric circuit theory

1-dimensional wave equation $u_{xx} - u_{tt} = 0$ for $u(x,t)$

D'Alembert's solution
$f(x-t) + g(x+t)$

what is analogy ?
$u_{xx} + u_{yy} = 0.$

analytic functions

$$\begin{aligned} f(x,y) &= f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \\ &= \phi_1(z) + \phi_2(\bar{z}) + \phi_3(z,\bar{z}) \end{aligned}$$

power series in z
$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$

complex derivative

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z+\triangle z) - f(z)}{\triangle z}$$

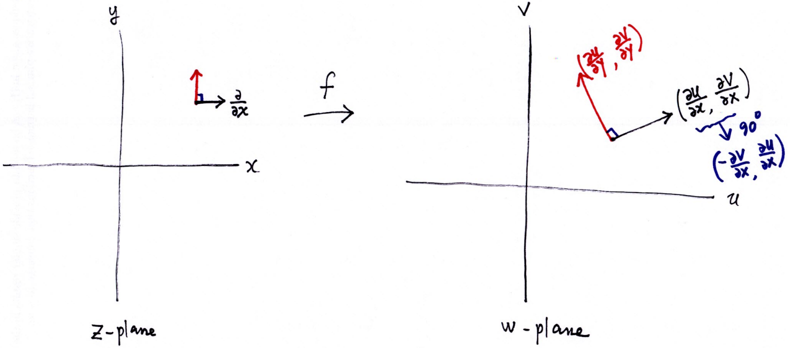
$\triangle z = \triangle x + i\triangle y$

Complex differentiable

Cauchy-Riemann

$$f(z) = u(x,y) + iv(x,y)$$

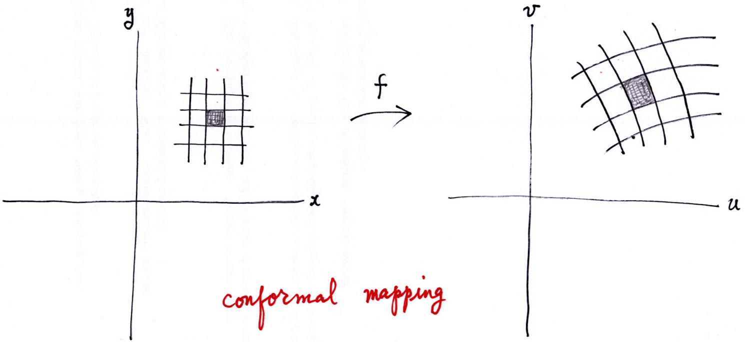
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$



Vector $(a,b) \xrightarrow{90^\circ} (-b,a)$

C.R.
Conformal.

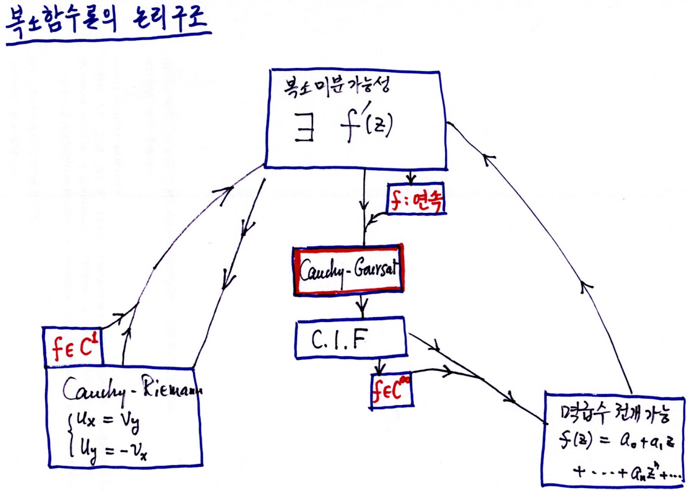
conformal



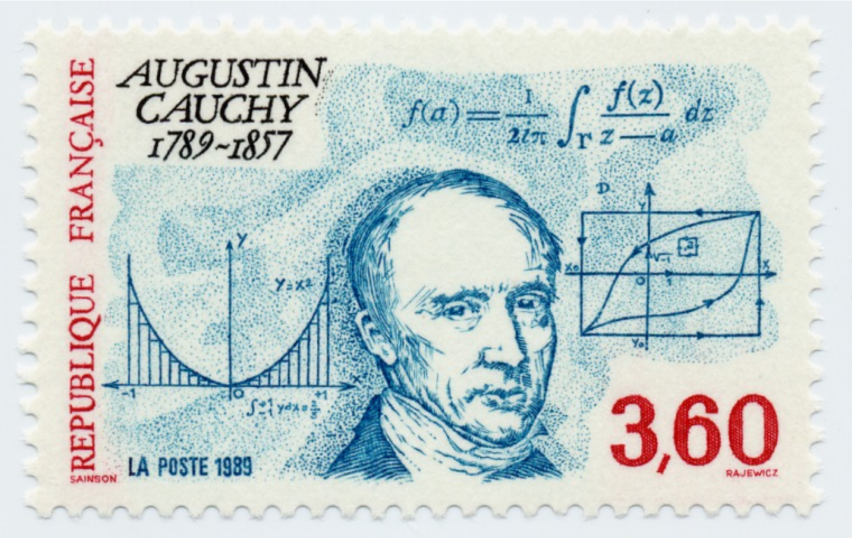
Conformal

frame of logic

Eduard J.-B. Goursat (1858-1936), Course d'anlyse mathématique, 1903±



19th Century: rigorous proof



Augustin Louis Cauchy (1789 - 1857)

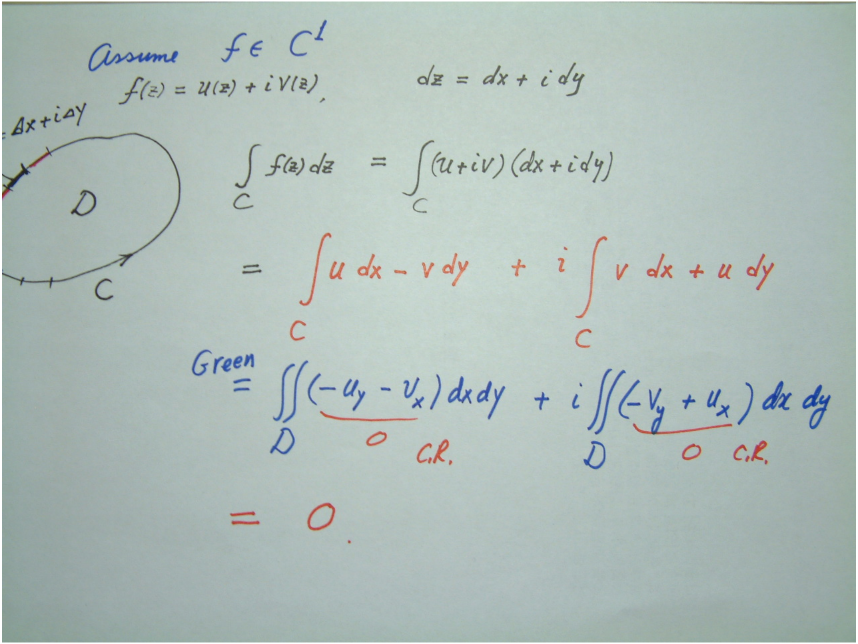
1814 - 1849 (35 years)

$\int_C f(z)dz = 0$

George Green (1793-1841)
Edouard Jean Baptiste Goursat (1858-1936) Cauchy-Goursat theorem

Cauchy's integral formula

1821, Cours d'analyse de l'École Polytechnique , $\epsilon - \delta$ method



Assume $f \in C^1$

$f(z) = u(z) + iv(z)$, $\quad dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$\overset{\text{Green}}{=} \iint_D (\underbrace{-u_y - v_x}_{0}) \,dx\,dy \quad \text{C.R.} \quad + i\iint_D (\underbrace{-v_y + u_x}_{0}) \,dx\,dy \quad \text{C.R.}$$

$$= 0.$$

Convergence of power series

Karl Theodor Wilhelm Weierstrass
1815 - 1897

Convergence of power series, after Abel,
uniform convergence, termwise differentiation, integration

Convergence of infinite products

students: H.A. Schwarz (1848-1921), Reflection principle, Schwarz lemma

Schur, Runge, Schottky, Kovalevskaya, Killing, Fuchs, Frobenius, Cantor

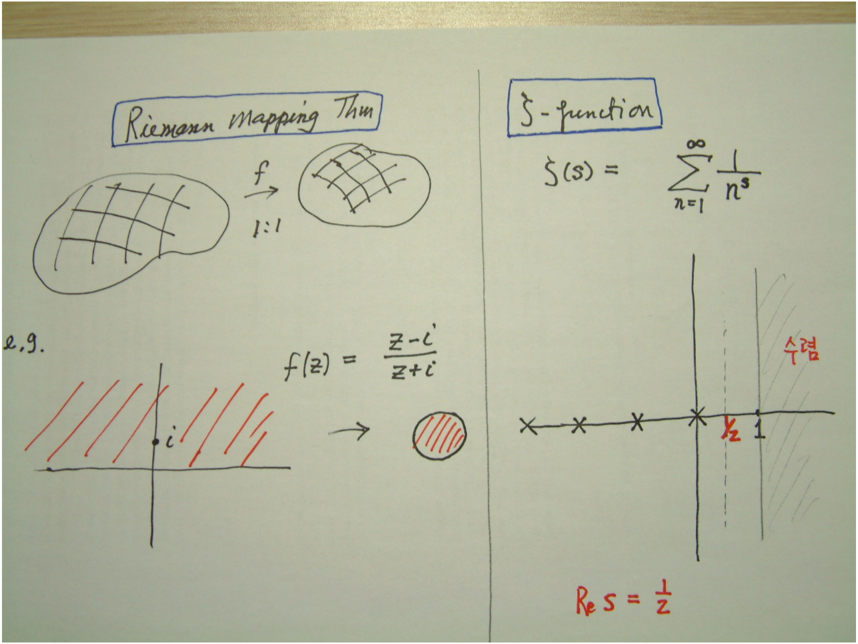
Riemann

Georg Friedrich Bernhard Riemann
1826 - 1866

Cauchy-Riemann equations, conformal mapping

Riemann mapping theorem

Zeta function, Riemann's hypothesis



Riemann mapping Thm

$$\xrightarrow[1:1]{f}$$

e.g.

$$f(z) = \frac{z-i}{z+i}$$

ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\mathrm{Re}\, s = \frac{1}{2}$

applications

to name a few out of hundreds

1) integral of real functions

2)numerical values of infinite series and products

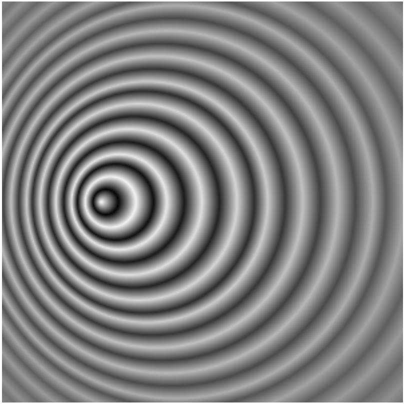
3) electric circuit theory

4) special relativity

5) Schrödinger equation

Lorenz metric

1) 1842 Doppler Effect



2) Maxwell's equations

1861 (James Clerk Maxwell 1831-1879), On physical lines of force

1884 Oliver Heaviside

Lorenz metric 2

1887 Voiqt, in connection with Doppler effect
1895 Lorenz
1897, 1900, Lamor
1904, Lorenz

1905 Poincaré:
named Lorenz transformation,
a symmetry group of Maxwell's equations
quadratic form $x^2 + y^2 + z^2 - c^2t^2$ invariant
fourth coordinate $ct\sqrt{-1}$
analogous to $O(3, \mathbb{R})$ invariant $ds^2 = dx^2 + dy^2 + dz^2$

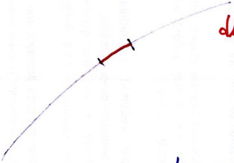
Lorenz metric 3

Lorenz transformation $(x,y,z,t)\mapsto(x',y',z',t')$, $\gamma := \frac{1}{\sqrt{1-(v/c)^2}}$, Lorenz factor

$$\begin{cases} x' &= \gamma(x-vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t-\frac{vx}{c^2}) \end{cases}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbb{R}^4 = \{(x, y, z, \tau)\}$$



$$ds^2 = dx^2 + dy^2 + dz^2 + \underbrace{d\tau^2}$$

$$(ic\,dt)^2 = -c^2\,dt^2$$

$$= -c^2\,dt^2 + dx^2 + dy^2 + dz^2$$

Lorenz metric

Schrödinger equation 1

$\Psi(\vec{x},t)$ wave function,

$$E\Psi = \omega\Psi, \quad E = \frac{1}{2}|\vec{p}|^2 + V$$

$$\Psi(\vec{x},t) = Ae^{i(\vec{p}\bullet\vec{x}-\omega t)}$$

$$\Psi_t = -i\omega\Psi, \quad \triangle\Psi = -\sum_{k=1}^{3}(p_k)^2\Psi.$$

$$i\Psi_t = (-\triangle + V)\Psi, \quad \text{Schrödinger equation}$$

Schrödinger equation 2

$$\Psi_t = i \quad \triangle\Psi$$
$$\Psi_t = \quad\quad \triangle\Psi$$

Initial condition:
$\Psi_0(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}}, \quad \epsilon \to 0$

$$K(x,t) = \frac{1}{\sqrt{4\pi i t}} e^{-\frac{x^2}{4it}}$$
$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

References

1. Paul Nahin, The story of $\sqrt{-1}$, Princeton U. Press, 1998

2. Eric Temple Bell, Men of Mathematics, 1937

translations into Korean

3. Michael Atiyah, Mind, matter and mathematics, 2008. 10.2, Royal Society of Edinburgh, Presidential address

4. Freeman Dyson, Birds and Frogs, Notices of AMS, 2009. 2.