# Complex Analysis and Applications

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**Complex Analysis and Applications**

You can download this textbook from the web site

`http://eul.ink/complex-analysis`

Its contents – lecture notes, exercises with answers and workshops notes – are also available as standalone documents in PDF, HTML and markdown formats.

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# Preface

The textbook "Complex Analysis and Applications" is used in 2017 for Mines ParisTech complex analysis course (C1223 / S1224). This course has now been programmed for more than forty years as a week-long session in Paris and at a ski resort (Les Arcs, for as long as I can remember).

In 2016, I have introduced "workshops" to conclude this session (the last three chapters of the book). Workshops are based on case studies; their main purpose is to bridge the gap between a rather short introduction to complex analysis and some of its applications in engineering. There are many such modern applications and they belong to a wide spectrum of fields such as scientific computing, image editing, signal processing, etc. Many of them may be stated as simple problems but with complex(-analytic) solutions which fortunately require only the modest amount of theory exposed in this course.

In 2017, this textbook also contains a new set of lectures and tutorials, started to "scratch my own itches" (such as an open design process, documents in English, a liberal license, web/digital/print publishing, etc). The new contents – based on my own experience of this course and the valuable feedback of former students and colleagues – should hopefully be pedagogically sound for my usual audience; actually the most non-standard parts were designed for pedagogical reasons. But ultimately, the contents probably reveal my taste and perspective more than anything else. We will see how this new edition goes!

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March 23, 2017

# Chapter 1

# Complex-Differentiability

## Core Definitions

**Definition – Complex-Differentiability & Derivative.** Let $f : A \subset \mathbb{C} \to \mathbb{C}$. The function $f$ is *complex-differentiable* at an interior point $z$ of $A$ if the *derivative* of $f$ at $z$, defined as the limit of the difference quotient

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in $\mathbb{C}$.

**Remark – Why Interior Points?** The point $z$ is an interior point of $A$ if

$$\exists\, r > 0,\ \forall\, h \in \mathbb{C},\ |h| < r \to z + h \in A.$$

In the definition above, this assumption ensures that $f(z+h)$ – and therefore the difference quotient – are well defined when $|h|$ is (nonzero and) small enough. Therefore, the derivative of $f$ at $z$ is defined as the limit in "all directions at once" of the difference quotient of $f$ at $z$. To question the existence of the derivative of $f : A \subset \mathbb{C} \to \mathbb{C}$ at every point of its domain, we therefore require that every point of $A$ is an interior point, or in other words, that $A$ is open.

**Definition – Holomorphic Function.** Let $\Omega$ be an open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is *complex-differentiable* – or *holomorphic* – in $\Omega$ if it is complex-differentiable at every point $z \in \Omega$. If additionally $\Omega = \mathbb{C}$, the function is *entire*.

**Examples – Elementary Functions.**

1. Every constant function $f : z \in \mathbb{C} \mapsto \lambda \in \mathbb{C}$ is holomorphic as

$$\forall\, z \in \mathbb{C},\ f'(z) = \lim_{h \to 0} \frac{\lambda - \lambda}{h} = 0.$$

2. The identity function $f : z \in \mathbb{C} \mapsto z$ is holomorphic:

$$\forall\, z \in \mathbb{C},\ f'(z) = \lim_{h \to 0} \frac{(z+h) - z}{h} = 1.$$

3. The inverse function $f : z \in \mathbb{C}^* \to 1/z$ is holomorphic: the set $\mathbb{C}^*$ is open and for any $z \in \mathbb{C}^*$ and any $h \in \mathbb{C}$ such that $z + h \neq 0$, we have

$$\frac{1/(z+h) - 1/z}{h} = -\frac{1}{z(z+h)},$$

hence

$$f'(z) = \lim_{h \to 0} -\frac{1}{z(z+h)} = -\frac{1}{z^2}.$$

4. The complex conjugate function $f : z \in \mathbb{C} \to \overline{z}$ is nowhere complex-differentiable. Its difference quotient satisfies

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h},$$

therefore when $t \in \mathbb{R}$,

$$\lim_{t \to 0} \frac{f(z+t) - f(z)}{t} = +1,$$

but

$$\lim_{t \to 0} \frac{f(z+it) - f(z)}{it} = -1,$$

hence the difference quotient has no limit when $h \to 0$.

**Remark – Derivatives are Continuous.** The derivative of a holomorphic function is always continuous. This similar result doesn't hold in the context of real analysis: there are some real-valued functions of a real variable that are differentiable and whose derivative is *not* continous[1].

We mention this property now because we will use it to simplify the statements of some results of the current and subsequent chapters. Unfortunately, we cannot prove it yet; it is a consequence of Cauchy's integral theory which is not trivial. To make sure that we won't develop a circular argument, we flag the results that use this property with the symbol [†], until we can prove it.

**Remark – Historical Perspective.** We should not feel too bad about the (temporary) assumption that the derivative is continuous; after all, it was good enough for the best mathematicians of the 19th century.

[1] In the context of real analysis, derivatives can't be totally arbitrary either. They satisfy for example the intermediate value theorem (a property which is weaker than continuity). Refer to (Freiling 1999) for a complete characterization.

A major result of complex analysis, Cauchy's integral theorem, was originally formulated under the assumption that the derivative exists and is continuous[2] (Cauchy 1825). We have to wait for the paper "Sur la définition générale des fonctions analytiques, d'après Cauchy" (Goursat 1900) to officially get rid of this assumption:

> J'ai reconnu depuis longtemps que la démonstration du théorème de Cauchy, que j'ai donnée en 1883, ne supposait pas la continuité de la dérivée. Pour répondre au désir qui m'a été exprimé par M. le Professeur W. F. Osgood, je vais indiquer ici rapidement comment on peut faire cette extension.

which means:

> I have long recognized that the proof of Cauchy's theorem, that I have given in 1883, did not assume the continuity of the derivative. To meet the desire which was expressed to me by Professor W. F. Osgood, I'll tell here quickly how we can make this extension.

Because of this improvement, Cauchy's integral theorem is also known as the "Cauchy-Goursat theorem". Refer to (Hille 1973, footnote p.163) for a broader historical perspective on this subject.

## Derivative and Complex-Differential

**Definition – Real/Complex Vector Space.** A vector space is *real* or *on* $\mathbb{R}$ if its field of scalars is the real line; it is *complex* or *on* $\mathbb{C}$ if its field of scalars is the complex plane.

**Remark – Complex Vector Spaces are Real Vector Spaces.** If the set $E$ is endowed with a structure of complex vector space, it is automatically endowed with a structure of real vector space. Because of this ambiguity, it may be necessary to qualify the usual concepts of linear algebra – for example to say that a function is *real-linear* or *complex-linear* instead of simply linear – to be totally explicit about the structure to which we refer.

**Example – The Complex Plane.** The set $\mathbb{C}$ is a complex vector space with the sum

$$(x+iy)+(v+iw)=(x+v)+i(y+w)$$

and scalar-vector multiplication

$$(\mu+i\nu)(x+iy)=(\mu x-\nu y)+i(\mu y+\nu x)$$

It is of dimension 1 with for example $\{1\}$ (the single vector $1\in\mathbb{C}$) as a basis; indeed every complex number $z\in\mathbb{C}$ is a linear combination of the vectors of

[2] To be honest, this assumption is only implicit, but you can't really blame Cauchy for this lack of precision. The standards of quality for Mathematics in the 19th century were quite different from the present ones.

$\{1\}$ (as $z = z1$), and the vectors of $\{1\}$ are linearly independent (the only scalar $\lambda \in \mathbb{C}$ such that $\lambda 1 = 0$ is $\lambda = 0$).

Note how things change if we consider the plane as a real vector space: it is of dimension 2 with for example $\{1, i\}$ as a basis. In particular, the vectors 1 and $i$ which are complex-colinear and not real-colinear.

**Definition – Complex-Linearity.** Let $E$ and $F$ be complex normed vector spaces. A function $\ell : E \to F$ is *complex-linear* if it is additive and *complex-homogeneous*:

$$\forall\, u \in E,\ \forall\, v \in E,\ \ell(u+v) = \ell(u) + \ell(v),$$

$$\forall\, \lambda \in \mathbb{C},\ \forall\, u \in E,\ \ell(\lambda u) = \lambda \ell(u).$$

**Definition – Complex-Differential.** Let $f : A \subset E \to F$ where $E$ and $F$ are complex normed vector spaces. Let $z$ be an interior point of $A$; the *complex-differential* of $f$ at $z$ is a complex-linear continuous operator $df_z : E \to F$ such that

$$\lim_{h \to 0} \frac{\|f(z+h) - f(z) - df_z(h)\|}{\|h\|} = 0,$$

or equivalently in expanded form

$$f(z+h) = f(z) + df_z(h) + \epsilon_z(h)\|h\| \ \text{ with } \ \lim_{h \to 0} \epsilon_z(h) = \epsilon_z(0) = 0.$$

If such an operator exists, it is unique.

**Remark – Real/Complex-Differentiability.** Since the complex vector spaces $E$ and $F$ are real vector spaces, we may also use the classic concept of differential from Real Analysis for the function $f : E \to F$. We call this operator *real-differential* to avoid any ambiguity with the complex-differential. The definitions of both operators are identical, except that the complex-differential is required to be a complex-linear operator when the real-differential is only required to be real-linear.

**Proof – Uniqueness.** If for small values of $h$ the function $f$ satisfies

$$f(z+h) = f(z) + \ell_z^1(h) + \epsilon_z^1(h)\|h\| = f(z) + \ell_z^2(h) + \epsilon_z^2(h)\|h\|$$

with continuous linear operators $\ell_z^1$ and $\ell_z^2$ and functions $\epsilon_z^1$ and $\epsilon_z^2$ such that

$$\lim_{h \to 0} \epsilon_z^1(h) = \epsilon_z^1(0) = 0 \ \text{ and } \ \lim_{h \to 0} \epsilon_z^2(h) = \epsilon_z^2(0) = 0,$$

then for any $u \in E$

$$(\ell_z^1 - \ell_z^2)(u) = \lim_{t \to 0} \frac{(\ell_z^1 - \ell_z^2)(tu)}{t} = \lim_{t \to 0} (\epsilon_z^2(tu) - \epsilon_z^1(tu))\|u\| = 0$$

and consequently $\ell_z^1 = \ell_z^2$. ■

**Theorem – Derivative and Differential.** Let $f : A \to \mathbb{C}$ with $A \subset \mathbb{C}$ and let $z$ be an interior point of $A$. The complex-differential $df_z$ exists if and only if the derivative $f'(z)$ exists. In this case, we have

$$\forall\, h \in \mathbb{C},\, df_z(h) = f'(z)h.$$

**Proof.** If $f'(z)$ exists, the mapping $h \in \mathbb{C} \mapsto f'(z)h$ is complex-linear and

$$\lim_{h\to 0} \frac{|f(z+h) - f(z) - f'(z)h|}{|h|} = \lim_{h\to 0} \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| = 0,$$

hence, it is the differential of $f$ at $z$. Conversely, if $df_z$ exists, its complex-linearity yields $df_z(h) = df_z(1)h$. Therefore,

$$\lim_{h\to 0} \left| \frac{f(z+h) - f(z)}{h} - df_z(1) \right| = \lim_{h\to 0} \frac{|f(z+h) - f(z) - df_z(h)|}{|h|} = 0$$

thus $f'(z)$ exists and is equal to $df_z(1)$. ■

# Calculus

**Theorem – Sum and Product Rules.** Let $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ with $A \subset \mathbb{C}$ and let $z$ an interior point of $A$. If $f$ and $g$ are complex-differentiable at $z$, the derivative of $f + g$ at $z$ exists and

$$(f + g)'(z) = f'(z) + g'(z),$$

the derivative of $f \times g$ at $z$ exists and

$$(f \times g)'(z) = f'(z) \times g(z) + f(z) \times g'(z).$$

**Proof.** For any $h \in \mathbb{C}$ such that $z + h \in A$, we have

$$\frac{(f+g)(z+h) - (f+g)(z)}{h} = \frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h},$$

hence the derivative of $f + g$ at $z$ exists and satisfies the sum rule. On the other hand,

$$\frac{(f\times g)(z+h) - (f\times g)(z)}{h} = \frac{f(z+h) - f(z)}{h} g(z) + f(z+h) \frac{g(z+h) - g(z)}{h},$$

hence the derivative of $f \times g$ exists and satisfies the product rule. ■

**Theorem – Chain Rule.** Let $f : A \to \mathbb{C}$, $g : B \to \mathbb{C}$ with $A, B$ two subsets of $\mathbb{C}$. If $z$ is an interior point of $A$, $f$ is complex-differentiable at $z$, $f(z)$ is an interior point of $B$ and $g$ is complex-differentiable at $f(z)$, then $g \circ f$ is complex-differentiable at $z$ and

$$(g \circ f)'(z) = g'(f(z)) \times f'(z).$$

**Proof.** Given the assumption, we have for $h$ small enough

$$f(z+h) - f(z) = f'(z)h + \epsilon_z^1(h)|h|$$

and

$$g(f(z)+h) - g(f(z)) = g'(f(z))h + \epsilon_{f(z)}^2(h)|h|$$

with

$$\lim_{h\to 0} \epsilon_z^1(h) = \epsilon_z^1(h) = 0 \text{ and } \lim_{h\to 0} \epsilon_{f(z)}^2(h) = \epsilon_{f(z)}^2(0) = 0.$$

Consequently,

$$\begin{aligned} g(f(z+h)) - g(f(z)) = g'(f(z))&(f(z+h) - f(z)) \\ &+ \epsilon_{f(z)}^2(f(z+h) - f(z))|f(z+h) - f(z)|, \end{aligned}$$

which can be expanded into

$$g(f(z+h)) - g(f(z)) = g'(f(z))f'(z)h + \epsilon_z^3(h)|h|,$$

where

$$\epsilon_z^3(h) = g'(f(z))\epsilon_z^1(h) + \epsilon_{f(z)}^2(f(z+h) - f(z)) \left| f'(z)\frac{h}{|h|} + \epsilon_z^1(h) \right|$$

and satisfies

$$\lim_{h\to 0} \epsilon_z^3(h) = \epsilon_z^3(0) = 0.$$

This decomposition proves the existence of the complex-differential of $g \circ f$ at $z$ as well as the equality $(g \circ f)'(z) = g'(f(z))f'(z)$. ■

**Corollary – Quotient Rule.** Let $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ with $A \subset \mathbb{C}$ and let $z$ be an interior point of $A$ such that $g(z) \neq 0$. If $f$ and $g$ are complex-differentiable at $z$, then $f/g$ is complex-differentiable at $z$ and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

**Proof.** By the chain rule applied to the function $g$ and $z \mapsto 1/z$, the derivative of $1/g$ is $-g'/g^2$. The desired result then follows from the product rule. ■

**Examples – Polynomials & Rational Functions.** Any polynomial $p$ with complex coefficients

$$p : z \in \mathbb{C} \mapsto a_0 + a_1 z + \cdots + a_n z^n$$

is holomorphic on $\mathbb{C}$ as the sum of products of holomorphic functions. By the quotient rule, the quotient of two polynomials $p$ and $q$ – with a non-zero $q$ – is also holomorphic on the open set $\{z \in \mathbb{C} \mid q(z) \neq 0\}$.

# Cauchy-Riemann Equations

It is sometimes convenient to remember that the set $\mathbb{C}$ is only $\mathbb{R}^2$, or in other words that we can always identify the complex number $z = x + iy$ with the pair of real numbers $(x, y)$.

For complex-valued functions of a complex variable, we can perform this identification for the variables $z = x + iy$ and/or for the values $f = u + iv$. Both options are actually interesting and lead to slightly different characterizations of holomorphic functions.

**Theorem – Cauchy-Riemann Equations.** Let $\Omega$ be an open subset of $\mathbb{C}$. The function $f = u + iv : \Omega \to \mathbb{C}$ is complex-differentiable on $\Omega$ if and only if:

- it is real-differentiable on $\Omega$ and
- for every $z$ in $\Omega$, $df_z$ is complex-linear.

The second clause may be replaced by any of the following:

1. the function $f$ satisfies

$$\forall\, z \in \Omega,\ df_z(i) = i df_z(1),$$

2. the function $f$ satisfies the *(complex) Cauchy-Riemann equation*:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y},$$

3. the functions $u$ and $v$ satisfy the *(scalar) Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \ \text{ and } \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If the complex-differentiability holds, we have

$$\begin{aligned} f' = (z \mapsto d_z f(1)) &= \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \end{aligned}$$

**Remark – A Geometric Insight.** We may rewrite the scalar Cauchy-Riemann equations as

$$\left[\begin{array}{c} \partial v/\partial x \\ \partial v/\partial y \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ +1 & 0 \end{array}\right] \left[\begin{array}{c} \partial u/\partial x \\ \partial u/\partial y \end{array}\right].$$

This formula provides the following insight: the gradient of the imaginary part of a holomorphic function is obtained by a rotation of $\pi/2$ of the gradient of its real part.

**Proof.** The equivalence between complex-differentiability and the combination of real-differentiability and complex-linearity of the differential is a direct consequence of the definitions.

Assume that $f$ is real-differentiable; the real-differential $\ell = df_z$ is real-linear, that is, additive and real-homogeneous. If additionally $\ell(i) = i\ell(1)$, then we have for any real numbers $\mu$, $\nu$, $x$ and $y$

- $\ell(\mu x) = \mu\ell(x)$,
- $\ell(i\nu x) = \nu x\ell(i) = \nu x i\ell(1) = i\nu\ell(x)$,
- $\ell(i\mu y) = \mu\ell(iy)$,
- $\ell(-\nu y) = -\nu y\ell(1) = i^2\nu y\ell(1) = i\nu y\ell(i) = i\nu\ell(iy)$.

The function $\ell$ is additive, hence $\ell((\mu + i\nu)(x + iy)) = (\mu + i\nu)\ell(x + iy)$: the function $\ell$ is complex-homogeneous and therefore complex-linear. Hence, property 1 yields the complex-linearity of the differential.

As additionally

$$\frac{\partial f}{\partial x}(z) = df_z(1),\ \frac{\partial f}{\partial y}(z) = df_z(i),$$

properties 1 and 2 are equivalent.

The function $f = (u, v) = u + iv$ is real-differentiable if and only if $u$ and $v$ are real-differentiable. In this case, $df = (du, dv) = du + idv$, hence

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},\ \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

which yields the equivalence between properties 2 and 3. ■

There is a variant of this theorem that does not require to check explicitly for the existence of the real-differential:

**Corollary – Cauchy-Riemann Equations (Alternate)** [†]. Let $\Omega$ be an open subset of $\mathbb{C}$. The function $f = u + iv : \Omega \to \mathbb{C}$ is complex-differentiable in $\Omega$ if and only if any of the following conditions holds:

1. The partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist, are continuous and

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

2. The partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ exist, are continuous and

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

**Proof.** If the partial derivatives of $f$ exist and are continous, or equivalently the partial derivatives of $u$ and $v$ exist and are continuous, then $f$ is continuously real-differentiable and we can apply the previous theorem to get our conclusion.

Reciprocally, if the derivative of $f$ exist, then it is continuous, hence the partial derivatives of $f$ (or of $u$ and $v$) are continous. The previous theorem also shows that the Cauchy-Riemann equations are satisfied. ■

**Definition & Example – Exponential.** The *exponential* function $\exp : \mathbb{C} \to \mathbb{C}$ is defined as

$$\exp(x + iy) = e^x \times (\cos y + i \sin y).$$

The exponential function satisfies on one hand

$$\frac{\partial \exp(x + iy)}{\partial x} = e^x \times (\cos y + i \sin y) = \exp(x + iy)$$

and on the other hand

$$\frac{1}{i} \frac{\partial \exp(x + iy)}{\partial y} = \frac{1}{i} e^x \times (-\sin y + i \cos y) = \exp(x + iy).$$

Both partial derivatives exist and are continuous. They also satisfy the Cauchy-Riemann equation, hence exp is complex-differentiable and

$$\exp'(z) = \exp(z).$$

**Definition & Example – Logarithm.** The *principal value of the logarithm* is the function $\log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ defined by

$$\log re^{i\theta} = (\ln r) + i\theta, \ r > 0, \ \theta \in \left]-\pi, \pi\right[.$$

It is a bijection from $\mathbb{C} \setminus \mathbb{R}_-$ into $\mathbb{R} \times \left]-\pi, +\pi\right[$ and for every $z \in \mathbb{C} \setminus \mathbb{R}_-$

$$\exp \circ \log(z) = z.$$

The exponential function is continuously real-differentiable and

$$d\exp_z(h) = (\exp z) \times h$$

hence its differential $d\exp_z$ is invertible. By the inverse function theorem, log is real-differentiable on $\mathbb{C} \setminus \mathbb{R}_-$ and

$$d\exp_{\log z} \circ d\log_z(h) = z \times d\log_z(h) = h.$$

Consequently, $d\log_z(h) = h/z$, which is a complex-linear function of $h$. Hence, log is complex-differentiable and

$$\log'(z) = \frac{1}{z}.$$

## Appendix – Terminology and Notation

It is common to use of the word "holomorphic" and the notation $\mathcal{H}(\Omega)$ to refer to functions that are complex-differentiable on some open set $\Omega$; it is for example the convention of the classic "Real and Complex Analysis" book (Rudin 1987).

The term "holomorphic" appears in "Théorie des fonctions elliptiques" (1875), by Charles Briot & Claude Bouquet, two students of Augustin-Louis Cauchy:

> Lorsqu'une fonction est continue, monotrope, et a une dérivée, quand la variable se meut dans une certaine partie du plan, nous dirons qu'elle est *holomorphe* dans cette partie du plan. Nous indiquons par cette dénomination qu'elle est semblable aux fonctions entières qui jouissent de cette propriété dans toute l'étendue du plan.

In essence, a function is holomorphic if it is continuous, single-valued and differentiable in a subset of the complex plane. The prefix "holo" (from ancient Greek) means "entire"; it makes sense because such a function is similar to polynomials, which have these properties in the full complex plane and were called "entire functions" in the 19th century[3].

The most common alternate notations and terms used in the literature to refer to holomorphic functions probably are:

- $\mathcal{A}(\Omega)$. The symbol "A" refers to the term "analytic"; it is often used interchangeably with the term "holomorphic". Originally, "analytic" means "locally defined as a power series", but both concepts actually refer to the same class of functions. This is a classic – but not trivial – result of complex analysis.

- $C^\omega(\Omega)$. Another result of the theory of analytic functions: analytic functions are "more than smooth": they all belong to the set $C^\infty(\Omega)$ of smooth functions, but not every smooth function is analytic. Hence, it makes sense to use the symbol $\omega$ – that denotes the smallest infinite ordinal number – as an exponent.

- $\mathcal{O}(\Omega)$ (used e.g. in "Theory of Complex Functions" by Remmert (1991)). Jean-Pierre Demailly (2009) traces the origin of this notation to the word "olomorfico" ("holomorphic" in Italian), but Hans Grauert and Reinhold Remmert (1984) have a different analysis: the symbol "O" may have been chosen by Henri Cartan, which is quoted saying (in French) that:

  > Je m'étais simplement inspiré d'une notation utilisée par van der Waerden dans son classique traité "Moderne Algebra" (cf. par exemple §16 de la 2e édition allemande, p.52)

  which means

[3] Nowadays, a function is entire if is defined and holomorphic on $\mathbb{C}$, which is a more consistent definition. Polynomials are still entire functions, but they are not the only ones.

> I simply took inspiration from a notation used by van der Waerden in his classic treatise "Modern Algebra" (see e.g. §16 of the 2nd german edition, p.52)

If this interpretation is correct, then the symbol "O" probably comes from the word "ordnüng" ("order" in German).

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# Exercises

## Antiholomorphic Functions

A function $f : \Omega \to \mathbb{C}$ is *antiholomorphic* if its complex conjugate $\overline{f}$ is holomorphic.

1. Is the complex conjugate function $c : z \in \mathbb{C} \mapsto \overline{z}$ real-linear? complex-linear? Is it real-differentiable? holomorphic? antiholomorphic?
2. Show that any antiholomorphic function $f$ is real-differentiable. Relate the differential of such a function and the differential of its complex conjugate.
3. Find the variant of the Cauchy-Riemann equation applicable to antiholomorphic functions.
4. What property has the composition of two antiholomorphic functions?
5. Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function; show that the function

$$g : z \in \overline{\Omega} \mapsto \overline{f(\overline{z})}$$

is holomorphic and compute its derivative.

## Principal Value of the Logarithm

According to the definition of log, for any $x + iy \in \mathbb{C} \setminus \mathbb{R}_-$,

$$\log(x+iy) = \ln \sqrt{x^2+y^2} + i \arg(x+iy),$$

where $\arg : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ is *the principal value of the argument*:

$$\forall\, z \in \mathbb{C} \setminus \mathbb{R}_-,\ \arg z \in \left]-\pi, \pi\right[ \ \wedge\ e^{i \arg z} = \frac{z}{|z|}.$$

1. Show that

$$\arg(x+iy) = \left| \begin{array}{cl} \arctan y/x & \text{if } x > 0, \\ +\pi/2 - \arctan x/y & \text{if } y > 0, \\ -\pi/2 - \arctan x/y & \text{if } y < 0. \end{array} \right.$$

2. Show that the function log is holomorphic and compute its derivative.

## Conformal Mappings

A $\mathbb{R}$-linear mapping $L : \mathbb{C} \to \mathbb{C}$ is *angle-preserving* if $L$ is invertible and

$$\forall\, \theta \in \mathbb{R},\ \exists\, \alpha_\theta > 0,\ L(e^{i\theta}) = \alpha_\theta \times e^{i\theta} L(1).$$

A $\mathbb{R}$-differentiable function $f : \Omega \to \mathbb{C}$ *(locally) angle-preserving* – or *conformal* – if its differential is angle-preserving everywhere.

1. Show that an invertible $\mathbb{R}$-linear mapping $L : \mathbb{C} \to \mathbb{C}$ is angle-preserving if and only if it is $\mathbb{C}$-linear.
2. Identify the class of conformal mappings defined on $\Omega$.

## Directional Derivative

Source: Mathématiques III, Francis Maisonneuve, Presses des Mines.

Let $f$ be a complex-valued function defined in a neighbourhood of a point $z_0 \in \mathbb{C}$. Assume that $f$ is $\mathbb{R}$-differentiable at $z_0$.

1. Let $\alpha \in \mathbb{R}$ and $z_{r,\alpha} = z_0 + re^{i\alpha}$ for $r \in \mathbb{R}$. Show that

$$\ell_\alpha = \lim_{r \to 0} \frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0}$$

   exists and determine its value as a function of $df_{z_0}$ and $\alpha$.

2. What is the geometric structure of the set $A = \{\ell_\alpha \mid \alpha \in \mathbb{R}\}$ ?
3. For which of these sets $A$ is $f$ $\mathbb{C}$-differentiable at $z_0$?

# Chapter 2

# Line Integrals & Primitives

## Introduction

The main goal of this chapter is to derive the fundamental theorem of calculus for functions of a complex variable. This theorem characterizes the relation between functions and their primitives with the help of integrals. A version of this theorem for functions of a real variable is the following:

**Theorem – Fundamental Theorem of Calculus (Real Analysis).** Let $I$ be an open interval of $\mathbb{R}$, $f : I \to \mathbb{R}$ be a continuous function and $a \in I$. A function $g : I \to \mathbb{R}$ is a primitive of $f$ if and only if it satisfies

$$\forall\, x \in I,\ g(x) = g(a) + \int_a^x f(t)\, dt.$$

**Proof.** Suppose that the function $g$ satisfies the integral equation of the theorem. For any $x \in I$ and any real number $h$ such that $x + h \in I$,

$$\begin{aligned}\frac{g(x+h) - g(x)}{h} &= \frac{1}{h} \int_x^{x+h} f(t)\, dt \\ &= \frac{1}{h} \int_x^{x+h} f(x)\, dt + \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt \\ &= f(x) + \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt,\end{aligned}$$

Let $\epsilon > 0$; by continuity of $f$ at $x$, there is a $\delta > 0$ such that

$$\forall\, t \in I,\ (|t - x| \leq \delta \Rightarrow |f(t) - f(x)| < \epsilon)$$

thus if $|h| < \delta$,

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| \leq \frac{1}{|h|} |h| \times \epsilon = \epsilon.$$

The difference quotient tends to $f(x)$ when $h$ tends to zero: $g'(x)$ exists and is equal to $f(x)$.

Conversely, suppose that $e : I \to \mathbb{R}$ is a primitive of $f$. The difference $d$ between $e$ and the function

$$g : x \in I \mapsto e(a) + \int_a^x f(t)\, dt$$

is zero at $a$ and has a zero derivative on $I$. By the mean value theorem, for any $x \in I$ such that $x \neq a$, there is a $b \in I$ such that

$$\frac{d(x) - d(a)}{x - a} = d'(b) = 0,$$

hence $d(x) = d(a) = 0$ and therefore $e = g$. ■

## Paths

**Definition – Path.** A *path* $\gamma$ is a continuous function from $[0, 1]$ to $\mathbb{C}$. If $A$ is a subset of the complex plane, $\gamma$ is a *path of* $A$ if additionally $\gamma([0, 1]) \subset A$.

**Definition – Image of a Path.** The *image* or *trajectory* or *trace* of the path $\gamma$ is the image $\gamma([0, 1])$ of the interval $[0, 1]$ by the function $\gamma$.

**Definition – Path Endpoints.** The complex numbers $\gamma(0)$ and $\gamma(1)$ are the *initial point* and *terminal point* of $\gamma$ – they are its *endpoints*; the path $\gamma$ *joins* its initial and terminal points. The path is *closed* if the initial and terminal point are the same. The paths $\gamma_1, \dots, \gamma_n$ are *consecutive* if for $k = 1, \dots, n-1$, the terminal point of $\gamma_k$ is the initial point of $\gamma_{k+1}$.

**Example – Oriented Line Segment.** The oriented line segment (or simply oriented segment) with initial point $a \in \mathbb{C}$ and terminal point $b \in \mathbb{C}$ is denoted $[a \to b]$ and defined as

$$[a \to b] : t \in [0, 1] \mapsto (1 - t)a + tb.$$

Its image is the line segment $[a, b]$.

**Example – Oriented Circle.** The oriented circle of radius one centered at the origin traversed once in the positive sense (counterclockwise) is denoted $[\circlearrowleft]$ and defined as

$$[\circlearrowleft] : t \in [0, 1] \to e^{i2\pi t}.$$

The circle of radius $r \geq 0$ centered at $c \in \mathbb{C}$ traversed $n \in \mathbb{Z}^*$ times in the positive sense is the path

$$c + r[\circlearrowleft]^n : t \in [0, 1] \to c + re^{i2\pi nt}.$$

Its image is the circle centered on $c$ with radius $r$; its initial and terminal points are both $c + r$, hence it is closed.

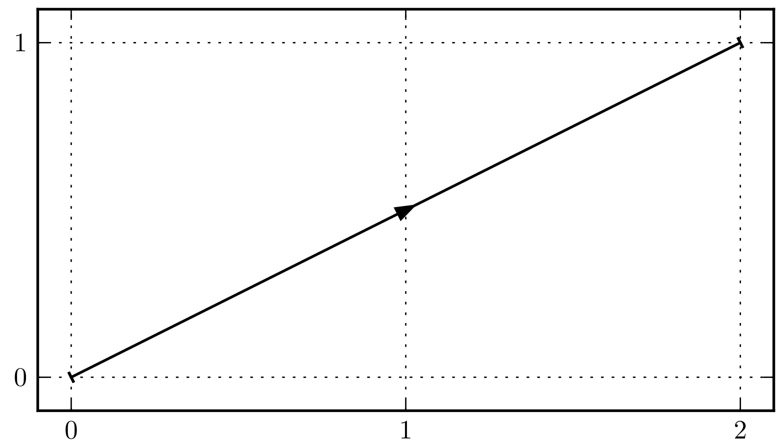


Figure 2.1: Representation of the oriented line segment $[0 \to 2 + i]$

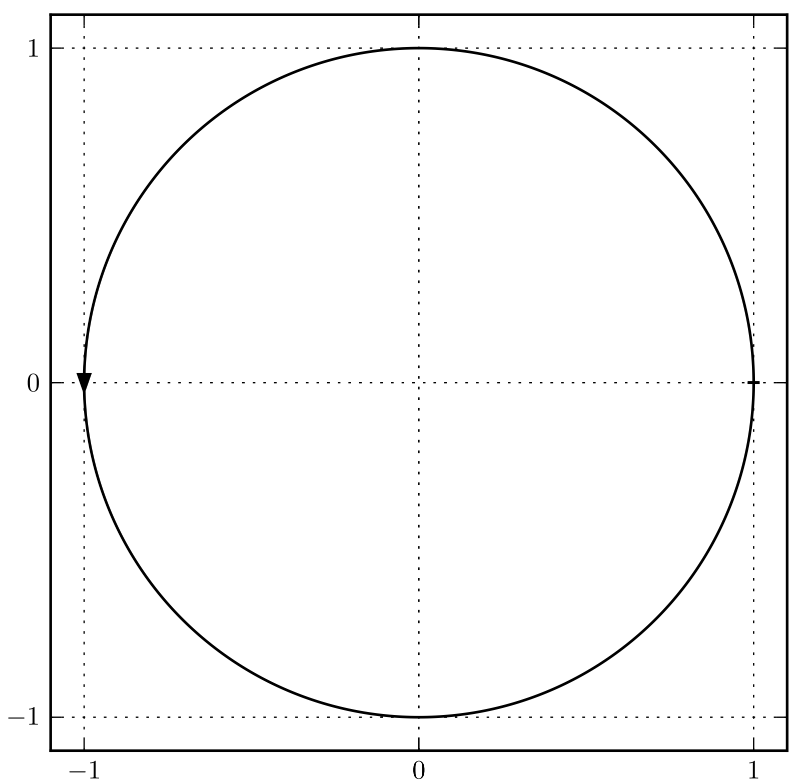


Figure 2.2: Representation of the oriented circle $[\circlearrowleft]$

**Definition – Open (Path-)Connected Sets.** An open subset $\Omega$ of the complex-plane is *(path-)connected* if for any points $x$ and $y$ of $\Omega$, there is a path of $\Omega$ that joins $x$ and $y$.

**Definition – Reverse Path.** The *reverse* (or *opposite*) of the path $\gamma$ is the path $\gamma^{\leftarrow}$ defined by

$$\forall\, t \in [0,1],\ \gamma^{\leftarrow}(t) = \gamma(1-t).$$

**Definition – Path Concatenation.** Let $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ be a partition of the interval $[0,1]$. The *concatenation* of consecutive paths $\gamma_1, \dots, \gamma_n$ associated to this partition is the path $\gamma$ denoted

$$\gamma_1 \,|_{t_1}\, \cdots \,|_{t_{n-1}}\, \gamma_n$$

such that

$$\forall\, k \in \{1, \dots, n\},\ \gamma|_{[t_{k-1}, t_k]} = \gamma_k \left( \frac{t - t_{k-1}}{t_k - t_{k-1}} \right).$$

If the partition of $[0,1]$ is uniform, that is, if

$$\forall\, k \in \{0, \dots, n\},\ t_k = k/n,$$

we denote the concatenated path with the simpler notation

$$\gamma_1 \,|\, \cdots \,|\, \gamma_n.$$

**Example – Oriented Polyline.** An oriented polyline (or piecewise linear path) is the concatenation of consecutive oriented line segments. When the associated partition of $[0,1]$ is uniform, we use the notation

$$[a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n] = [a_0 \to a_1] \,|\, \cdots \,|\, [a_{n-1} \to a_n].$$

**Definition – Rectifiable Path.** A path $\gamma : [0,1] \to \mathbb{C}$ is *rectifiable* if the function $\gamma$ is piecewise continuous differentiable, that is, if there are consecutive continuously differentiable paths $\gamma_1, \dots, \gamma_n$ and a partition $(t_0, \dots, t_n)$ of the interval $[0,1]$ such that

$$\gamma = \gamma_1 \,|_{t_1}\, \cdots \,|_{t_{n-1}}\, \gamma_n.$$

We characterized initially connected sets via merely continuous paths. However, when such sets are open, we can use rectifiable paths instead:

**Lemma – Connectedness & Rectifiable Paths.** An open subset $\Omega$ of the complex plane is *connected* if and only if every pair of points of $\Omega$ may be joined by a rectifiable path of $\Omega$.

**Proof.** If any pair of points of $\Omega$ can be joined by a rectifiable path of $\Omega$, then $\Omega$ is connected. Conversely, assume that a (merely continuous) path $\gamma$ of $\Omega$ joins $x$ and $y$. Its image $\gamma([0,1])$ is a compact subset of $\Omega$ – as the image of a compact set by a continuous function – thus the distance $r$ between $\gamma([0,1])$ and the closed set $\mathbb{C} \setminus \Omega$ is positive. Additionally, the function $\gamma$ is uniformly continuous – as a continuous function with a compact domain of definition; there is a positive integer $n$ such that

$$\forall\, t \in [0,1],\ \forall\, s \in [0,1],\ (|t-s| \leq 1/n \,\Rightarrow\, |\gamma(t) - \gamma(s)| < r).$$

For any $k \in \{0, \dots, n\}$, the point $\gamma(k/n)$ belongs to $\Omega$; the path $\mu$ defined as

$$\mu = [\gamma(0) \to \cdots \to \gamma(k/n) \to \cdots \to \gamma(1)]$$

is rectifiable and joins $x$ and $y$. Now, for any $t \in [0,1]$, let $k \in \{0, \dots, n-1\}$ be such that $t \in [k/n, (k+1)/n]$. We have

$$|\mu(t) - \gamma(k/n)| \leq |\gamma((k+1)/n) - \gamma(k/n)| < r,$$

therefore $\mu$ is a path of $\Omega$. ■

# Line Integrals

**Definition – Length of a Rectifiable Path.** The length of a continuously differentiable path $\gamma$ is the nonnegative real number

$$\ell(\gamma) = \int_0^1 |\gamma'(t)|\, dt.$$

The length of a rectifiable path $\gamma = \gamma_1 \,|_{t_1} \cdots |_{t_{n-1}} \gamma_n$ – where every $\gamma_k$ is continuously differentiable – is the nonnegative real number

$$\ell(\gamma) = \sum_{k=1}^{n} \ell(\gamma_k).$$

**Example – Length of an Oriented Segment.** The oriented segment $[a \to b]$ is continuously differentiable and thus rectifiable. For any $t \in [0,1]$, $[a \to b]'(t) = b - a$, hence its length is

$$\ell([a \to b]) = \int_0^1 |b-a|\, dt = |b-a|.$$

**Example – Length of an Oriented Circle.** The oriented circle $c + r[\circlearrowleft]^n$ centered at $c$ with radius $r \geq 0$ traversed $n$ times in the positive sense is continuously differentiable and thus rectifiable. For any $t \in [0,1]$,

$$[c + r[\circlearrowleft]^n]'(t) = (i2\pi n) r e^{i2\pi n t},$$

hence the length of this path is

$$\ell(c + r[\circlearrowleft]^n) = \int_0^1 |(i2\pi n) r e^{i2\pi n t}|\, dt = \int_0^1 |2\pi n r|\, dt = 2\pi r \times |n|.$$

It differs from the length of its circle image – which is $2\pi r$ – unless the circle is traversed exactly once in the positive or negative sense.

**Definition – Line Integral.** The *line integral* along a continuously differentiable path $\gamma$ of a complex-valued function $f$ defined and continuous on the image of $\gamma$ is the complex number defined by

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt.$$

If $\gamma$ is the rectifiable path $\gamma_1 \,|_{t_1} \cdots |_{t_{n-1}} \gamma_n$ – where every $\gamma_k$ is continuously differentiable – then the line integral along $\gamma$ of $f$ is defined as the sum of the line integrals of $f$ along the $\gamma_k$:

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$

**Example – Integration along an Oriented Segment.** The line integral of the continuous function $f : [a,b] \mapsto \mathbb{C}$ along the oriented segment $[a \to b]$ is

$$\begin{aligned}\int_{[a\to b]} f(z)\,dz &= \int_0^1 f((1-t)a + tb)(b-a)\,dt \\ &= (b-a)\int_0^1 f((1-t)a+tb)\,dt.\end{aligned}$$

**Example – Integration along an Oriented Circle.** The line integral of a continuous function $f : \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C}$ on the oriented circle $[\circlearrowleft]$ is

$$\begin{aligned}\int_{[\circlearrowleft]} f(z)\,dz &= \int_0^1 f(e^{i2\pi t})(i2\pi e^{i2\pi t}dt) \\ &= i\int_0^1 f(e^{i2\pi t})e^{i2\pi t}\,(2\pi dt) \\ &= i\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta.\end{aligned}$$

**Theorem – M-L Inequality.** For any rectifiable path $\gamma$ and any continuous function $f : \gamma([0,1]) \to \mathbb{C}$,

$$\left|\int_\gamma f(z)\,dz\right| \leq \left(\max_{z\in\gamma([0,1])} |f(z)|\right) \times \ell(\gamma).$$

**Proof.** Let $\gamma_1 \,|_{t_1} \cdots |_{t_{n-1}} \gamma_n$ be a continuously differentiable decomposition of

$\gamma$. For any $\ell \in \{1, \dots, n\}$,

$$\begin{aligned}\left|\int_{\gamma_\ell} f(z)\,dz\right| &= \left|\int_0^1 f(\gamma_\ell(t))\gamma_\ell'(t)\,dt\right| \\ &\leq \int_0^1 |f(\gamma_\ell(t))||\gamma_\ell'(t)|\,dt \\ &\leq \left(\max_{t\in[0,1]} |f(\gamma_\ell(t))|\right) \times \int_0^1 |\gamma_\ell'(t)|\,dt \\ &= \left(\max_{z\in\gamma_\ell([0,1])} |f(z)|\right) \times \ell(\gamma_\ell).\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\int_{\gamma} f(z)\,dz\right| &\leq \sum_{\ell=1}^{n} \left|\int_{\gamma_\ell} f(z)\,dz\right| \\ &\leq \left(\max_{z\in\gamma([0,1])} |f(z)|\right) \times \sum_{\ell=1}^{n} \ell(\gamma_\ell) \\ &\leq \left(\max_{z\in\gamma([0,1])} |f(z)|\right) \times \ell(\gamma)\end{aligned}$$

which is the desired inequality. ■

A practical consequence of the M-L inequality:

**Corollary – Convergence in Line Integrals.** For any rectifiable path $\gamma$ and any sequence of continuous function $f_n : \gamma([0,1]) \to \mathbb{C}$ which converges uniformly to the function $f$, we have

$$\lim_{n\to+\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

**Proof.** The M-L inequality provides

$$\begin{aligned}\left|\int_{\gamma} f_n(z)\,dz - \int_\gamma f(z)\right| &= \left|\int_{\gamma} (f_n(z) - f(z))\,dz\right| \\ &\leq \left(\max_{z\in\gamma([0,1])} |f_n - f(z)|\right) \times \ell(\gamma)\end{aligned}$$

which yields the desired result. ■

**Theorem – Invariance By Reparametrization.** Let $\gamma : [0,1] \to \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0,1] \to [0,1]$ be an increasing $C^1$-diffeomorphism – a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0,1]$. The following statements hold:

- The path $\mu = \gamma \circ \phi$ is a continuously differentiable path.

- It has the same initial point, terminal point and image as $\gamma$.
- The length of $\mu$ and $\gamma$ are identical.
- For any continuous function $f : \gamma([0,1]) \to \mathbb{C}$,

$$\int_\mu f(z)\, dz = \int_\gamma f(z)\, dz.$$

**Proof.** The function $\mu$ is continuously differentiable as the composition of continuously differentiable functions. We have

$$\mu(0) = \gamma(\phi(0)) = \gamma(0),\ \mu(1) = \gamma(\phi(1)) = \gamma(1),$$

hence the endpoints of $\gamma$ and $\mu$ are identical. The function $\phi$ is a bijection from $[0,1]$ into itself, therefore

$$\mu([0,1]) = \gamma(\phi([0,1])) = \gamma([0,1])$$

and the images of $\gamma$ and $\mu$ are identical.

The length of $\mu$ is

$$\ell(\mu) = \int_0^1 |\mu'(t)|\, dt = \int_0^1 |\gamma'(\phi(t))\phi'(t)|\, dt = \int_0^1 |\gamma'(\phi(t))|\, \phi'(t) dt$$

The change of variable $s = \phi(t)$ provides

$$\int_0^1 |\gamma'(\phi(t))|\, \phi'(t) dt = \int_0^1 |\gamma'(s)|\, ds,$$

hence the lengths of $\gamma$ and $\mu$ are equal. We also have

$$\int_\mu f(z)\, dz = \int_0^1 (f \circ \mu)(t)\mu'(t)\, dt = \int_0^1 (f\circ\gamma)(\phi(t))\gamma'(\phi(t))\, (\phi'(t) dt).$$

The same change of variable leads to

$$\int_\mu f(z)\, dz = \int_0^1 (f\circ\gamma)(s)\gamma'(s)\, ds = \int_\gamma f(z)\, dz,$$

which concludes the proof. ■

**Definition – Image of a Path by a Function.** Let $\gamma : [0,1] \to \mathbb{C}$ be a path and $f : \gamma([0,1]) \to \mathbb{C}$ be a continuous function. The *image of* $\gamma$ *by* $f$ is the path $f \circ \gamma$.

**Theorem – Change of Variable.** Let $\Omega$ be an open subset of $\mathbb{C}$, let $\gamma$ be a rectifiable path of $\Omega$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. The path $f \circ \gamma$ is rectifiable and for any continuous function $g : (f \circ \gamma)([0,1]) \to \mathbb{C}$,

$$\int_{f\circ\gamma} g(z)\, dz = \int_\gamma g(f(w))f'(w)\, dw.$$

**Proof.** Assume that $\gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, \gamma_n$ is a decomposition of $\gamma$ into continuously differentiable paths. We have

$$f \circ \gamma = f \circ \gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, f \circ \gamma_n,$$

and for any $k \in \{1, \dots, n\}$, the function $f \circ \gamma_k$ is continuously (real-)differentiable with

$$(f \circ \gamma_k)'(t) = f'(\gamma_k(t))\gamma_k'(t),$$

hence the path $f \circ \gamma$ is rectifiable. Moreover,

$$\int_{\gamma_k} g(f(w))f'(w)\,dw = \int_0^1 g(f(\gamma_k(t))f'(\gamma_k(t))\gamma_k'(t)\,dt,$$

hence

$$\begin{aligned} \int_0^1 g(f(\gamma_k(t))f'(\gamma_k(t))\gamma_k'(t)\,dt &= \int_0^1 g(f(\gamma_k(t))(f \circ \gamma_k)'(t)\,dt \\ &= \int_{f \circ \gamma_k} g(w)\,dw \end{aligned}$$

which proves the desired result. ■

# Primitives

**Definition – Primitive.** Let $f : \Omega \to \mathbb{C}$ where $\Omega$ is an open subset of $\mathbb{C}$. A *primitive* (or *antiderivative)* of $f$ is a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g' = f$.

**Theorem – Fundamental Theorem of Calculus (Complex Analysis).** Let $\Omega$ be an open connected subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a continuous function and let $a \in \Omega$. A function $g : \Omega \to \mathbb{C}$ is a primitive of $f$ if and only if for any $z \in \Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins $a$ and $z$,

$$g(z) = g(a) + \int_\gamma f(w)\,dw.$$

**Proof.** Let $g$ be a primitive of $f$ and $\gamma$ be a rectifiable path of $\Omega$ that joins $a$ and $z$. Let $\gamma = \gamma_1 \,|_{t_1} \, \dots \, |_{t_{n-1}} \, \gamma_n$ where every $\gamma_k$ is continuously differentiable. For any $k \in \{1, \dots, n\}$, the function

$$\phi : t \in [0, 1] \mapsto g(\gamma_k(t))$$

is differentiable as a composition of real-differentiable functions, with

$$\phi'(t) = dg_{\gamma_k(t)}(\gamma_k'(t)) = g'(\gamma_k(t))\gamma_k'(t).$$

The function $\phi'$ is continuous, hence, by the real analysis version of the fundamental theorem of calculus, applied to the real and imaginary parts of $\phi'$ on $]0,1[$, we have for any positive number $\epsilon$ smaller than 1,

$$\phi(1-\epsilon) - \phi(\epsilon) = \int_{\epsilon}^{1-\epsilon} \phi'(t)\,dt,$$

hence by continuity of $\phi$ and $\phi'$

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t)\,dt,$$

which is equivalent to

$$g(\gamma_k(1)) - g(\gamma_k(0)) = \int_0^1 g'(\gamma_k(t))\gamma_k'(t)\,dt = \int_{\gamma_k} f(w)\,dw.$$

The sum of these equations for all $k \in \{1, \dots, n\}$ provides

$$g(z) - g(a) = \int_{\gamma} f(w)\,dw.$$

Conversely, assume that $g$ satisfies the theorem property. Let $\gamma$ be a rectifiable path of $\Omega$ that joins $a$ and $z$ and let $r > 0$ be such that the open disk centered at $z$ with radius $r$ is included in $\Omega$. Consider the concatenation $\mu$ of $\gamma$ and of the oriented segment $[z \to z+h]$ for $h$ such that $|h| < r$. It is a rectifiable path of $\Omega$, hence

$$\begin{aligned} g(z+h) &= g(a) + \int_{\mu} f(w)\,dw \\ &= g(a) + \int_{\gamma} f(w)\,dw + h\int_0^1 f(z+th)\,dt \\ &= g(z) + h\int_0^1 f(z+th)\,dt \end{aligned}$$

hence

$$\frac{g(z+h) - g(z)}{h} = \int_0^1 f(z+th)\,dt.$$

The right-hand side of this equation converges to $f(z)$ by continuity when $h$ goes to zero, therefore $g$ is a primitive of $f$. ■

**Corollary – Existence of Primitives [†].** Let $\Omega$ be an open connected subset of $\mathbb{C}$. The function $f : \Omega \to \mathbb{C}$ has a primitive if and only if it is continuous and for any closed rectifiable path $\gamma$

$$\int_{\gamma} f(z)\,dz = 0.$$

**Proof – Existence of Primitives.** If the function $f$ has primitives, it is the derivative of a holomorphic function, thus it is continuous. Additionally, for any closed rectifiable path $\gamma$ of $\Omega$, the fundamental theorem of calculus provides

$$g(\gamma(1)) = g(\gamma(0)) + \int_\gamma f(w)\, dw,$$

hence as $\gamma(1) = \gamma(0)$,

$$\int_\gamma f(w)\, dw = 0.$$

Conversely, assume that any such integral is zero. Select any $a$ in $\Omega$ and define for any point $z$ in $\Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins them the function

$$g(z) = g(a) + \int_\gamma f(w)\, dw.$$

This definition is non-ambiguous: if we select a different path $\mu$, the difference between the right-hand sides of the definitions would be

$$\left(g(a) + \int_\gamma f(w)\, dw\right) - \left(g(a) + \int_\mu f(w)\, dw\right) = \int_{\gamma \,|\, \mu^{\leftarrow}} f(w)\, dw = 0$$

as $\gamma \,|\, \mu^{\leftarrow}$ is a closed rectifiable path of $\Omega$. Consequently, $g$ is uniquely defined and by the fundamental theorem of calculus, it is a primitive of $f$. ■

**Corollary – Set of Primitives.** Let $\Omega$ be an open connected subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. If $g : \Omega \to \mathbb{C}$ is a primitive of $f$, the function $h : \Omega \to \mathbb{C}$ is also a primitive of $f$ if and only iff it differs from $g$ by a constant.

**Proof.** It is clear that a function $h$ that differs from $g$ by a constant is a primitive of $f$. Conversely, if $g$ and $h$ are both primitives of $f$, $g - h$ is a primitive of the zero function. The fundamental theorem of calculus shows that for any $a$ and $z$ in $\Omega$ and any rectifiable path $\gamma$ of $\Omega$ that joins them,

$$g(z) - h(z) = g(a) - h(a) + \int_\gamma 0\, dw = g(a) - h(a)$$

hence their difference is a constant. ■

**Corollary – Integration by Parts [†].** Let $\Omega$ be an open connected subset of $\mathbb{C}$ and let $\gamma$ be a rectifiable path of $\Omega$. For any pair of holomorphic functions $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$,

$$\int_\gamma f' g(z)\, dz = [fg(\gamma(1)) - fg(\gamma(0))] - \int_\gamma f g'(z)\, dz.$$

**Proof.** The derivative of the holomorphic function $fg$ is $f'g + fg'$. It is continuous as a sum and product of continuous functions, thus the fundamental theorem of calculus provides

$$fg(\gamma(1)) = fg(\gamma(0)) + \int_\gamma (f'g + fg')(z)\, dz,$$

which is equivalent to the conclusion of the corollary. ■

**Remark & Definition – Variation of a Function on a Path.** The difference between the value of a function $f$ at the terminal value and at the initial value of a path $\gamma$ may be denoted $[f]_\gamma$. With this convention, the formula that connects a function $f$ and its primitive $g$ is

$$[g]_\gamma = \int_\gamma f(z)\,dz$$

and the integration by parts formula becomes

$$\int_\gamma f'g(z)\,dz = [fg]_\gamma - \int_\gamma fg'(z)\,dz.$$

# Appendix – A Better Theory of Rectifiability

## Rectifiable Paths

The definition we used so far for "rectifiable" is a conservative one. In this section, we come up with a more general definition of the concept that still meets the requirements for the definition of line integrals.

To "rectify" a path (from Latin *rectus* "straight" and *facere* "to make") is to straighten – or by extension to compute its length, which is a trivial operation once a path has been straightened.

The general definition of the length of a path does not require line integrals. Instead, consider any partition $(t_0, \ldots, t_n)$ of the interval $[0,1]$ and the path $\mu = \mu_1 \,|_{t_1}\, \ldots \,|_{t_{n-1}} \mu_n$ where

$$\mu_k(t) = (1-t)\gamma(t_{k-1}) + t\gamma(t_k).$$

We may define the length of such a combination of straight lines as

$$\ell(\mu) = \sum_{k=1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|.$$

As the straight line is the shortest path between two points, this number should provide a lower bound of the length of $\gamma$. On the other hand, using finer partitions of the interval $[0,1]$ should also provide better approximations of the length of $\gamma$. Following this idea, we may *define* the length of $\gamma$ as the supremum of the length of $\mu$ for all possible partitions of $[0,1]$:

$$\ell(\gamma) = \sup \left\{ \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \;\middle|\; n \in \mathbb{N}^*,\, t_0 = 0 < \cdots < t_n = 1 \right\}$$

Not every path has a finite length; those who have are by definition *rectifiable*. In general, a function $\gamma : [0,1] \mapsto \mathbb{C}$ whose length is finite – even if it is not continuous – is of *bounded variation*.

## The Line Integral

To define line integrals along the path $\gamma$, it is enough that $\gamma$ is of bounded variation. For any such function $\gamma$, we may build a (complex-valued, Borel) measure on $[0,1]$ denoted $d\gamma$. This measure is defined by its integral of any continuous function $\phi : [0,1] \to \mathbb{C}$, as a limit of Riemann(-Stieltjes) sums

$$\int_{[0,1]} \phi \, d\gamma = \lim \sum_{m=0}^{n-1} \phi(t_m)(\gamma(t_{m+1}) - \gamma(t_m)).$$

The limit is taken over the partitions of the interval $[0,1]$ with

$$\max \{|t_{m+1} - t_m| \mid m \in \{0, \ldots, n-1\}\} \to 0.$$

The line integral of a continuous function $f : \gamma([0,1]) \to \mathbb{C}$ is then defined by

$$\int_\gamma f(z) \, dz = \int_{[0,1]} (f \circ \gamma) \, d\gamma.$$

The *total variation* $|d\gamma|$ of $d\gamma$ is the positive measure defined by

$$|d\gamma|(A) = \sup_{\mathfrak{P}} \sum_{B \in \mathfrak{P}} |d\gamma(B)|$$

where the supremum is taken over all finite partitions $\mathfrak{P}$ of $A$ into measurable sets. This measure provides an integral expression for the length of $\gamma$:

$$\ell(\gamma) = \int_{[0,1]} |d\gamma|.$$

## A Non-Rectifiable Curve

The Koch snowflake (Koch 1904) is an example of a continuous curve which is is nowhere differentiable; it is also a non-rectifiable closed path. It is defined as the limit of a sequence of polylines $\gamma_n$. The first element of this sequence is an oriented equilateral triangle:

$$\gamma_1 = [0 \to 1 \to e^{i\pi 3} \to 0].$$

Then, $\gamma_{n+1}$ is defined as a transformation of $\gamma_n$: every oriented line segment $[a \to a + h]$ that composes $\gamma_n$ is replaced by the polyline:

$$\left[a \to a + \frac{h}{3} \to a + \left(1 + e^{-i\pi/3}\right)\frac{h}{3} \to a + 2\frac{h}{3} \to a + h\right]$$

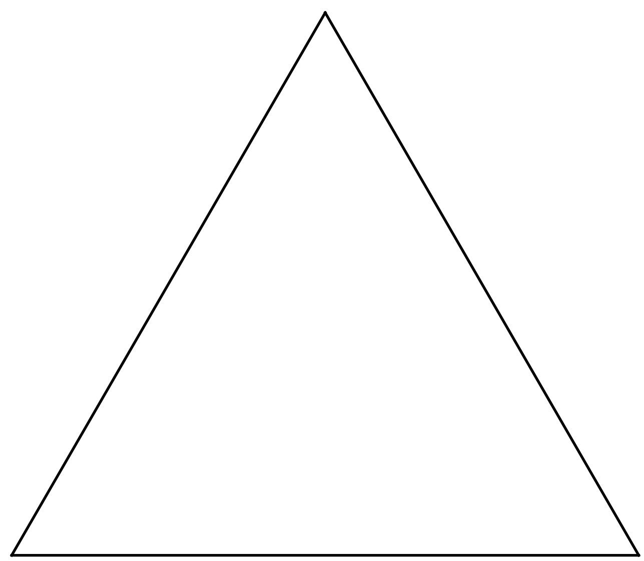

Figure 2.3: Image of the Koch snowflake, first iteration.

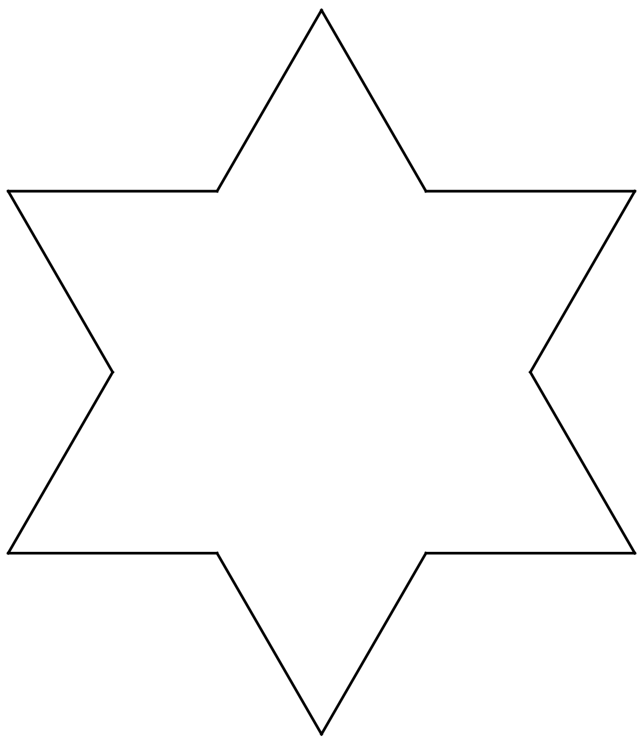

Figure 2.4: Image of the Koch snowflake, second iteration.

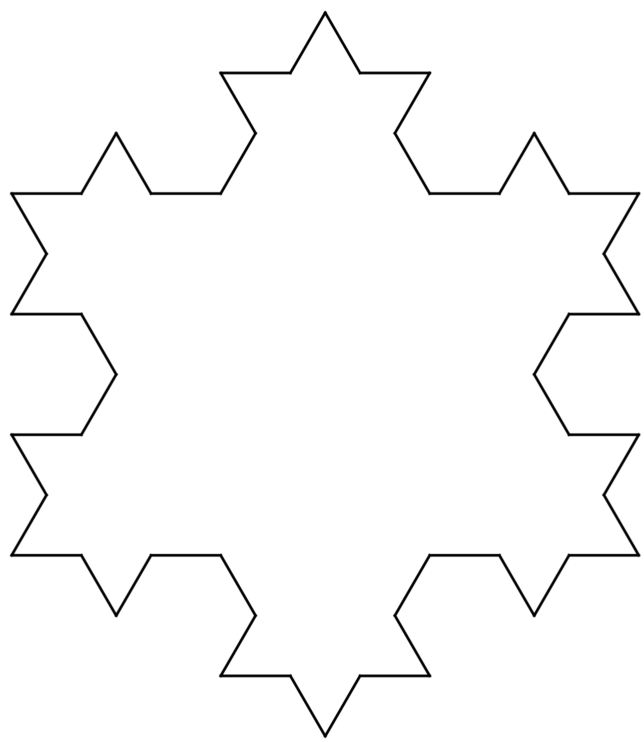

Figure 2.5: Image of the Koch snowflake, third iteration.

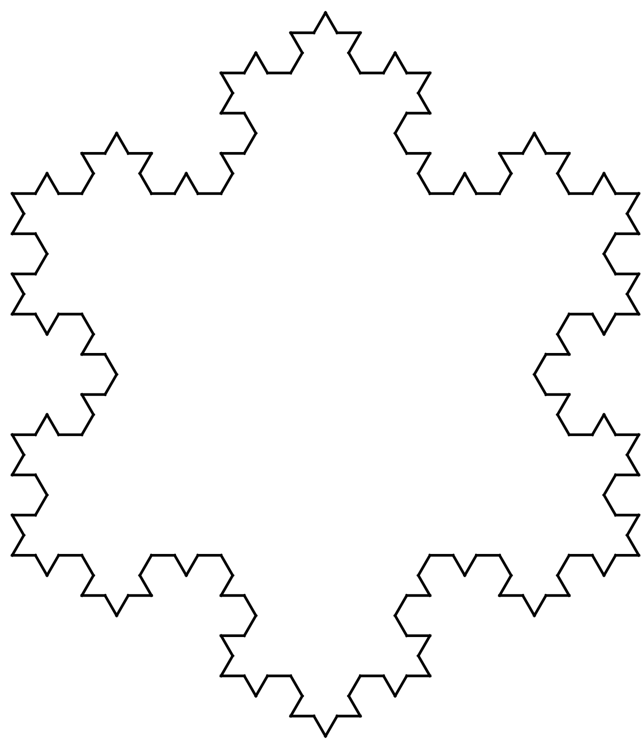

Figure 2.6: Image of the Koch snowflake, fourth iteration

The Koch snowflake $\gamma$ is defined as the limit of the $\gamma_n$ sequence. The geometric construction yields that for any $n$ greater than zero,

$$\forall\, t \in [0,1],\ |\gamma_{n+1}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{\sqrt{3}}{2}.$$

As $\sum_{p=0}^{+\infty} \left(\frac{1}{3}\right)^p = \frac{1}{1-1/3} = \frac{3}{2}$, for any positive integer $p$ we have

$$\forall\, t \in [0,1],\ |\gamma_{n+p}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{3}{2}\frac{\sqrt{3}}{2}.$$

The sequence $\gamma_n$ is a Cauchy sequence in the space of continuous and complex-valued functions defined on $[0,1]$; its uniform limit exists and is also continuous.

On the other hand, the curve is not rectifiable. First, the definition of the sequence $\gamma_n$ makes it plain that every iteration increases the initial length of the path by one-third:

$$\ell(\gamma_n) = 3 \times \left(\frac{4}{3}\right)^{n-1}.$$

The length of $\gamma_n$ tends to $+\infty$ when $n \to +\infty$. Now, every point at the junction of the segments of the polyline $\gamma_n$ also belongs to the Koch snowflake; more precisely

$$\forall\, m \in \{0, \dots, 3 \times 4^{n-1}\},\ \gamma\left(\frac{m}{3 \times 4^{n-1}}\right) = \gamma_n\left(\frac{m}{3 \times 4^{n-1}}\right).$$

Therefore

$$\ell(\gamma) \geq \sum_{m=0}^{3\times 4^{n-1}-1} \left|\gamma\left(\frac{m+1}{3 \times 4^{n-1}}\right) - \gamma\left(\frac{m}{3 \times 4^{n-1}}\right)\right| = \ell(\gamma_n)$$

and thus $\ell(\gamma) = +\infty$: the path $\gamma$ is not rectifiable.

# References

Koch, Helge von. 1904. "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire." *Arkiv för Matematik, Astronomi och Fysik* 1. Kungliga Svenska Vetenskapsakademien.: 681–702.

# Exercises

## Primitives of Power Functions

Determine the primitives of the power $z \mapsto z^n$ – defined on $\mathbb{C}$ if $n$ nonnegative and on $\mathbb{C}^*$ otherwise – or prove that no such function exist.

## Primitive of a Rational Function

Let $\Omega = \mathbb{C} \setminus \{0, 1\}$ and let $f : \Omega \to \mathbb{C}$ be defined by

$$f(z) = \frac{1}{z(z-1)}.$$

Show that $f$ has no primitive on $\Omega$, but that it has a primitive on $\mathbb{C} \setminus [0, 1]$ and determine its expression.

## Reparametrization of Paths

Let $\alpha : [0, 1] \to \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0, 1] \to [0, 1]$ be a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0, 1]$.

1. Show that $\beta = \alpha \circ \phi$ is a rectifiable path which has the same initial point, terminal point and image as $\alpha$.
2. Prove that for any continuous function $f : \alpha([0, 1]) \to \mathbb{C}$,

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$

3. Prove that the paths $\alpha$ and $\beta$ have the same length.

## The Logarithm: Alternate Choices

Show that for any $\alpha \in \mathbb{R}$, the function $z \in \mathbb{C}_\alpha \mapsto 1/z$ defined on

$$\mathbb{C}_\alpha = \mathbb{C} \setminus \{re^{i\alpha} \mid r \geq 0\}.$$

has a primitive; describe the set of all its primitives.

# Chapter 3

# Connected Sets

## Introduction

We characterize the subsets of the plane that are "in one piece". Two slightly different mathematical properties can play this role: *path-connectedness*, whose definition is quite elementary, and *connectedness*, a slightly weaker – and arguably more convoluted – property, but also a more robust and powerful one. The difference matters only when one deals with "pathological" sets; for "well-behaved" sets – and that includes all open sets – the two properties are equivalent.

In this document, we use the word "set" to mean "subset of the complex plane" because this is what we need most of the time. However, the theory still works if we interpret "set" as "subset of a given normed vector space" instead; the only adaption that is required is the replacement of open disks by open balls.

## Path-Connected/Connected Sets

**Definition – Path-Connected Set.** A set $A$ is *path-connected* if any pair of points of $A$ can be joined by a path of $A$:

$$\forall\,(w, z) \in A^2,\ \exists\,\gamma \in C^0([0,1], A),\ \gamma(0) = w \ \text{ and } \ \gamma(1) = z.$$

**Definition – Dilation.** A set $B$ is a *dilation* of a set $A$ if it is the union of a collection of non-empty open disks whose centers are the points of $A$:

$$B = \bigcup_{a \in A} D(a, r_a) \ \text{ and } \ \forall\, a \in A,\ r_a > 0.$$

**Remark – Non-Uniformity of Dilations.** We borrowed the word *dilation* from mathematical morphology, but our use of the word is not completely standard. The dilation of a set $A$ by the non-empty open disk $D(0, r)$ would be classically defined as

$$B = A + D(0,r) = \{a + b \mid a \in A,\, b \in D(0,r)\} = \bigcup_{a \in A} D(a,r).$$

By contrast, the definition that we use allows *non-uniform* dilations: the radius of the disks may change with their centers.

**Definition – Connected Set.** A set is *connected* if all its dilations are path-connected. A set which is not connected is *disconnected*.

**Theorem – Path-Connected/Connected Set.** Every path-connected set is connected. Conversely, every open connected set is path-connected.

**Proof.** Let $A$ be a path-connected set and $B = \cup_{a \in A} D_a$ be a dilation of $A$. For any points $w$ and $z$ in $B$, there are points $a$ and $b$ in $A$ such that $w \in D_a$ and $z \in D_b$. There is a path that joins $w$ and $a$ in $D_a$, a path that joins $a$ and $b$ in $A$ and a path that joins $b$ and $z$ in $D_b$. The concatenation of these paths joins $w$ and $z$ in $B$, hence $A$ is connected.

Conversely, let $A$ be an open connected set. For any $a \in A$, the distance $r_a$ between $a$ and the complement of $A$ – which is a closed set – is positive, hence the disk $D_a = D(a, r_a)$ is a non-empty subset of $A$ and $A = \cup_{a \in A} D_a$. The set $A$ is one of its dilations, hence it is path-connected. ■

**Corollary – Open Connected Sets.** An open set is connected if and only if it is path-connected.

## Set Operations

Many properties of connected sets are similar to properties of path-connected sets, so many statements exist in two variants. For example:

**Theorem – Union of Sets With a Non-Empty Intersection.** if $\mathcal{A}$ is a collection of path-connected/connected sets whose intersection $\cap\mathcal{A}$ is non-empty, then the union $\cup\mathcal{A}$ is path-connected/connected.

**Proof.** For path-connected sets: let $a$ and $b$ in $\cup\mathcal{A}$. There are some sets $A$ and $B$ in $\mathcal{A}$ such that $a \in A$ and $b \in B$. The intersection $\cap\mathcal{A}$ is included in $A \cap B$, hence $A \cap B$ is not empty; let $c \in A \cap B$. There is a path of $A$ that joins $a$ and $c$ and a path of $B$ that joins $c$ and $b$; their concatenation joins $a$ and $b$ in $\cup\mathcal{A}$. Hence, this set is path-connected.

For connected sets: let $\cup_{a \in \cup\mathcal{A}} D_a$ be a dilation of $\cup\mathcal{A}$. We have

$$\bigcup_{a \in \cup\mathcal{A}} D_a = \bigcup_{A \in \mathcal{A}} \cup_{a \in A} D_a.$$

For any $A \in \mathcal{A}$, the set $\cup_{a\in A} D_a$ is a dilation of $A$, hence it is path-connected; the inclusion $A \subset \bigcup_{a\in A} D_a$ provides

$$\cap\mathcal{A} = \bigcap_{A\in\mathcal{A}} A \subset \bigcap_{A\in\mathcal{A}} \cup_{a\in A} D_a,$$

hence the intersection of all $\cup_{a\in A} D_a$ over $A \in \mathcal{A}$ is not empty. We may therefore apply the result of the theorem for path-connected sets to the collection $\{\cup_{a\in A} D_a \mid A \in \mathcal{A}\}$. Our arbitrary dilation of $\cup\mathcal{A}$ is path-connected, hence $\cup\mathcal{A}$ is connected. ■

**Theorem – Disjoint Union of Open Sets.** If $A$ and $B$ are two non-empty open sets such that $A \cap B = \varnothing$, then $A \cup B$ is not path-connected/connected.

**Proof.** Assume that $\gamma$ is a path of $A \cup B$ that joins a point $a \in A$ and a point $b \in B$. Consider the function

$$\phi : t \in [0, 1] \mapsto d(\gamma(t), A) - d(\gamma(t), B).$$

If $z = \gamma(t) \in A$, for example when $t = 0$, $d(z, A) = 0$ and as $A$ is open and $A \cap B = \varnothing$, $d(z, B) > 0$, hence $\phi(t) < 0$. Otherwise, for example when $t = 1$, $z = \gamma(t) \in B$, $d(z, B) = 0$ and as $B$ is open and $A \cap B = \varnothing$, $d(z, A) > 0$, hence $\phi(t) > 0$. But the function $\phi$ is also continuous; the intermediate value theorem asserts the existence of a $t \in \,]0, 1[$ such that $\phi(t) = 0$, which is a contradiction. Hence no such path $\gamma$ can exist and $A \cup B$ is not path-connected; as $A \cup B$ is open, it is not connected either. ■

Connected sets also have some interesting properties that are not shared by all path-connected sets; for example:

**Theorem – Closure of Connected Sets.** The closure of a connected set is connected.

**Proof.** Let $A$ be a connected set and let $\cup_{b\in B} D_b$ be a dilation of its closure $B = \overline{A}$. For any $b \in B$, let $r_b$ be the distance between $b$ and the complement of this dilation. We have

$$\cup_{b\in B} D_b = \cup_{b\in B} D(b, r_b).$$

Consider the dilation $\cup_{a\in A} D(a, r_a)$ of $A$. It is a clearly a subset of the dilation of $B$; actually, we can prove that both sets are equal. Assume that $z$ belongs to the dilation of $B$: there is a $b \in B$ such that $|z - b| < r_b$. As $B$ is the closure of $A$, there is a point $a \in A$ such that $|a - b| < (r_b - |z - b|)/2$; we have

$$|z - a| \leq |z - b| + |a - b| < r_b - |a - b| \leq r_a,$$

hence the point $z$ also belongs to the dilation of $A$. As the dilation of $A$ is path-connected, so is the dilation of $B$: $B$ is connected. ■

The equivalent statement is false for some path-connected sets. Actually, we may leverage this difference to build a connected set which is not path-connected:

**Example – The Topologist's Sine Curve.** Consider

$$A = \{(x, \sin 1/x) \mid x \in \,]0,1]\}.$$

This set is path-connected – as the image by a continuous function of a path-connected set – hence its closure

$$\overline{A} = A \cup \{(0,y) \mid y \in [-1,+1]\}$$

is connected; however, it is not path-connected.

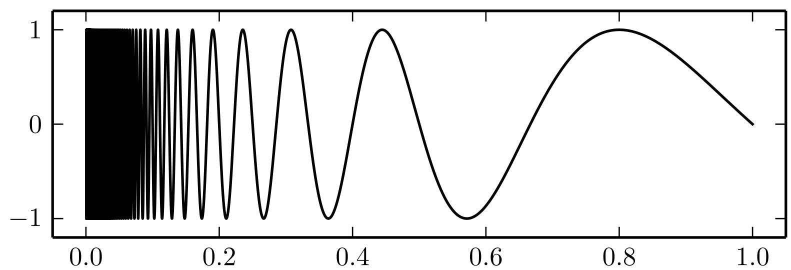


Figure 3.1: The Topologist's Sine Curve.

Assume on the contrary that $\gamma : [0,1] \to \mathbb{C}$ is a path of $\overline{A}$ that joins the points $a_0 = (2/\pi, 1)$ and $a_\infty = (0,0)$; it has to go through every point

$$a_n = (x_n, y_n), \; n \in \mathbb{N} \;\text{ where } \left| \begin{array}{lcl} x_n & = & 1/((n+1/2)\pi) \\ y_n & = & \sin 1/x_n = (-1)^n \end{array} \right.$$

in this specific order. Indeed, given some $t_n \in [0,1[$ such that $\gamma(t_n) = a_n$, we have $\mathrm{Re}(\gamma(t_n)) = x_n$. As $\mathrm{Re}(\gamma(1)) = \mathrm{Re}(a_\infty) = 0$, by continuity of $t \in [0,1] \mapsto \mathrm{Re}(\gamma(t))$, there is a $t_{n+1} \in \,]t_n, 1[$ such that $\mathrm{Re}(\gamma(t_{n+1})) = x_{n+1}$. Since for any $x > 0$, there is a unique real number $y$ such that $(x,y) \in A$, this yields $\gamma(t_{n+1}) = a_{n+1}$. Now, since the sequence $t_n$ is increasing and bounded from above, necessarily $|t_{n+1} - t_n| \to 0$ when $n \to +\infty$. But on the other hand, for any $n \in \mathbb{N}$,

$$\begin{aligned} |\gamma(t_{n+1}) - \gamma(t_n)| &= |a_{n+1} - a_n| \\ &\geq |y_{n+1} - y_n| \\ &= 2 \end{aligned}$$

Hence the function $\gamma$, despite being continuous and defined on the compact set $[0,1]$ cannot be uniformly continuous, which is a contradiction.

## Components

We define two concepts of components based respectively on path-connectedness and connectedness.

**Definition – Component.** A *(path-connected/connected) component* of a non-empty set $A$ is a subset of $A$ which is path-connected/connected and maximal with respect to inclusion among such sets – that is, included in no other path-connected/connected subset of $A$.

**Theorem – Partition into Components.** The path-connected/connected components of a non-empty set $A$ are a partition of $A$: they are a collection of non-empty and pairwise disjoint subsets of $A$ whose union is $A$.

**Proof.** The proof is identical for path-connected and connected components. Let $a \in A$. Consider the collection $\mathcal{A}_a$ of all connected subsets of $A$ that contain the point $a$. The set $A_a = \cup \mathcal{A}_a$ is connected. By construction, the set $A_a$ is maximal: it is a component of $A$. As every component of $A$ is maximal, it contains at least one point $a \in A$: it is therefore non-empty and equal to $A_a$. Hence the union of all components of $A$ is $\cup_{a \in A} A_a = A$. Finally, if two such components $A_a$ and $A_b$ have a non-empty intersection $c \in A$, the set $A_a \cup A_b$ is connected and contains $A_a$ and $A_b$, therefore $A_a = A_b$. ■

**Corollary – Connectedness & Components.** A non-empty set is path-connected/connected if and only if it has a single path-connected/connected component.

**Proof.** If a set is path-connected/connected, it is one of its components, because it is clearly connected and maximal. As the components form a partition of the set, it is the only component. Conversely, if there is a unique component, again because the components form a partition of the set, it is the set itself, which is therefore path-connected/connected.

**Theorem – Components of Open Sets.** The partitions of a non-empty open set into path-connected components and connected components are identical. All such components are open.

**Proof.** Let $A$ be an open set and let $B$ be a path-connected component of $A$. For any $b \in B$, there is a non-empty open disk $D$ centered on $b$ which is included in $A$. The disk $D$ is a path-connected subset of $A$ that contains $a$; it is therefore included in the unique maximal path-connected subset of $A$ that contains $a$: the set $B$. Therefore, $B$ is open.

The path-connected components of $A$ are open and path-connected, hence they are also connected. They are also maximal among the connected sets of $A$: a connected component of $A$ contains a path-connected component of $A$ if it contains a point of it; if it were to contain more than one path-connected component, it would be the union several disjoint open sets and hence could not be connected. ■

## Locally Constant Functions

**Definition – Locally Constant Function.** A function $f$ defined on a set $A$ is locally constant if for any $a$ in $A$, there is a non-empty open disk $D$ centered on $a$ such that $f$ is constant on $A \cap D$:

$$\forall\, a \in A,\ \exists\, \epsilon > 0,\ \forall\, b \in A,\ |b - a| < \epsilon \Rightarrow f(b) = f(a).$$

**Theorem – Locally Constant Functions & Connected Sets.** A set $A$ is connected if and only if every locally constant function defined on $A$ is constant.

**Proof.** Let $f$ be a locally constant function defined on $A$. Let $a \in A$ and $B = \{b \in A \mid f(b) = f(a)\}$. Assume that $f$ is not constant, that is, that $C = A \setminus B$ is non-empty. As $f$ is locally constant, the distance between any point $b$ of $B$ and the set $C$ is positive; we may define $D_b = D(b, r_b)$ where $r_b = d(b, C)/2 > 0$. We may perform a similar construction for any point $c$ of $C$ and define a disk $D_c = D(c, r_c)$ with $r_c = d(c, B)/2 > 0$. By construction, the sets $\cup_{b \in B} D_b$ and $\cup_{c \in C} D_c$ are open, non-empty and disjoints, hence the dilation $\cup_{a \in A} D_a$ of $A$ is not path-connected. Therefore, $A$ is not connected.

Conversely, if $A$ is not connected, let $\cup_{a \in A} D_a$ be a dilation of $A$ which is not path-connected. Select one component $B$ of it and define $C$ as the union of all other components. Then, the function $f$ defined by $f(z) = 1$ if $z \in B$ and $f(z) = 0$ if $z \in C$ is locally constant as $B$ and $C$ are both open sets. However, it is not constant. ■

## Exercises

### Image of Path-Connected/Connected Sets

Let $f : A \subset \mathbb{C} \to \mathbb{C}$ be a continuous function.

Show that if $A$ is path-connected/connected, its image $f(A)$ is path-connected/connected.

### Complement of a Compact Set

Prove that the complement $\mathbb{C} \setminus K$ of a compact subset $K$ of the complex plane has a single unbounded component.

### Union of Separated Sets

Source: "Sur les ensembles connexes" (Knaster and Kuratowski 1921)

Let $A$ and $B$ be two non-empty subsets of the complex plane.

1. If $A \cap B = \varnothing$, is $A \cup B$ always disconnected ?
2. Assume that $d(A, B) > 0$; show that $A \cup B$ is not connected.
3. Assume that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$; show that $A \cup B$ is not connected.

## Anchor Set

1. Prove that if $\mathcal{A}$ is a collection of path-connected/connected sets and there is a set $A^* \in \mathcal{A}$ such that $\forall\, A \in \mathcal{A},\ A \cap A^* \neq \varnothing$, then the union $\cup\mathcal{A}$ is path-connected/connected.
2. A *deformation retraction* of a subset $A$ of the complex plane onto a subset $B$ of $A$ is a "continuous shrinking process" of $A$ into $B$; formally, it is a collection of paths $\gamma_a$ of $A$, indexed by $a \in A$, such that:

   - $\forall\, a \in A,\ \gamma_a(0) = a$ and $\gamma_a(1) \in B$,
   - $\forall\, a \in B,\ \forall\, t \in [0, 1],\ \gamma_a(t) = a$,
   - the function $(t, a) \in [0, 1] \times A \mapsto \gamma_a(t)$ is continuous.

   (see e.g. (Hatcher 2002)). Show that if there is a deformation retraction of $A$ onto $B$ and $B$ is path-connected/connected, then $A$ is also path-connected/connected.

## References

Hatcher, Allen. 2002. *Algebraic Topology.* Cambridge University Press. `https://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

Knaster, B., and C. Kuratowski. 1921. "Sur les ensembles connexes." *Fundamenta Mathematicae* 2. Polish Academy of Sciences (Polska Akademia Nauk - PAN), Institute of Mathematics (Instytut Matematyczny), Warsaw: 206–55.

# Chapter 4

# Cauchy's Integral Theorem – Local Version

## Introduction

We derive in this document a first version of Cauchy's integral theorem:

**Theorem – Cauchy's Integral Theorem (Local Version).** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. For any $a \in \Omega$, there is a radius $r > 0$ such that the open disk $D(a, r)$ is included in $\Omega$ and for any rectifiable closed path $\gamma$ of $D(a, r)$,

$$\int_\gamma f(z)\, dz = 0.$$

We will actually state and prove a slightly stronger version – one that does not require the restriction to small disks if $\Omega$ is *star-shaped*.

In a subsequent document, we will prove an even more general result, the global version of Cauchy's integral theorem. It will be applicable if $\Omega$ is merely *simply connected* (that is "without holes").

## Integral Lemma for Polylines

**Lemma – Integral Lemma for Triangles.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $\Delta$ is a triangle with vertices $a$, $b$ and $c$ which is included in $\Omega$

$$\Delta = \{\lambda a + \mu b + \nu c \mid \lambda \geq 0,\ \mu \geq 0,\ \nu \geq 0 \ \text{ and } \ \lambda + \mu + \nu = 1\} \subset \Omega$$

and if $\gamma = [a \to b \to c \to a]$ is an oriented boundary of $\Delta$ then

$$\int_\gamma f(z)\, dz = 0.$$

**Proof.** Let $a_0 = a$, $b_0 = b$, $c_0 = c$; consider the midpoints of the triangle edges:

$$d_0 = \frac{b_0 + c_0}{2}, \ e_0 = \frac{a_0 + c_0}{2}, \ f_0 = \frac{a_0 + b_0}{2}.$$

The sum of the integrals of $f$ along the four paths $[a_0 \to f_0 \to e_0 \to a_0]$, $[f_0 \to b_0 \to d_0 \to f_0]$, $[e_0 \to d_0 \to c_0 \to e_0]$, $[d_0 \to e_0 \to f_0 \to d_0]$ is equal to the integral of $f$ along $\gamma$. By the triangular inequality, there is at least one path in this set, that we denote $\gamma_1$, such that

$$\left| \int_{\gamma_1} f(z)\, dz \right| \geq \frac{1}{4} \left| \int_\gamma f(z)\, dz \right|.$$

We can iterate this process and come up with a sequence of paths $\gamma_n$ such that

$$\left| \int_{\gamma_n} f(z)\, dz \right| \geq \frac{1}{4^n} \left| \int_\gamma f(z)\, dz \right|.$$

Denote $\Delta_n$ the triangles associated to the $\gamma_n$; they form a sequence of non-empty and nested compact sets. By Cantor's intersection theorem, there is a point $w$ such that $w \in \Delta_n$ for every natural number $n$. The differentiability of $f$ at $w$ provides a complex-valued function $\epsilon_w$, defined in a neighbourhood of 0, such that $\lim_{h \to 0} \epsilon_w(h) = \epsilon_w(0) = 0$ and

$$f(z) = f(w) + f'(w)(z - w) + \epsilon_w(z - w)|z - w|$$

Consequently, for any $\epsilon > 0$ and for any number $n$ large enough,

$$\left| \int_{\gamma_n} [f(z) - f(w) - f'(w)(z - w)]\, dz \right| \leq \epsilon \operatorname{diam} \Delta_n \times \ell(\gamma_n),$$

where the diameter of a subset $A$ of the complex plane is defined as

$$\operatorname{diam} A = \sup \{ |z - w| \mid z \in A,\, w \in A \}.$$

We have $\ell(\gamma_n) = \ell(\gamma)/2^n$ and $\operatorname{diam} \Delta_n = \operatorname{diam} \Delta_0 / 2^n$. Additionally,

$$\int_{\gamma_n} f(w)\, dz = \int_{\gamma_n} f'(w)(z - w)\, dz = 0$$

since the functions $z \in \mathbb{C} \mapsto f(w)$ and $z \in \mathbb{C} \mapsto f'(w)(z - w)$ have primitives. Consequently, for any $\epsilon > 0$, for $n$ large enough,

$$\frac{1}{4^n} \left| \int_\gamma f(z)\, dz \right| \leq \left| \int_{\gamma_n} f(z)\, dz \right| \leq \frac{1}{4^n} \epsilon \operatorname{diam} \Delta_0 \times \ell(\gamma),$$

which is only possible if the integral of $f$ along $\gamma$ is zero. ∎

**Definition – Star-Shaped Set.** A subset $A$ of the complex plane is *star-shaped* if it contains at least one point $c$ – a *(star-)center*, the set of which is called the *kernel* of $A$ – such that for any $z$ in $A$, the segment $[c, z]$ is included in $A$.

**Lemma – Integral Lemma for Polylines.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function where $\Omega$ is an open star-shaped subset of $\mathbb{C}$. For any closed path $\gamma = [a_0 \to \cdots \to a_{n-1} \to a_0]$ of $\Omega$,

$$\int_\gamma f(z)\, dz = 0.$$

**Proof.** Let $c$ be a star-center of $\Omega$ and define $a_n = a_0$; for any $k \in \{0, \dots, n-1\}$, the triangle with vertices $c$, $a_k$ and $a_{k+1}$ is included in $\Omega$. Hence, by the integral lemma for triangles, the integral along the path $\gamma_k = [c \to a_k \to a_{k+1} \to c]$ of $f$ is zero. Now, as

$$\int_\gamma f(z)\, dz = \sum_{k=0}^{n-1} \int_{\gamma_k} f(z)\, dz,$$

the integral of $f$ along $\gamma$ is zero as well. ∎

# Approximations of Rectifiable Paths by Polylines

To extend the integral lemma beyond closed polylines, we prove that polylines provide appropriate approximations of rectifiable paths:

**Lemma – Polyline Approximations of Rectifiable Paths.** Let $\gamma$ be a rectifiable path. For any $\epsilon_\ell > 0$ and $\epsilon_\infty > 0$, there is an oriented polyline $\mu$, with the same endpoints as $\gamma$, such that

$$\ell(\mu - \gamma) \leq \epsilon_\ell \;\text{ and }\; \forall\, t \in [0,1],\ |(\mu - \gamma)(t)| \leq \epsilon_\infty.$$

**Proof – Polyline Approximations of Rectifiable Paths.** Suppose that the path $\gamma$ is continuously differentiable. Let $(t_0, \dots, t_n)$ be a partition of the interval $[0, 1]$ and let $\mu$ be the associated polyline:

$$\mu = [\gamma(t_0) \to \gamma(t_1)] \,|_{t_1} \cdots |_{t_{n-1}} [\gamma(t_{n-1}) \to \gamma(t_n)]$$

The path $\gamma$ and $\mu$ have the same endpoints. The path $\gamma$ may be considered as the concatenation $\gamma = \gamma_1 \,|_{t_1} \dots |_{t_{n-1}} \gamma_n$ with the paths $\gamma_k$ defined by

$$\forall\, k \in \{1, \dots, n\},\ \forall t \in [0,1],\ \gamma_k(t) = \gamma\left(t_{k-1} + t(t_k - t_{k-1})\right),$$

hence we have

$$\ell(\mu - \gamma) = \sum_{k=1}^{n} \int_0^1 |\gamma(t_k) - \gamma(t_{k-1}) - \gamma_k'(t)|\; dt.$$

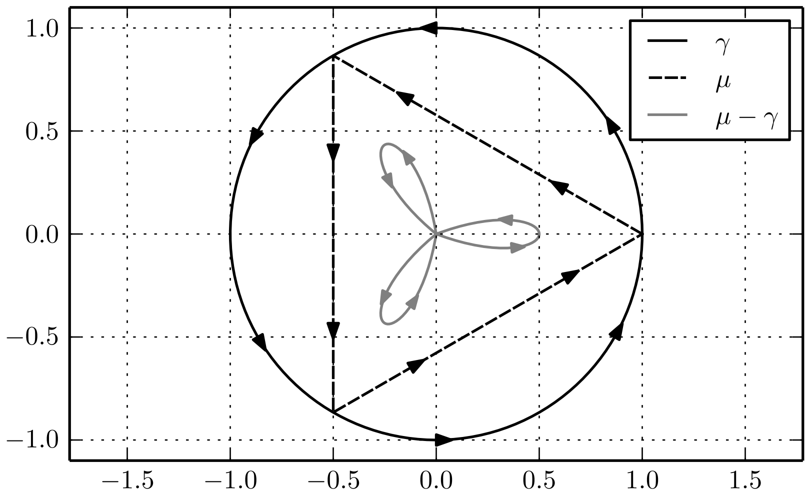


Figure 4.1: A 3-line approximation of the oriented unit circle.

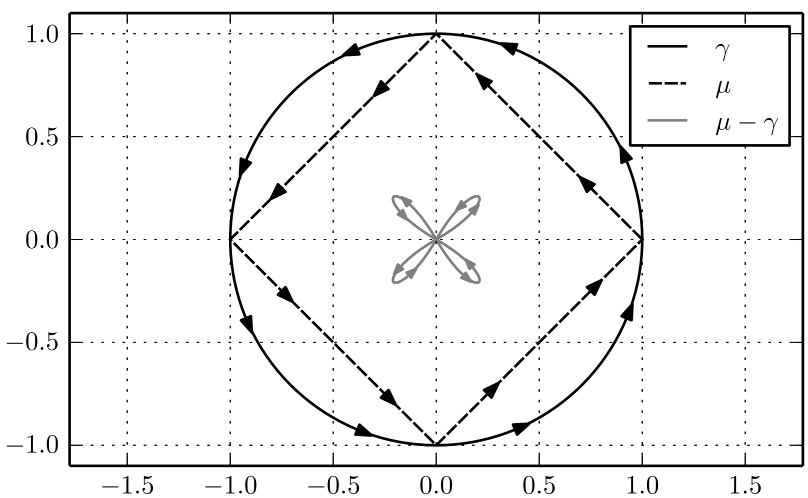


Figure 4.2: A 4-line approximation of the oriented unit circle.

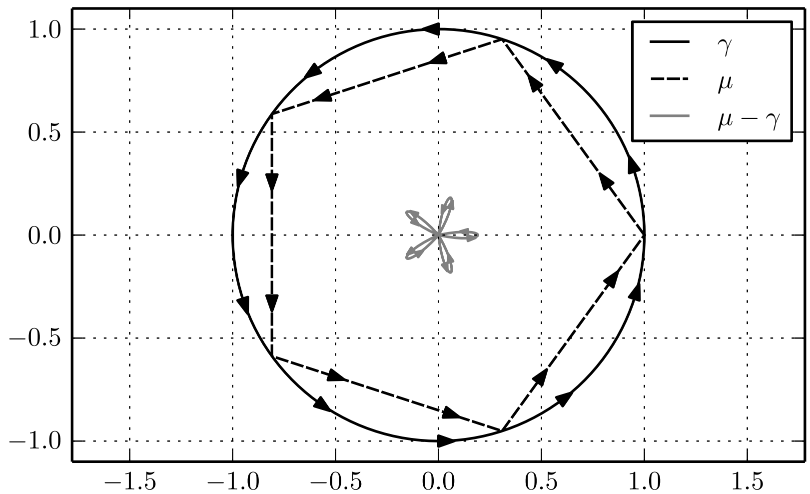


Figure 4.3: A 5-line approximation of the oriented unit circle.

As

$$\gamma(t_k) - \gamma(t_{k-1}) = \int_{t_{k-1}}^{t_k} \gamma'(s)\, ds$$

and

$$\begin{aligned} \gamma_k'(t) &= (t_k - t_{k-1})\gamma'\left(t_{k-1} + t(t_k - t_{k-1})\right) \\ &= \int_{t_{k-1}}^{t_k} \gamma'\left(t_{k-1} + t(t_k - t_{k-1})\right)\, ds, \end{aligned}$$

we have the inequality

$$\ell(\mu - \gamma) \leq \int_0^1 \left[\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(s) - \gamma'(t_{k-1} + t(t_k - t_{k-1}))|\, ds\right] dt$$

The function $\gamma'$ is by assumption continuous, and hence uniformly continuous, on $[0, 1]$, therefore for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that, $|\gamma'(s) - \gamma'(t)| < \epsilon$ whenever $|s - t| < \delta(\epsilon)$. For any $\epsilon_\ell > 0$, for any partition $(t_0, \ldots, t_n)$ such that $|t_k - t_{k-1}| < \delta(\epsilon_\ell)$ for any $k \in \{1, \ldots, n\}$, we have

$$\ell(\mu - \gamma) \leq \int_0^1 \left[\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \epsilon_\ell\, ds\right] dt = \epsilon_\ell.$$

For any $\epsilon_\infty > 0$, as

$$\forall\, t \in [0, 1],\ |\mu(t) - \gamma(t)| \leq |\mu(0) - \gamma(0)| + \ell(\mu - \gamma) = \ell(\mu - \gamma),$$

any partition $(t_0, \dots, t_n)$ such that $|t_k - t_{k-1}| < \delta(\epsilon_\infty)$ ensures that

$$\forall\, t \in [0,1],\ |(\mu - \gamma)(t)| \leq \epsilon_\infty.$$

If $\gamma$ is merely rectifiable, the same approximation process, applied to each of its continuously differentiable components provides the result. ■

## Cauchy's Integral Theorem

We finally get rid of the polyline assumption:

**Theorem – Cauchy's Integral Theorem (Star-Shaped Version).** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function where $\Omega$ is an open star-shaped subset of $\mathbb{C}$. For any rectifiable closed path of $\Omega$,

$$\int_\gamma f(z)\, dz = 0.$$

**Proof.** Let $\epsilon > 0$. Let $r > 0$ be smaller than the distance between $\gamma([0,1])$ and $\mathbb{C} \setminus \Omega$. The set

$$K = \{z \in \mathbb{C} \,|\, d(z, \gamma([0,1])) \leq r\},$$

is compact and included in $\Omega$. Consequently, the restriction of $f$ to $K$ is bounded and uniformly continuous: there is a $M > 0$ such that

$$\forall\, z \in K,\ |f(z)| \leq M,$$

and a there is a $\eta_\epsilon > 0$ – smaller than or equal to $r$ – such that

$$\forall\, z \in K, \forall\, w \in \gamma([0,1]),\ |z - w| \leq \eta_\epsilon \Rightarrow |f(z) - f(w)| \leq \frac{\epsilon}{2(\ell(\gamma) + 1)}.$$

Now, let $\gamma_\epsilon$ be a closed polyline approximation of $\gamma$ such that

$$\ell(\gamma_\epsilon - \gamma) \leq \frac{\epsilon}{2M} \ \text{ and } \ \forall\, t \in [0,1],\ |(\gamma_\epsilon - \gamma)(t)| \leq \eta_\epsilon.$$

By construction, $\gamma_\epsilon$ belongs to $K$, hence it is a closed path of $\Omega$. Therefore, the integral lemma for polylines provides

$$\int_{\gamma_\epsilon} f(z)\, dz = 0.$$

The rectifiable $\gamma$ and $\gamma_\epsilon$ have a decomposition into continuously differentiable paths associated to a common partition $(t_0, \dots, t_n)$ of the interval $[0,1]$:

$$\gamma = \gamma_1 \,|_{t_1} \cdots |_{t_n}\, \gamma_n \ \text{ and } \ \gamma_\epsilon = \gamma_{1\epsilon} \,|_{t_1} \cdots |_{t_n}\, \gamma_{n\epsilon}$$

The difference between the integral of $f$ along $\gamma$ and $\gamma_\epsilon$ satisfies

$$\left|\int_\gamma f(z)\,dz - \int_{\gamma_\epsilon} f(z)\,dz\right| = \left|\sum_{k=1}^{n}\int_0^1 [(f\circ\gamma_k)\gamma_k' - (f\circ\gamma_{\epsilon k})\gamma_{\epsilon k}'](t)\,dt\right|$$

Since for any $k \in \{1,\dots,n\}$

$$(f\circ\gamma_k)\gamma_k' - (f\circ\gamma_{\epsilon k})\gamma_{\epsilon k}' = (f\circ\gamma_k - f\circ\gamma_{\epsilon k})\gamma_k' - (f\circ\gamma_{\epsilon k})(\gamma_k' - \gamma_{\epsilon k}'),$$

we have

$$\begin{aligned}&\left|\int_\gamma f(z)\,dz - \int_{\gamma_\epsilon} f(z)\,dz\right| \\ &\leq \left|\sum_{k=1}^{n}\int_0^1 [(f\circ\gamma_k - f\circ\gamma_{\epsilon k})\gamma_k'](t)\,dt\right| \\ &+ \left|\sum_{k=1}^{n}\int_0^1 [(f\circ\gamma_{\epsilon k})(\gamma_k' - \gamma_{\epsilon k}')](t)\,dt\right|\end{aligned}$$

and thus

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &\leq \max_{t\in[0,1]} |f(\gamma(t)) - f(\gamma_\epsilon(t))| \times \ell(\gamma) \\ &\quad + \max_{t\in[0,1]} |f(\gamma_\epsilon(t))| \times \ell(\gamma_\epsilon - \gamma) \\ &\leq \frac{\epsilon}{2(\ell(\gamma)+1)} \times \ell(\gamma) + M \times \frac{\epsilon}{2M} \\ &\leq \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, the integral of $f$ along $\gamma$ is zero. ■

# Consequences

**Theorem – Cauchy's Integral Formula for Disks.** Let $\Omega$ be an open subset of the complex plane and $\gamma = c + r[\circlearrowleft]$ be an oriented circle such that the closed disk $\overline{D}(c,r)$ is included in $\Omega$. For any holomorphic function $f : \Omega \to \mathbb{C}$,

$$\forall\, z \in D(c,r),\ f(z) = \frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw.$$

**Proof.** Refer to the answers of exercise "Cauchy's Integral Formula for Disks" ■

**Corollary – Derivatives are Complex-Differentiable.** The derivative of any holomorphic function is holomorphic.

**Proof.** Refer to the answers of exercise "Cauchy's Integral Formula for Disks" ■

**Theorem – Morera's Theorem.** Let $\Omega$ be an open subset of $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if it is continuous and locally, its line integrals along rectifiable closed paths are zero: for any $c \in \Omega$, there is a $r > 0$ such that $D(c, r) \subset \Omega$ and for any rectifiable closed path $\gamma$ of $D(c, r)$,

$$\int_\gamma f(z)\, dz = 0.$$

**Proof.** If $f$ is holomorphic, then it is continuous and by Cauchy's integral theorem, its line integrals along rectifiable closed paths are locally zero. Conversely, if $f$ is continuous and all its line integrals along closed rectifiable paths are zero in some non-empty open disk $D(c, r)$ of $\Omega$, then it has a primitive in $D(c, r)$, which is holomorphic. Its derivative is therefore holomorphic too and $f$ is holomorphic in some neighbourhood of $c$; when the initial assumption holds for any $c \in \Omega$, we can conclude that $f$ is holomorphic on $\Omega$. ■

**Theorem – Limit of Holomorphic Functions.** Let $\Omega$ be an open subset of $\mathbb{C}$. If a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges locally uniformly to a function $f : \Omega \to \mathbb{C}$, that is if for any $c \in \Omega$, there is a $r > 0$ such that $D(c, r) \subset \Omega$ and

$$\lim_{n\to+\infty} \sup_{z\in D(c,r)} |f_n(z) - f(z)| = 0,$$

then $f$ is holomorphic.

**Proof.** The function $f$ is continuous as a locally uniform limit of continuous functions. Now, let $c \in \Omega$ and let $r > 0$ be such that $D(c, r) \subset \Omega$ and the functions $f_n$ converge uniformly to $f$ in $D(c, r)$. By Cauchy's integral theorem, for any rectifiable closed path $\gamma$ of $D(c, r)$, the integral of $f_n$ along $\gamma$ is zero. Thus

$$\int_\gamma f(z)\, dz = \lim_{n\to+\infty} \int_{\gamma_n} f(z)\, dz = 0.$$

By Morera's theorem, $f$ is holomorphic. ■

**Theorem – Liouville's Theorem.** Any holomorphic function defined on $\mathbb{C}$ (any *entire* function) which is bounded is constant.

**Proof.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq \kappa$ for any $z \in \mathbb{C}$. We apply Cauchy's integral formula for disks to the function $f'$ which is holomorphic and to the oriented circle $\gamma = z + r[\circlearrowleft]$ for $r > 0$ and $z \in \mathbb{C}$. We have

$$f'(z) = \frac{1}{i2\pi} \int_\gamma \frac{f'(w)}{w - z} dw$$

and by integration by parts,

$$f'(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w - z)^2} dw,$$

which yields by the M-L estimation lemma

$$|f'(z)| \leq \frac{\kappa}{r}.$$

This inequality holds for any $r > 0$, thus $f'(z) = 0$. The function $f$ is a primitive of 0 on the connected set $\mathbb{C}$, therefore it is a constant. ■

# Exercises

## A Fourier Transform

We wish to compute for any real number $\omega$ the value of the integral

$$\hat{x}(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}\, dt$$

when $x : \mathbb{R} \to \mathbb{R}$ is the Gaussian function defined by

$$\forall\, t \in \mathbb{R},\ x(t) = e^{-t^2/2}.$$

We remind you of the value of the *Gaussian integral* (see e.g. Wikipedia):

$$\hat{x}(0) = \int_{-\infty}^{+\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$$

1. Show that for any pair of real numbers $\tau$ and $\omega$, we can compute

$$\int_{-\tau}^{\tau} x(t)e^{-i\omega t}\, dt$$

   from the line integral of a fixed holomorphic function on a path $\gamma$ that depends on $\tau$ and $\omega$.

2. Use Cauchy's integral theorem to evaluate $\hat{x}(\omega)$.

## Cauchy's Integral Formula for Disks

Let $\Omega$ be an open subset of the complex plane and $\gamma = c + r[\circlearrowleft]$. We assume that the closed disk $\overline{D}(c, r)$ is included in $\Omega$ (this is stronger than the requirement that $\gamma$ is a path of $\Omega$).

We wish to prove that for any holomorphic function $f : \Omega \to \mathbb{C}$,

$$\forall\, z \in \Omega,\ |z - c| < r \ \Rightarrow\ f(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z} dw.$$

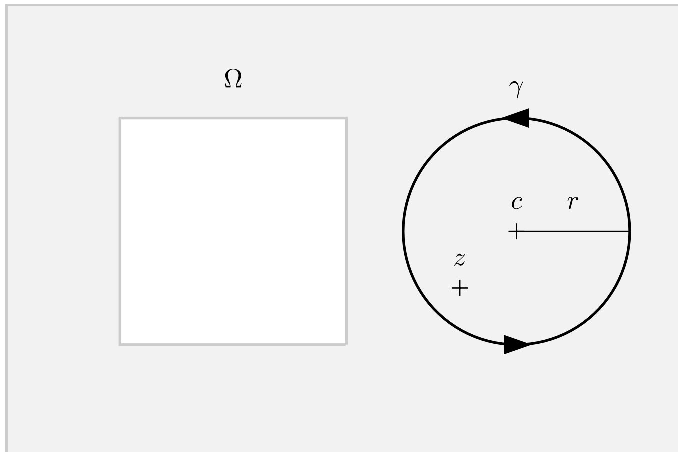


Figure 4.4: Geometry of Cauchy's integral formula for disks.

1. What is the value of the line integral above when $|z - c| > r$?

2. Compute the line integral above when $z = c$ as an integral with respect to a real variable. When happens in this case when $r \to 0$ ?

3. Let $\epsilon > 0$ be such that $|z| + \epsilon < r$ and let $\lambda = z + \epsilon[\circlearrowleft]$. Provide two paths $\mu$ and $\nu$ whose images belong to (different) star-shaped subsets of $\Omega \setminus \{z\}$ and such that for any continuous function $g : \Omega \setminus \{z\} \to \mathbb{C}$,

$$\int_\gamma g(w)\, dw = \int_\lambda g(w) dw + \int_\mu g(w)\, dw + \int_\nu g(w) dw.$$

4. Prove Cauchy's integral formula for disks.

5. Show that $f'(z)$ can be computed as a line integral on $\gamma$ of an expression that depends on $f(w)$ and not on $f'(w)$. What property of $f'$ does this expression shows?

## The Fundamental Theorem of Algebra

Prove that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

## Image of Entire Functions

Show that any non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has an image which is dense in $\mathbb{C}$:

$$\forall\, w \in \mathbb{C},\ \forall\, \epsilon > 0,\ \exists\, z \in \mathbb{C},\ |f(z) - w| < \epsilon.$$

# Chapter 5

# The Winding Number

## Definitions

The argument of a non-zero complex number is only defined modulo $2\pi$. A convenient way to describe mathematically this relationship is to associate to any such number the set of admissible values of its argument:

**Definition – The Argument Function.** The *set-valued* (or *multi-valued*) function Arg, defined on $\mathbb{C}^*$ by

$$\operatorname{Arg} z = \left\{ \theta \in \mathbb{R} \;\middle|\; e^{i\theta} = \frac{z}{|z|} \right\},$$

is called the *argument* function.

If we need a classic *single-valued* function instead, we have for example:

**Definition – Principal Value of the Argument.** The *principal value of the argument* is the unique continuous function

$$\arg : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{R}$$

such that

$$\arg 1 = 0$$

which is a *choice* of the argument on its domain:

$$\forall \, z \in \mathbb{C} \setminus \mathbb{R}_-, \; \arg z \in \operatorname{Arg} z.$$

**Proof (existence and uniqueness).** Define arg on $\mathbb{C} \setminus \mathbb{R}_- \to \mathbb{R}$ by:

$$\arg(x+iy) = \left| \begin{array}{cl} \arctan y/x & \text{if } x > 0, \\ +\pi/2 - \arctan x/y & \text{if } y > 0, \\ -\pi/2 - \arctan x/y & \text{if } y < 0. \end{array} \right.$$

This definition is non-ambiguous: if $x > 0$ and $y > 0$, we have

$$\arctan x/y + \arctan y/x = \pi/2$$

and a similar equality holds when $x > 0$ and $y < 0$. As each of the three expressions used to define arg has an open domain and is continuous, the function itself is continuous. It is a choice of the argument thanks to the definition of arctan: for example, if $x > 0$, with $\theta = \arg(x + iy)$, we have

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \tan(\arctan y/x) = \frac{y}{x},$$

hence, as $\cos\theta > 0$ and $x > 0$, there is a $\lambda > 0$ such that

$$x + iy = \lambda(\cos\theta + i\sin\theta) = \lambda e^{i\theta},$$

This equation yields $\arg x + iy \in \operatorname{Arg} x + iy$. The proof for the half-planes $y > 0$ and $y < 0$ is similar.

If $f$ is another continuous choice of the argument on $\mathbb{C} \setminus \mathbb{R}_-$ such that $f(1) = 0$, the image of $\mathbb{C} \setminus \mathbb{R}_-$ by the difference $f - \arg$ is a subset of $2\pi\mathbb{Z}$ that contains 0, and it's also path-connected as the image of a path-connected set by a continuous function. Consequently, it is the singleton $\{0\}$: $f$ and arg are equal. ■

We cannot avoid the introduction of a *cut* in the complex plane when we search for a continuous choice of the argument: there is no continuous choice of the argument on $\mathbb{C}^*$. However, for a continuous choice of the argument along a path of $\mathbb{C}^*$, there is no such restriction:

The following theorem is a special case of the path lifting property (in the context of covering spaces; refer to (Hatcher 2002) for details).

**Theorem – Continuous Choice of the Argument.** Let $a \in \mathbb{C}$ and $\gamma$ be a path of $\mathbb{C} \setminus \{a\}$. Let $\theta_0 \in \mathbb{R}$ be a value of the argument of $\gamma(0) - a$:

$$\theta_0 \in \operatorname{Arg}(\gamma(0) - a).$$

There is a unique continous function $\theta : [0, 1] \mapsto \mathbb{R}$ such that $\theta(0) = \theta_0$ which is a *choice* of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$:

$$\forall\, t \in [0, 1],\ \theta(t) \in \operatorname{Arg}(\gamma(t) - a).$$

**Proof.** Let $(x(t), y(t))$ be the cartesian coordinates of $\gamma(t)$ in the system with origin $a$ and basis $(e^{i\theta_0}, ie^{i\theta_0})$. As long as $x(t) > 0$, the function

$$t \mapsto \theta_0 + \arg(x(t) + iy(t))$$

is a continuous choice of the argument of $\gamma(t) - a$. Let $d$ be the distance between $a$ and $\gamma([0, 1])$ and let $n \in \mathbb{N}$ such that

$$|t - s| \leq 2^{-n} \ \Rightarrow\ |\gamma(t) - \gamma(s)| < d.$$

The condition $x(t) > 0$ is ensured for any $t$ in $[0, 2^{-n}]$. This construction of a continuous choice may be iterated locally on every interval $[k2^{-n}, (k+1)2^{-n}]$ with a new coordinate system to provide a global continuous choice of the argument on $[0, 1]$.

The uniqueness of a continuous choice is a consequence of the intermediate value theorem: if we assume that there are two such functions $\theta_1$ and $\theta_2$ with the same initial value $\theta_0$, as $\theta_1(0) - \theta_2(0) = 0$, if $\theta_1(t) - \theta_2(t) \neq 0$ for some $t \in [0, 1]$, then either $|\theta_1(t) - \theta_2(t)| < \pi$, or there is a $\tau \in ]0, t[$ such that $\theta_1(\tau) - \theta_2(\tau) \neq 0$ and $|\theta_1(\tau) - \theta_2(\tau)| < \pi$. In any case, there is a contradiction since all values of the argument differ of a multiple of $2\pi$. ■

**Definition – Variation of the Argument.** Let $a \in \mathbb{C}$ and $\gamma$ be a path of $\mathbb{C} \setminus \{a\}$. The *variation* of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$ is defined as

$$[z \mapsto \operatorname{Arg}(z - a)]_\gamma = \theta(1) - \theta(0)$$

where $\theta$ is a continous choice of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$.

**Proof (unambiguous definition).** If $\theta_1$ and $\theta_2$ are two continuous choices of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$, for any $t \in [0, 1]$, they differ of a multiple of $2\pi$. As the function $\theta_1 - \theta_2$ is continuous, by the intermediate value theorem, it is constant. Hence

$$(\theta_1 - \theta_2)(1) = (\theta_1 - \theta_2)(0),$$

and $\theta_1(1) - \theta_1(0) = \theta_2(1) - \theta_2(0)$. ■

**Definition – Winding Number / Index.** Let $a \in \mathbb{C}$ and $\gamma$ be a closed path of $\mathbb{C} \setminus \{a\}$. The *winding number* – or *index* – of $\gamma$ around $a$ is the integer

$$\operatorname{ind}(\gamma, a) = \frac{1}{2\pi}[z \mapsto \operatorname{Arg}(z - a)]_\gamma.$$

**Proof – The Winding Number is an Integer.** Let $\theta$ be a continuous choice function of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$; as the path $\gamma$ is closed, $\theta(0)$ and $\theta(1)$, which are values of the argument of $\gamma(0) - a = \gamma(1) - a$, are equal modulo $2\pi$, hence $(\theta(1) - \theta(0))/2\pi$ is an integer. ■

**Definition – Path Exterior & Interior.** The *exterior* and *interior* of a closed path $\gamma$ are the subsets of the complex plane defined by

$$\operatorname{Ext} \gamma = \{z \in \mathbb{C} \setminus \gamma([0, 1]) \mid \operatorname{ind}(\gamma, z) = 0\}.$$

and

$$\operatorname{Int} \gamma = \mathbb{C} \setminus (\gamma([0, 1]) \cup \operatorname{Ext} \gamma) = \{z \in \mathbb{C} \setminus \gamma([0, 1]) \mid \operatorname{ind}(\gamma, z) \neq 0\}.$$

## Properties

**Theorem – The Winding Number is Locally Constant.** Let $a \in \mathbb{C}$ and $\gamma$ be a closed path of $\mathbb{C} \setminus \{a\}$. There is a $\epsilon > 0$ such that, for any $b \in \mathbb{C}$ and any closed path $\beta$, if

$$|b - a| < \epsilon \text{ and } (\forall\, t \in [0,1],\ |\beta(t) - \gamma(t)| < \epsilon)$$

then $\beta$ is a path of $\mathbb{C} \setminus \{b\}$ and

$$\mathrm{ind}(\gamma, a) = \mathrm{ind}(\beta, b).$$

**Proof.** Let $\epsilon = d(a, \gamma([0,1]))/2$. If $|b-a| < \epsilon$ and for any $t \in [0,1]$, $|\gamma(t)-\beta(t)| < \epsilon$, then clearly $b \in \mathbb{C} \setminus \beta([0,1])$. Additionally, for any $t \in [0,1]$ there are values $\theta_1$ of $\mathrm{Arg}(\gamma(t) - a)$ and $\theta_2$ of $\mathrm{Arg}(\beta(t) - b)$ such that $|\theta_1 - \theta_2| < \pi/2$. If we select some values $\theta_{1,0}$ of $\mathrm{Arg}(\gamma(0) - a)$ and $\theta_{2,0}$ of $\mathrm{Arg}(\beta(0) - b)$ such that $|\theta_{1,0} - \theta_{2,0}| < \pi/2$, then the corresponding continuous choices $\theta_1$ et $\theta_2$ satisfy $|\theta_1(t) - \theta_2(t)| < \pi/2$ for any $t \in [0,1]$([1]). Consequently

$$|\mathrm{ind}(\gamma, a) - \mathrm{ind}(\beta, b)| = \left| \frac{\theta_1(1) - \theta_1(0)}{2\pi} - \frac{\theta_2(1) - \theta_2(0)}{2\pi} \right| < \frac{1}{2}.$$

As both winding numbers are integers, they are equal. ■

**Corollary – The Winding Number is Constant on Components.** Let $\gamma$ be a closed path. The function

$$z \in \mathbb{C} \setminus \gamma([0,1]) \mapsto \mathrm{ind}(\gamma, z)$$

is constant on each component of $\mathbb{C} \setminus \gamma([0,1])$. If additionally the component is unbounded, the value of the winding number is zero.

**Proof.** The mapping $z \mapsto \mathrm{ind}(\gamma, z)$ is locally constant – and hence constant – on every connected component of $\mathbb{C} \setminus \gamma([0,1])$. If $a$ belongs to some unbounded component of this set, there is a $b$ in the same component such that $|b| > r = \max_{t\in[0,1]} |\gamma(t)|$. It is possible to connect $b$ to any point $c$ such that $|c| = r$ by a circular path in $\mathbb{C} \setminus \gamma([0,1])$, thus we may assume that $b \in \mathbb{R}_-$. The function

$$\theta : t \in [0,1] \mapsto \arg(\gamma(t) - b)$$

is a continuous choice of $z \mapsto \mathrm{Arg}(z - b)$ along $\gamma$ and it satisfies

$$\forall\, t \in [0,1],\ |\theta(t)| = \arctan \frac{\mathrm{Im}(\gamma(t) - b)}{\mathrm{Re}(\gamma(t) - b)} < \arctan \frac{r}{|b| - r} < \frac{\pi}{2}.$$

[1] Otherwise, by the intermediate value theorem, we could find some $t \in \,]0,1]$ such that $|\theta_1(t) - \theta_2(t)| = \pi/2$, but then, for every value $\theta_{1,t}$ of $\mathrm{Arg}(\gamma(t) - a)$ and $\theta_{2,t}$ of $\mathrm{Arg}(\beta(t) - b)$, we would have

$$\theta_{1,t} - \theta_{2,t} = \theta_1(t) - \theta_2(t) + 2\pi k$$

for some $k \in \mathbb{Z}$. Therefore, the choice of $\theta_{1,t}$ and $\theta_{2,t}$ such that $|\theta_{1,t} - \theta_{2,t}| < \pi/2$ would be impossible.

As $\gamma$ is a closed path, $\theta(0)$ and $\theta(1)$ – which are equal modulo $2\pi$ – are actually equal and

$$\text{ind}(\gamma, a) = \text{ind}(\gamma, b) = \frac{\theta(1) - \theta(0)}{2\pi} = 0$$

as expected. ∎

# Simply Connected Sets

**Definition – Simply/Multiply Connected Set & Holes.** Let $\Omega$ be an open subset of the plane. A *hole* of $\Omega$ is a bounded component of its complement $\mathbb{C} \setminus \Omega$. The set $\Omega$ is *simply connected* if it has no hole (if every component of its complement is unbounded) and *multiply connected* otherwise.

**Examples.**

1. The open set $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ is not connected but it is simply connected: its complement has a unique component which is unbounded, hence it has no holes.
2. The open set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \mid n \in \mathbb{N}\}}$ is multiply connected: its holes are exactly the singletons of its complement.

Intuitively, we should be able to circle around any hole of $\Omega$ without leaving the set; this idea leads to an alternate characterization of simply connected sets.

**Theorem – Simply Connected Sets & The Winding Number.** An open subset $\Omega$ of the complex plane is simply connected if and only if the interior of any closed path $\gamma$ of $\Omega$ is included in $\Omega$:

$$\forall\, z \in \mathbb{C} \setminus \gamma([0, 1]),\ \text{ind}(\gamma, z) \neq 0 \ \Rightarrow\ z \in \Omega,$$

or equivalently, if the complement of $\Omega$ is included in the exterior of $\gamma$:

$$\forall\, z \in \mathbb{C} \setminus \Omega,\ \text{ind}(\gamma, z) = 0.$$

**Examples.**

1. If $\gamma$ is a closed path of $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } x > 1\}$ and $z \in \mathbb{C} \setminus \Omega$, since $\mathbb{C} \setminus \Omega$ is connected and unbounded, $z$ belongs to an unbounded component of $\mathbb{C} \setminus \gamma([0, 1])$. Thus $\text{ind}(\gamma, z) = 0$ for any $z \in \mathbb{C} \setminus \Omega$.
2. The open set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \mid n \in \mathbb{N}\}}$ is open and multiply connected: for example $\gamma = 1 + 1/4[\circlearrowleft]$ is a path of $\Omega$, $z = 1$ is a point of $\mathbb{C} \setminus \Omega$ and $\text{ind}(\gamma, 1) = 1$.

**Remark.** Note that we may not always be able to encircle only one hole at a time. For example, in the case of the set $\Omega = \mathbb{C} \setminus \overline{\{2^{-n} \mid n \in \mathbb{N}\}}$, we can find a closed path $\gamma$ of $\mathbb{C} \setminus \Omega$ such that $\text{ind}(\gamma, 0) = 1$, but then we also have

$\text{ind}(\gamma, 2^{-n}) = 1$ for $n$ large enough: we cannot encircle the hole $\{0\}$ of $\Omega$ unless we also encircle an infinity of extra holes.

**Lemma.** The compact set $K$ is a hole of the open set $\Omega$ if and only if there is a compact subset $L$ of $\mathbb{C} \setminus \Omega$ such that $K \subset L$ and $\Omega \cup L$ is open.

**Proof of the Lemma.** If the subset $L$ of $\mathbb{C} \setminus \Omega$ is compact and $\Omega \cup L$ is open, then $L$ and $\mathbb{C} \setminus (\Omega \cup L)$ form a partition of $\mathbb{C} \setminus \Omega$ into a compact and a closed set. The distance between them is positive, thus any connected subset of $\mathbb{C} \setminus \Omega$ that contains a point of $L$ is actually included in $L$ and therefore bounded: it is a hole of $\Omega$.

Conversely, if $K$ is a hole of $\Omega$, then it is a compact set: $K$ is connected, hence its closure, which is a subset of the closed set $\mathbb{C} \setminus \Omega$, is also connected and a superset of $K$; as $K$ is maximal among these sets, $\overline{K} = K$. The set $K$ is therefore closed and bounded, thus is is compact.

Let $r > 0$ such that $K \subset D(0, r)$. The set $K$ is a component of the closed set $A = (\mathbb{C} \setminus \Omega) \cap \overline{D(0, r)}$. For any point $a \in A$ on the boundary $\partial D(0, r)$ of $D(0, r)$, there is a cover of $A$ into disjoint open set $U_a$ and $V_a$ such that $a \in U_a$ and $K \subset V_a$. Now, the boundary $\partial D(0, r)$ is compact, thus there is a finite collection of points $a_1, \ldots, a_n$ such that $U = \cup_{i=1}^n U_a$ covers $A \cap \partial D(0, r)$. The sets $U$ and $V = (\cap_{i=1}^n V_a) \cap D(0, r)$ are disjoints open sets that cover $A$ and $K \subset V$, thus the set $\mathbb{C} \setminus \Omega$ is the disjoint union of the compact set $L = A \setminus U$ that contains $K$ and of the closed set $(\mathbb{C} \setminus \Omega) \setminus V$. Therefore, the distance between $L$ and $\mathbb{C} \setminus (\Omega \cup L)$ is positive which means that every point of $L$ is an interior point of $\Omega \cup L$. Since every point of $\Omega$ is also an interior point of $\Omega \cup L$, this set is open. ■

**Proof – Simply Connected Sets & The Winding Number.** Assume that $\Omega$ is simply connected and let $\gamma$ be a closed path of $\Omega$. Let $z \in \mathbb{C} \setminus \Omega$; this point belongs to an unbounded connected component of $\mathbb{C} \setminus \Omega$ and therefore to an unbounded connected component of $\mathbb{C} \setminus \gamma([0, 1])$, thus $\text{ind}(\gamma, z) = 0$.

Conversely, if $\Omega$ is not simply connected, the set $\mathbb{C} \setminus \Omega$ has a hole $K$ which is contained in some compact subset $L$ of $\mathbb{C} \setminus \Omega$ such that $\Omega \cup L$ is open. The distance $\epsilon$ between $L$ and $\mathbb{C} \setminus (\Omega \cup L)$ is positive. Let $r < \epsilon/\sqrt{2}$; Define for any pair $(k, l)$ of integers the node $n_{k,l} = (k + il)r$ and $S_{k,l}$ as the closed square with vertices $n_{k,l}$, $n_{k+1,l}$, $n_{k+1,l+1}$ and $n_{k,l+1}$. The (positively) oriented boundary of the square $S_{k,l}$ is the polyline

$$[n_{k,l} \to n_{k+1,l} \to n_{k+1,l+1} \to n_{k,l+1} \to n_{k,l}]$$

The collection of squares that intersect $L$ is finite and covers $L$. Additionally, all of its squares are included in $\Omega \cup L$.

For any square $S$ in the cover of $L$ and any interior point $a$ of $S$ if $\gamma$ is the oriented boundary of $S$, then $\text{ind}(\gamma, a) = 1$. Additionally, $\text{ind}(\mu, a) = 0$ for the oriented boundary $\mu$ of any other square in the collection. Consequently, if $\Gamma$ denotes the collection of oriented line segments that composes the oriented boundaries of all squares of the cover of $L$, we have

$$\sum_{\gamma \in \Gamma} \frac{1}{2\pi} [z \mapsto \operatorname{Arg}(z-a)]_\gamma = 1.$$

Now if the line segment $\gamma$ belongs to $\Gamma$ and $\gamma([0,1]) \cap L \neq \varnothing$, then $\gamma^{\leftarrow}$ also belongs to $\Gamma$; if we remove all such pairs from $\Gamma$, the resulting collection $\Gamma'$ also satisfies

$$\sum_{\gamma \in \Gamma'} \frac{1}{2\pi} [z \mapsto \operatorname{Arg}(z-a)]_\gamma = 1.$$

and by construction the image of any $\gamma$ in $\Gamma'$ is included in $\Omega$. The original collection $\Gamma$ is balanced: for any square vertice $n$, the number of line segments with $n$ as an initial point and with $n$ as a terminal point is the same. The collection $\Gamma'$ has the same property. Consequently, the line segments of $\Gamma'$ may be assembled in a finite sequence of closed paths $\gamma_1, \ldots, \gamma_n$ and

$$\sum_{k=1}^{n} \operatorname{ind}(\gamma_k, a) = 1.$$

Every point of $L$ is either an interior point of some square of the collection, or the limit of such point; anyway, that means that

$$\forall\, z \in L,\ \sum_{k=1}^{n} \operatorname{ind}(\gamma_k, z) = 1$$

and thus that there is at least one path $\gamma_k$ such that $\operatorname{ind}(\gamma_k, z) \neq 0$. ■

## A Complex Analytic Approach

If a closed path is rectifiable, we may compute its winding number as a line integral; to prove this, we need the:

**Lemma.** Let $a \in \mathbb{C}$ and $\gamma$ be a rectifiable path of $\mathbb{C} \setminus \{a\}$. For any $t \in [0,1]$, let $\gamma_t$ be the path such that for any $s \in [0,1]$, $\gamma_t(s) = \gamma(ts)$. The function $\mu : [0,1] \to \mathbb{C}$, defined by

$$\mu(t) = \int_{\gamma_t} \frac{dz}{z-a}$$

satisfies

$$\exists\, \lambda \in \mathbb{C}^*,\ \forall\, t \in [0,1],\ e^{\mu(t)} = \lambda \times (\gamma(t) - a).$$

**Proof.** We only prove the lemma under the assumption that $\gamma$ is continuously differentiable; the rectifiable case is a straightforward extension.

We have for any $t \in [0, 1]$

$$\mu(t) = \int_{\gamma_t} \frac{dz}{z - a} = \int_0^1 \frac{\gamma'(ts) \times t}{\gamma(ts) - a} ds = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} ds,$$

hence

$$\mu'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

and the derivative of the quotient $\phi(t) = e^{\mu(t)}/(\gamma(t) - a)$ satisfies

$$\phi'(t) = \mu'(t)\phi(t) - \frac{\gamma'(t)}{\gamma(t) - a}\phi(t) = 0$$

which yields the result. ■

**Theorem – The Winding Number as a Line Integral.** Let $a \in \mathbb{C}$ and $\gamma$ be a rectifiable path of $\mathbb{C} \setminus \{a\}$. Then

$$[z \mapsto \operatorname{Arg}(z - a)]_\gamma = \operatorname{Im}\left(\int_\gamma \frac{dz}{z - a}\right).$$

If the path $\gamma$ is closed, then

$$\operatorname{ind}(\gamma, a) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z - a}.$$

**Proof.** We use the function $\mu$ of the previous lemma. Applying the modulus to both sides of the equation $e^{\mu(t)} = \lambda \times (\gamma(t) - a)$ provides $e^{\operatorname{Re}(\mu(t))} = |\lambda| \times |\gamma(t) - a|$, hence

$$e^{i\operatorname{Im}(\mu(t))} = \frac{\lambda}{|\lambda|} \frac{\gamma(t) - a}{|\gamma(t) - a|}.$$

The function $t \in [0, 1] \mapsto \operatorname{Im}(\mu(t))$ is – up to a constant – a continuous choice of $z \mapsto \operatorname{Arg}(z - a)$ on $\gamma$. Consequently,

$$[z \mapsto \operatorname{Arg}(z - a)]_\gamma = \operatorname{Im}(\mu(1)) - \operatorname{Im}(\mu(0)) = \operatorname{Im}(\mu(1)),$$

which is the desired result.

If additionally $\gamma$ is a closed path, the equations

$$\gamma(0) = \gamma(1) \text{ and } e^{\operatorname{Re}(\mu(t))} = |\lambda| \times |\gamma(t) - a|$$

yield $e^{\operatorname{Re}(\mu(0))} = e^{\operatorname{Re}(\mu(1))}$ and hence $\operatorname{Re}(\mu(1)) = \operatorname{Re}(\mu(0)) = 0$. Thus,

$$\operatorname{ind}(\gamma, a) = \frac{1}{2\pi}\operatorname{Im}(\mu(1)) = \frac{1}{i2\pi}\mu(1),$$

which concludes the proof. ■

# References

Hatcher, Allen. 2002. *Algebraic Topology.* Cambridge University Press. `https://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

# Exercises

## Star-Shaped Sets

Prove that every open star-shaped subset of $\mathbb{C}$ is simply connected.

## The Argument Principle for Polynomials

Let $p$ be the polynomial

$$p(z) = \lambda \times (z - a_1)^{n_1} \times \cdots \times (z - a_m)^{n_m}$$

where $\lambda$ is a nonzero complex number, $a_1, \dots, a_m$ are distinct complex numbers (the *zeros* or *roots* of the polynomial) and $n_1, \dots, n_m$ are positive natural numbers (the roots *orders* or *multiplicities*). Let $\gamma$ be a closed path whose image contains no root of $p$:

$$\forall\, t \in [0, 1],\ p(\gamma(t)) \neq 0.$$

The argument principle then states that

$$\mathrm{ind}(p \circ \gamma, 0) = \sum_{k=1}^{m} \mathrm{ind}(\gamma, a_k) \times n_k.$$

1. **Application: Finding the Roots of a Polynomial.**

   Use the figures below to determine – according to the argument principle – the number of roots $z$ of the polynomial $p(z) = z^3 + z + 1$ in the open unit disk centered on the origin.

2. **Argument Principle Proof (Elementary).** For any $k \in \{1, \dots, m\}$, we denote $\theta_k$ a continous choice of $z \mapsto \mathrm{Arg}(z - a_k)$ on $\gamma$. Use the functions $\theta_k$ to build a continuous choice of $z \mapsto \mathrm{Arg}\, z$ on $p \circ \gamma$; then, prove the argument principle.

3. **Argument Principle Proof (Complex Analysis).** Assume that $\gamma$ is rectifiable; write the winding number $\mathrm{ind}(p \circ \gamma, 0)$ as a line integral, then find another way to prove the argument principle in this context.

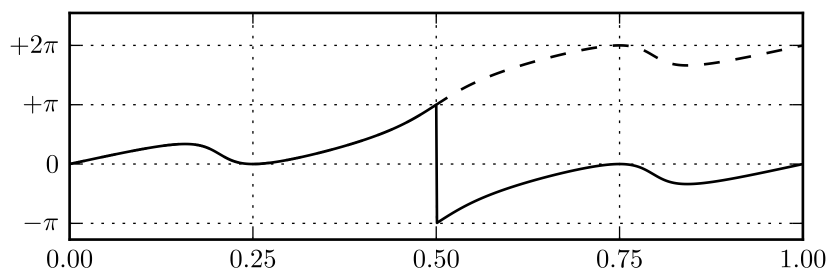


Figure 5.1: Graph of $t \in [0,1] \mapsto \arg\left[(e^{i2\pi t})^3 + (e^{i2\pi t}) + 1\right]$; this function has a jump of $-2\pi$ at $t = 0.5$ (where it is undefined). The dashed line represents a continuous choice of the argument of $t \in [0,1] \mapsto (e^{i2\pi t})^3 + (e^{i2\pi t}) + 1$.

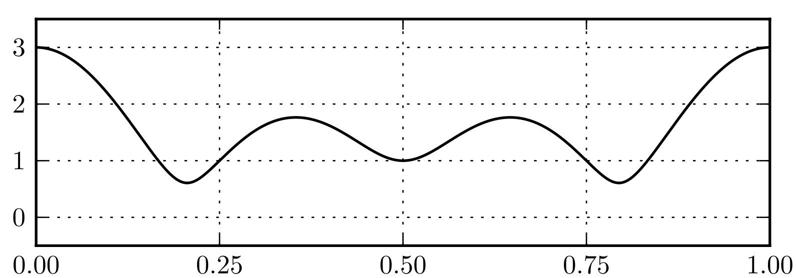


Figure 5.2: Graph of $t \in [0,1] \mapsto |(e^{i2\pi t})^3 + (e^{i2\pi t}) + 1|$.

## Set Operations & Simply Connected Sets

Suppose that $A$, $B$ and $\mathbb{C} \setminus C$ are open subsets of $\mathbb{C}$. For each of the three statements below,

- determine whether or not the statement is true (either prove it or provide a counter-example);
- if the statement is false, find a sensible assumption that makes the new statement true (and provide a proof).

The statements are:

1. **Intersection.** The intersection $A \cap B$ of two simply connected sets $A$ and $B$ is simply connected.
2. **Complement.** The relative complement $A \setminus C$ of a connected set $C$ in a simply connected set $A$ is simply connected.
3. **Union.** The union $A \cup B$ of two connected and simply connected sets $A$ and $B$ is simply connected.

# Chapter 6

# Cauchy's Integral Theorem – Global Version

## Path Sequences

It is convenient to state Cauchy's integral theorem for finite sequences of paths instead of paths. To this end, we generalize some of the concepts initially defined for paths.

**Definition – Opposite & Concatenation.** The opposite of the path sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the path sequence

$$\gamma^{\leftarrow} = (\gamma_n^{\leftarrow}, \ldots, \gamma_1^{\leftarrow}).$$

The concatenation of the path sequences

$$\alpha = (\alpha_1, \ldots, \alpha_k) \text{ and } \beta = (\beta_1, \ldots, \beta_l)$$

is the path sequence

$$\alpha \,|\, \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l).$$

**Definition – Image.** The image of the path sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the set

$$\gamma([0,1]) = \bigcup_{k=1}^{n} \gamma_k([0,1]).$$

**Definition – Winding Number, Exterior, Interior.** If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a sequence of closed paths and $a \in \mathbb{C}$ is not on its image, the winding number of

$\gamma$ around $a$ is defined by

$$\operatorname{ind}(\gamma, a) = \sum_{k=1}^{n} \operatorname{ind}(\gamma_k, a).$$

The exterior of $\gamma$ is the set

$$\operatorname{Ext}\gamma = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) = 0\}$$

and its interior is the set

$$\operatorname{Int}\gamma = \mathbb{C} \setminus (\gamma([0,1]) \cup \operatorname{Ext}\gamma) = \{z \in \mathbb{C} \setminus \gamma([0,1]) \mid \operatorname{ind}(\gamma, z) \neq 0\}.$$

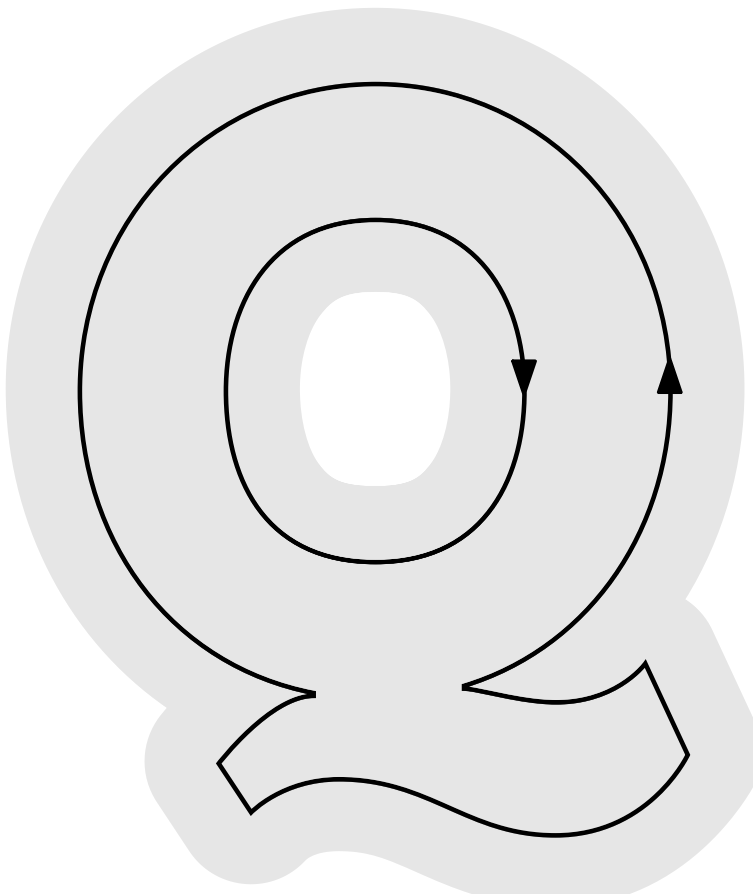

Figure 6.1: A pair $\gamma$ of two rectifiable closed paths in an open set $\Omega$ represented in light grey. Both paths are concatenations of quadratic Bézier curves.

Figure 6.2: This path sequence forms the outline of the capital “Q” letter in the League Spartan typeface. The interior of the path sequence is represented in dark grey; as the two paths are oriented in opposite directions, the interior of the path sequence is included in $\Omega$. The interior of the inner path does *not* belong to the interior of the path sequence; a typographist would say that it is *a closed counter* of the letter.

**Definition – Length & Line Integral.** Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a sequence of rectifiable paths. The length of $\gamma$ is defined as

$$\ell(\gamma) = \sum_{k=1}^{n} \ell(\gamma_k).$$

The integral along $\gamma$ of a complex-valued function $f$ which is defined and continuous on the image of $\gamma$ is

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz.$$

## Cauchy's Theorem & Corollaries

For the global version of Cauchy's integral theorem, the star-shaped assumption is replaced by a weaker geometric requirement:

**Theorem – Cauchy's Integral Theorem (Global Version).** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let $\gamma$ be a sequence of rectifiable closed paths of $\Omega$. If $\operatorname{Int} \gamma \subset \Omega$ then

$$\int_\gamma f(z)\, dz = 0.$$

**Remark – One or Two Paths.** This version of Cauchy's integral theorem is clearly applicable for a single path $\gamma$ instead of a path sequence. Now, the next common use case involves two rectifiable closed paths $\gamma$ and $\mu$ of $\Omega$. If they have the same winding number with respect to any point which is not in $\Omega$:

$$\forall\, z \in \mathbb{C} \setminus \Omega,\ \operatorname{ind}(\gamma, z) = \operatorname{ind}(\mu, z),$$

then the interior of the path sequence $(\gamma, \mu^{\leftarrow})$ is included in $\Omega$ and Cauchy's integral theorem is applicable. Its conclusion provides

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

**Remark – Simply Connected Sets.** If $\Omega$ is simply connected, Cauchy's theorem is applicable for any sequence of rectifiable closed paths $\gamma$ of $\Omega$. Indeed in any such set $\Omega$, for any path $\gamma$ – and thus for any sequence of paths $\gamma$ – we have $\operatorname{Int} \gamma \subset \Omega$. Since any star-shaped set is simply connected, the local version of Cauchy's theorem is a special case of the global version.

Cauchy's residue theorem is a generalization of his integral theorem. It covers the case where the interior of the path sequence $\gamma$ is included in the domain $\Omega$ of the holomorphic function $f$, except for a set of *isolated singularities.* The integral of $f$ along $\gamma$ in this case can be computed in terms of the *residues* of the function at these singularities.

**Definition – Singularity.** Let $\Omega$ be an open subset of $\mathbb{C}$. A *singularity* of a function $f : \Omega \to \mathbb{C}$ is a point $a$ of $\mathbb{C} \setminus \Omega$. It is *isolated* if its distance to the other singularities of $f$ is positive:

$$\exists \epsilon > 0, \ \forall z \in \mathbb{C}, \ (|z-a| < \epsilon \text{ and } z \neq a\,) \ \Rightarrow \ z \in \Omega.$$

**Definition – Residue.** Let $a$ be an isolated singularity of the holomorphic function $f : \Omega \to \mathbb{C}$. Let $d$ be the distance between $a$ and the other singularities of $f$ ($+\infty$ if $a$ is the only singularity of $f$). The integral of $f$ along $\gamma = a + r[\circlearrowleft]$ is defined and independent of $r$ as long as $0 < r < d$. We define the *residue* of $f$ at $a$ as

$$\operatorname{res}(f, a) = \frac{1}{i2\pi} \int_\gamma f(z)\, dz$$

for any such $r$.

**Examples – Singularity & Residue.** Let $a \in \mathbb{C}$ and

$$f : z \in \mathbb{C} \setminus \{a\} \mapsto \frac{1}{z-a}.$$

The point $a$ is the only singularity of $f$; it is clearly isolated. For any $r > 0$ and $\gamma = a + r[\circlearrowleft]$ we have

$$\int_\gamma f(z)\, dz = \int_\gamma \frac{dz}{z-a} = i2\pi \times \operatorname{ind}(\gamma, a) = i2\pi,$$

thus $\operatorname{res}(f, a) = 1$. Now, let $a \in \mathbb{C}$ and let

$$f : z \in \mathbb{C} \setminus \{a\} \mapsto (z-a)^n \ \text{ where } \ n \in \mathbb{Z} \setminus \{-1\}.$$

Since $z \in \mathbb{C} \setminus \{a\} \mapsto (z-a)^{n+1}/(n+1)$ is a primitive of $f$,

$$\int_\gamma f(z)\, dz = \int_\gamma (z-a)^n\, dz = 0,$$

thus $\operatorname{res}(f, a) = 0$.

**Proof – Residue: independence with respect to the radius.** Let $r_1$ and $r_2$ be two real numbers in $]0, d[$ and let $\gamma_1 = z + r_1[\circlearrowleft]$ and $\gamma_2 = z + r_2[\circlearrowleft]$. If $z \in \mathbb{C} \setminus \Omega$, either $z = a$ and

$$\operatorname{ind}(\gamma_1, z) = \operatorname{ind}(\gamma_2, z) = 1,$$

or $|z-a| \geq d$ and

$$\operatorname{ind}(\gamma_1, z) = \operatorname{ind}(\gamma_2, z) = 0.$$

In any case the winding numbers are equal. The "One or Two Paths" remark therefore provides

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

which concludes the proof.

■

**Theorem – Cauchy's Residue Theorem.** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let $\gamma$ be a sequence of rectifiable closed paths of $\Omega$. If $A$ is a finite set of isolated singularities of $f$ such that

$$\operatorname{Int}\gamma \subset \Omega \cup A$$

then

$$\int_{\gamma} f(z)\, dz = i2\pi \sum_{a \in A} \operatorname{ind}(\gamma, a) \times \operatorname{res}(f, a).$$

**Remark – Infinite Set of Singularities.** Note that if we drop the assumption that $A$ is finite, the conclusion of the theorem still holds since only a finite number of singularities of $A$ may be in the interior of $\gamma$([1]); the sum in the right-hand side of the theorem equation may then have an infinite number of terms, but only a finite number of them are non-zero.

**Proof – Cauchy's Residue Theorem.** We may assume that the set $A$ is included in $\operatorname{Int}\gamma$. If this assumption is not satisfied, replace $A$ with $A \cap \operatorname{Int}\gamma$; this new set $A$ still satisfies $\operatorname{Int}\gamma \subset \Omega \cup A$ and the conclusion of the theorem for the new set does provide the result for the original set.

Let $\epsilon > 0$ be such that for any $a \in A$, $D(a, \epsilon) \subset \Omega \cup \{a\}$ and let $0 < r < \epsilon$. Define for every $a$ in $A$ the path $\gamma_a$ by

$$\gamma_a(t) = a + r[\circlearrowleft]^{-\operatorname{ind}(\gamma, a)}.$$

We clearly have $\operatorname{ind}(\gamma_a, a) = -\operatorname{ind}(\gamma, a)$.

Let $\lambda$ be the concatenation of $\gamma$ and the sequence of all $\gamma_a$ for $a \in A$. We now prove that $\operatorname{Int}\lambda \subset \Omega$; we need to establish that $\operatorname{ind}(\lambda, z) = 0$ for every $z \in \mathbb{C} \setminus \Omega$. For such a point $z$, either

1. $z \in A$.
   In this case, $\operatorname{ind}(\gamma_z, a) = 0$ for any other singularity $a \in A$. Therefore,

$$\operatorname{ind}(\lambda, z) = \operatorname{ind}(\gamma, z) + \operatorname{ind}(\gamma_z, z) = 0.$$

[1] Indeed, assume instead that there is a infinite sequence of distincts points of $A$ in $\operatorname{Int}\gamma$; by compactness a subsequence of it converges to some point $a$ in its closure. The singularities of $f$ are a closed set, thus $a$ is itself a singularity. Since the boundary of $\operatorname{Int}\gamma$ is included in the image of $\gamma$ and hence in $\Omega$, the point $a$ actually belongs to $\operatorname{Int}\gamma$. Now $\operatorname{Int}\gamma \subset \Omega \cup A$, therefore $a \in A$, but by construction it is not isolated, which is a contradiction.

2. $z \notin \Omega \cup A$.
   We have $\operatorname{ind}(\gamma_a, z) = 0$ for any $a \in A$. Additionally, as $\operatorname{Int}\gamma \subset \Omega \cup A$, $\operatorname{ind}(\gamma, z) = 0$. Finally, $\operatorname{ind}(\lambda, z) = 0$.

Cauchy's integral theorem then provides

$$\int_\lambda f(z)dz = \int_\gamma f(z)\,dz + \sum_{a\in A}\int_{\gamma_a} f(z)\,dz = 0,$$

By construction of the $\gamma_a$ and the definition of residues, we have

$$\int_{\gamma_a} f(z)\,dz = -\operatorname{ind}(\gamma, a) \times i2\pi \operatorname{res}(f, a).$$

The result of the theorem follows. ■

There is a third equivalent form of Cauchy's integral theorem: Cauchy's integral formula[2]. It gives the value of $f$ at any point of the interior of $\gamma$ as a function of its values on the image of $\gamma$.

**Theorem – Cauchy's Integral Formula.** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Let $\gamma$ be a sequence of rectifiable closed paths of $\Omega$ and $a \in \Omega \setminus \gamma([0, 1])$. If $\operatorname{Int}\gamma \subset \Omega$, then

$$\int_\gamma \frac{f(z)}{z-a}\,dz = i2\pi \times \operatorname{ind}(\gamma, a) \times f(a).$$

**Proof.** The function

$$g : z \in \Omega \setminus \{a\} \mapsto \frac{f(z)}{z-a}$$

is holomorphic. The point $a$ is one of its isolated singularities. For $A = \{a\}$, we have

$$\operatorname{Int}\gamma \subset (\Omega \setminus \{a\}) \cup A = \Omega.$$

Additionally, if $\mu = a + r[\circlearrowleft]$,

$$\operatorname{res}(g, a) = \lim_{r\to 0} \frac{1}{i2\pi} \int_\mu g(z)\,dz = \lim_{r\to 0} \int_0^1 f(a + re^{i2\pi t})\,dt = f(a).$$

Therefore, Cauchy's residue theorem provides

$$\int_\gamma \frac{f(z)}{z-a}\,dz = i2\pi \times \operatorname{ind}(\gamma, a) \times f(a)$$

which is Cauchy's integral formula. ■

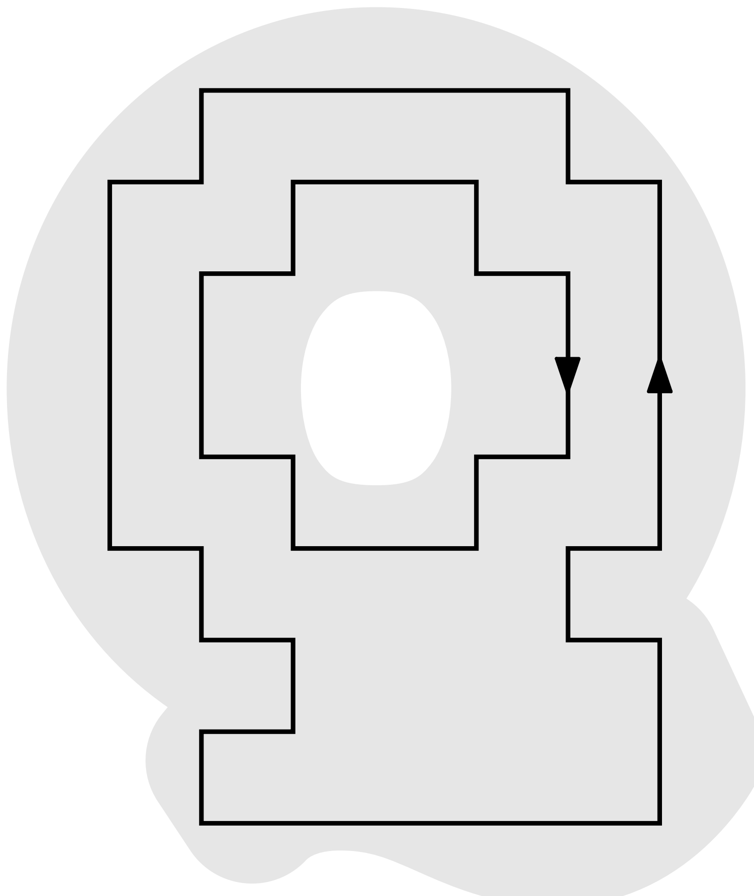

Figure 6.3: A path sequence made of arrows whose interior is included in $\Omega$.

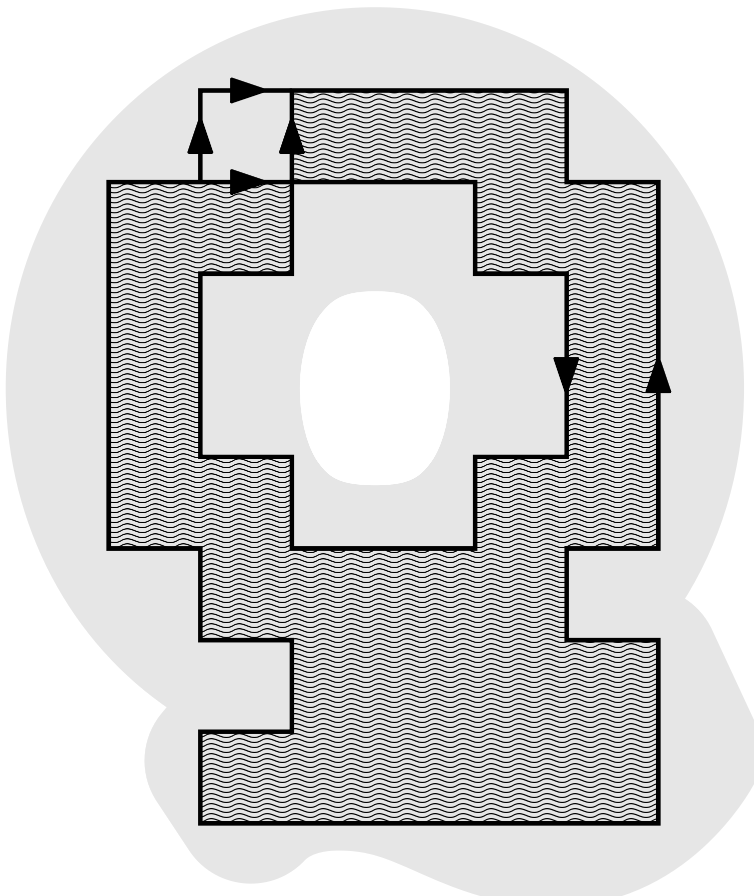

Figure 6.4: We can apply the local version of Cauchy's integral theorem "cell-by-cell" to such a path to prove the global version of the theorem.

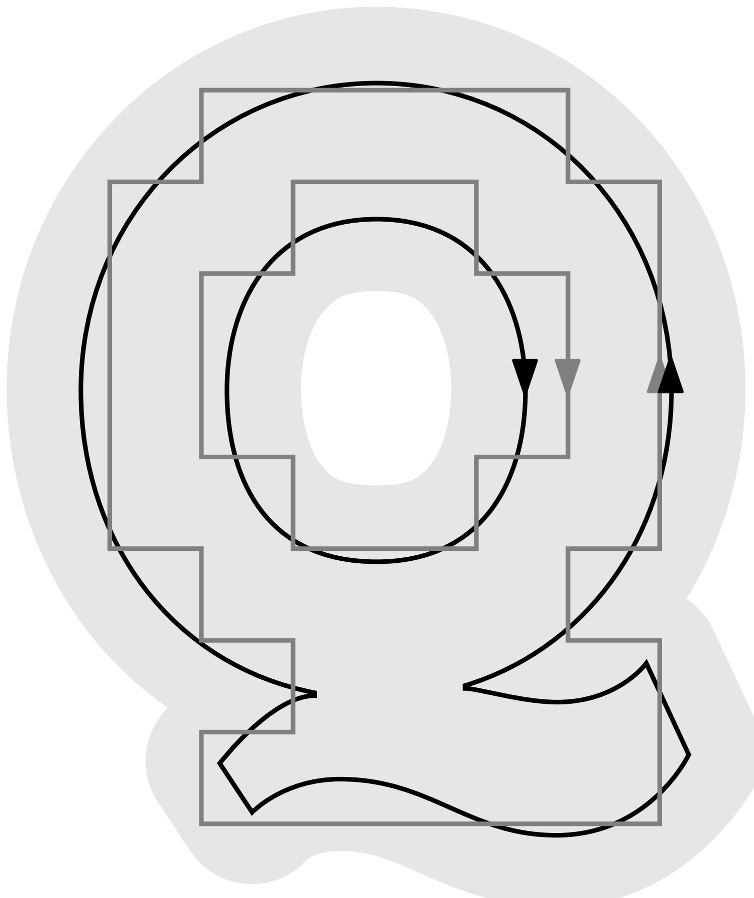

Figure 6.5: The path sequence made of arrows (in grey) is a suitable approximation of the original outline: the integral on both path sequence of holomorphic functions defined in $\Omega$ are equal.

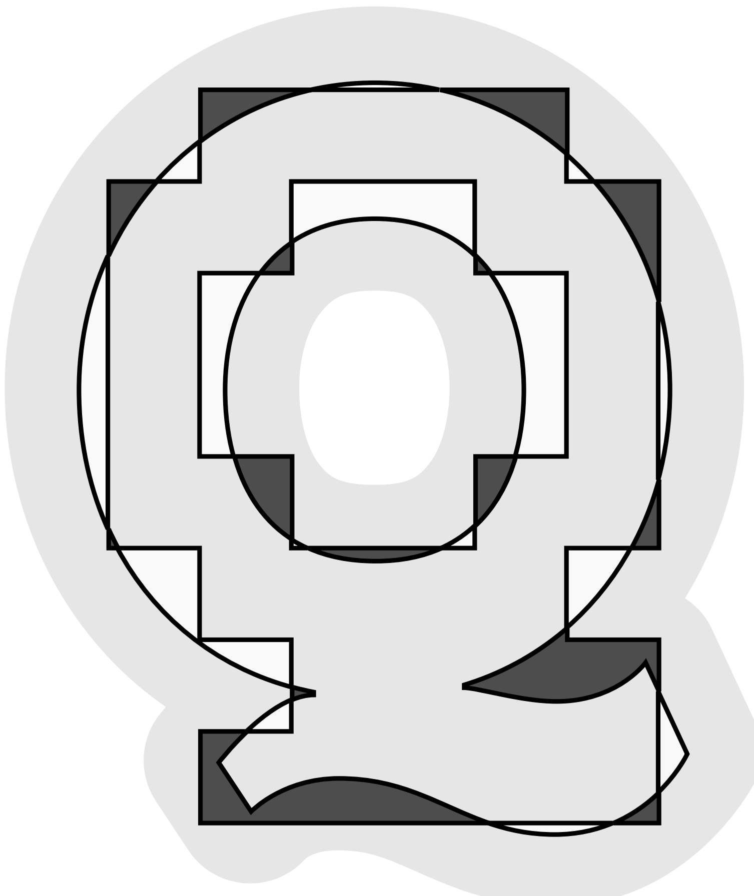

Figure 6.6: Indeed, the integral on the "difference" between the two paths sequence (the concatenation of the original and the opposite of the grid approximation) is an integral on a collection of "small" closed paths for which the local version of Cauchy's integral theorem can be applied: all these integrals are equal to zero.

## The Proof

**Definition – Path Sequence Decomposition.** A *decomposition* of a sequence of rectifiable paths $\gamma = (\gamma_1, \dots, \gamma_n)$ is a sequence of rectifiable paths

$$\gamma^* = (\gamma_1^*, \dots, \gamma_{p_1}^*, \gamma_{p_1+1}^*, \dots, \gamma_{p_n}^*, \gamma_{p_n+1}^*, \dots, \gamma_p^*)$$

such that for a suitable set of partitions of the unity

$$\begin{array}{rcl} \gamma_1 & = & \gamma_1^* \,|_{t_1^1} \cdots |_{t_{p_1-1}^1} \gamma_{p_1}^* \\ \gamma_2 & = & \gamma_{p_1+1}^* \,|_{t_1^2} \cdots |_{t_{p_2-p_1-1}^2} \gamma_{p_2}^* \\ & \vdots & \\ \gamma_p & = & \gamma_{p_n+1}^* \,|_{t_1^n} \cdots |_{t_{p-p_n-1}^n} \gamma_p^* \end{array}$$

**Definition – Equivalent Path Sequences.** Let $n_\gamma(\mu)$ be the number of occurences of the path $\mu$ in the path sequence $\gamma$. Two sequences of rectifiable paths $\gamma$ and $\lambda$ are *equivalent* if they have decompositions $\gamma^*$ and $\lambda^*$ such that for any path $\mu$

$$n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^\leftarrow) = n_{\lambda^*}(\mu) - n_{\lambda^*}(\mu^\leftarrow).$$

**Remark – Integral along Equivalent Paths.** If the sequence of rectifiable paths $\gamma$ has a decomposition into a sequence of rectifiable paths $\gamma^*$, then for every continuous and complex-valued function $f$ defined on the image of $\gamma$,

$$\int_\gamma f(z)\,dz = \frac{1}{2}\sum_\mu (n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^\leftarrow)) \times \left(\int_\mu f(z)\,dz\right)$$

where the sum is taken over the paths $\mu$ such that $\mu$ or $\mu^\leftarrow$ has at least one occurrence in $\gamma^*$. Consequently, if $\gamma$ and $\lambda$ are equivalent and $f$ is defined and continuous on the images of $\gamma$ and $\lambda$,

$$\int_\gamma f(z)\,dz = \int_\lambda f(z)\,dz.$$

**Definition – Path Diameter.** The *diameter* of a path $\gamma : [0,1] \to \mathbb{C}$ is defined as the diameter of its image: it is the nonnegative real number

$$\operatorname{diam}(\gamma) = \operatorname{diam}(\gamma([0,1])) = \max\{|z-w| \mid z \in \gamma([0,1]),\, w \in \gamma([0,1])\}.$$

**Theorem – Small Closed Paths Theorem.** Let $\Omega$ be an open subset of $\mathbb{C}$ and $\gamma$ be a sequence of rectifiable closed paths of $\Omega$ such that $\operatorname{Int}\gamma \subset \Omega$. There is

[2]The integral theorem implies the residue theorem which in turn implies the integral formula. Finally, the proof of the integral theorem is straightforward if we assume that the integral formula holds.

a sequence of rectifiable closed paths $\mu$ of $\Omega$ of arbitrarily small diameter which is equivalent to $\gamma$.

This geometric result and the local Cauchy theory yield the global version of Cauchy's integral theorem:

**Proof – Cauchy's Integral Theorem.** Assume that $\Omega$, $f$ and $\gamma$ satisfy the assumptions of Cauchy's integral theorem. Since the union of the image of $\gamma$ and its interior is a compact set, there is a $\epsilon > 0$ such that the open set

$$\Omega' = \{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) > \epsilon\}$$

contains the image of $\gamma$ and its interior. By the small closed paths theorem, there is a sequence of rectifiable closed paths $\mu = (\mu_1, \dots, \mu_n)$ of $\Omega'$ of diameter less than $\epsilon$ which is equivalent to $\gamma$, and therefore such that

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

The image of every path $\mu_k$ of diameter less than $\epsilon$ is included in the disk centered on $\mu_k(0)$ and of radius $\epsilon$. This disk belongs to $\Omega$ by construction and thus the local version of Cauchy's theorem is applicable. Finally,

$$\int_\mu f(z)\, dz = \sum_{k=1}^{n} \int_{\mu_k} f(z)\, dz = 0,$$

which provides the desired result. ■

The proof of the small closed paths theorem itself requires several lemmas.

**Definition – Arrow.** An *arrow* is an oriented line segment

$$[(k + il)2^{-n} \to (k' + il')2^{-n}]$$

for some $n \in \mathbb{N}$ and $k$, $l$, $k'$, $l' \in \mathbb{Z}$ such that

$$|k' - k| + |l' - l| = 1.$$

**Lemma – Small Paths & Path of Arrows.** Let $\Omega$ be an open subset of $\mathbb{C}$ and $\gamma$ be a sequence of rectifiable closed paths of $\Omega$. For any $\epsilon > 0$, there are two sequences $\lambda_1$ and $\lambda_2$ of rectifiable closed paths of $\Omega$ such that

1. the path sequences $\gamma$ and $\lambda_1 \mid \lambda_2$ are equivalent.
2. the diameter of every path of $\lambda_1$ is smaller than $\epsilon$.
3. $\lambda_2$ has a decomposition into arrows.

**Proof.** We prove the result for a rectifiable closed path $\gamma$; the result for path sequences is a simple corollary. Let $n$ be a natural number and let $(\gamma_1, \dots, \gamma_m)$

be a sequence of rectifiable paths such that $\gamma = \gamma_1 \,|\, \cdots \,|\, \gamma_m$ and $\ell(\gamma_k) < 2^{-n}$ for any $k \in \{1, \dots, m\}$. Denote $\pi_n$ the function defined on $\mathbb{C}$ by

$$\pi_n(z) = [\mathrm{Re}(2^n z)]2^{-n} + i[\mathrm{Im}(2^n z)]2^{-n};$$

where the function $[\cdot]$ rounds a real number to (one of) the nearest integer(s). For any $z \in \mathbb{C}$, $|\pi_n(z) - z| < 2^{-n}$. The points $\pi_n(\gamma_k(0))$ and $\pi_n(\gamma_k(1))$ are distant by less than $3 \times 2^{-n}$ and thus may always be joined by a path $\lambda_{2,k}$ which is the concatenation of at most four consecutive arrows of length $2^{-n}$.

Define the rectifiable closed path $\lambda_{1,k}$ as the concatenation:

$$\lambda_{1,k} = \gamma_k \,|\, [\gamma_k(1) \to \pi_n(\gamma_k(1))] \,|\, \lambda_{2,k}^{\leftarrow} \,|\, [\pi_n(\gamma_k(0)) \to \gamma_k(0)]$$

The length of the closed path $\lambda_{1,k}$ is smaller than $7 \times 2^{-n}$, hence its diameter is smaller than $7/2 \times 2^{-n}$. A suitable choice of $n$ provides $\mathrm{diam}(\lambda_{1,k}) < \epsilon$. The paths $\lambda_1 = (\lambda_{1,1}, \dots, \lambda_{1,m})$ and $\lambda_2 = (\lambda_{2,1}, \dots, \lambda_{2,m})$ satisfy the statement of the lemma. ■

**Lemma – Small Closed Paths Theorem (Arrow Version).** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $\gamma$ be a sequence of rectifiable closed paths of $\Omega$ *with a decomposition into arrows* and such that $\mathrm{Int}\,\gamma \subset \Omega$. There is a sequence of rectifiable closed paths $\mu$ of $\Omega$ of arbitrarily small diameter which is equivalent to $\gamma$.

**Proof – Small Closed Paths Theorem (Arrow Version).** Any sequence $\gamma$ of rectifiable closed paths made of arrows may be decomposed further into a sequence $\gamma^*$ of arrows of the same length $2^{-n}$ for an arbitrary large $n$. Now, we may associate to this level of decomposition a family indexed by integers $k$ and $l$ of square cells

$$C_{k,l} = \{(k + il + s + it)2^{-n} \mid (s,t) \in [0,1]^2\}.$$

with centers

$$c_{k,l} = k + 0.5 + i(l + 0.5).$$

For every arrow $\mu$ of length $2^{-n}$, the number $n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow})$ only depends on the numbers $\mathrm{ind}(\gamma, c)$ where $c$ is the center of a cell (or actually with respect to any other point of the cell – the index is constant in each cell) and thus, two path sequences with the same set of winding numbers are equivalent. Consider for example the vertical arrow

$$\mu = [(k + il)2^{-n} \to (k + i(l+1))2^{-n}]$$

and the associated left cell $C_{k-1,l}$ and $C_{k,l}$. If we edit $\gamma^*$ to replace every occurence of $\mu$ with the polyline

$$\mu_2 = [(k + il)2^{-n} \to (k + 1 + il)2^{-n} \to (k + 1 + i(l+1))2^{-n} \to (k + i(l+1))2^{-n}]$$

and every occurence of $\mu^{\leftarrow}$ by $\mu_2^{\leftarrow}$, we have increased the index of the right cell (with center $c_{k,l}$) by $n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow})$ and the index of every other cell remains the same. By construction, for this new path sequence $\gamma_2$, we have $n_{\gamma_2^*}(\mu) = 0$ and $n^*_{\gamma_2}(\mu^{\leftarrow}) = 0$; the left and right cells belongs to the same component of $\mathbb{C} \setminus \gamma_2([0,1])$. Therefore the index of $\gamma_2$ around both cells is the same which means that

$$\mathrm{ind}(\gamma, c_{k-1,l}) = \mathrm{ind}(\gamma, c_{k,l}) + n_{\gamma^*}(\mu) - n_{\gamma^*}(\mu^{\leftarrow}).$$

The treatment of horizontal arrows is similar.

Now, consider the sequence of centers $(c_1, \dots, c_m)$ such that $\mathrm{ind}(\gamma, c_p) \neq 0$ and the path sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ where $\lambda_p$ is either the concatenation of $\mathrm{ind}(\gamma, c_p)$ times the boundary of the cell with center $c_p$ oriented counterclockwise if this winding number is positive, or the concatenation of $-\mathrm{ind}(\gamma, c_p)$ times the boundary of the cell of center $c_p$ oriented clockwise if it is negative. Every path $\lambda_p$ is rectifiable; the corresponding cell with center $c_p$ is included in $\mathrm{Int}\,\gamma$ and therefore $\lambda_p([0,1])$ is included in $\Omega$. Additionally, by construction, for every cell center $c$, $\mathrm{ind}(\gamma, c) = \mathrm{ind}(\lambda, c)$ and therefore $\gamma$ and $\lambda$ are equivalent. The diameter of $\lambda_p$ is smaller than $2^{-n}$; a suitably large choice of $n$ makes the diameter as small as required and this concludes the proof. ■

**Proof – Small Closed Paths Theorem.** Let $\epsilon > 0$ such that that the open set

$$\Omega' = \{z \in \Omega \mid d(z, \mathbb{C} \setminus \Omega) > \epsilon\}$$

contains the image of $\gamma$ and its interior. Let $\lambda_1$ and $\lambda_2$ be the path sequences provided by the small paths & path of arrows lemma with $\Omega = \Omega'$.

The image of every path $\mu$ of the sequence $\lambda_1$ is included in the disk centered on $\mu(0)$ of radius $\epsilon$ which is itself included in $\Omega$. Any point $z \in \mathbb{C} \setminus \Omega$ belongs to the unbounded component of $\mathbb{C} \setminus \mu([0,1])$, thus $\mathrm{ind}(\mu, z) = 0$. Consequently, for any such $z$,

$$\mathrm{ind}(\lambda_2, z) = \mathrm{ind}(\gamma, z)$$

and thus $\mathrm{Int}\,\lambda_2 \subset \Omega$. The conclusion of the theorem then follows from the application of the arrow version of the small closed paths theorem to the sequence of paths $\lambda_2$. ■

# Exercises

## Cauchy's Converse Integral Theorem

Let $\Omega$ be an open subset of $\mathbb{C}$.

Suppose that for every holomorphic function $f : \Omega \to \mathbb{C}$,

$$\int_\gamma f(z)\, dz = 0$$

for some sequence $\gamma$ of closed rectifiable paths of $\Omega$. What conclusion can we draw? What if the property holds for every sequence $\gamma$ of closed rectifiable paths of $\Omega$?

## Cauchy Transform of Power Functions

Compute for any $n \in \mathbb{Z}$ and any $z \in \mathbb{C}$ such that $|z| \neq 1$ the line integral

$$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{w^n}{w - z} dw.$$

# Chapter 7

# Power Series

## Convergence of Power Series

**Definition & Theorem – Radius of Convergence.** Let $c \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. The *radius of convergence* of the power series

$$\sum_{n=0}^{+\infty} a_n (z-c)^n$$

is the unique $r \in [0, +\infty]$ such that the series converges if $|z-c| < r$ and diverges if $|z-c| > r$. The disk $D(c, r)$ – the largest open disk centered on $c$ where the series converges – is the *open disk of convergence* of the series.

The radius of convergence $r$ is the inverse of the *growth ratio* of the sequence $a_n$, defined as the infimum in $[0, +\infty]$ of the set of values $\sigma \in [0, +\infty)$ such that $a_n$ is eventually dominated by $\sigma^n$:

$$\exists\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ (n \geq m) \Rightarrow |a_n| \leq \sigma^n.$$

(or equivalently, such that $\exists\, \kappa > 0$, $\forall\, n \in \mathbb{N}$, $|a_n| \leq \kappa \sigma^n$.) This growth ratio is equal to $\limsup_{n \to +\infty} |a_n|^{1/n}$, which leads to the Cauchy-Hadamard formula[1]:

$$r = \frac{1}{\limsup\limits_{n \to +\infty} |a_n|^{1/n}}.$$

By convention here, $1/0 = +\infty$ and $1/(+\infty) = 0$.

[1] to compute the *limit superior* of a sequence of (extended) real numbers, consider all subsequences that converge (as extended real numbers: in $[-\infty, +\infty]$) and take the supremum of their limits.

**Proof.** Let $\rho$ be the growth ratio of the sequence $a_n$. If a complex number $z$ satisfies $|z-c| < \rho^{-1}$, $\rho$ is finite and there is a $\sigma > \rho$ such that $|z-c| < \sigma^{-1}$. Eventually, we have $|a_n| \leq \sigma^n$ and thus

$$|a_n(z-c)^n| \leq (\sigma|z-c|)^n.$$

As $\sigma|z-c| < 1$, the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is convergent. Conversely, if $|z-c| > \rho^{-1}$, $\rho > 0$ and there is a $\sigma < \rho$ such that $|z-c| > \sigma^{-1}$. As $\sigma < \rho$, there is a strictly increasing sequence of $n \in \mathbb{N}$ such that $|a_n| > \sigma^n$ and thus $|a_n(z-c)^n| > (\sigma\sigma^{-1})^n = 1$. Since its terms do not converge to zero, the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is divergent.

We now prove that the growth ratio of $|a_n|$ is equal $\limsup_n |a_n|^{1/n}$. Indeed, for any $\sigma$ greater than the growth ratio $\rho$, eventually $|a_n| \leq \sigma^n$, hence $|a_n|^{1/n} \leq \sigma$ and $\limsup_n |a_n|^{1/n} \leq \sigma$, therefore $\limsup_n |a_n|^{1/n} \leq \rho$. Conversely, if $\sigma$ is smaller than the growth ratio, there is a strictly increasing sequence of $n \in \mathbb{N}$ such that $|a_n| > \sigma^n$, hence $|a_n|^{1/n} > \sigma$ and $\limsup_n |a_n|^{1/n} \geq \sigma$, thus $\limsup_n |a_n|^{1/n} \geq \rho$. ■

**Example – A Geometric Series.** Consider the power series

$$\sum_{n=0}^{+\infty} (-1/2)^n z^n.$$

Since $|(-1/2)^n| = 1/2^n \leq \sigma^n$ eventually if and only if $\sigma \geq 1/2$, the growth bound of the geometric sequence $(-1/2)^n$ is $1/2$. Thus the open disk of convergence of this power series is $D(0,2)$.

**Example – A Lacunary Series.** Consider the power series:

$$\sum_{n=0}^{+\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots.$$

The "lacunary" adjective refers to the large gaps between nonzero coefficients; These coefficients are defined by

$$a_n = \left| \begin{array}{ll} 1 & \text{if } \exists\, p \in \mathbb{N},\ n = 2^p, \\ 0 & \text{otherwise.} \end{array} \right.$$

It is plain that $|a_n| \leq \sigma^n$ eventually if and only if $\sigma \geq 1$. Hence the growth bound of the sequence if 1 and the open disk of convergence of the power series is $D(0,1)$.

**Lemma – Multiplication of Power Series Coefficients.** The radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n b_n (z-c)^n$ is at least the product of the radii of convergence of the series $\sum_{n=0}^{+\infty} a_n(z-c)^n$ and $\sum_{n=0}^{+\infty} b_n(z-c)^n$. In particular, for any nonzero polynomial sequence

$$a_n = \alpha_0 + \alpha_1 n + \cdots + \alpha_p n^p,$$

the radii of convergence of $\sum_{n=0}^{+\infty} a_n b_n (z-c)^n$ and $\sum_{n=0}^{+\infty} b_n (z-c)^n$ are identical.

**Proof.** Denote by $\rho_a$ and $\rho_b$ the respective growth bounds of the sequences $a_n$ and $b_n$; the growth bound of the product sequence $a_n b_n$ is at most $\rho_a \rho_b$: for any $\sigma > \rho_a \rho_b$, we may find some $\sigma_a > \rho_a$ and $\sigma_b > \rho_b$ such that $\sigma = \sigma_a \sigma_b$. Since $|a_n| \leq (\sigma_a)^n$ and $|b_n| \leq (\sigma_b)^n$ eventually, $|a_n b_n| \leq \sigma^n$ eventually.

The growth bound of any polynomial sequence $a_n$ is at most 1: the inequality

$$|\alpha_0 + \alpha_1 n + \cdots + \alpha_p n^p| \leq \rho^n$$

holds for any $\rho > 1$ eventually. Now, for any nonzero polynomial sequence $a_n$ and any sequence $b_n$, eventually $|b_n|$ is dominated by a multiple of $|a_n b_n|$, thus the growth bound of $|b_n|$ is at most the growth bound of $|a_n b_n|$. Reciprocally, the growth bound of $|a_n b_n|$ is at most the product of the growth bound of $|a_n|$ – at most one – and the growth bound of $|b_n|$ and thus at most the growth bound of $|b_n|$. ■

**Theorem – Locally Normal Convergence.** The convergence of the power series $\sum_{n=0}^{+\infty} a_n (z-c)^n$ in its open disk of convergence $D(c,r)$ is *locally normal*: for any $z \in D(c,r)$, there is an open neighbourghood $U$ of $z$ in $D(c,r)$ such that

$$\exists\, \kappa > 0,\ \forall\, z \in U,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \kappa$$

or equivalently, for every compact subset $K$ of $D(c,r)$,

$$\exists\, \kappa > 0,\ \forall\, z \in K,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \kappa.$$

**Proof.** If $K$ is compact subset of $D(c,r)$ and $\rho = \sup \{|z-c| \mid z \in K\}$,

$$\forall\, z \in K,\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| \leq \sum_{n=0}^{+\infty} |a_n| \rho^n.$$

Since the growth bound of the sequence $a_n$ and $|a_n|$ are identical, the radius of convergence of the series $\sum_{n=0}^{+\infty} |a_n| z^n$ is $r$. Given that $|\rho| < r$, the series $\sum_{n=0}^{+\infty} |a_n| \rho^n$ is convergent; all its terms are non-negative real numbers, thus the sum is finite: there is a $\kappa > 0$ such that $\sum_{n=0}^{+\infty} |a_n| \rho^n \leq \kappa$. ■

**Remark – Other Types of Convergence.** The locally normal convergence implies the *absolute convergence*:

$$\forall\, z \in D(c,r),\ \sum_{n=0}^{+\infty} |a_n (z-c)^n| < +\infty.$$

It also provides the *locally uniform convergence*: on any compact subset $K$ of $D(c,r)$, the partial sums $\sum_{n=0}^{p} a_n(z-c)^n$ converge uniformly to the sum $\sum_{n=0}^{+\infty} a_n(z-c)^n$:

$$\lim_{p\to+\infty} \sup_{z\in K} \left| \sum_{n=0}^{p} a_n(z-c)^n - \sum_{n=0}^{+\infty} a_n(z-c)^n \right| = 0.$$

## Power Series and Holomorphic Functions

**Theorem – Power Series Derivative.** A power series and its *formal derivative*

$$\sum_{n=0}^{+\infty} a_n(z-c)^n \text{ and } \sum_{n=1}^{+\infty} na_n(z-c)^{n-1}.$$

have the same radius of convergence $r$. The sum

$$f : z \in D(c,r) \mapsto \sum_{n=0}^{+\infty} a_n(z-c)^n$$

is holomorphic; its derivative is the sum of the formal derivative:

$$\forall\, z \in D(c,r),\ f'(z) = \sum_{n=1}^{+\infty} na_n(z-c)^{n-1}.$$

More generally, the $p$-th order derivative of $f$ is defined for any $p \in \mathbb{N}$ and

$$\forall\, z \in D(c,r),\ f^{(p)}(z) = \sum_{n=p}^{+\infty} n(n-1)\cdots(n-p+1)a_n(z-c)^{n-p}.$$

**Lemma.** For any $z \in \mathbb{C}$, $h \in \mathbb{C}^*$ and $n \geq 2$,

$$\left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| \leq \frac{n(n-1)}{2}(|z|+|h|)^{n-2}|h|.$$

**Proof – Lemma.** Using the identity $a^n - b^n = (a-b)\sum_{m=0}^{n-1} a^m b^{n-1-m}$ yields

$$(z+h)^n - z^n = h\sum_{m=0}^{n-1}(z+h)^m z^{n-1-m},$$

hence

$$\begin{aligned} \frac{(z+h)^n - z^n}{h} - nz^{n-1} &= \sum_{m=0}^{n-1}(z+h)^m z^{n-1-m} - \sum_{m=0}^{n-1} z^m z^{n-1-m} \\ &= \sum_{m=0}^{n-1}\left[(z+h)^m - z^m\right] z^{n-1-m}. \end{aligned}$$

By the same identity, we also have

$$|(z+h)^m - z^m| = \left| h \sum_{l=0}^{m-1} (z+h)^l z^{m-1-l} \right| \leq m(|z|+|h|)^{m-1}|h|.$$

Therefore

$$\begin{aligned} \left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| &\leq \left[ \sum_{m=0}^{n-1} m \; (|z|+|h|)^{m-1}(|z|+|h|)^{n-1-m} \right] |h| \\ &\leq \frac{n(n-1)}{2}(|z|+|h|)^{n-2}|h| \end{aligned}$$

as expected. ■

**Proof – Power Series Derivative.** Let $D(c,r)$ be the open disk of convergence of the series

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n.$$

The radii of convergence of the series

$$\sum_{n=1}^{+\infty} n a_n (z-c)^{n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} n a_n (z-c)^n$$

are equal. Since the coefficient sequence of the latter series is the product of $a_n$ and a nonzero polynomial sequence, the open radius of convergence of $f$ and of its the formal derivative are identical. For any $z \in D(c,r)$ and $h \in \mathbb{C}$, define $e(z,h)$ as

$$e(z,h) = \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

A straightforward calculation leads to

$$e(z,h) = \sum_{n=1}^{+\infty} a_n \left[ \frac{(z+h-c)^n - (z-c)^n}{h} - n(z-c)^{n-1} \right],$$

hence, using the lemma, we obtain

$$|e(z,h)| \leq \left[ \sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| (|z-c|+|h|)^{n-2} \right] \times |h|.$$

The power series

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| z^{n-2}$$

has the same radius of convergence than

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} a_n (z-c)^{n-2}$$

which is the the formal derivative of order 2 of the original series, hence the three series have the same radius of convergence $r$. Consequently, for any $h$ such that $|z-c| + |h| < r$,

$$\sum_{n=2}^{+\infty} \frac{n(n-1)}{2} |a_n| (|z-c| + |h|)^{n-2} < +\infty$$

and therefore

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \sum_{n=1}^{+\infty} n a_n (z-c)^{n-1}.$$

The statement about the $p$-th order derivative of $f$ can be obtained by a simple induction on $p$. ■

**Theorem & Definition – Taylor Series.** If the complex-valued function $f$ has a power series expansion centered at $c$ inside the non-empty open disk $D(c,r)$, it is the *Taylor series* of $f$:

$$\forall\, z \in D(c,r),\ f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!} (z-c)^n.$$

**Proof.** If $f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n$, then for any $p \in \mathbb{N}$, the $p$-th order derivative of $f$ inside $D(c,r)$ is given by

$$f^{(p)}(z) = \sum_{n=p}^{+\infty} n(n-1)\dots(n-p+1) a_n (z-c)^{n-p}$$

and consequently, $f^{(p)}(c) = p! a_p$. ■

Note that the above theorem is only a uniqueness result; it says nothing about the existence of the power series expansion. This is the role of the following theorem.

**Theorem – Power Series Expansion.** Let $\Omega$ be an open subset of $\mathbb{C}$, let $c \in \Omega$ and $r \in \,]0, +\infty]$ such that the open disk $D(c,r)$ is included in $\Omega$. For any holomorphic function $f : \Omega \to \mathbb{C}$, there is a power series with coefficients $a_n$ such that

$$\forall\, z \in D(c,r),\ f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n.$$

Its coefficients are given by

$$\forall\, \rho \in \,]0, r[\,,\ a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft].$$

**Proof – Power Series Expansion.** For any $n \in \mathbb{N}$, the complex number

$$a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft]$$

is independent of $\rho$ as long as $0 < \rho < r$. Indeed, if $\rho_1$ and $\rho_2$ are two such numbers, denote $\gamma_1 = c + \rho_1[\circlearrowleft]$ and $\gamma_2 = c + \rho_2[\circlearrowleft]$. The interior of the sequence of paths $\mu = \gamma_1 \,|\, \gamma_2^{\leftarrow}$ is included in $D(c, r) \setminus \{c\}$ where the function $z \mapsto f(z)/(z-c)^{n+1}$ is holomorphic. Hence, by Cauchy's integral theorem,

$$\int_\mu \frac{f(z)}{(z-c)^{n+1}}\, dz = \int_{\gamma_1} \frac{f(z)}{(z-c)^{n+1}}\, dz - \int_{\gamma_2} \frac{f(z)}{(z-c)^{n+1}}\, dz = 0.$$

Now, let $z \in D(c, r)$ and let $\rho \in \,]0, r[$ such that $|z - c| < \rho$. Cauchy's integral formula provides

$$f(z) = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

For any $w \in \gamma([0, 1])$, we have

$$\frac{1}{w-z} = \frac{1}{(w-c) - (z-c)} = \frac{1}{w-c} \frac{1}{1 - \frac{z-c}{w-c}}.$$

Since

$$\left| \frac{z-c}{w-c} \right| = \frac{|z-c|}{\rho} < 1,$$

we may expand $f(w)/(w - z)$ into

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-c} \frac{1}{1 - \frac{z-c}{w-c}} = \sum_{n=0}^{+\infty} \frac{f(w)}{w-c} \left( \frac{z-c}{w-c} \right)^n.$$

The term of this series is dominated by

$$\frac{\sup_{|w-c|=\rho} |f(w)|}{\rho} \left( \frac{|z-c|}{\rho} \right)^n;$$

the convergence of the series is normal – and thus uniform – with respect to the variable $w$. Finally

$$\begin{aligned} f(z) &= \int_\gamma \left[ \sum_{n=0}^{+\infty} \frac{f(w)}{(w-c)^{n+1}} (z-c)^n \right] dw \\ &= \sum_{n=0}^{+\infty} \left[ \int_\gamma \frac{f(w)}{(w-c)^{n+1}} (z-c)^n \, dw \right] \\ &= \sum_{n=0}^{+\infty} \left[ \int_\gamma \frac{f(w)}{(w-c)^{n+1}} \, dw \right] (z-c)^n \end{aligned}$$

which is the desired expansion. ■

## Laurent Series

**Definition – Annulus.** Let $c \in \mathbb{C}$ and $r_1, r_2 \in [0, +\infty]$. We denote by

$$A(c, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - c| < r_2\}$$

the *open annulus* with *center* $c$, *inner radius* $r_1$ and *outer radius* $r_2$.

**Examples – Annuli.**

1. The open annulus $A(0, 0, +\infty)$, centered on the origin, with inner radius 0 and outer radius $+\infty$, is the set $\mathbb{C}^*$.
2. The sets $A(0, 0, 1)$, $A(0, 1, 2)$ and $A(0, 2, +\infty)$ are three open annuli centered on the origin and included in the open set $\Omega = \mathbb{C} \setminus \{i, 2\}$. They are maximal in $\Omega$ – if we decrease their inner radius and/or increase their outer radius the resulting annulus is not a subset of $\Omega$ anymore.

**Definition – Laurent Series.** The *Laurent series* centered on $c \in \mathbb{C}$ with coefficients $a_n \in \mathbb{C}$ for every $n \in \mathbb{Z}$ is

$$\sum_{n=-\infty}^{+\infty} a_n (z - c)^n.$$

It is *convergent* for some $z \in \mathbb{C} \setminus \{c\}$ if the series

$$\sum_{n=0}^{+\infty} a_n (z - c)^n \;\text{ and }\; \sum_{n=1}^{+\infty} a_{-n} (z - c)^{-n}$$

are both convergent – otherwise it is *divergent*. When the Laurent series is convergent its *sum* is defined as

$$\sum_{n=-\infty}^{+\infty} a_n (z - c)^n = \sum_{n=0}^{+\infty} a_n (z - c)^n + \sum_{n=1}^{+\infty} a_{-n} (z - c)^{-n}.$$

**Theorem – Convergence of Laurent Series.** Let $c \in \mathbb{C}$ and let $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. The *inner radius of convergence* $r_1 \in [0, +\infty]$ and *outer radius of convergence* $r_2 \in [0, +\infty]$ of the Laurent series $\sum_{n=-\infty}^{+\infty} a_n (z - c)^n$ defined by

$$r_1 = \limsup_{n \to +\infty} |a_{-n}|^{1/n} \;\text{ and }\; r_2 = \frac{1}{\limsup_{n \to +\infty} |a_n|^{1/n}}.$$

are such that the series converges in $A(c, r_1, r_2)$ and diverges if $|z - c| < r_1$ or $|z - c| > r_2$. In this *open annulus of convergence*, the convergence is locally normal.

**Proof – Convergence of Laurent Series.** The first series converges if $|z-c|$ is smaller than the radius of convergence $r_2$ of this power series and diverges if it is greater. We may rewrite the second series as:

$$\sum_{n=1}^{+\infty} a_{-n}(z-c)^{-n} = \sum_{n=1}^{+\infty} a_{-n}\left(\frac{1}{z-c}\right)^n.$$

Consequently, it converges if $|1/(z-c)|$ is smaller than the radius of convergence $1/r_1$ of the power series $\sum_{n=1}^{+\infty} a_{-n}z^n$, that is if $|z-c| > r_1$, and diverges if $|1/(z-c)|$ is greater than $1/r_1$, that is $|z-c|$ is smaller than $r_1$.

Now, for any $z \in A(c, r_1, r_2)$, there is an open neighbourhood $U$ of $z$ where $\sum_{n=0}^{+\infty} a_n(z-c)^n$ is normally convergent and an open neighbourhood $V$ of $(z-c)^{-1}$ in $\mathbb{C}^*$ where $\sum_{n=1}^{+\infty} a_{-n}w^n$ is normally convergent. The Laurent series $\sum_{n=-\infty}^{+\infty} a_n(z-c)^n$ is normally convergent in the open neighbourhood $U \cap \{w^{-1} + c \mid w \in V\}$ of $z$. ■

**Theorem – Laurent Series Expansion.** Let $\Omega$ be an open subset of $\mathbb{C}$, let $c \in \mathbb{C}$ and $r_1, r_2 \in [0, +\infty]$ such that $r_1 < r_2$ and the open annulus $A(c, r_1, r_2)$ is included in $\Omega$. For any holomorphic function $f : \Omega \to \mathbb{C}$, there is a Laurent series with coefficients $a_n$ such that

$$\forall\, z \in A(c, r_1, r_2),\ f(z) = \sum_{n=-\infty}^{+\infty} a_n(z-c)^n.$$

Its coefficients are given by

$$\forall\, \rho \in \,]r_1, r_2[\,,\ a_n = \frac{1}{i2\pi}\int_\gamma \frac{f(z)}{(z-c)^{n+1}}\,dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft].$$

**Proof – Laurent Series Expansion.** For any integer $n$, the coefficient

$$a_n = \frac{1}{i2\pi}\int_\gamma \frac{f(z)}{(z-c)^{n+1}}\,dz \ \text{ with } \ \gamma = c + \rho[\circlearrowleft]$$

is independent of $\rho \in \,]r_1, r_2[$ – refer to the proof of “Power Series Expansion” for a detailled argument.

Let $z \in A(c, r_1, r_2)$ and $\rho_1, \rho_2 \in \,]r_1, r_2[$ such that $\rho_1 < |z-c| < \rho_2$. Let $\gamma_1 = c + \rho_1[\circlearrowleft]$ and $\gamma_2 = c + \rho_2[\circlearrowleft]$; Cauchy’s integral formula provides

$$f(z) = \frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{i2\pi}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

As in the proof of “Power Series Expansion”, we can establish that

$$\frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{+\infty}\left[\frac{1}{i2\pi}\int_{\gamma_2} \frac{f(w)}{(w-c)^{n+1}}\,dw\right](z-c)^n.$$

A similar argument, based on a series expansion of

$$\frac{1}{w-z} = -\frac{1}{(z-c)-(w-c)} = -\frac{1}{z-c}\frac{1}{1-\frac{w-c}{z-c}}$$

yields

$$\frac{1}{i2\pi}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw = -\sum_{n=-1}^{-\infty}\left[\frac{1}{i2\pi}\int_{\gamma_1}\frac{f(w)}{(w-c)^{n+1}}\,dw\right](z-c)^n.$$

The combination of both expansions provides the expected result. ■

# Exercises

## The Fibonacci Sequence

We search for a closed form of the Fibonacci sequence $a_n$, defined by

$$a_0 = 0,\ a_1 = 1,\ \forall\, n \in \mathbb{N},\, a_{n+2} = a_n + a_{n+1}.$$

1. Show that the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

   is the largest solution of the equation $x^2 = x+1$ and that the other solution is $\psi = -1/\phi$.

2. Establish that for any $n \in \mathbb{N}$, $a_n \leq \phi^n$.

3. Show that the radius of convergence of the generating function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

   is at least $1/\phi$.

4. Compute $f(z)$ when $|z| < 1/\phi$.

5. Find a closed form for $a_n$, $n \in \mathbb{N}$.

## Entire Functions Dominated By Polynomials

Show that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is dominated by a polynomial $P$ of order $p$

$$\forall\, z \in \mathbb{C},\ |f(z)| \leq |P(z)|$$

then it is a polynomial whose degree is at most $p$.

## Existence of Primitives

Show that the function

$$f : z \in \mathbb{C} \setminus [-1, 1] \mapsto \frac{\pi}{z} \frac{1}{\sin \pi / z}$$

has a primitive.

## A Removable Set

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{C} \setminus \mathbb{U}$ (where $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$).

Prove that $f$ is an entire function.

## Derivative of Power Series

Provide an alternate proof of the existence and value of the derivative of the sum $\sum_{n=0}^{+\infty} a_n (z - c)^n$ in its open disk of convergence.

Hint: a locally uniform limit of a sequence of holomorphic functions is holomorphic.

# Chapter 8

# Zeros & Poles

## Preamble

In this chapter, we study the behavior of any holomorphic functions in the neighbourhood of a point that may or may not be in its domain of definition. When the function is not defined at the point of interest – in other words when it is a singularity – we only study the case where it is isolated.

**Definition – Isolated Point.** A point $c$ of a subset $C$ of $\mathbb{C}$ is *isolated* (in $C$) if it is in some neighbourhood of $c$ the only point of $C$:

$$\exists\, r > 0,\ \forall\, z \in C,\ |z - c| < r \ \Rightarrow\ z = c.$$

**Remark – Isolated Points in Closed Sets.** Note that a point $c$ is isolated in the closed set $C$ if and only if $C \setminus \{c\}$ is still closed. This is directly applicable to singularities of a function defined on an open set $\Omega$, which belongs by definition to the closed set $C = \mathbb{C} \setminus \Omega$.

**Definition – Isolated Singularity.** A *singularity* of a function $f : \Omega \to \mathbb{C}$ defined on an open subset $\Omega$ of $\mathbb{C}$ is a point $c$ of $\mathbb{C} \setminus \Omega$. It is *isolated* if

$$\exists\, r > 0,\ \forall\, z \in \mathbb{C},\ (z \neq c \ \text{and}\ |z - c| < r) \Rightarrow z \in \Omega,$$

in other words if there is a radius $r > 0$ such that the annulus $A(c, 0, r)$ is a subset of $\Omega$. Alternatively, it is isolated if and only if $\Omega \cup \{c\}$ is open.

## Zeros of Holomorphic Functions

**Definition – Zero & Multiplicity.** Let $\Omega$ be an open subset of the complex plane. A *zero* (or *root*) $c$ of a function $f : \Omega \to \mathbb{C}$ is a point $c \in \Omega$ such that

$f(c) = 0$; it is of *multiplicity* $p$ for some positive number $p$ if

$$\exists\, a^* \in \mathbb{C}^*,\; f(z) \underset{c}{\sim} a^*(z-c)^p$$

or equivalently

$$\exists\, a \in \mathbb{C}^*,\; \lim_{z \to c} \frac{f(z)}{(z-c)^p} = a^*.$$

A zero of multiplicity 1 is *simple*; zeros of higher multiplicity (*double*, *triple*, etc.) are *multiple*.

**Example – Simple Zero.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $c$ is a zero of $f$ but not of its derivative $f'$, then

$$\lim_{z \to c} \frac{f(z)}{(z-c)^1} = \lim_{z \to c} \frac{f(z) - f(c)}{z - c} = f'(c) \neq 0,$$

thus $f(z) \underset{c}{\sim} f'(c)(z-c)^1$ and $c$ is a simple zero of $f$.

**Theorem – Characterization of Zero Multiplicity.** A zero $c$ of a holomorphic function $f : \Omega \to \mathbb{C}$ is of multiplicity $p$ if and only if one of the equivalent condition holds:

1. The function $f$ and exactly its first $p-1$ derivatives are zero at $c$:

$$f(c) = 0,\; f'(c) = 0, \ldots,\; f^{(p-1)}(c) = 0 \;\text{ and }\; f^{(p)}(c) \neq 0.$$

2. The Taylor expansion of $f$ at $c$ is

$$f(z) = \sum_{n=p}^{+\infty} a_n (z-c)^n \;\text{ with }\; a_p \neq 0.$$

3. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\; f(z) = a(z)(z-c)^p \;\text{ and }\; a(c) \neq 0.$$

**Proof.** The formula $n! a_n = f^{(n)}(c)$ holds for any $n \in \mathbb{N}$ hence condition 1 and 2 are equivalent. The theorem statement is otherwise a direct consequence of the local behavior of holomorphic functions lemma (refer to the appendix). ■

Zeros of a holomorphic functions have finite multiplicities, except in very specific circumstances:

**Lemma – Zero With No (Finite) Multiplicity.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected set $\Omega$. If $c$ is a zero of $f$ but has no finite multiplicity, then $f$ is identically zero.

**Proof.** By assumption, $f^{(n)}(c) = 0$ for every number $n$. Consider the function $\chi : \Omega \to \mathbb{C}$ defined by

$$\chi(z) = \left| \begin{array}{ll} 0 & \text{if } \forall\, n \in \mathbb{N},\; f^{(n)}(z) = 0, \\ 1 & \text{otherwise.} \end{array} \right.$$

We can prove that the function $\chi$ is locally constant. Indeed, if $\chi(z) = 1$, there is a number $n$ such that $f^{(n)}(z) \neq 0$; by continuity, the function $f^{(n)}$ has no zero in some neighbourghood of $z$; thus the function $\chi$ is equal to 1 in this neighbourghood. If instead $\chi(z) = 0$, then the Taylor expansion of $f$ at $z$ shows that $f$ is zero in a suitable neighbourhood of $z$, where $\chi$ is also zero.

As the set $\Omega$ is connected, the function $\chi$ is actually constant. On the other hand, $\chi(c) = 0$, hence the function $\chi$ is identically zero on $\Omega$, which means that the function $f$ is also identically zero on $\Omega$. ■

**Lemma – Zeros of Finite Multiplicity are Isolated.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $c$ is a zero of multiplicity $p$ of $f$, it is isolated:

$$\exists\, r > 0,\ \forall\, z \in \Omega,\ (f(z) = 0 \ \text{ and } \ |z - c| < r) \ \Rightarrow\ z = c.$$

**Proof.** Let $c$ be a zero of multiplicity $p$ of $f$ for some positive number $p$. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that $f(z) = a(z)(z - c)^p$ on $\Omega$ and $a(c) \neq 0$. By continuity of $a$ at $c$, there is a $r > 0$ such that the disk $D(c, r)$ is a subset of $\Omega \cup \{c\}$ on which $a$ has no zero. The function $f$ therefore has no zero on $D(c, r) \setminus \{c\}$ either: the point $c$ is an isolated zero of $f$. ■

**Theorem – Isolated Zeros Theorem I.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected open set $\Omega$. Unless $f$ is identically zero, each of its zeros is isolated.

**Proof.** A direct consequence of the two above lemmas. ■

**Remark.** More often than not, we leverage the isolated zeros theorem to prove that some holomorphic function is identically zero. In other words, we rely on the contraposition of the theorem. The statement of this contraposition may be slightly rephrased with the introduction of the concept of limit point.

**Definition – Limit Point.** A point $c \in \mathbb{C}$ is a *limit point* of a set $C \subset \mathbb{C}$ if every open annulus $A(c, 0, r)$ intersects $C$:

$$\forall\, r > 0,\ A(c, 0, r) \cap C \neq \varnothing$$

or equivalently, if the distance between $c$ and $C \setminus \{c\}$ is zero.

**Theorem – Isolated Zeros Theorem II.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected set $\Omega$. If the set of zeros of $f$ has a limit point in $\Omega$, then $f$ is identically zero.

**Proof.** If the set of zeros of $f$ has a limit point in $\Omega$, then by continuity of $f$ at this point, it is a zero of $f$, which is clearly not isolated; thus, $f$ is identically zero. ■

**Remark.** Despite its apparent simplicity, the importance of the isolated zeros theorem is difficult to overestimate. However, it is a rather low-level tool; the corresponding high-level tool is a permanence principle for functional equations.

**Theorem – Principle of Permanence.** Let $F$ be a complex-valued function of $n$ complex variables, defined and complex-differentiable on some open subset of $\mathbb{C}^n$. Let $\Omega$ be an open connected subset of $\mathbb{C}$ and $(f_1, \dots, f_n) : \Omega \mapsto \mathbb{C}^n$ be a $n$-uple of holomorphic functions whose image is included in the domain of definition of $F$. If the set of points $z \in \Omega$ such that $F(f_1(z), \dots, f_n(z)) = 0$ has a limit point in $\Omega$, then

$$\forall \, z \in \Omega, \; F(f_1(z), \dots, f_n(z)) = 0.$$

**Proof.** Under the assumptions of the theorem, the function

$$z \in \Omega \mapsto F(f_1(z), \dots, f_n(z))$$

is defined and holomorphic on $\Omega$ (it is complex-differentiable as the composition of complex-differentiable functions) on $\Omega$. Its set of zeros has a limit point in $\Omega$, which is connected, thus it is identically zero. ■

**Corollary – Uniqueness Principle.** Two functions $f_1 : \Omega \to \mathbb{C}$ and $f_2 : \Omega \to \mathbb{C}$ defined and holomorphic on some open connected subset of $\Omega$ with the same values on a set with a limit point in $\Omega$ are identical.

**Proof.** Set $F(w_1, w_2) = w_1 - w_2$ and apply the permanence principle. ■

**Example – A Trigonometric Identity.** The identity

$$\sin^2 z + \cos^2 z = 1$$

holds for every $z \in \mathbb{C}$. Indeed, it is satisfied on the real line: every real number is a zero of the holomorphic function $f : z \in \mathbb{C} \mapsto \sin^2 z + \cos^2 z - 1$. Every real number is a limit point of $\mathbb{R}$ in $\mathbb{C}$, hence $f$ is identically zero and the identity may be extended to the whole complex plane. Alternatively, apply the permanence principle with $F(w_1, w_2) = w_1^2 + w_2^2 - 1$, $f_1(z) = \sin z$ and $f_2(z) = \cos z$.

## Isolated Singularities of Holomorphic Functions

**Definition – Typology of Isolated Singularities.** Let $\Omega$ be an open subset of the complex plane and $f : \Omega \to \mathbb{C}$ be a holomorphic function. An isolated singularity $c$ of $f$ is:

- a *removable singularity* if there is a holomorphic extension of $f$ over $c$, that is, a holomorphic function $a : \Omega \cup \{c\} \to \mathbb{C}$ such that

$$\forall \, z \in \Omega, \; f(z) = a(z).$$

- a *pole of multiplicity* $p$ for some $p \in \mathbb{N}^*$ if there is a $a^* \in \mathbb{C}^*$ such that

$$f(z) \underset{c}{\sim} \frac{a^*}{(z-c)^p} \quad \text{or equivalently} \quad \lim_{z \to c} f(z)(z-c)^p = a^*.$$

- an *essential singularity* otherwise.

**Theorem – Characterization of Removable Singularities.** An isolated singularity $c$ of a holomorphic function $f : \Omega \to \mathbb{C}$ is removable if and only if one of the following conditions holds:

1. The Laurent expansion of $f$ in some non-empty open annulus $A(c, 0, r)$ is a power series (its coefficients $a_n$ are zero if $n < 0$).
2. The value $f(z)$ has a limit in $\mathbb{C}$ when $z \to c$.
3. The function $f$ is bounded in some non-empty open annulus $A(c, 0, r)$.

**Proof.** The validity of the criteria 1 and 2 is a direct consequence of the local behavior of holomorphic functions lemma (refer to the appendix). It is also plain that condition 3 is a consequence of condition 2. Conversely, if condition 2 holds and

$$|f(z)| \leq m \;\text{ whenever }\; 0 < |z - c| < r,$$

define $\gamma = c + \rho[\circlearrowleft]$ for $0 < \rho < r$. For any $n < 0$,

$$|a_n| = \left| \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}} \, dz \right| \leq m\rho^{-n} \to 0 \;\text{ when }\; \rho \to 0$$

thus condition 1 holds. ■

**Theorem – Characterization of Pole Multiplicity.** An isolated singularity of a holomorphic function $f : \Omega \to \mathbb{C}$ is a pole of multiplicity $p$ if and only if one of the equivalent condition holds:

1. The Laurent expansion of $f$ in some non-empty open annulus $A(c, 0, r)$ is

$$f(z) = \sum_{n=-p}^{+\infty} a_n (z - c)^n \;\text{ with }\; a_{-p} \neq 0.$$

2. There is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\; f(z) = \frac{a(z)}{(z-c)^p} \;\text{ and }\; a(c) \neq 0.$$

**Proof.** A straightforward consequence of the local behavior of holomorphic functions lemma. ■

**Theorem – Characterization of Poles.** An isolated singularity $c$ of a holomorphic function $f : \Omega \to \mathbb{C}$ is a pole (of multiplicity $p$) if and only if the inverse $1/f$ is defined in some open annulus $A(c, 0, r)$, has a holomorphic extension to $D(c, r)$ and $c$ is a zero (of multiplicity $p$) of this extension. Alternatively, $c$ is a pole of $f$ if and only if

$$|f(z)| \to +\infty \;\text{ when }\; z \to c.$$

**Proof.** If the point $c$ is a pole of order $p$ of $f$, there is a holomorphic function $a : \Omega \to \mathbb{C}$ such that

$$\forall\, z \in \Omega,\ f(z) = \frac{a(z)}{(z-c)^p} \ \text{ and } \ a(c) \neq 0.$$

Let $r > 0$ be such that $a(z) \neq 0$ on $D(c,r)$. The function $z \mapsto 1/f(z)$ is holomorphic on $\in D(c,r) \setminus \{c\}$, the function $b : z \mapsto 1/a(z)$ is holomorphic on $D(c,r)$, $b(c) \neq 0$ and

$$\forall\, z \in D(c,r) \setminus \{c\},\ \frac{1}{f(z)} = b(z)(z-c)^p$$

Thus the point $c$ is a removable singularity of $1/f$ and a zero of order $p$ of its holomorphic extension over $c$. The converse statement may be proved by a similar method.

The condition $|f(z)| \to +\infty$ when $z \to c$ is equivalent to

$$1/f(z) \to 0 \ \text{ when } \ z \to c.$$

This property holds if and only if $f$ has a holomorphic extension over $c$ and $c$ is a zero of this extension. As $1/f$ is not identically zero locally, this zero has a finite multiplicity $p$ and hence $c$ is a pole of order $p$ of $f$. ■

# Computation of Residues

**Theorem – Computation of Residues.** Let $\Omega$ be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and $c$ be an isolated singularity of $f$. If the Laurent series expansion of $f$ in some non-empty annulus $A(c,0,r) \subset \Omega$ is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-c)^n$$

then

$$\mathrm{res}(f,c) = a_{-1}.$$

**Proof.** By definition of the residue, for any $0 < \rho < r$,

$$\mathrm{res}(f,c) = \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} f(z)dz = \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} \left[ \sum_{n=-\infty}^{+\infty} a_n (z-c)^n \right] dz.$$

The convergence of the Laurent series expansion is uniform on any compact subset of $A(c,0,r)$, hence

$$\mathrm{res}(f,c) = \sum_{n=-\infty}^{+\infty} \left[ a_n \frac{1}{i2\pi} \int_{\rho[\circlearrowleft]+c} (z-c)^n dz \right].$$

When $n \neq -1$, the function $z \mapsto (z-c)^n$ has a primitive in $\mathbb{C}^*$, hence all the terms but one in the right-hand side of the equation are equal to zero. Finally,

$$\operatorname{res}(f,c) = a_{-1}\left[\frac{1}{i2\pi}\int_{\rho[\circlearrowleft]+c}\frac{dz}{z-c}\right] = a_{-1},$$

as expected. ■

**Corollary – Residue of Poles.** Let $\Omega$ be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and $c$ be an isolated singularity of $f$. If $c$ is a pole of $f$ whose multiplicity is at most $p$:

$$\exists\, a \in \mathbb{C},\ \lim_{z\to c} f(z)(z-c)^p = a,$$

then

$$\operatorname{res}(f,c) = \lim_{z\to c}\frac{1}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}\left(f(z)(z-c)^p\right)$$

**Proof.** Given the assumption on the multiplicity of $c$, the Laurent series expansion of $f$ on $D(c,r)\setminus\{c\}$ for $r$ small enough is

$$f(z) = \sum_{n=-p}^{+\infty} a_n(z-c)^n,\ 0 < |z-c| < r$$

thus

$$f(z)(z-c)^p = \sum_{m=0}^{+\infty} a_{m-p}(z-c)^m,\ 0 < |z-c| < r.$$

The right-hand side of this equation displays no negative power of $z$; this series is therefore convergent on the whole disk $D(c,r)$ where the function $z \mapsto f(z)(z-p)^p$ can be extended to a function $g$ which is holomorphic. As the residue $a_{-1}$ of $f$ at $c$ is the coefficient of $(z-c)^{p-1}$ in this Taylor expansion, we have

$$a_{-1} = \frac{g^{(p-1)}(c)}{(p-1)!} = \lim_{z\to c}\frac{g^{(p-1)}(z)}{(p-1)!}$$

which provides the expected formula. ■

**Corollary – Residue of Simple Poles I.** Let $\Omega$ be an open set of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and $c$ be an isolated singularity of $f$. The point $c$ is a simple pole of $f$ if and only if

$$\exists\, a \in \mathbb{C}^*,\ \lim_{z\to c} f(z)(z-c) = a,$$

and then

$$\operatorname{res}(f,c) = \lim_{z\to c} f(z)(z-c)$$

**Proof.** Trivial. ■

**Corollary – Residue of Simple Poles II.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let $c$ be an isolated singularity of $f$. If there are two holomorphic functions $g$ and $h$ on $\Omega$ such that

$$f = \frac{g}{h} \;\text{ where }\; g(c) \neq 0,\; h(c) = 0,\; h'(c) \neq 0,$$

then $c$ is a simple pole of $f$ and

$$\operatorname{res}(f, c) = \frac{g(c)}{h'(c)}.$$

**Proof.** Given the assumptions, we have

$$\lim_{z \to c} (z - c) f(z) = \lim_{z \to c} \frac{z - c}{h(z) - h(c)} g(z) = \frac{g(c)}{h'(c)} \neq 0,$$

hence $c$ is a simple pole of $f$ whose residue is $g(c)/h'(c)$. ■

# Appendix – Local Behavior of Holomorphic Functions

**Lemma – Local Behavior of Holomorphic Functions.** Let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function defined on some open subset $\Omega$ of $\mathbb{C}$. Let $c \in \mathbb{C}$ be a point which is either in the domain of definition of $f$ or an isolated singularity of $f$; in any case, there is a $r > 0$ such that $D(c, r) \subset \Omega \cup \{c\}$.

For any $p \in \mathbb{Z}$ and $a^* \in \mathbb{C}$, the following properties are equivalent:

1. We have
$$\lim_{z \to c} \frac{f(z)}{(z - c)^p} = a^*.$$

2. There is a function $a : \Omega \cup \{c\} \to \mathbb{C}$ such that
$$\forall\, z \in \Omega,\; f(z) = a(z)(z - c)^p \;\text{ and }\; \lim_{z \to c} a(z) = a(c) = a^*.$$

3. There are some $a_n \in \mathbb{C}$ defined for $n \geq p$ such that
$$\forall\, z \in \Omega \cap D(c, r),\; f(z) = \sum_{n=p}^{+\infty} a_n (z - c)^n \;\text{ and }\; a_p = a^*.$$

4. There is a holomorphic function $a : \Omega \cup \{c\} \to \mathbb{C}$, such that
$$\forall\, z \in \Omega,\; f(z) = a(z)(z - c)^p \;\text{ and }\; a(c) = a^*.$$

**Proof.** If condition 1 holds, the function $a : \Omega \cup \{c\} \to \mathbb{C}$ defined by

$$a(z) = \frac{f(z)}{(z-c)^p} \;\text{ if }\; z \in \Omega \setminus \{c\} \;\text{ and }\; a(c) = a^*$$

satisfies condition 2.

If condition 2 holds, the function $a$ is continuous, thus for any compact set $K \subset D(c,r)$, there is a finite $m$ such that

$$\forall\, z \in K \cap \Omega,\; |a(z)| = \left|\frac{f(z)}{(z-c)^p}\right| \le m.$$

Hence, if $0 < \rho < r$ and $\gamma = c + \rho[\circlearrowleft]$, the $n$-th coefficient $a_n$ of the Laurent expansion of $f$ in $D(c,r) \setminus \{c\}$ satisfies

$$a_n = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^p} (z-c)^{p-n-1}\, dz$$

and by the M-L inequality,

$$|a_n| \le \left[\sup_{|z|=\rho} \left|\frac{f(z)}{(z-c)^p}\right|\right] \rho^{p-n}.$$

If $n < p$, the right-hand side of this inequality tends to zero when $\rho \to 0$, therefore $a_n = 0$. If $n = p$ on the other hand,

$$a_p = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{(z-c)^{p+1}}\, dz = \int_0^1 a(c + \rho e^{i2\pi t})\, dt$$

and hence $a_p = \lim_{z \to c} a(z) = a^*$. Now if $c \in \Omega$, the Taylor expansion of $f$ in $D(c,r)$ provides a Laurent expansion of $f$ in $D(c,r) \setminus \{c\}$; this expansion is unique, hence the coefficient sequences are equal and the initial Laurent expansion is valid in $D(c,r)$.

If condition 3 holds, the series

$$\sum_{k=0}^{+\infty} a_{k+p} (z-c)^k$$

is convergent in $D(c,r) \setminus \{c\}$ and hence in $D(c,r)$. Its sum $a_c(z)$ satisfies $f(z) = a_c(z)(z-c)^p$ in $\Omega \cap D(c,r)$. Consequently, the function $a : \Omega \cup \{c\} \to \mathbb{C}$ may be defined unambiguously by

$$a(z) = a_c(z) \;\text{ if }\; z \in D(c,r) \;\text{ and }\; a(z) = \frac{f(z)}{(z-c)^p} \;\text{ otherwise}$$

and it is holomorphic.

Finally if condition 4 holds, it is plain that condition 1 holds. ■

## Exercises

### The Weierstrass-Casorati Theorem

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and let $a \in \mathbb{C}$ be an essential singularity of $f$. Show that the image of $f$ is dense in $\mathbb{C}$:

$$\forall\, w \in \mathbb{C},\ \forall\, \epsilon > 0,\ \exists\, z \in \Omega,\ |f(z) - w| < \epsilon.$$

Hint: assume instead that some complex number $w$ is *not* in the closure of the image of $f$; study the function $z \mapsto 1/(f(z) - w)$ in a neighbourhood of $a$.

### The Maximum Principle

Let $\Omega$ be an open connected subset of the complex plane and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Show that if $|f|$ has a local maximum at some $a \in \Omega$, then $f$ is constant.

### The $\Pi$ Function

We introduce the $\Pi$ function, a holomorphic extension of the factorial.

1. Find the domain in the complex plane of the function

   $$\Pi : z \mapsto \int_0^{+\infty} t^z e^{-t}\, dt$$

   and show that it is holomorphic.

2. Prove that whenever $\Pi(z)$ is defined, $\Pi(z+1)$ is also defined and

   $$\Pi(z+1) = (z+1)\Pi(z).$$

   Compute $\Pi(n)$ for every $n \in \mathbb{N}$.

3. Let $\Omega$ be an open connected subset of the complex plane that contains the domain of $\Pi$ and such that $\Omega + 1 \subset \Omega$. Prove that if $\Pi$ has a holomorphic extension on $\Omega$ (still denoted $\Pi$), it is unique and satisfies the functional equation

   $$\forall\, z \in \Omega,\ \Pi(z+1) = (z+1)\Pi(z).$$

4. Prove the existence of such an extension $\Pi$ on

   $$\Omega = \mathbb{C} \setminus \{k \in \mathbb{Z} \mid k < 0\}.$$

5. Show that every negative integer is a simple pole of $\Pi$; compute the associated residue.

## Singularities and Residues

Analyze the singularities (location, type, residues) of

$$z \mapsto \frac{\sin \pi z}{\pi z},\ z \mapsto \frac{1}{(\sin \pi z)^2},\ z \mapsto \sin \frac{\pi}{z},\ z \mapsto \frac{1}{\sin \frac{\pi}{z}}.$$

## Integrals of Functions of a Real Variable

See "Technologie de calcul des intégrales à l'aide de la formule des résidus" (Demailly 2009, chap. III, sec. 4) for a comprehensive analysis of the computation of integrals with the the residue theorem.

1. For any $n \geq 2$, compute
$$\int_0^{+\infty} \frac{dx}{1+x^n}.$$
2. Compute
$$\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\, dx.$$

## References

Demailly, Jean-Pierre. 2009. *Fonctions holomorphes et surfaces de riemann.* `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/variable_complexe.pdf`.

# Chapter 9

# Analytic Functions

## Analytic Functions

**Definition – Analytic Function.** A function $f : A \subset \mathbb{C} \to \mathbb{C}$ is *analytic* if it is locally the sum of a (convergent) power series: for any $c \in A$, there is a $r > 0$ and a sequence of complex numbers $a_n$ such that

$$\forall\, z \in A \cap D(c,r),\ f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n.$$

The characterization of analytic functions defined on open sets is simple: we know that the sum of every power series is holomorphic in its open disk of convergence and conversely that every holomorphic function defined on an open set is locally the sum of a power series. Hence, we have the:

**Theorem – Analyticity & Holomorphicity in Open Sets.** A function defined in an open set is analytic if and only if it is holomorphic.

The definition of analytic function is not limited to open sets; we may also extend the definition of holomorphic function to sets that may not be open:

**Definition – Holomorphic Function (Non-Open Sets).** A function $f : A \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* if there is an open subset $\Omega$ of $\mathbb{C}$ such that $A \subset \Omega$ and a holomorphic function $g : \Omega \to \mathbb{C}$ that extends $f$.

This definition is appropriate because – at least on "well-behaved" sets – the classes of analytic and holomorphic functions are identical, as they are on open sets. For example, convex sets are well-behaved:

**Theorem – Analyticity & Holomorphicity in Convex Sets.** A function $f : A \subset \mathbb{C} \to \mathbb{C}$ defined in a convex set $A$ is analytic if and only if it is holomorphic.

**Proof.** It is clear that such restrictions of holomorphic functions are analytic (the convexity assumption is not needed here).

Conversely, let $A$ be a convex subset of $\mathbb{C}$ and let $f : A \to \mathbb{C}$ be an analytic function. There is a positive function $a \in A \mapsto r_a$ and a family $a \in A \mapsto f_a$ of holomorphic functions defined on $D(a, r_a)$ such that $f$ and $f_a$ are equal on $A \cap D(a, r_a)$. Let $\phi$ be the complex-valued function defined on the union $\Omega$ of all disks $D(a, r_a)$ by

$$\phi(z) = f_a(z) \;\text{ if }\; a \in A \;\text{ and }\; z \in D(a, r_a).$$

This definition is non-ambiguous: if $a \in A$, $b \in A$ and $z$ belongs to $D(a, r_a)$ and $D(b, r_b)$, then by convexity of $A$, the domain of $f$ contains the set $L = [a, b] \cap D(a, r_a) \cap D(b, r_b)$. By the isolated zeros theorem, the functions $f_a$ and $f_b$, equal on $L$, are also equal on $D(a, r_a) \cap D(b, r_b)$, therefore $f_a(z) = f_b(z)$. The function $\phi$ is by construction an extension of $f$ which is holomorphic. ■

**Example – Closed Unit Disk.** If the function $f$ is analytic on the closed unit disk

$$\overline{D}(0, 1) = \{z \in \mathbb{C} \mid |z| \leq 1\},$$

there is a holomorphic extension of $f$ on an open superset $\Omega$ of the closed unit disk. This function may be restricted to an open disk $D(0, r)$ for some radius $r > 1$ and expanded into a Taylor series: there are complex coefficients $a_n$ such that $\sum_{n=0}^{+\infty} a_n z^n$ is convergent in $D(0, r)$ and

$$\forall\, z \in \overline{D}(0, 1),\; f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

Conversely it is plain that such a function is holomorphic – and thus analytic – in the closed unit disk.

**Definition – Projection.** A *projection* onto a non-empty closed subset $A$ of $\mathbb{C}$ of a point $z \in \mathbb{C}$ is a minimizer of the distance between $z$ and $A$. In other words, the set of all projections of $z$ onto $A$ is

$$\Pi(z) = \{a \in A \mid |z - a| = \inf_{w \in A} |z - w|\}.$$

**Theorem – Analyticity & Holomorphicity in Regular Compact Sets.** Let $A$ be a compact subset of the complex plane with no isolated point and such that the projection onto $A$ is unique in some neighbourhood of $A$. A function $f : A \to \mathbb{C}$ is analytic if and only if it is holomorphic.

**Proof.** It is plain that restrictions to $A$ of holomorphic functions defined in a open neighbourhood of $A$ are analytic on $A$ : holomorphic functions are analytic.

Conversely, assume that $f : A \to \mathbb{C}$ is analytic. Let $r_z$ be a collection of positive radii indexed by $z \in A$ and $f_z : D(z, r_z) \to \mathbb{C}$ the corresponding collection of holomorphic functions such that

$$\forall\, z \in A,\; \forall\, w \in A \cap D(z, r_z),\; f_z(w) = f(w).$$

There is another collection with the same property, but associated with a constant and positive radius $r$. It is easy to build this collection – by a simple restriction of the $f_z$ – if we can first extend the functions of the original collection to make sure that $z \in A \mapsto r_z$ has a positive lower bound. To do so, we may consider for any $z \in A$ the set of holomorphic functions defined on open disks centered on $z$ with the same values as $f$ on the points that also belong to $A$. This set is not empty, as it contains $f_z$. Additionally, if two functions belong to this set, as $z$ is not isolated in $A$, they are equal on the intersection of their domain of definition. Hence, we may select as a new $f_z$ the function in the set whose domain of definition has the largest radius. Now, if we assume that the infimum of the $r_z$ is 0, there is a sequence $z_n$ in $A$ such that $r_{z_n} \to 0$ and thus a subsequence $w_n$ that converges to some $z \in A$. But for any $w \in D(z, r_z)$, the radius of the largest disk at $w$ satisfies

$$r_w + |w - z| \geq r_z,$$

thus we have a contradiction eventually when $|w_n - z| < r_z/2$ and $r_{w_n} < r_z/2$.

Let $V$ be an open neighbourhood of $A$ where the projection onto $A$ is unique. We may assume that $r$ is smaller than the distance between $A$ and $\mathbb{C} \setminus V$. For any $z \in \Omega = A + D(0, r)$, denote $\pi(z)$ the projection of $z$ onto $A$ and define $g(z) = f_{\pi(z)}(z)$; the function $g : \Omega \mapsto \mathbb{C}$ obviously extends $f$. Additionally, it is holomorphic, because it satisfies

$$\forall\, z \in \Omega,\ \exists\, \epsilon > 0,\ \forall\, w \in D(z, \epsilon),\ g(w) = f_{\pi(z)}(w).$$

Indeed, if $|w - z| < \epsilon = r - |z - \pi(z)|$, then

$$|w - \pi(z)| \leq |w - z| + |z - \pi(z)| < r,$$

so $w$ that belongs to $D(\pi(w), r)$ by construction also belongs to $D(\pi(z), r)$. By construction, the functions $f_{\pi(z)}$ and $f_{\pi(w)}$ have the same values on every point of $A$ that belongs to both of their domains of definition. Naturally, $\pi(z)$ belongs to the domain of $f_{\pi(z)}$; since

$$|\pi(w) - \pi(z)| \leq |w - z| < r$$

it also belongs to the domain of $f_{\pi(w)}$. As none of the points of $A$ is isolated, by the isolated zeros theorem, the functions $f_{\pi(w)}$ and $f_{\pi(z)}$ are identical on $D(\pi(z), r) \cap D(\pi(w), r)$, which leads to $g(w) = f_{\pi(w)}(w) = f_{\pi(z)}(w)$. ■

**Example – Unit Circle.** The unit circle

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$$

is not convex, but it is compact and has no isolated point. Additionally, unless $z = 0$, the point $z/|z|$ is the unique projection of $z$ onto $\mathbb{C}$; the set $\mathbb{C}^*$ is a neighbourhood of $\mathbb{U}$. Thus, the classes of analytic and holomorphic functions defined on $\mathbb{U}$ are identical. Any holomorphic extension of a holomorphic function defined on $\mathbb{U}$ has a restriction to $A(0, r, 1/r)$ for some radius $r \in \left]0, 1\right[$, thus a function $f$ is analytic on $\mathbb{U}$ if and only if there are some coefficients $a_n$ such that $\sum_{n=-\infty}^{+\infty} a_n z^n$ is convergent on some annulus $A(0, r, 1/r)$ and

$$\forall\, z \in \mathbb{U},\ f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

## Real Analytic Functions

**Definition – Interval.** In the sequel, an interval may be open, closed or half-open, bounded or unbounded; in other words, it is a synonym for "convex subset of the real line".

**Definition – Real Analytic Function.** A real-valued function defined on an interval of the real line which is analytic is *real analytic*.

The characterization of analytic functions on convex sets provides:

**Theorem – Characterization of Real Analytic Functions.** A real-valued function defined on an interval of the real line is real analytic if and only if it has a holomorphic extension on some open neighbourhood of its domain of definition in the complex plane.

We also have:

**Theorem – (Real) Taylor Series Expansion.** A function $f : I \to \mathbb{R}$ defined on an interval $I$ of $\mathbb{R}$ is real analytic if and only it is smooth – the $n$-th order (real) derivative defined by $f^{(0)} = f$ and

$$f^{(n+1)}(x) = \lim_{y \in I \to x} \frac{f^{(n)}(y) - f^{(n)}(x)}{y - x}$$

exists at every order – and for any $a \in I$, there is a $r > 0$ such that

$$\forall\, x \in I \cap \left]a - r, a + r\right[\,,\ f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

**Proof.** Assume that the function $f$ is smooth and that for any $a$ in $I$, it is equal to the sum of its (convergent) Taylor expansion on $]a - r, a + r[$ for some $r > 0$. Then, the power series

$$z \mapsto \sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n$$

is convergent on $D(a, r)$ and its restriction to

$$I \cap D(a, r) = I \cap \left]a - r, a + r\right[$$

is equal to $f$. Hence, $f$ is real analytic.

Conversely, if $f$ is real analytic, let $g$ be one of its holomorphic extensions. At every $x$ in $I$,

$$f'(x) = \lim_{y \in I \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \in I \to x} \frac{g(y) - g(x)}{y - x} = g'(x);$$

the real derivative of $f$ exists and is equal to the complex derivative of $g$, hence $g'$ is a holomorphic extension of $f'$. By induction, at every order $n$, $f^{(n)}$ exists and accepts $g^{(n)}$ as a holomorphic extension. Consequently, the local expansion of $g$ as a Taylor series provides a local expansion of $f$ as a (real) Taylor series. ■

## Analytic Continuation

The term "analytic continuation" refers to two related concepts; the first one is straightforward: "continuation" is simply used as a synonym of "function extension".

**Definition – Analytic Continuation to a Set.** Let $\Omega$ be an non-empty open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function. An *analytic continuation* of $f$ on a superset $\Sigma$ of $\Omega$ which is open and connected is a holomorphic function $g : \Sigma \to \mathbb{C}$ such that the restriction of $g$ to $\Omega$ is $f$.

We define the continuation for open and connected sets only so that:

**Theorem – Uniqueness of Analytic Continuation to Sets.** There is at most one analytic continuation of an analytic function to a given set.

**Proof.** It is a consequence of the isolated zeros theorem. Indeed, if there are two holomorphic extensions, they are identical on $\Omega$ which is open and non-empty. Every point of $\Omega$ is a limit point of the zeros of their difference; this difference is holomorphic and defined on a connected set, thus it is identically zero and the two holomorphic extensions are equal. ■

The second concept of continuation applies to holomorphic functions along paths. Although the concept may be defined for arbitrary paths, for the sake of simplicity we will only consider the case of regular analytic paths.

**Definition – Regular Analytic Path.** A path $\gamma : [0, 1] \mapsto \mathbb{C}$ is *analytic* if it is an analytic function. It is *regular* if $\gamma'$ has no zero on $[0, 1]$.

**Definition – Analytic Continuation Along a Path.** Let $\Omega$ be an open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic function and $\gamma : [0, 1] \mapsto \mathbb{C}$ be a regular analytic path such that $\gamma(0) \in \Omega$. An *analytic continuation* of $f$ along $\gamma$ is an analytic function $\phi : [0, 1] \mapsto \mathbb{C}$ such that for some $0 < \epsilon \leq 1$,

$$\forall \, t \in [0, \epsilon[\,, \ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

**Example – Analytic Continuations of the Logarithm.** The principal value of the logarithm

$$\log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$$

is an analytic continuation of the function $f : D(1,1) \mapsto \mathbb{C}$ defined by

$$f(z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} (z-1)^n.$$

On the other hand, the function

$$\phi : t \in [0,1] \mapsto i2\pi t$$

is an analytic continuation of log along the regular analytic path

$$[\circlearrowleft] : t \in [0,1] \mapsto e^{i2\pi t}.$$

Indeed, the function $\phi$ is analytic – the holomorphic function $z \in \mathbb{C} \mapsto i2\pi z$ is an extension of it – and

$$\forall\, t \in [0, 1/2[\,,\ [\circlearrowleft](t) = e^{i2\pi t} \in \mathbb{C} \setminus \mathbb{R}_- \ \text{ and } \ \phi(t) = \log e^{i2\pi t} = \log([\circlearrowleft](t)).$$

Like analytic continuations to sets, analytic continuations along paths are also unique whenever they exist:

**Theorem – Uniqueness of Analytic Continuation along Paths.** There is at most one analytic continuation of an analytic function along a given regular analytic path.

**Proof.** If $\phi_1 : [0,1] \to \mathbb{C}$ and $\phi_2 : [0,1] \to \mathbb{C}$ are two analytic continuations of the same holomorphic function along the same path, they have holomorphic extensions – still denoted $\phi_1$ and $\phi_2$ – defined on some shared open tubular neighbourhood of $[0,1]$

$$\{z \in \mathbb{C} \mid d(z, [0,1]) < \epsilon\},$$

which is connected. The point 0 is a limit point of the zeros of the holomorphic function $\phi_1 - \phi_2$. By the isolated zeros theorem, $\phi_1$ and $\phi_2$ are identical. ■

**Remark – Comparison of Analytic Continuations Concepts.** Continuation along paths is a more flexible tool than continuation to sets: if $\phi : \Sigma \mapsto \mathbb{C}$ is an analytic continuation of $f : \Omega \mapsto \mathbb{C}$ to the open connected set $\Sigma$ and $\gamma$ is a regular analytic path of $\Sigma$ whose initial point is in $\Omega$, then $\phi \circ \gamma$ is always an analytic continuation of $f$ along $\gamma$. However, on the other hand, there may exist an analytic continuation of $f$ along a path $\gamma$ but no analytic continuation $\phi : \Sigma \mapsto \mathbb{C}$ such that $\gamma([0,1]) \subset \Sigma$ (refer e.g. to the logarithm example).

**Remark – Values of Analytic Continuations.** The analytic continuation of a function $f$ to an open connected set $\Sigma$ provides a unique "natural" value of the function $f$ associated to a point $z \in \Sigma$ that may be outside of the original function domain. An analytic continuation along a path $\phi$ also provides a new value associated to the terminal point $z = \gamma(1)$: the terminal value $\phi(1)$ of the continuation $\phi$. However, several analytic continuations of $f$ along paths with the same initial and terminal points may have different terminal values; thus, analytic continuations along paths may define multi-valued functions.

**Example – Values of the Logarithm.** The analytic continuation of log along the path $[\circlearrowleft]$ – whose initial and terminal point is $z = 1$ – is the function $t \in [0,1] \mapsto i2\pi t$; its terminal value is $i2\pi$. On the other hand, the path $\gamma = 2 - [\circlearrowleft]$ has the same endpoints as $[\circlearrowleft]$, but its image is included in $\mathbb{C} \setminus \mathbb{R}_-$. Thus, the function $t \in [0,1] \mapsto \log \gamma(t)$ is the analytic continuation of log along $\gamma$ and its terminal value is $\log 1 = 0$. Both paths have the same initial and terminal points but the terminal values of the corresponding continuations are different.

Actually, an analytic continuation along a path carries more information than a simple value from the initial point to the terminal point of the path: it defines a new holomorphic function in the some open neighbourhood of the terminal point. To formalize this, we introduce a new definition.

**Definition – Germ of Holomorphic Function.** Two holomorphic functions $f : \Omega \mapsto \mathbb{C}$ and $g : \Sigma \to \mathbb{C}$ *define the same germ* at $z \in \Omega \cap \Sigma$ if the functions are identical in some neighbourghood of $z$. We denote this relation

$$f \sim_z g$$

A *germ of holomorphic function* at $z$ is an equivalence class for this relation between holomorphic functions:

$$[f]_z = \{g \mid f \sim_z g\}.$$

**Theorem – Analytic Continuation of Germs.** The existence and definition of an analytic continuation of a holomorphic function $f : \Omega \to \mathbb{C}$ along the regular analytic path $\gamma$ depends only on the germ of $f$ at $\gamma(0)$.

**Proof.** Assume that $g : \Sigma \to \mathbb{C}$ defines the same germ as $f$ at $\gamma(0)$; let $\Gamma \subset \Omega \cap \Sigma$ be some open set that contains $\gamma(0)$ and where both functions are identical. Let $\epsilon > 0$ be such that

$$\forall\, t \in [0, \epsilon[\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

and $\eta$ be some positive number, smaller than $\epsilon$, such that

$$\forall\, t \in [0, \eta[\,,\ \gamma(t) \in \Gamma.$$

Then, by construction of $\Gamma$, it is plain that

$$\forall\, t \in [0, \eta[\,,\ \gamma(t) \in \Sigma \ \text{ and } \ \phi(t) = g(\gamma(t)),$$

thus $\phi$ is also a continuation of $g$. ■

There is a converse statement that holds true: a candidate continuation function does determine uniquely the germ that it continues. We encapsulate that result in a definition:

**Definition – Initial Germ.** If $\gamma : [0,1] \mapsto \mathbb{C}$ is a regular analytic path and $\phi : [0,1] \mapsto \mathbb{C}$ is an analytic function, there is a unique germ of holomorphic function at $\gamma(0)$ of which $\phi$ is an analytic continuation: the *initial germ* of the continuation.

**Proof.** A function $f : \Omega \mapsto$ is an element of the germ of which $\phi$ is an analytic continuation along $\gamma$ if there is a $\epsilon \in \,]0,1]$ such that

$$\forall\, t \in [0, \epsilon[\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

The uniqueness of such a germ is a consequence of the isolated zeros theorem; let's prove its existence. The functions $\phi$ and $\gamma$ are analytic, therefore there are holomorphic functions, still denoted by the symbols $\phi$ and $\gamma$, that extend the original functions to some non-empty open tubular neighbourhood $\Sigma$ of $[0,1]$:

$$\Sigma = \{z \in \mathbb{C} \mid d(z, [0,1]) < r\}.$$

Since the path $\gamma$ is regular and analytic, the function $\gamma : D(0,r) \mapsto \mathbb{C}$ is continuously real-differentiable and its real-differential $d\gamma$ is invertible on some non-empty neighbourhood of 0. By the inverse function theorem, there is an open neighbourhood $U$ of 0 included in $D(0,r)$ and an open neighbourhood $V$ of $\gamma(0)$ such that the restriction $\gamma|_U$ of $\gamma$ to $U$ is a $C^1$-diffeomorphism. Its inverse satisfies $d(\gamma|_U^{-1})_{\gamma(z)} = (d\gamma_z)^{-1}$; this differential is complex-linear, therefore $\gamma|_U^{-1}$ is holomorphic. Thus we may define a holomorphic function $f$ on $V$ by

$$\forall\, w \in V,\ f(w) = \phi(\gamma|_U^{-1}(w))$$

Now let $\epsilon \in \,]0,1]$ be such that $[0, \epsilon[\, \subset U$. For any value of $t \in [0, \epsilon[\,$, the point $w = \gamma(t)$ belongs to $V$ and $f(\gamma(t)) = \phi(t)$. ∎

**Definition – Terminal Germ.** Let $\gamma$ be a regular analytic path and let $\phi : [0,1] \mapsto \mathbb{C}$ be an analytic continuation of a germ of holomorphic function at $\gamma(0)$. The *terminal germ* of the continuation $\phi$ is the initial germ of the continuation $\phi^{\leftarrow} : t \in [0,1] \mapsto \phi(1-t)$ along $\gamma^{\leftarrow}$; in other words, an holomorphic function $f : \Omega \mapsto \mathbb{C}$ belongs to the terminal germ if and only if, there is a $\epsilon \in [0,1]$ such that

$$\forall\, t \in \,]1-\epsilon, 1]\,,\ \gamma(t) \in \Omega \ \text{ and } \ \phi(t) = f(\gamma(t)).$$

**Theorem – Monodromy Theorem (Local).** Let $\Omega$ be an open star-shaped subset of $\mathbb{C}$ that contains the non-empty open disk $D(z_0, r)$ and let $f : D(z_0, r) \to \mathbb{C}$ be a holomorphic function. If for any regular analytic path $\gamma$ of $\Omega$ whose initial point is $z_0$ an analytic continuation of $f$ along $\gamma$ exists, then there is an analytic continuation of $f$ on $\Omega$.

**Proof.** We will assume in the sequel that $z_0$ is a center $c$ of the star-shaped set $\Omega$: if this is not true, consider a center $c$ of $\Omega$ and the germ $f_c$ of the holomorphic function defined at $c$ by the analytic continuation of $f$ along $[z_0, c]$. A function defined on $\Omega$ is an analytic continuation of $f_c$ if and only if it is an analytic continuation of the original germ $f$.

Let $z \in \Omega$ and let $\phi_z$ be the analytic continuation of $f$ along $[c \to z]$. By construction, there is a $\epsilon \in \,]0,1]$ such that $\phi_z(t) = f((1-t)c + tz)$ for any $0 \leq t < \epsilon$. If we still denote $\phi_z$ the holomorphic extension of $\phi_z$ to some open neighbourghood $\{w \in \mathbb{C} \mid d(w, [0,1]) < r\}$ of $[0,1]$, then for any complex number $w$ in some non-empty open disk centered at 0, we have

$$\phi_z(w) = f((1-w)c + wz).$$

Now, for any $z \in \Omega \setminus D(z_0, r)$ and any complex number $h$ such that $|h| < r|z - c|$, the function

$$\psi_{z,h} : t \in [0,1] \mapsto \phi_z(w) \;\text{ with }\; w = t\frac{z+h-c}{z-c}$$

is defined, analytic and as $w$ satisfies

$$(1-t)c + t(z+h) = (1-w)c + wz,$$

for small values of $t$, we have

$$\psi_{z,h}(t) = \phi_z(w) = f((1-w)c + wz) = f((1-t)c + t(z+h)),$$

therefore $\psi_{z,h}$ is the analytic continuation of $f_c$ along $[c \to z+h]$:

$$\psi_{z,h} = \phi_{z+h}.$$

This leads for small values of $h$ to the equality

$$\phi_{z+h}(1) = \psi_{z,h}(1) = \phi_z\left(\frac{z+h-c}{z-c}\right).$$

The function $z \in \Omega \mapsto \phi_z(1)$ is an extension of $f$ to $\Omega$; by the above equality it is also holomorphic, thus it is the analytic extension of $f$ to $\Omega$. ■

# Exercises

## Taylor Series of a Rational Function

1. Show that the function

$$f : x \in \mathbb{R} \mapsto \frac{1}{1+x^2}$$

   is analytic.

2. Determine for any $x_0 \in \mathbb{R}$ the open interval of convergence of its Taylor series expansion at $x_0$.

## Analytic Functions of a Real Variable

1. Show that the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(x) = \left| \begin{array}{rl} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{array} \right.$$

is smooth but is not analytic.

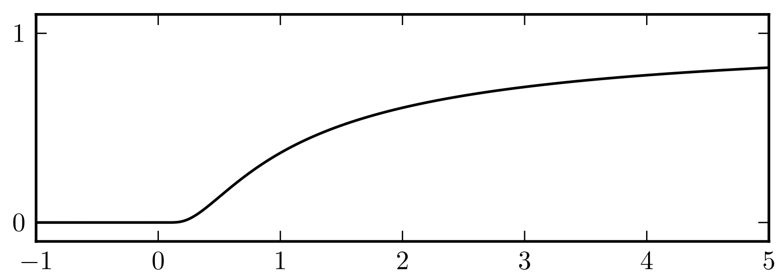


Figure 9.1: The graph of a function which is smooth, but not analytic.

2. Let $K$ be an compact interval of $\mathbb{R}$ and $f : K \to \mathbb{C}$ be a smooth function. Show that $f$ is analytic if and only if there are positive constants $\alpha > 0$ and $r > 0$ such that

$$\forall \, x \in K, \forall \, n \in \mathbb{N}, \; |f^{(n)}(x)| \leq \alpha \, r^n n!$$

## Periodic Analytic Functions

**Notations.** In this exercise, $\mathbb{U}$ is the unit circle centered at the origin:

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

For any radius $0 \leq r < 1$, we define the annulus

$$A_r = A(0, r, 1/r) = \{z \in \mathbb{C} \mid r < |z| < 1/r\}$$

(with the convention that $A_0 = \mathbb{C}^*$). For any $0 < \epsilon \leq +\infty$, the notation $\Omega_\epsilon$ refers to the horizontal strip

$$\Omega_\epsilon = \{z \in \mathbb{C} \mid |\text{Im } z| < \epsilon\}.$$

Let $f : \mathbb{U} \to \mathbb{C}$ be a function with an analytic extension in some open neighbourhood $U$ of $\mathbb{U}$.

1. Prove that there is an annulus $A_r$ such that $A_r \subset U$.

2. Let $g : t \in \mathbb{R} \mapsto f(e^{it})$; show that $g$ is a $2\pi$-periodic analytic function.

Conversely, let $g : t \in \mathbb{R} \to \mathbb{C}$ be a $2\pi$-periodic analytic function.

3. Show there is an analytic extension $g^*$ of $g$ on some strip $\Omega_\epsilon$, such that

$$\forall\, z \in \Omega_\epsilon,\ g^*(z + 2\pi) = g^*(z).$$

4. Show that there exist a function $f : \mathbb{U} \to \mathbb{C}$ with an analytic extension in some open neighbourhood of $\mathbb{U}$ such that

$$\forall\, t \in \mathbb{R},\ g(t) = f(e^{it}).$$

# Chapter 10

# Integral Representations

## Complex Differentiation of Integrals

**Theorem – Complex-Differentiation under the Integral Sign.** Let $\Omega$ be an open subset of $\mathbb{C}$ and $(X, \mu)$ be a measurable space. Let $f : \Omega \times X \to \mathbb{C}$ be a function such that:

1. for every $z$ in $\Omega$, $x \in X \mapsto f(z, x)$ is $\mu$-measurable,
2. for any $z_0 \in \Omega$, there is a neighborhood $V$ of $z_0$ in $\Omega$ and a $\mu$-integrable function $g : X \to \mathbb{R}_+$ such that

$$\forall\, z \in V,\ |f(z, x)| \leq g(x)\ \mu\text{-a.e.}$$

3. for $\mu$-almost every $x \in X$, the function $z \in \Omega \mapsto f(z, x)$ is holomorphic.

Then the function $z \in \Omega \mapsto \int_X f(z, x)\, d\mu(x)$ is holomorphic and its derivative at any order $n$ is

$$\frac{\partial^n}{\partial z^n} \left[ \int_X f(z, x)\, d\mu(x) \right] = \int_X \partial_z^n f(z, x)\, d\mu(x).$$

**Proof.** Let $z_0$ in $\Omega$ and $V$ be as in assumption 2; let $r > 0$ be a radius such that $\overline{D}(z_0, r) \subset V$ and let $\gamma = z_0 + r[\circlearrowleft]$. The Cauchy formula, followed by an integration by parts, yields for $\mu$-almost every $x \in X$ and any $z \in D(z_0, r/2)$

$$\partial_z f(z, x) = \frac{1}{i2\pi} \int_\gamma \frac{\partial_z f(w, x)}{w - z}\, dw = \frac{1}{i2\pi} \int_\gamma \frac{f(w, x)}{(w - z)^2}\, dw,$$

which by the M-L estimation lemma provides the bound

$$|\partial_z f(z, x)| \leq \frac{4|g(x)|}{r}.$$

The difference quotient of $z \mapsto \int_X f(z,x)\,d\mu(x)$ at $z_0$ is equal to

$$\int_X \frac{f(z_0+h,x)-f(z_0,x)}{h}\,d\mu(x).$$

Let $h$ be a complex number such that $|h|<r/2$. For $\mu$-almost every $x \in X$, the function $\phi : t \in [0,1] \mapsto f(z_0+th,x)$ is continuous on $[0,1]$, differentiable on $]0,1[$ and satisfies

$$|\phi'(t)| = |\partial_z f(z_0+th,x)||h| \leq \frac{g(x)}{r}|h|.$$

Hence, the mean value inequality yields

$$\left|\frac{f(z_0+h,x)-f(z_0,x)}{h}\right| = \frac{|\phi(1)-\phi(0)|}{|h|} \leq \frac{4g(x)}{r}.$$

Since

$$\lim_{h\to 0} \frac{f(z_0+h,x)-f(z_0,x)}{h} = \partial_z f(z_0,x) \quad \mu\text{-a.e.},$$

Lebesgue's dominated convergence theorem provides the result for $n=1$. Now, the function $\partial_z f$ also satisfies the three assumptions required by the theorem, hence by induction, the theorem statement holds at any order $n$. ■

**Corollary – Complex-Differentiation of Line Integrals.** Let $f : \Omega \times \Lambda \to \mathbb{C}$ where $\Omega$ and $\Lambda$ are two subsets of $\mathbb{C}$ and $\Omega$ is open. Assume that

1. $f$ is a continuous function.
2. for any $w \in \Lambda$, the function $z \in \Omega \mapsto f(z,w)$ is holomorphic.

Then, for any sequence of rectifiable paths $\gamma$ of $\Lambda$, the function $z \in \Omega \mapsto \int_\gamma f(z,w)\,dw$ is holomorphic and

$$\frac{\partial}{\partial z}\left[\int_\gamma f(z,w)\,dw\right] = \int_\gamma \partial_z f(z,w)\,dw.$$

**Proof.** We prove the result for any continuously differentiable path $\gamma$ of $\Lambda$ (the case of a sequence of rectifiable paths is a simple corollary). By definition of the line integral,

$$\int_\gamma f(z,w)\,dw = \int_{[0,1]} f(z,\gamma(t))\gamma'(t)\,dt.$$

Now,

1. For any $z \in \Omega$, the function $t \in [0,1] \mapsto f(z,\gamma(t))\gamma'(t)$ is continuous and therefore Lebesgue measurable.
2. Let $z_0 \in \Omega$ and let $r>0$ be such that $K = \overline{D}(z_0,r) \subset \Omega$. The restriction of $f$ to the compact set $K \times \gamma([0,1])$ is bounded by some constant $\kappa$. Therefore, for any $z \in D(z_0,r)$, the function $t \in [0,1] \mapsto f(z,\gamma(t))\gamma'(t)$ is dominated by $t \in [0,1] \mapsto \kappa|\gamma'(t)|$ which is Lebesgue integrable.

3. For any $t \in [0,1]$, the function $z \in \Omega \mapsto f(z, \gamma(t))\gamma'(t)$ is holomorphic; its derivative is $\partial_z f(z, \gamma(t))\gamma'(t)$.

Consequently, the differentiation of Lebesgue integrals theorem provides the existence of $\partial_z \left[\int_\gamma f(z,w)\,dw\right]$ and its value:

$$\frac{\partial}{\partial z}\left[\int_\gamma f(z,w)\,dw\right] = \int_{[0,1]} \partial_z f(z,\gamma(t))\gamma'(t)\,dt.$$

The right-hand side is equal to $\int_\gamma \partial_z f(z,w)\,dw$. ■

# The Laplace Transform

**Definition – The Laplace Transform.** Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a Lebesgue measurable function. We denote by $\sigma$ the extended real number defined by

$$\sigma \in [-\infty, +\infty] = \inf \left\{ \sigma^+ \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}_+} |f(t)|e^{-\sigma^+ t}\,dt < +\infty \right\}.$$

If $s \in \mathbb{C}$ and $\mathrm{Re}(s) > \sigma$, the function $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue integrable. The *Laplace transform* of $f$ is the function

$$\mathcal{L}[f] : \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\} \to \mathbb{C}$$

defined by

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} f(t)e^{-st}\,dt.$$

**Proof – Definition of the Laplace Transform.** For any $s \in \mathbb{C}$, the function $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue measurable . If additionally $\mathrm{Re}(s) > \sigma$, then there is some $\sigma^+$ such that $\sigma < \sigma^+ < \mathrm{Re}(s)$ and $t \mapsto |f(t)|e^{-\sigma^+ t}$ is Lebesgue integrable. Thus,

$$\int_{\mathbb{R}_+} |f(t)e^{-st}|\,dt = \int_{\mathbb{R}_+} |f(t)|e^{-\mathrm{Re}(s)t}\,dt \le \int_{\mathbb{R}_+} |f(t)|e^{-\sigma^+ t}\,dt < +\infty.$$

and therefore $t \in \mathbb{R}_+ \mapsto f(t)e^{-st}$ is Lebesgue integrable. ■

**Example – Laplace Transform of Exponential Functions.** For any $\lambda \in \mathbb{C}$, the function $t \in \mathbb{R}_+ \mapsto e^{\lambda t}$ is Lebesgue measurable. Additionally,

$$\forall\, t \ge 0,\ |f(t)|e^{-\sigma^+ t} = e^{-(\sigma^+ - \mathrm{Re}(\lambda))t},$$

hence the function $t \in \mathbb{R}_+ \mapsto |f(t)|e^{-\sigma^+}$ is Lebesgue integrable if and only if $\sigma^+ > \mathrm{Re}(\lambda)$. The infimum $\sigma$ of all such $\sigma^+$ is therefore $\mathrm{Re}(\lambda)$. Now, if $\mathrm{Re}(s) > \mathrm{Re}(\lambda)$,

$$\mathcal{L}[f](s) = \int_{\mathbb{R}_+} e^{(\lambda - s)t}\,dt = \left[\frac{e^{(\lambda-s)t}}{\lambda - s}\right]_0^{+\infty} = \frac{1}{s-\lambda}.$$

**Theorem – Derivative of the Laplace Transform.** The Laplace transform of a Lebesgue measurable function $f : \mathbb{R}_+ \to \mathbb{C}$ is holomorphic on its domain of definition and

$$(\mathcal{L}[f])'(s) = \mathcal{L}[t \mapsto -tf(t)](s).$$

**Proof.** Let $\Omega = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma\}$.

1. For any $s \in \Omega$, the function $t \mapsto f(t)e^{-st}$ is Lebesgue measurable.
2. Let $s \in \Omega$ and let $r > 0$ be such that $\epsilon = \mathrm{Re}(s) - \sigma - r > 0$. For any $w \in D(s, r)$, we have $\mathrm{Re}(w) > \mathrm{Re}(s) - r = \sigma + \epsilon$, thus

$$\int_{\mathbb{R}_+} |f(t)e^{-wt}|\, dt = \int_{\mathbb{R}_+} |f(t)|e^{-\mathrm{Re}(w)t}\, dt \leq \int_{\mathbb{R}_+} |f(t)|e^{-(\sigma+\epsilon)t}\, dt < +\infty.$$

3. For almost any $t \geq 0$, $s \mapsto f(t)e^{-st}$ is holomorphic and

$$\partial_s[f(t)e^{-st}] = -tf(t)e^{-st}.$$

We can therefore differentiate under the integral sign and obtain

$$\frac{\partial}{\partial s} \int_0^{+\infty} f(t)e^{-st}\, dt = \int_0^{+\infty} -tf(t)e^{-st}\, dt = \mathcal{L}[t \mapsto -tf(t)](s)$$

as expected. ■

**Example – Laplace Transform of Polynomials.** The constant function defined by $f(t) = 1$ for $t \geq 0$ is an exponential function (as $1 = e^{0\times t}$); its Laplace transform is defined for $\mathrm{Re}(s) > 0$ and equal to $1/s$. Now, this Laplace transform has a derivative at every of order $n$ which is

$$\frac{(-1)^n n!}{s^{n+1}}.$$

It is also the Laplace transform of $t \in \mathbb{R}_+ \mapsto (-t)^n$. Thus, by linearity, the Laplace transform of the polynomial $f(t) = \sum_{p=0}^{n} a_p t^p$ is

$$\mathcal{L}[f](s) = \sum_{p=0}^{n} a_p p! \frac{1}{s^{p+1}}.$$

# Cauchy's Integral Theorem – Dixon's Proof

In (Dixon 1971), John D. Dixon provides a short proof of the global version of Cauchy's Formula, using the local Cauchy theory. The proof relies on the following key result:

**Lemma – Integral of the Difference Quotient.** Let $\Omega$ be an open subset of the complex plane, $f$ be a holomorphic function on $\Omega$ and $\gamma$ be a sequence of rectifiable closed paths of $\Omega$. The function

$$z \in \Omega \setminus \gamma([0,1]) \mapsto \int_\gamma \frac{f(z) - f(w)}{z - w} dw$$

has a holomorphic extension on $\Omega$.

**Proof.** We may define the function $g : \Omega \times \Omega \to \mathbb{C}$ by

$$g(z, w) = \frac{f(z) - f(w)}{z - w} \text{ if } z \neq w \text{ and } g(w, w) = f'(w).$$

The continuity and complex-differentiability of $g$ at any point $(z, w) \in \Omega^2$ such that $z \neq w$ is plain. Now, let $c \in \Omega$ and let $r > 0$ be a radius such that the closure of the disk $D = D(c, r)$ is included in $\Omega$. Using the Taylor expansion of $f$ in this disk, we derive for any $z \in D$ and $w \in D$:

$$\begin{aligned} \frac{f(z) - f(w)}{z - w} &= \frac{1}{z - w} \sum_{n=0}^{+\infty} a_n((z - c)^n - (w - c)^n) \\ &= \sum_{n=1}^{+\infty} a_n \left[ \sum_{p=0}^{n-1} (z - c)^{n-1-p} (w - c)^p \right] \end{aligned}$$

The right-hand side of this equation is a uniformly convergent sum of continuous functions of $(w, z) \in D^2$ . Thus, its limit is a continuous function of $(w, z)$ and we have

$$\lim_{(w,z) \to (c,c), w \neq z} \frac{f(z) - f(w)}{z - w} = \sum_{n=1}^{+\infty} n a_n (w - c)^{n-1} = f'(w) = g(w, w),$$

thus this continuous function is actually $g$. Additionally, for every $w \in D$, every function of the sum is a holomorphic function with respect to $z$, hence its uniform limit $z \in D \mapsto g(z, w)$ is also holomorphic.

Now the function

$$z \in \Omega \mapsto \int_\gamma g(z, w) dw$$

clearly extends the function of the lemma statement. It also satisfies the assumptions of the complex-differentiation of line integrals result, thus it is holomorphic. ■

For completeness, here is Dixon's proof of Cauchy's formula:

**Proof – Cauchy's Integral Formula.** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f : \Omega \mapsto \mathbb{C}$ be a holomorphic function. Let $\gamma$ be a sequence of rectifiable closed paths of $\Omega$ such that $\operatorname{Int} \gamma \subset \Omega$.

Introduce the holomorphic extension $h$ to $\Omega$ of

$$z \in \Omega \setminus \gamma([0,1]) \mapsto \frac{1}{i2\pi} \int_\gamma \frac{f(z) - f(w)}{z - w} dw$$

and define the function $\phi : \mathbb{C} \mapsto \mathbb{C}$ by

$$\phi(z) = h(z) \;\text{ if }\; z \in \Omega, \; \phi(z) = -\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{w - z}\, dw \;\text{ if }\; z \in \operatorname{Ext} \gamma.$$

This definition is unambiguous: if $z \in \Omega \cap \operatorname{Ext} \gamma$, then

$$\begin{aligned} h(z) &= \frac{1}{i2\pi} \int_\gamma \frac{f(z) - f(w)}{z - w} dw \\ &= f(z) \operatorname{ind}(\gamma, z) - \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw. \\ &= -\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw \end{aligned}$$

The function $\phi$ is holomorphic on $\Omega$ and also on $\operatorname{Ext} \gamma$ by the complex-differentiation of line integrals theorem. Hence, it is holomorphic on $\mathbb{C}$. Additionally, if $|z| > r = \max\{|w| \mid w \in \gamma([0,1])\}$, then $z \in \operatorname{Ext} \gamma$, thus if $M$ is an upper bound of $f$ on the image of $\gamma$,

$$|\phi(z)| \leq \frac{1}{2\pi} \frac{M}{|z| - r} \times \ell(\gamma)$$

and $|\phi(z)| \to 0$ when $|z| \to +\infty$. By Liouville's Theorem, $\phi$ is identically zero; hence, if $z \in \Omega$,

$$\frac{1}{i2\pi} \int_\gamma \frac{f(w)}{z - w} dw = \frac{1}{i2\pi} \int_\gamma \frac{f(z)}{z - w} dw = \operatorname{ind}(\gamma, z) f(z),$$

which is Cauchy's integral formula. ■

## The Π Function

**Definition – Π Function.** The Π function is defined for all complex numbers $z$ such that $\operatorname{Re}(z) > -1$ by

$$\Pi(z) = \int_0^{+\infty} t^z e^{-t}\, dt$$

It is a holomorphic function whose $n$-th order derivative is given by

$$\Pi^{(n)}(z) = \int_0^{+\infty} (\ln t)^n t^z e^{-t}\, dt.$$

**Proof** – Π **Function.** For any $z \in \mathbb{C}$ and any $t > 0$,

$$t^z e^{-t} = e^{z \ln t - t} \text{ and } |t^z e^{-t}| = e^{\mathrm{Re}(z) \ln t - t} = t^{\mathrm{Re}(z)} e^{-t}.$$

Thus, if $\mathrm{Re}(z) > -1$, the function $t \in \mathbb{R}_+^* \mapsto t^z e^{-t}$ is Lebesgue integrable. Let $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > -1$ and let $r = (\mathrm{Re}(z) + 1)/2 > 0$. For any $h \in \mathbb{C}$ such that $|h| < r$ and any $t > 0$,

$$|t^{(z+h)} e^{-t}| = t^{\mathrm{Re}(z+h)} e^{-t} < \max(t^{\mathrm{Re}(z)-r}, t^{\mathrm{Re}(z)+r}) e^{-t}$$

and the right-hand side of this inequality is a Lebesgue integrable function of $t$. Finally, for any $t > 0$, the function $z \mapsto t^z e^{-t}$ is holomorphic on the domain of the Π function and at any order $n$,

$$\partial_z^n t^z e^{-t} = \partial_z^n e^{z \ln t - t} = (\ln t)^n t^z e^{-t}.$$

The assumptions of differentiation under the integral sign are met and the application of this theorem provides the desired result. ■

# References

Dixon, John D. 1971. "A Brief Proof of Cauchy's Integral Theorem." *Proceedings of the American Mathematical Society* 29. American Mathematical Society (AMS), Providence, RI: 625–26. doi:10.2307/2038614.

# Exercises

## Functions of Several Complex Variables

Let $n \geq 2$, let $\Omega$ be an open subset of $\mathbb{C}^n$ and let $f : \Omega \mapsto \mathbb{C}$ a continuous function. Show that $f$ is complex-differentiable in $\Omega$ if and only if for any $(z_1, \ldots, z_n) \in \Omega$, the partial function

$$f_{k,z} : w \mapsto f(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)$$

is holomorphic.

# Chapter 11

# Complex-Step Differentiation

## Introduction

You may already have used numerical differentiation to estimate the derivative of a function, using for example Newton's finite difference approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The implementation of this scheme in Python is straightforward:

```
def FD(f, x, h):
    return (f(x + h) - f(x)) / h
```

However, the relationship between the value of the step $h$ and the accuracy of the numerical derivative is more complex. Consider the following sample data:

| Expression | Value |
|---|---|
| $\exp'(0)$ | 1 |
| `FD(exp, 0, 1e-4)` | `1.000050001667141` |
| `FD(exp, 0, 1e-8)` | `0.99999999392252903` |
| `FD(exp, 0, 1e-12)` | `1.000088900582341` |

The most accurate value of the numerical derivative is obtained for $h = 10^{-8}$ and only 8 digits of the result are significant. For the larger value of $h = 10^{-4}$, the accuracy is limited by the quality of the Taylor development of exp at the first order; this truncation error decreases linearly with the step size. For the smaller value of $h = 10^{-12}$, the accuracy is essentially undermined by round-off errors in computations.

In this document, we show that *complex-step differentiation* may be used to get rid of the influence of the round-off error for the computation of the first derivative. For higher-order derivatives, we introduce a *spectral method*, a fast algorithm with an error that decreases exponentially with the number of function evaluations.

## Computer Arithmetic

You may skip this section if you are already familiar with the representation of real numbers as "doubles" on computers and with their basic properties. At the opposite, if you wish to have more details on this subject, it is probably a good idea to have a look at the classic "What every computer scientist should know about computer arithmetic" (Goldberg 1991).

In the sequel, the examples are provided as snippets of Python code that often use the Numerical Python (NumPy) library; first of all, let's make sure that all NumPy symbols are available:

```
>>> from numpy import *
```

### Floating-Point Numbers: First Contact

The most obvious way to display a number is to print it:

```
>>> print pi
3.14159265359
```

This is a lie of course: `print` is not supposed to display an accurate information about its argument, but something readable. To get something unambiguous instead, we can do:

```
>>> pi
3.141592653589793
```

When we say "unambiguous", we mean that there is enough information in this sequence of digits to compute the original floating-point number; and indeed:

```
>>> pi == eval("3.141592653589793")
True
```

Actually, this representation is *also* a lie: it is not an exact decimal representation of the number `pi` stored in the computer memory. To get an exact representation of `pi`, we can request the display of a large number of the decimal digits:

```
>>> def all_digits(number):
...     print "{0:.100g}".format(number)
>>> all_digits(pi)
3.141592653589793115997963468544185161590576171875
```

Asking for 100 digits was actually good enough: only 49 of them are displayed anyway, as the extra digits are all zeros.

Note that we obtained an exact representation of the floating-point number `pi` with 49 digits. That does *not* mean that all – or even most – of these digits are significant in the representation the real number of $\pi$. Indeed, if we use the Python library for multiprecision floating-point arithmetic mpmath, we see that

```
>>> import mpmath
>>> mpmath.mp.dps = 49; mpmath.mp.pretty = True
>>> +mpmath.pi
3.141592653589793238462643383279502884197169399375
```

and both representations are identical only up to the 16th digit.

## Binary Floating-Point Numbers

Representation of floating-point numbers appears to be complex so far, but it's only because we insist on using a *decimal* representation when these numbers are actually stored as *binary* numbers. In other words, instead of using a sequence of *(decimal) digits* $f_i \in \{0, 1, \dots, 9\}$ to represent a real number $x$ as

$$x = \pm(f_0.f_1 f_2 \dots f_i \dots) \times 10^e$$

we should use *binary digits* – aka *bits* – $f_i \in \{0, 1\}$ to write:

$$x = \pm(f_0.f_1 f_2 \dots f_i \dots) \times 2^e.$$

These representations are *normalized* if the leading digit of the *significand* $(f_0.f_1 f_2 \dots f_i \dots)$ is non-zero; for example, with this convention, the rational number 999/1000 would be represented in base 10 as $9.99 \times 10^{-1}$ and not as $0.999 \times 10^0$. In base 2, the only non-zero digit is 1, hence the significand of a normalized representation is always $(1.f_1 f_2 \dots f_i \dots)$.

In scientific computing, real numbers are usually approximated to fit into a 64-bit layout named "double"[1]. In Python standard library, doubles are available as instances of `float` – or alternatively as `float64` in NumPy.

A triple of

- *sign bit* $s \in \{0, 1\}$,

[1]**"Double"** is a shortcut for "double-precision floating-point format", defined in the IEEE 754 standard, see (IEEE Task P754 1985). A single-precision format is also defined, that uses only 32 bits. NumPy provides it under the name `float32`.

- *biased exponent* $e \in \{1, \dots, 2046\}$ (11-bit),
- *fraction* $f = (f_1, \dots, f_{52}) \in \{0, 1\}^{52}$.

represents a normalized double

$$x = (-1)^s \times 2^{e-1023} \times (1.f_1 f_2 \dots f_{52}).$$

The doubles that are not normalized are not-a-number (`nan`), infinity (`inf`) and zero (`0.0`) (actually *signed* infinities and zeros), and denormalized numbers. In the sequel, we will never consider such numbers.

## Accuracy

Almost all real numbers cannot be represented exactly as doubles. It makes sense to associate to a real number $x$ the nearest double $[x]$. A "round-to-nearest" method that does this is fully specified in the IEE754 standard (see IEEE Task P754 1985), together with alternate ("directed rounding") methods.

To have any kind of confidence in our computations with doubles, we need to be able to estimate the error in the representation of $x$ by $[x]$. The *machine epsilon*, denoted $\epsilon$ in the sequel, is a key number in this respect. It is defined as the gap between 1.0 – that can be represented exactly as a double – and the next double in the direction $+\infty$.

```
>>> after_one = nextafter(1.0, +inf)
>>> after_one
1.0000000000000002
>>> all_digits(after_one)
1.0000000000000002220446049250313080847263336181640625
>>> eps = after_one - 1.0
>>> all_digits(eps)
2.220446049250313080847263336181640625e-16
```

This number is also available as an attribute of the `finfo` class of NumPy that gathers machine limits for floating-point data types:

```
>>> all_digits(finfo(float).eps)
2.220446049250313080847263336181640625e-16
```

Alternatively, the examination of the structure of normalized doubles yields directly the value of $\epsilon$: the fraction of the number after 1.0 is $(f_1, f_2, \dots, f_{51}, f_{52}) = (0, 0, \dots, 0, 1)$, hence $\epsilon = 2^{-52}$, a result confirmed by:

```
>>> all_digits(2**-52)
2.220446049250313080847263336181640625e-16
```

The machine epsilon matters so much because it provides a simple bound on the relative error of the representation of a real number as a double. Indeed, for any sensible rounding method, the structure of normalized doubles yields

$$\frac{|[x] - x|}{|x|} \leq \epsilon.$$

If the "round-to-nearest" method is used, you can actually derive a tighter bound: the inequality above still holds with $\epsilon/2$ instead of $\epsilon$.

### Significant Digits

This relative error translates directly into how many significant decimal digits there are in the best approximation of a real number by a double. Consider the exact representation of $[x]$ in the scientific notation:

$$[x] = \pm(f_0.f_1 \ldots f_{p-1} \ldots) \times 10^e.$$

We say that it is significant up to the $p$-th digit if

$$|x - [x]| \leq \frac{10^{e-(p-1)}}{2}.$$

On the other hand, the error bound on $[x]$ yields

$$|x - [x]| \leq \frac{\epsilon}{2}|x| \leq \frac{\epsilon}{2} \times 10^{e+1}.$$

Hence, the desired precision is achieved as long as

$$p \leq -\log_{10} \epsilon/2 = 52 \log_{10} 2 \approx 15.7.$$

Consequently, doubles provide a 15-th digit approximation of real numbers.

### Functions

Most real numbers cannot be represented exactly as doubles; accordingly, most real functions of real variables cannot be represented exactly as functions operating on doubles either. The best we can hope for are *correctly rounded* approximations. An approximation $[f]$ of a function $f$ of $n$ variables is *correctly rounded* if for any $n$-uple $(x_1, \ldots, x_n)$, we have

$$[f](x_1, \ldots, x_n) = [f([x_1], \ldots, [x_n])].$$

The IEEE 754 standard (see IEEE Task P754 1985) mandates that some functions have a correctly rounded implementation; they are:

> add, substract, multiply, divide, remainder and square root.

Other standard elementary functions – such as sine, cosine, exponential, logarithm, etc. – are usually *not* correctly rounded; the design of computation algorithms that have a decent performance and are *provably* correctly rounded is a complex problem (see for example the documentation of the Correctly Rounded mathematical library).

# Complex Step Differentiation

## Forward Difference

Let $f$ be a real-valued function defined in some open interval. In many concrete use cases, we can make the assumption that the function is actually analytic and never have to worry about the existence of derivatives. As a bonus, for any real number $x$ in the domain of the function, the (truncated) Taylor expansion

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots + \frac{f^{(n)}}{n!}h^n + \mathcal{O}(h^{n+1})$$

is locally valid[2]. A straighforward computation shows that

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

The asymptotic behavior of this *forward difference* scheme – controlled by the term $\mathcal{O}(h^1)$ – is said to be of order 1. An implementation of this scheme is defined for doubles $x$ and $h$ as

$$\mathrm{FD}(f, x, h) = \left[\frac{[[f]([x] + [h]) - [f](x)]}{[h]}\right].$$

or equivalently, in Python as:

```
def FD(f, x, h):
    return (f(x + h) - f(x)) / h
```

## Round-Off Error

We consider again the function $f(x) = \exp(x)$ used in the introduction and compute the numerical derivative based on the forward difference at $x = 0$ for several values of $h$. The graph of $h \mapsto \mathrm{FD}(\exp, 0, h)$ shows that for values of $h$ near or below the machine epsilon $\epsilon$, the difference between the numerical derivative and the exact value of the derivative is *not* explained by the classic asymptotic analysis.

If we take into account the representation of real numbers as doubles however, we can explain and quantify the phenomenon. To focus only on the effect of the round-off errors, we'd like to get rid of the truncation error. To achieve this, in the following computations, instead of exp, we use $\exp_0$, the Taylor expansion of exp of order 1 at $x = 0$; we have $\exp_0(x) = 1 + x$.

[2]**Bachmann-Landau notation.** For a real or complex variable $h$, we write $\psi(h) = \mathcal{O}(\phi(h))$ if there is a suitable deleted neighbourhood of $h = 0$ where the functions $\psi$ and $\phi$ are defined and the inequality $|\psi(h)| \leq \kappa|\phi(h)|$ holds for some $\kappa > 0$. When $N$ is a natural number, we write $\psi(N) = \mathcal{O}(\phi(N))$ if there is a $n$ such that $\psi$ and $\phi$ are defined for $N \geq n$ and for any such $N$, the inequality $|\psi(N)| \leq \kappa|\phi(N)|$ holds for some $\kappa > 0$.

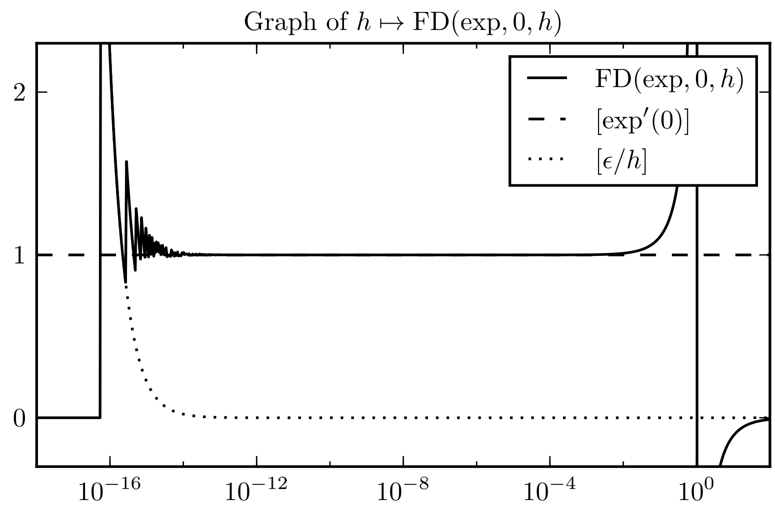


Figure 11.1: Forward Difference Scheme Values.

Assume that the rounding scheme is "round-to-nearest"; select a floating-point number $h > 0$ and compare it to the machine epsilon:

- If $h \ll \epsilon$, then $1 + h$ is close to 1, actually, closer to 1 than from the next binary floating-point value, which is $1 + \epsilon$. Hence, the value is rounded to $[\exp_0](h) = 1$, and a *catastrophic cancellation* happens:

$$\mathrm{FD}(\exp_0, 0, h) = \left[\frac{[[\exp_0](h) - 1]}{h}\right] = 0.$$

- If $h \approx \epsilon$, then $1 + h$ is closer from $1 + \epsilon$ than it is from 1, hence we have $[\exp_0](h) = 1 + \epsilon$ and

$$\mathrm{FD}(\exp_0, 0, h) = \left[\frac{[[\exp_0](h) - 1]}{h}\right] = \left[\frac{\epsilon}{h}\right].$$

- If $\epsilon \ll h \ll 1$, then $[1 + h] = 1 + h \pm \epsilon(1 + h)$ (the symbol $\pm$ is used here to define a confidence interval[3]). Hence

$$[[\exp_0](h) - 1] = h \pm \epsilon \pm \epsilon(2h + \epsilon + \epsilon h)$$

and

$$\left[\frac{[[\exp_0](h) - 1]}{h}\right] = 1 \pm \frac{\epsilon}{h} + \frac{\epsilon}{h}(3h + 2\epsilon + 3h\epsilon + \epsilon^2 + \epsilon^2 h)$$

[3] **Plus-minus sign and confidence interval.** The equation $a = b \pm c$ should be interpreted as the inequality $|a - b| \leq |c|$.

therefore

$$\mathrm{FD}(\exp_0, 0, h) = \exp_0'(0) \pm \frac{\epsilon}{h} \pm \epsilon',\ \epsilon' \ll \frac{\epsilon}{h}.$$

Going back to $\mathrm{FD}(\exp, 0, h)$ and using a log-log scale to display the total error, we can clearly distinguish the region where the error is dominated by the round-off error – the curve envelope is $\log(\epsilon/h)$ – and where it is dominated by the truncation error – a slope of 1 being characteristic of schemes of order 1.

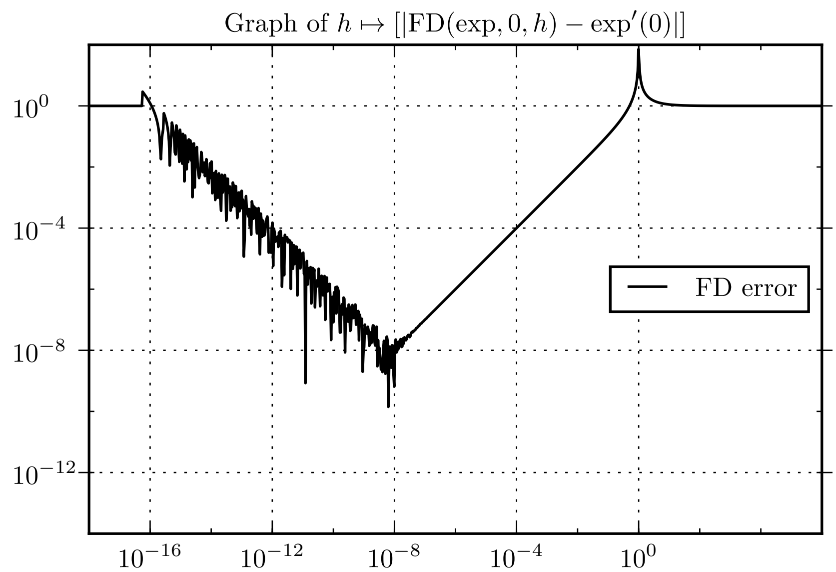


Figure 11.2: Forward Difference Scheme Error.

## Higher-Order Scheme

The theoretical asymptotic behavior of the forward difference scheme can be improved, for example if instead of the forward difference quotient we use a central difference quotient. Consider the Taylor expansion at the order 2 of $f(x+h)$ and $f(x-h)$:

$$f(x+h) = f(x) + f'(x)(+h) + \frac{f''(x)}{2}(+h)^2 + \mathcal{O}\left(h^3\right)$$

and

$$f(x-h) = f(x) + f'(x)(-h) + \frac{f''(x)}{2}(-h)^2 + \mathcal{O}\left(h^3\right).$$

We have

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2),$$

hence, the *central difference* scheme is a scheme of order 2, with the implementation:

$$\mathrm{CD}(f, x, h) = \left[\frac{[[f]([x] + [h]) - [f]([x] - [h])]}{[2 \times [h]]}\right].$$

or equivalently, in Python:

```
def CD(f, x, h):
    return 0.5 * (f(x + h) - f(x - h)) / h
```

The error graph for the central difference scheme confirms that a truncation error of order two may be used to improve the accuracy. However, it also shows that a higher-order actually *increases* the region dominated by the round-off error, making the problem of selection of a correct step size $h$ even more difficult.

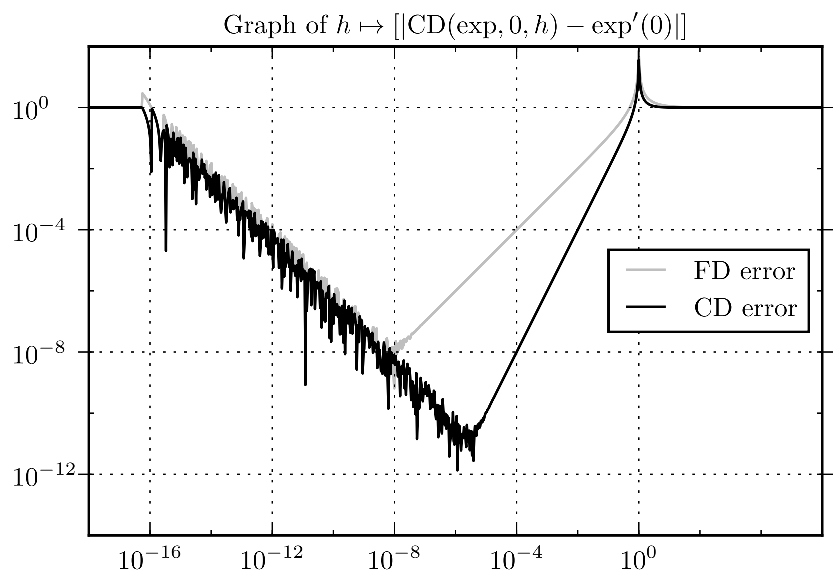


Figure 11.3: Central Difference Scheme Error.

## Complex Step Differentiation

If the function $f$ is analytic at $x$, the Taylor expansion is also valid for (small values of) complex numbers $h$. In particular, if we replace $h$ by a pure imaginary number $ih$, we end up with

$$f(x + ih) = f(x) + f'(x)ih + \frac{f''(x)}{2}(ih)^2 + \mathcal{O}(h^3)$$

If $f$ is real-valued, using the imaginary part yields:

$$\mathrm{Im}\left(\frac{f(x+ih)}{h}\right) = f'(x) + \mathcal{O}(h^2).$$

This is a method of order 2. The straightforward implementation of the complex-step differentiation is

$$\mathrm{CSD}(f, x, h) = \left[\frac{\mathrm{Im}([f]([x] + i[h]))}{[h]}\right].$$

or equivalently, in Python:

```
def CSD(f, x, h):
    return imag(f(x + 1j * h)) / h
```

The distinguishing feature of this scheme: it almost totally gets rid of the truncation error. Indeed, let's consider again $\exp_0$; when $x$ and $y$ are floating-point real numbers, the sum $x + iy$ can be computed with any round-off, hence, if $h$ is a floating-point number, $[\exp_0](ih) = [1 + ih] = 1 + ih$ and consequently, $\mathrm{Im}([\exp_0](ih)) = h$, which yields

$$\mathrm{CSD}(\exp_0, 0, h) = \left[\frac{h}{h}\right] = 1 = \exp_0'(0).$$

## Spectral Method

The complex step differentiation is a powerful method but it also has limits. We can use it to compute the first derivative of a real analytic function $f$, but not its second derivative because our estimate $x \mapsto \mathrm{CSD}(f, x, h)$ of the first derivative is only available for real values of $x$, hence the method cannot be iterated. We cannot use it either if we know that $f$ is analytic but not real-valued.

We introduce in this section an alternate method to compute first, second and higher-order derivatives of – real or complex-valued – analytic functions. More details may be found in (Fornberg 2006) and (Trefethen 2000).

### Computation Method

Let $f$ be a function that is holomorphic in an open neighbourhood of $x \in \mathbb{R}$ that contains the closed disk with center $x$ and radius $r$. In this disk, the values of $f$ can be computed by the Taylor series

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - x)^n, \;\; a_n = \frac{f^{(n)}(x)}{n!}.$$

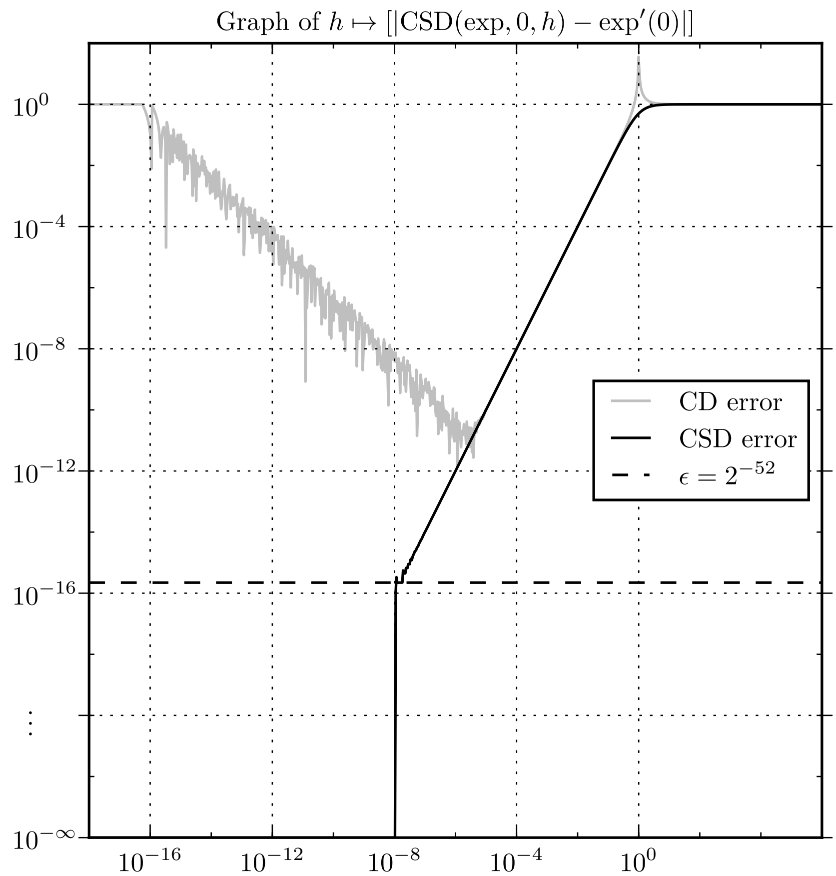


Figure 11.4: Complex Step Difference Scheme Error.

The open disk of convergence of the series has a radius that is larger than $r$, thus the growth bound of the sequence $a_n$ is smaller than $1/r$. Hence,

$$\exists \kappa > 0, \forall n \in \mathbb{N}, \ |a_n| \leq \kappa r^{-n}.$$

Let $h \in (0, r)$ and $N$ be a positive integer; let $f_k$ be the sequence of $N$ values of $f$ on the circle with center $x$ and radius $h$ defined by

$$f_k = f(x + hw^k), \ w = e^{-i2\pi/N}, \ k = 0, \ldots, N-1.$$

**Estimate and Accuracy**

The values $f_k$ can be computed as

$$f_k = \sum_{n=0}^{+\infty} a_n (hw^k)^n = \sum_{n=0}^{N-1} \left[ \sum_{m=0}^{+\infty} a_{n+mN} h^{n+mN} \right] w^{k(n+mN)}.$$

Notice that we have $w^{k(n+mN)} = w^{kn}(w^N)^{km} = w^{kn}$. Hence, if we define

$$c_n = a_n h^n + a_{n+N} h^{n+N} + \cdots = \sum_{m=0}^{+\infty} a_{n+mN} h^{n+mN},$$

we end up with the following relationship between the values $f_k$ and $c_n$:

$$f_k = \sum_{n=0}^{N-1} w^{kn} c_n.$$

It is useful because the coefficients $c_n/h^n$ provide an approximation of $a_n$:

$$\left| a_n - \frac{c_n}{h^n} \right| \leq \kappa r^{-n} \sum_{m=1}^{+\infty} (h/r)^{mN} = \kappa r^{-n} \frac{(h/r)^N}{1 - (h/r)^N}$$

There are two ways to look at this approximation: if we freeze $N$ and consider the behavior of the error when the radius $h$ approaches 0, we derive

$$a_n = \frac{c_n}{h^n} + \mathcal{O}(h^N)$$

and conclude that the approximation of $a_n$ is of order $N$ with respect to $h$; on the other hand, if we freeze $h$ and let $N$ grow to $+\infty$, we obtain instead

$$a_n = \frac{c_n}{h^n} + \mathcal{O}(e^{-\alpha N}) \ \text{ with } \ \alpha = -\log(h/r) > 0,$$

in other words, the approximation is exponential with respect to $N$.

**Computation of the Estimate**

The right-hand side of the equation

$$f_k = \sum_{n=0}^{N-1} w^{kn} c_n.$$

can be interpreted as a classic matrix-vector product; the mapping from the $c_n$ to the $f_k$ is known has the *discrete Fourier transform* (DFT). The inverse mapping – the *inverse discrete Fourier transform* – is given by

$$c_n = \frac{1}{N} \sum_{n=0}^{N-1} w^{-kn} f_k.$$

Both the discrete Fourier transform and it inverse can be computed by algorithms having a $\mathcal{O}(N \log N)$ complexity (instead of the $\mathcal{O}(N^2)$ of the obvious method), the aptly named *fast Fourier transform* (FFT) and *inverse fast Fourier transform* (IFFT). Some of these algorithms actually deliver the minimal complexity only when $N$ is a power of two, so it is safer to pick only such numbers if you don't know exactly what algorithm you are actually using.

The implementation of this scheme is simple:

```
from numpy.fft import ifft
from scipy.misc import factorial

def SM(f, x, h, N):
    w = exp(-1j * 2 * pi / N)
    k = n = arange(N)
    f_k = f(x + h * w**k)
    c_n = ifft(f_k)
    a_n = c_n / h ** n
    return a_n * factorial(n)
```

## Error Analysis

The algorithm introduced in the previous section provides approximation methods with an arbitrary large order for $n$-th order derivatives. However, the region in which the round-off error dominates the truncation error is large and actually *increases* when the integer $n$ grows. A specific analysis has to be made to control both kind of errors.

We conduct the detailled error analysis for the function

$$f(z) = \frac{1}{1-z}$$

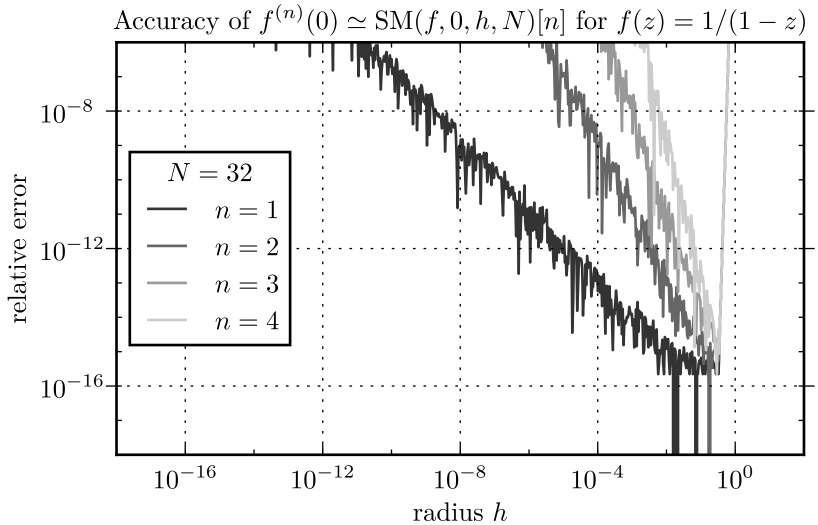


Figure 11.5: Spectral Method Error

at $x = 0$ and attempt to estimate the derivatives up to the fourth order. We have selected this example because $a_n = 1$ for every $n$, hence the computation of the relative errors of the results are simple.

### Round-off error

We assume that the main source of round-off errors is in the computations of the $f_k$. The distance between $f_k$ and its approximation $[f_k]$ is bounded by $|f_k| \times \epsilon/2$; the coefficients in the IFFT matrix are of modulus $1/N$, hence, if the sum is exact, we end up with an absolute error on $c_n$ bounded by $M(h) \times \epsilon/2$ with

$$M(h) = \max_{|z-x|=h} |f(z)|.$$

Hence the absolute error on $a_n = c_n/h^n$ is bounded by $M(h)\epsilon/(2h^n)$. Using the rough estimate $|a_n| \simeq \kappa r^{-n}$, we end up with a relative error for $a_n$ controlled by

$$\left(\frac{M(h)}{\kappa}\frac{\epsilon}{2}\right)\left(\frac{h}{r}\right)^{-n}$$

On our example, we can pick $M(h) = 1/(1-h)$, $\kappa = 1$, and $r = 1$, hence the best error bound we can hope for is obtained for the value of $h$ that minimizes $1/((1-h)h^n)\epsilon/2$; the best $h$ and round-off error bound are actually

$$h = \frac{n}{n+1} \quad \text{and} \quad \text{round-off}(a_n) \leq \frac{(n+1)^{n+1}}{n^n}\frac{\epsilon}{2}.$$

The error bound is always bigger than the structural relative error $\epsilon/2$ and increases with $n$, hence the worst case is obtained for the highest derivative order that we want to compute, that is $n = 4$. If for example we settle on a round-off relative error of 1000 times $\epsilon/2$, we can select $h = 0.2$.

### Truncation error

We have already estimated the difference between $a_n$ and $c_n/h^n$; if we again model $|a_n|$ as $\kappa r^{-n}$, the relative error of this estimate is bounded by

$$\frac{(h/r)^N}{1-(h/r)^N} \simeq \left(\frac{h}{r}\right)^N,$$

hence to obtain a truncation error of the same magnitude than the truncation error – that is $1000 \times \epsilon/2$, we may select $N$ such that $0.2^N \leq 1000 \times \epsilon/2$, that is

$$N \geq \left\lceil \frac{\log(1000 \times \epsilon/2)}{\log 0.2} \right\rceil = 19.$$

We pick for $N$ the next power of two after 19; the choices $h = 0.2$ and $N = 32$ yield the following estimates of the first 8 $n$-th order derivatives of $f$.

| $n$ | $f^{(n)}(0)$ estimate | relative error |
|---|---|---|
| 0 | 1.0000000000000000 | 0.0 |
| 1 | 0.9999999999999998 | $2.2 \times 10^{-16}$ |
| 2 | 1.9999999999999984 | $7.8 \times 10^{-16}$ |
| 3 | 6.0000000000000284 | $4.7 \times 10^{-15}$ |
| 4 | 23.999999999999996 | $1.1 \times 10^{-16}$ |
| 5 | 120.00000000001297 | $1.1 \times 10^{-13}$ |
| 6 | 720.00000000016007 | $2.2 \times 10^{-13}$ |
| 7 | 5040.0000000075588 | $1.5 \times 10^{-12}$ |

## Appendix

Augustin-Louis is the proud author of a very simple Python CSD code fragment that people cut-and-paste in their numerical code:

```
from numpy import imag
def CSD(f, x, h=1e-100):
    return imag(f(x + 1j * h)) / h
```

One day, he receives the following mail:

> Dear Augustin-Louis,

> We are afraid that your Python CSD code is defective; we used it to compute the derivative of $f(x) = \sqrt{|x|}$ at $x = 1$ and got 0. We're pretty sure that it should be 0.5 instead.
>
> Yours truly,
>
> Isaac & Gottfried Wilhelm.

1. Should the complex-step differentiation method work in this case?
2. How do you think that Isaac and Gottfried Wilhelm have implemented the function $f$? Would that explain the value of `CSD(f, x=1.0)` that they report?
3. Can you modify the CSD code fragment to "make it work" with the kind of function and implementation that Isaac and Gottried Wilhelm are using? Of course, you cannot change their code, only yours.

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# Chapter 12

# Poisson Image Editing

## Introduction

Poisson Image Editing refers to a family of methods introduced in (Pérez, Gangnet, and Blake 2003) that rely on the resolution of the Poisson equation to perform seamless editing of images. Typical use cases include: healing of images from damaged or missing data, image enhancements, e.g. the removal of skin blemishes in portraits, heavy image editing, for example the concealement of objects.

The GNU Image Manipulation Program (GIMP) "Heal tool" implements a Poisson Image Editing method[1], described in the documentation as "a smart clone tool on steroids", because:

> Pixels are not simply copied from source to destination, but the area around the destination is taken into account before cloning is applied.

Let's demonstrate Poisson image editing with GIMP. Say that we want to hide the watch in the image below, in a way one cannot guess that there was an item to begin with.

We first select a disk in a region of the image where there is no object, to capture the texture of the background behind the objects. GIMP displays this region with a dashed line (see "GIMP Heal tool in action" figure). We make sure that the disk is big enough to cover a significant part of the watch, and small enough to avoid the objects that are close to it. Then we graft this heal source on top of the lower part of the watch, then on its upper part. The final result is a seamless removal of the watch from the picture.

[1] the GIMP developers state that they use the method described in (Georgiev 2005), a variant of the original design that is invariant under relighting, but a closer examination of the source code shows that they actually use the original method.

Figure 12.1: GIMP Toolbox – The Heal tool is featured by the band-aid icon.

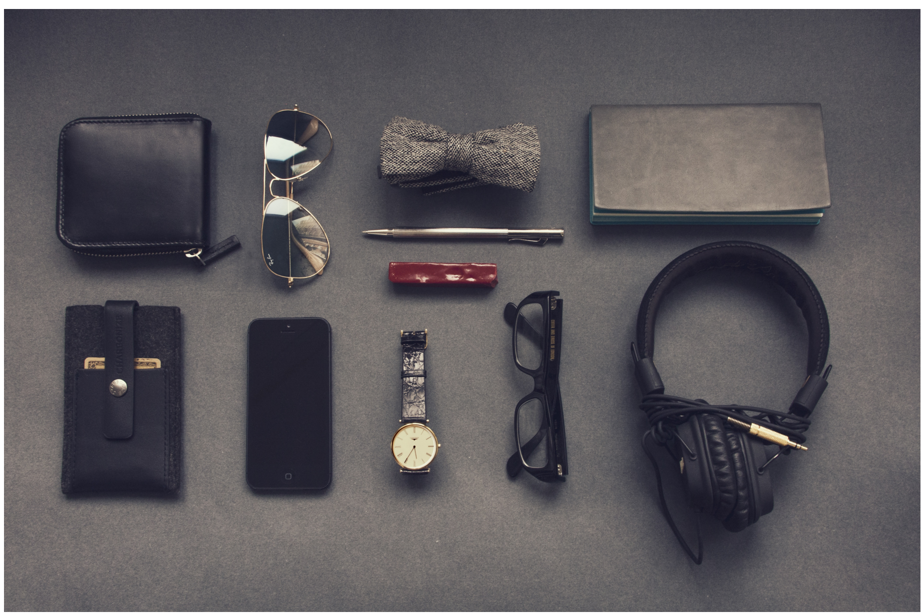

Figure 12.2: Photograph by Vadim Sherbakov, licensed under Unsplash Creative Commons Zero License.

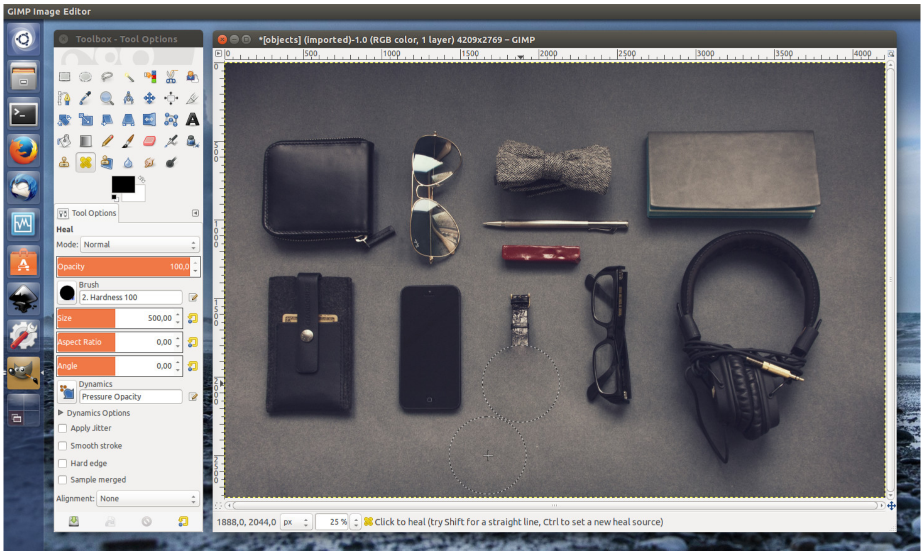


Figure 12.3: The GIMP Heal tool in action.

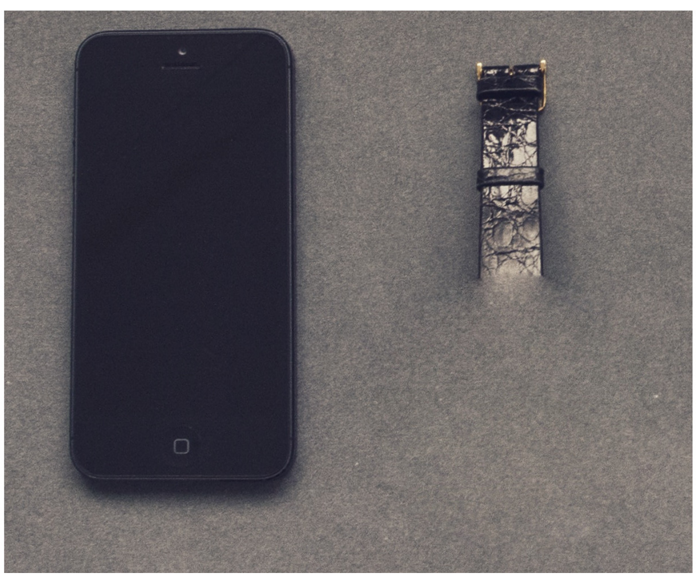

Figure 12.4: Work in progress (zoom in)

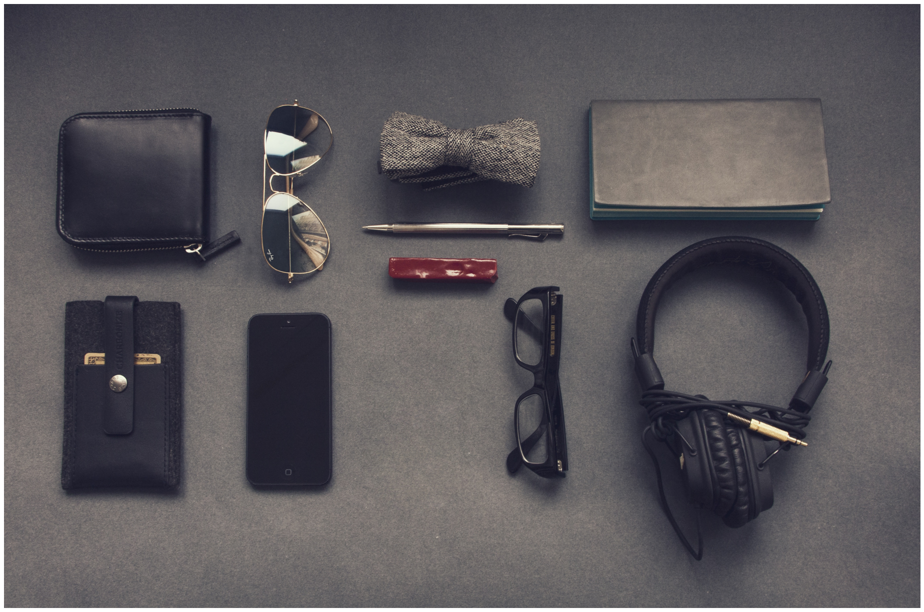

Figure 12.5: The Poisson edited image

## Modelling the Problem

Consider a circular image fragment $\phi$ that requires some editing. If the image resolution is good enough, it is sensible to model this fragment as a function $\phi$ of two continuous variables $x$ and $y$, defined inside the unit disk

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$$

and on its boundary $\partial D$.

If we deal with grayscale images, we can assume that $\phi$ is real-valued, even with values in $[0, 1]$ after a suitable normalization of the image intensity. Color images can easily be modelled too, for example as triples of real-valued functions, one for each channel in a RGB decomposition.

We search for a new image fragment $\psi$, that matches *exactly* the original fragment $\phi$ on the boundary $\partial D$ and whose texture matches *approximately* the texture of a third image fragment $\chi$ in $D$. We assume that the texture of an image fragment is essentially captured by its gradient field; for the sake of simplicity, we measure the dissimilarity of two gradients with the quadratic mean of their difference.

To summarize, we are trying to solve:

$$\min_{\psi} \int_D \|\nabla(\psi - \chi)\|^2 \ \text{ with } \ \psi|_{\partial D} = \phi|_{\partial D}.$$

If we set $u = \psi - \chi$ and $f = (\phi - \chi)|_{\partial D}$, we end up with a function $u$ that solves the *variational Dirichlet problem*: the minimization of the *Dirichlet energy* of $u$, subject to the *Dirichlet boundary condition* $f$:

$$\min_u \int_D \|\nabla u\|^2 \;\text{ with }\; u|_{\partial D} = f.$$

We may also search $u$ as a solution of the *Dirichlet problem*:

$$\Delta u = 0 \;\text{ on }\; D \;\text{ with }\; u|_{\partial D} = f$$

as both problems are – at least informally – equivalent. Indeed, given two candidate solutions $u$ and $u + \epsilon$ to the variational problem, we have

$$\int_D \|\nabla(u+\epsilon)\|^2 = \int_D \|\nabla u\|^2 + 2\int_D \nabla u \cdot \nabla \epsilon + \int_D \|\epsilon\|^2,$$

hence $u$ is a minimizer if and only if

$$\forall \epsilon \;\text{ such that }\; \epsilon|_{\partial D} = 0, \int_D \nabla u \cdot \nabla \epsilon = 0$$

Given Green's first identity, this condition holds if and only if $\Delta u = 0$ on $D$.

In the sequel we study the Dirichlet problem in the classic setting:

**Definition – Dirichlet Problem.** Given a continuous function $f : \partial D \to \mathbb{R}$, a *classic* solution of the Dirichlet problem is a function $u : \overline{D} \to \mathbb{R}$ which is continuous in $\overline{D}$, twice continously differentiable in $D$, and such that

$$\Delta u = 0 \;\text{ on }\; D \;\text{ with }\; u|_{\partial D} = f.$$

# Harmonic Functions

**Definition – Harmonic functions.** A function $u : \Omega \to \mathbb{R}$ defined in an open subset $\Omega$ of $\mathbb{C}$ is *harmonic* if it is twice continuously differentiable and $\Delta u = 0$ on $\Omega$.

**Theorem – The real part of a holomorphic function is harmonic.**

**Proof.** First we prove that $u = \operatorname{Re} f$ is twice continuously differentiable when $f : \Omega \to \mathbb{C}$ is holomorphic. The operator Re is real-linear and continuous hence it is real-differentiable and $d\mathrm{Re}_z = \mathrm{Re}$ for any $z \in \mathbb{C}$. On the other hand, $f$ is real-differentiable and $df_z(h) = f'(z) \times h$. Therefore, $u$ is real-differentiable and $du_z(h) = \operatorname{Re}(f'(z) \times h)$. Consequently,

$$\frac{\partial u}{\partial x}(x,y) = \operatorname{Re}(f'(z) \times 1) \;\text{ and }\; \frac{\partial u}{\partial y}(x,y) = \operatorname{Re}(f'(z) \times i).$$

The partial derivatives of first order of $u$ are continuous. They also appear as real parts of holomorphic functions, hence we can repeat the computation of partial derivatives with the same scheme to prove that $u$ is twice continuously differentiable.

Now, for any function $u$ that is twice continuously differentiable, we have

$$\Delta u = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right).$$

If $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ where $f$ is holomorphic, then

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

hence

$$\Delta u = \frac{\partial f'}{\partial x} + i\frac{\partial f'}{\partial y} = 0$$

by the complex form of the Cauchy-Riemann equation for $f'$. ■

There is a partial converse theorem:

**Theorem – A harmonic function is *locally* the real part of a holomorphic function.** If $u : \Omega \to \mathbb{R}$ is harmonic, for any $z \in \Omega$, there is an open neighbourhood $V$ of $z$ and a holomorphic function $f : V \to \mathbb{C}$ such that $u|_V = \operatorname{Re} f$.

Actually, we will show a stronger result: the *global* existence of a holomorphic function if the domain of definition of the harmonic function is simply connected. But first, we need the following lemma:

**Lemma.** If $u : \Omega \to \mathbb{R}$ is harmonic, the function $f : \Omega \to \mathbb{C}$ defined by

$$f(x + iy) = \frac{\partial u}{\partial x}(x, y) - i\frac{\partial u}{\partial y}(x, y)$$

is holomorphic.

**Proof.** As $u$ is twice continuously differentiable, $f$ is continuously differentiable and its partial derivatives are given by

$$\frac{\partial f}{\partial x} = \frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y^2}.$$

Consequently, the complex form of the Cauchy-Riemann equation holds:

$$\frac{\partial f}{\partial y} - i\frac{\partial f}{\partial x} = \left(\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}\right) - i\Delta u = 0.$$

It follows that the function $f$ is holomorphic. ■

Now, we may prove the converse theorem.

**Proof.** Assume that $u : \Omega \to \mathbb{R}$ is harmonic and that $\Omega$ is simply connected. By Cauchy's integral theorem, the integral along any closed rectifiable path of $\Omega$ of the holomorphic function $f$ of the lemma is zero, thus it has a primitive. Any such primitive $g$ satisfies:

$$g'(x+iy) = f(x+iy) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

Let $\tilde{u} = \operatorname{Re} g$ and $\tilde{v} = \operatorname{Im} g$; we have

$$g'(x+iy) = \frac{\partial \tilde{u}}{\partial x}(x,y) + i\frac{\partial \tilde{v}}{\partial x}(x,y) = \frac{\partial \tilde{u}}{\partial x}(x,y) - i\frac{\partial \tilde{u}}{\partial y}(x,y),$$

hence $d\tilde{u} = du$. Up to the correction of $f$ by a constant value in each connected component of $\Omega$, this result yields $\tilde{u} = u$. ■

A consequence of the converse theorem:

**Theorem – Harmonic functions are smooth (of class $C^\infty$).**

**Proof.** Every harmonic function is locally the real part of a holomorphic function. The same method that we used to prove that such a function is twice continuously differentiable can actually be used to prove that it is of class $C^\infty$.

**Theorem – The Maximum Principle.** Let $\Omega$ be an open connected subset of $\mathbb{C}$ and $u : \Omega \to \mathbb{R}$ be an harmonic function. If $u$ has a maximum or a minimum on $\Omega$, then $u$ is constant.

**Proof.** We use the maximum principle that holds for the modulus of holomorphic functions, first in a special case. Assume that $u$ has a maximum on $\Omega$ and additionally that $\Omega$ is simply connected. There is a holomorphic function $f$ on $\Omega$ such that $\operatorname{Re} f = u$. The function $g = \exp f$ is holomorphic on $\Omega$ and $|g| = \exp \operatorname{Re}(f) = \exp u$, hence $|g|$ has a maximum in $\Omega$, therefore $g$ is a constant $\lambda \in \mathbb{C}^*$. Let $\mu$ be a complex number such that $e^\mu = \lambda$; the image of $\Omega$ by $f$ is necessarily a subset of $\{\mu\} + i2\pi\mathbb{Z}$. Since $f(\Omega)$ is the image of a connected set by a continuous function, it is connected and thus, it is a singleton. Finally, $f$ – and therefore $u$ – are constant.

We now consider the general case: we only assume that $u$ has a maximum on $\Omega$ at some point $z_0$. Let $z \in \Omega$; we can find some connected and simply connected open subset $V$ of $\Omega$ that contains $z_0$ and $z$([2]). Using the result obtained in the special case proves that the function $u$ is constant on $V$ and in particular $u(z) = u(z_0)$. As $z$ is arbitrary, that proves that $u$ is constant on $\Omega$.

If $u$ has a minimum instead of a maximum, we may apply the previous result to to the function $-u$ that is also harmonic and has a maximum. ■

[2]for example, consider a polyline $\gamma = [z_0 \to z_1 \to \cdots \to z_{n-1} \to z_n]$ of $\Omega$ that joins $z_0$ and $z_n = z$; we may assume that it is simple (that the function $\gamma$ is injective). If $r > 0$ is small enough, the open connected set $V = \gamma([0,1]) + D(0,r)$ is included in $\Omega$, connected and simply connected.

## The Dirichlet Problem

We will prove soon that the Dirichlet problem has a unique solution; what we can already prove is the:

**Lemma – Uniqueness of the Solution.** There is at most one classic solution $u$ to the Dirichlet problem with continuous boundary condition $f$.

**Proof.** The Dirichlet problem is linear, hence we only need to prove that if $u = 0$ on the boundary of $D$, then $u = 0$ in $D$. The function $u$ is continous on $\overline{D}$, therefore it has a maximum and a minimum. By the maximum principle for harmonic functions, if the maximum or the minimum is attained in $D$, then $u$ is constant in $D$, and by continuity on the boundary, $u = 0$ on $D$. Otherwise, the minimum and maximum of $u$ are both attained on $\partial D$ and therefore they are both 0; we also conclude in this case that $u = 0$ on $D$. ■

### Harmonic/Fourier Analysis

**Lemma – Elementary Solutions.** Let $n \in \mathbb{N}$. If $f(e^{i\theta}) = \cos(n\theta + \phi)$, then

$$u(re^{i\theta}) = r^n \cos(n\theta + \phi)$$

solves the Dirichlet problem with boundary condition $f$.

**Proof.** The function $u$ is continuous and its restriction on $\partial D$ is $f$. Moreover, if $|z| < 1$, $u(z) = \mathrm{Re}(e^{i\phi} z^n)$; as the function $z \mapsto e^{i\phi} z^n$ is holomorphic in $D$, the function $u$ is harmonic in $D$. ■

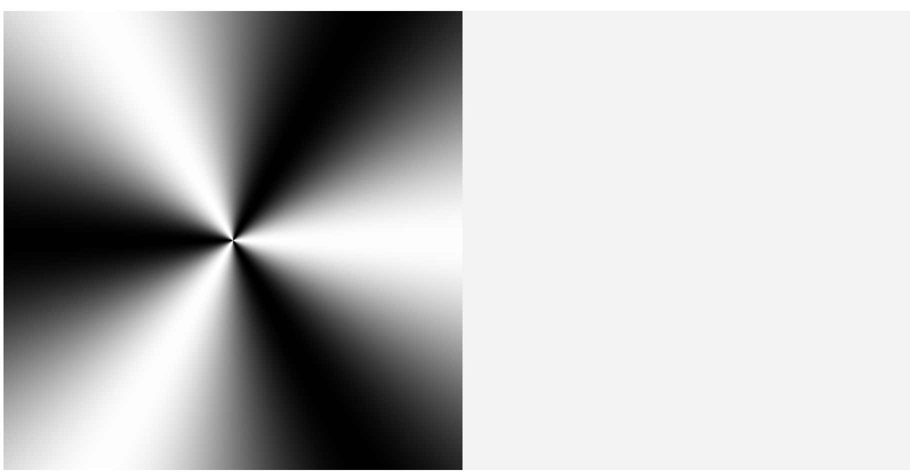

Figure 12.6: On the left, a greyscale representation of the function $re^{i\theta} \mapsto \cos 3\theta$. On the right, the uniform image used as a source for the Poisson editing.

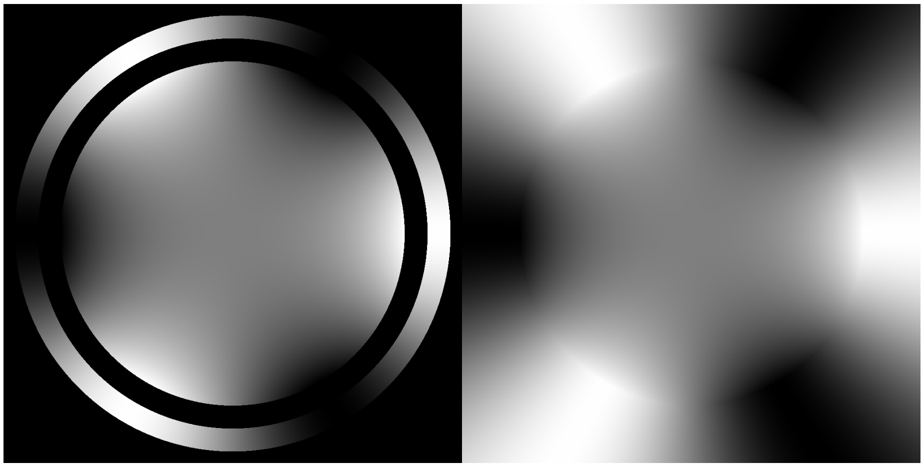

Figure 12.7: On the left, in the circle, (boundary) values extracted from the original image; in the disk, the solution ($re^{i\theta} \mapsto r^3 \cos 3\theta$) to the corresponding Dirichlet problem. On the right, the original image edited with this solution.

This result can be readily extended to finite trigonometric series: if for some finite sequence of real-valued coefficients $(a_n)$ and $(\phi_n)$, $f : \partial D \to \mathbb{R}$ can be decomposed as

$$f(\theta) = \sum_n a_n \cos(n\theta + \phi_n)$$

then by linearity, the function

$$u(re^{i\theta}) = \sum_n a_n r^n \cos(n\theta + \phi_n)$$

is a solution of the corresponding Dirichlet problem. Now, if $f^*$ is merely continuous but can be uniformly approximated to the precision $\epsilon$ by the finite trigonometric series $f$:

$$\sup_\theta |f^*(e^{i\theta}) - f(e^{i\theta})| \leq \epsilon$$

and if $u^*$ is a solution to the Dirichlet problem with boundary condition $f^*$, then by linearity, $u^* - u$ is a solution to the Dirichlet problem with boundary condition $f^* - f$; the maximum principle then provides

$$\sup_{|z| \leq 1} |u^*(z) - u(z)| \leq \epsilon.$$

Now, Fejér's theorem ensures that the approximation of $f^*$ by finite trigonometric series can be achieved with an arbitrary small precision $\epsilon > 0$. Hence, we can build an abritrarily precise uniform approximation of the solution to the Dirichlet problem.

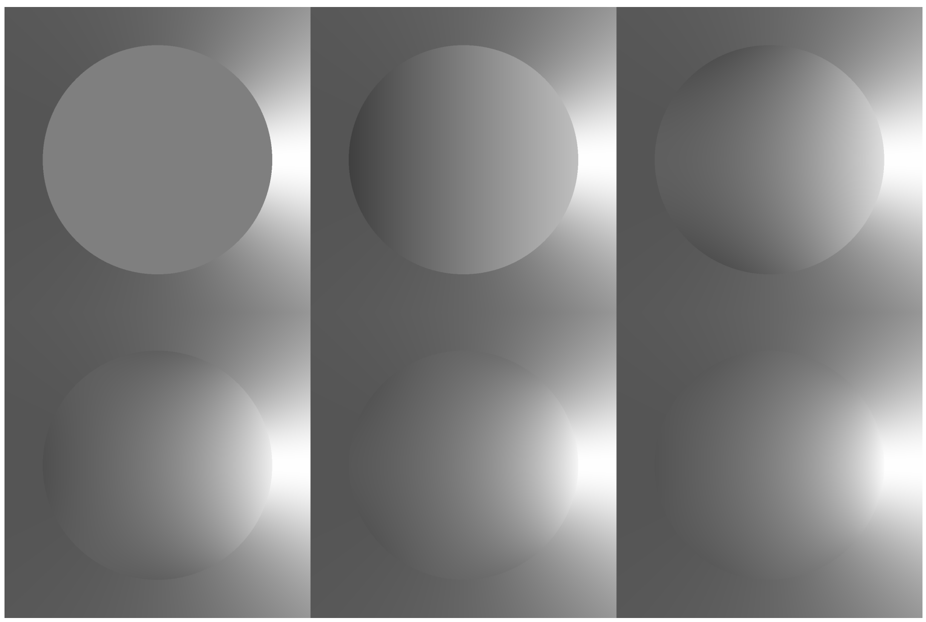

Figure 12.8: Approximations of the solution of the Dirichlet problem with boundary condition $f(e^{i\theta}) = 0.25 + 0.75/(5 - 4\cos\theta)$ based on Fourier series expansions of length 1 to 6. Refer to the appendix for details.

## The Poisson Kernel

The main result of this section:

**Theorem – Solution of the Dirichlet Problem.** The Dirichlet problem with a continuous boundary condition $f$ has a unique classic solution $u$, given in the unit disk by the Poisson integral

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2} f(e^{i\alpha})\, d\alpha.$$

Let's start with some definitions:

**Definition – Poisson Kernel.** The Poisson kernel is the function $P$ defined in the unit disk by

$$P(re^{i\theta}) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

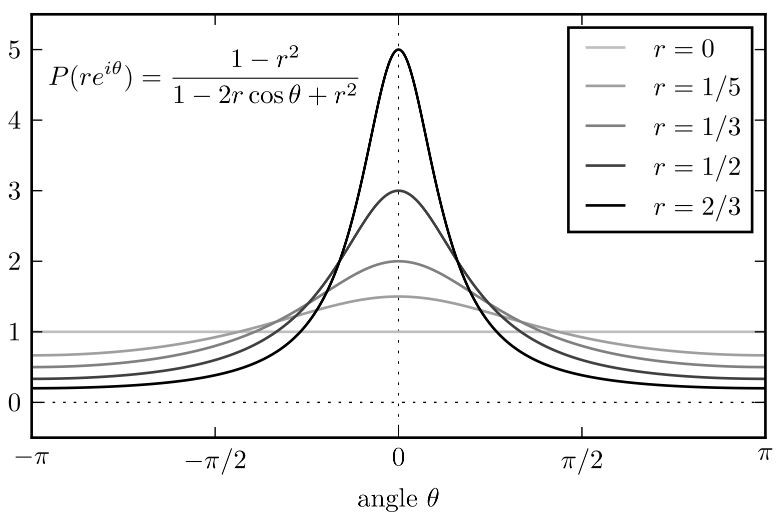


Figure 12.9: Poisson Kernel – Graphs of $[\theta \mapsto P(re^{i\theta})]$

**Definition – Poisson Integral Operator.** The Poisson integral operator $\mathcal{P}$ maps a continuous function $f$ defined on the boundary of the unit disk to the the function $\mathcal{P}[f]$ defined inside the unit disk by

$$\mathcal{P}[f](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-\alpha)}) f(e^{i\alpha})\, d\alpha.$$

Accordingly, the main result of this section may be restated as:

**Theorem – Solution of the Dirichlet Problem.** Let $f : \partial D \to \mathbb{R}$ be continuous. The function $u : \overline{D} \mapsto \mathbb{R}$ defined as $u|_D = \mathcal{P}[f]$ and $u|_{\partial D} = f$ is the unique classic solution of the Dirichlet problem with boundary condition $f$.

The proof of this fundamental result requires several lemmas.

**Lemma – Poisson Kernel, Alternate Representations.** The Poisson kernel satisfies:

$$P(z) = \frac{1}{1-z} - \frac{1}{1-1/\overline{z}} = \mathrm{Re}\left[\frac{1+z}{1-z}\right].$$

Hence, it is harmonic.

**Proof.** Start with the formula of the first alternate representation and write

$$\frac{1}{1-z} - \frac{1}{1-1/\overline{z}} = \left[\frac{1}{1-z} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{\overline{z}}{\overline{z}-1}\right].$$

This expression can be simplified into

$$\frac{1}{2}\left[\frac{2}{1-z} - \frac{1-z}{1-z}\right] + \frac{1}{2}\left[\frac{\overline{z}-1}{\overline{z}-1} - \frac{2\overline{z}}{\overline{z}-1}\right] = \frac{1}{2}\left[\frac{1+z}{1-z} + \frac{1+\overline{z}}{1-\overline{z}}\right],$$

which is the second alternate representation. On the other hand, the identity

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\overline{z})}{(1-z)(1-\overline{z})}$$

leads to the equation

$$\mathrm{Re}\left[\frac{1+z}{1-z}\right] = \frac{1-|z|^2}{1-2\mathrm{Re}\, z + |z|^2}$$

which is equivalent to the original definition of the Poisson kernel. ■

**Lemma – Poisson Kernel, Fourier Series.** The Poisson kernel has the following (locally uniformly convergent) Fourier expansion

$$P(re^{i\theta}) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta}.$$

**Proof.** Rewrite the first alternate representation of the Poisson kernel as

$$P(z) = \frac{1}{1-z} - \frac{1}{1-1/\overline{z}} = \frac{1}{1-z} + \overline{z}\frac{1}{1-\overline{z}}.$$

The two terms of the right-hand side are sums of geometric series with ratio $z$ and $\overline{z}$ respectively and can be expanded accordingly. This process yields the Fourier expansion formula. Both power series are locally uniformly convergent, hence the Fourier expansion also has this property. ■

**Lemma – Poisson Integral, Harmonicity.** Let $f : \partial D \to \mathbb{R}$ be continuous. The function $\mathcal{P}[f]$ is harmonic in $D$.

**Proof.** The definition of the Poisson integral operator and the second alternate representation of the Poisson kernel yield

$$\mathcal{P}[f](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}}\right] f(e^{i\alpha})\, d\alpha = \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}} f(e^{i\alpha})\, d\alpha\right].$$

The function

$$\left[z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}} f(e^{i\alpha})\, d\alpha\right]$$

is holomorphic by differentiation under the integral sign, hence $\mathcal{P}[f]$ is harmonic as the real part of a holomorphic function. ■

**Lemma – Poisson Integral, Boundary Values.** For any continuous function $f : \partial D \to \mathbb{R}$, the function $u : \overline{D} \mapsto \mathbb{R}$ defined as

$$u|_D = \mathcal{P}[f] \;\text{ and }\; u|_{\partial D} = f$$

is continous on $\overline{D}$.

**Proof (see e.g. Rudin (1987)).** We denote $\overline{\mathcal{P}}[f]$ the function defined in $D$ as $\mathcal{P}[f]$ and on $\partial D$ as $f$. The formula that defines the Poisson kernel shows that it is positive everywhere in $D$. Moreover, the expansion of the Poisson kernel as a Fourier series yields that for any $r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta})\, d\theta = 1.$$

Given these two properties, it is clear that for any $f \in C^0(\partial D, \mathbb{R})$,

$$\sup_{\overline{D}} |\overline{\mathcal{P}}[f]| \leq \sup_{\partial D} |f|.$$

Let $(f_p)$ be a sequence of real-valued finite trigonometric sums that converges to $f$ uniformly (see section Harmonic/Fourier Analysis); for any $p \in \mathbb{N}$, we can write

$$f_p(e^{i\theta}) = \sum_m c_{mp} e^{in\theta},\; \overline{c_{mp}} = c_{(-m)p},$$

hence

$$\mathcal{P}[f_p](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\sum_n r^{|n|} e^{in(\theta-\alpha)}\right]\left[\sum_m c_{mp} e^{im\alpha}\right] d\alpha,$$

which yields

$$\mathcal{P}[f_p](re^{i\theta}) = \sum_n \sum_m r^{|n|} c_{np} e^{in\theta} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)\alpha}\, d\alpha\right] = \sum_m r^{|m|} c_{mp} e^{im\theta}.$$

Consequently, for any $p \in \mathbb{N}$, $\overline{\mathcal{P}}[f_p]$ belongs to $C^0(\overline{D}, \mathbb{R})$. Moreover,

$$\sup_{\overline{D}} |\overline{\mathcal{P}}[f] - \overline{\mathcal{P}}[f_p]| = \sup_{\overline{D}} |\overline{\mathcal{P}}[f - f_p]| \leq \sup_{\overline{D}} |f - f_p| \to 0 \text{ when } p \to +\infty,$$

hence $\overline{\mathcal{P}}[f]$ can be uniformly approximated by a sequence of functions that are continuous on $\overline{D}$, therefore it is continuous. ■

# Appendix

## Linear Gradient

What happens if you heal a region in the left of the image below with a source taken from the right of the image?

Figure 12.10: On the left, a linear gradient image; on the right, a uniform image.

## Analytic Boundary Condition

We search for approximate solutions to the Dirichlet problem associated with

$$f(e^{i\theta}) = \frac{1}{4}\left[1 + \frac{3}{5 - 4\cos\theta}\right].$$

1. Check that $f$ is defined and continuous in $\partial D$; compute its range.
2. Find a holomorphic function, defined in $\mathbb{C}^*$, that extends $f$ and show that it is unique; we also denote $f$ this extension.

3. Show that
$$f(z) = \frac{1}{2}\left[\frac{1}{2-z} + \frac{1}{2-z^{-1}}\right];$$
determine the Laurent series expansion of $f$ in a neighbourghood of $\partial D$.

4. Find a finite trigonometric series that approximates $f$ with the precision $\epsilon = 10^{-2}$; provide an approximate solution of the Dirichlet problem associated to $f$ with the same precision.

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# Chapter 13

# Discrete-Time Signals in the Frequency Domain

## Introduction

Discrete-time signals are obtained by the sampling of continous-time signals – real-valued functions of a real-valued time – at a constant rate (see e.g. Strang (2000)). The analysis of many of their properties are simpler to carry out "in the frequency domain", once a Fourier transform has been applied to the original data (also called representation in the signal "in the time domain").

In the classical setting, the Fourier transform generates *functions* of the frequency. However, the signals with arguably the simplest frequency content, sinusoidal signals, then cannot be represented in the frequency domain. Hence, this theory should be considered a partial failure, or at least incomplete.

Extensions of the classical approach uses *generalized functions* of the frequency to represent discrete-time signals in the frequency domain. We introduce in this document a type of generalized functions called *hyperfunctions* (Sato 1959; Kaneko 1988), whose foundation is complex analysis, that fits perfectly the needs of discrete-time signal processing.

## Terminology & Notation

### Signals and Domains

We use in this document a convenient convention that is more popular among physicists than it is among mathematicians. In a given application domain –

for us, that is digital signal processing – when an object has several equivalent representations as functions, we use the same name for the object, and distinguish the representations by (a superset of the) domain of definition of the function. To every such domain we also associate fixed variable names.

In this document, we are dealing with discrete-time signals with a sample period $\Delta t$ (or sample rate $\Delta f = 1/\Delta t$). A signal $x$ is is represented *in the time domain* as a function $x(t)$ where $t \in \mathbb{Z}\Delta t$, *in the frequency (or Fourier) domain* as the function $x(f)$ where $f \in \mathbb{R}/\Delta f$, and *in the complex (or Laplace) domain* as a function $x(z)$ where $z \in \mathbb{C}$. We often implicitly favor some representation and refer for example to $x(t)$ as "the signal" instead of "the representation of the signal $x$ in the time domain".

If $t$ is a free variable, $x(t)$ denotes a function of the time $t$, if it is bound to some value, the value of the function. If there is some ambiguity in the choice of the representation, we use an assignment syntax, for example $x(t = 0)$ instead of $x(0)$, because it could be mistaken as $x(f = 0)$.

## Sets and Functions

The set of functions from $A$ to $B$ is denoted $A \to B$.

The set $\mathbb{C}$ is the complex plane; $\mathbb{U}$ refers to the unit circle centered on the origin:

$$\mathbb{U} = \{z \in \mathbb{C}\, ,\, |z| = 1\}.$$

The symbol $\partial\mathbb{U}$ denotes the boundary of $\mathbb{U}$. Its positively oriented boundary, the closed path $t \in [0, 1] \mapsto e^{i2\pi t}$, is denoted $[\circlearrowleft]$.

For any $r > 0$, $D_r$ is the open disk with radius $r$ centered on the origin:

$$D_r = \{z \in \mathbb{C}\, ,\, |z| < r\}$$

and for any $r \in [0, 1[$ , $A_r$ is the open annulus with internal radius $r$ and external radius $1/r$:

$$A_r = \{z \in \mathbb{C}\, ,\, r < |z| < 1/r\}.$$

## Iverson Bracket

We[1] denote $\{\,\cdot\,\}$ the function defined by:

$$\{b\} = \left|\begin{array}{ll} 1 & \text{if } b \text{ is true,} \\ 0 & \text{if } b \text{ is false.} \end{array}\right.$$

[1]Actually, Kenneth Iverson originally used the syntax $(\,\cdot\,)$ while Donald Knuth prefers $[\,\cdot\,]$ (see Knuth 1992).

This elementary notation supercedes many other ones. For example, we can use $\{x \in A\}$ instead of $\chi_A(x)$ to denote the characteristic function of the set $A$, $\{i = j\}$ instead of $\delta_{ij}$ to denote the Kronecker delta, $\{t \geq 0\}$ instead of $H(t)$ to denote the Heaviside function.

# Finite Signals

**Definition – Signal (Time Domain), Sampling Period/Rate** A *discrete-time signal* $x(t)$ is a real or complex-valued function defined on $\mathbb{Z}\Delta t$ for some $\Delta t > 0$, the signal *sampling period* (or *sampling time*); the number $\Delta f = 1/\Delta t$ is the signal *sampling rate* (or *sampling frequency*).

In the sequel, all signals are discrete-time, hence we often drop this qualifier. Also, in this introductory section, although many definitions and results are valid in a more general setting, for the sake of simplicity, we always assume that signals are finite:

**Definition – Finite Signal.** A discrete-time signal $x(t)$ is *of finite support* – or simply *finite* – if $x(t) = 0$ except for a finite set of times $t$.

## Fourier Transform

**Definition – Signal in the Frequency Domain, Fourier Transform.** A signal $x(t)$ is represented in the frequency domain as $x(f)$, the *(discrete-time) Fourier transform* of $x(t)$, defined for $f \in \mathbb{R}$ by:

$$x(f) = \Delta t \sum_{t \in \mathbb{Z}\Delta t} x(t) e^{-i2\pi f t}.$$

**Remark – Frequency Domain.** Note that $x(f)$ is $\Delta f$-periodic. Indeed, for any $f \in \mathbb{R}$ and $t = n\Delta t$ with $n \in \mathbb{Z}$,

$$e^{-i2\pi(f+\Delta f)t} = e^{-i2\pi f t}(e^{-i2\pi})^n = e^{-i2\pi f t}$$

and therefore

$$x(f + \Delta f) = \Delta t \sum_{t \in \mathbb{Z}\Delta t} x(t) e^{-i2\pi(f+\Delta f)t} = x(f).$$

As $x(f)$ does not really depend directly of the value of $f \in \mathbb{R}$, but only on the value of $f \in \mathbb{R}$ *modulo some multiple of* $\Delta f$, we may alternatively define $x(f)$ as a function defined on the frequency domain $\mathbb{R}/\Delta f$, and totally forget about the periodicity, because it is now captured by the domain definition. An alternate – arguably less contrived – way to deal with the periodicity is to consider only the values of $x(f)$ on one period, for example in the interval $[-\Delta f/2, \Delta f/2[$.

**Remark – Fourier Transform of Continuous-Time Signals.** The discrete-time Fourier transform formula is similar to the continous-time Fourier transform formula

$$x(f) = \int_{-\infty}^{+\infty} x(t) e^{-i2\pi f t} \, dt.$$

Actually, if $x(t)$ if defined for every $t \in \mathbb{R}$ and not only $t \in \mathbb{Z}\Delta t$ – if our discrete-time signal samples the continuous-time signal $x(t)$ – the discrete-time Fourier transform is the continuous one with the integral replaced by its Riemann sum. In many respects, the operator $\Delta t \sum_{t \in \mathbb{Z}\Delta t}$ plays the same role for discrete-time signals than the integral with respect to the time $t$ plays for continous-time signal.

**Theorem – Inversion Formula.** If $x(t)$ is a finite signal, represented in the frequency domain as $x(f)$, we have

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi f t} \, df.$$

**Remark – Continuous-Time Signals (Inverse Fourier Transform).** The Fourier inversion formula for discrete-time signals is also very similar to its counterpart for continuous-time signals, that is:

$$x(t) = \int_{-\infty}^{+\infty} x(f) e^{i2\pi f t} \, df.$$

Two differences are obvious: for continous-time time signals, the formula is meaningful for any $t \in \mathbb{R}$ while in discrete-time, it is only meaningful for $t \in \mathbb{Z}\Delta t$; for continous-time signals, the integral with respect to the frequency $f$ ranges over $\mathbb{R}$ while for discrete-time signals, it ranges over $[-\Delta f/2, \Delta f/2[$. Unlike continous-time signals, the information contained in discrete-time signals is structurally contained in a bounded frequency band of width $\Delta f$.

**Proof – Inversion Formula.** The right-hand side of the Fourier inversion formula is equal to

$$\int_{-\Delta f/2}^{+\Delta f/2} \left[ \Delta t \sum_{\tau \in \mathbb{Z}\Delta t} x(t = \tau) e^{-i2\pi f \tau} \right] e^{i2\pi f t} \, df.$$

With the integral and sum symbols swapped, $t = n\Delta t$ and $\tau = m\Delta t$, we get

$$\sum_{m \in \mathbb{Z}} x(t = m\Delta t) \left[ \frac{1}{\Delta f} \int_{-\Delta f/2}^{+\Delta f/2} e^{i2\pi f (n-m)\Delta t} \, df \right].$$

A straightforward computation yields

$$\frac{1}{\Delta f} \int_{-\Delta f/2}^{+\Delta f/2} e^{i2\pi f (n-m)\Delta t} \, df = \{n = m\},$$

hence we have

$$\int_{-\Delta f/2}^{+\Delta f/2} x(f)e^{i2\pi f n\Delta t}\, df = \sum_{m\in\mathbb{Z}} x(t = m\Delta t) \times \{n = m\} = x(t = n\Delta t)$$

which is the desired result. ■

## $z$-Transform

There is yet another useful representation of a finite signal – this time as a function of a complex variable $z$ – and it is closely related to the frequency-domain representation.

**Definition – Signal in the Complex Domain, $z$-Transform.** A signal $x(t)$ is *represented in the complex domain* as $x(z)$, the *z-transform* of $x(t)$, defined for some $z \in \mathbb{C}$ by:

$$x(z) = \Delta t \sum_{t\in\mathbb{Z}\Delta t} x(t)z^{-t/\Delta t} = \Delta t \sum_{n\in\mathbb{Z}} x(t = n\Delta t)z^{-n}.$$

**Remark – $z$-Transform Domain for Finite Signals.** When $x(t)$ is finite, the $z$-transform $x(z)$ is defined for any $z \in \mathbb{C}^*$ ; it can be extended to $\mathbb{C}$ if $x(t) = 0$ when $t > 0$.

We have the straightforward, but nevertheless very useful:

**Theorem – $z$-Transform to Fourier Transform.** The frequency domain representation of a signal $x(f)$ is related to the complex domain representation $x(z)$ by:

$$x(f) = x(z = e^{i2\pi f\Delta t}).$$

**Example – Unit Impulse.** The unit impulse signal $\mathbf{1}$ is defined in the time domain as

$$\mathbf{1}(t) = (1/\Delta t) \times \{t = 0\}.$$

It is equal to zero outside $t = 0$ and satisfies

$$\Delta t \sum_{t\in\mathbb{Z}\Delta} \mathbf{1}(t) = 1.$$

## Convolution and Filters

**Definition – Convolution.** The *convolution* of the signals $x(t)$ and $y(t)$ is the signal $(x * y)(t)$ defined by:

$$(x * y)(t) = \Delta t \sum_{\tau\in\mathbb{Z}\Delta t} x(\tau)y(t - \tau).$$

**Theorem – Representation of the Convolution in the Frequency Domain.** For finite signals, we have

$$(x * y)(f) = x(f) \times y(f).$$

**Proof.** The definition of the convolution yields

$$(x * y)(f) = \Delta t \sum_{t \in \mathbb{Z}\Delta t} \left[ \Delta t \sum_{\tau \in \mathbb{Z}\Delta t} x(\tau) y(t-\tau) \right] e^{-i2\pi f t}.$$

We may write the exponential $e^{-i2\pi f t}$ as $e^{-i2\pi f \tau} \times e^{-i2\pi f(t-\tau)}$. Using the change of variable $t' = t - \tau$ then leads to

$$(x * y)(f) = \left[ \Delta t \sum_{\tau \in \mathbb{Z}\Delta t} x(\tau) e^{-i2\pi f \tau} \right] \times \left[ \Delta t \sum_{t' \in \mathbb{Z}\Delta t} y(t') e^{-i2\pi f t'} \right]$$

which is the desired result. ■

**Example – Unit Impulse.** For any finite signal $x$, the definition of convolution yields

$$(\mathbf{1} * x) = x(t) = (x * \mathbf{1})(t).$$

In other words, the signal $\mathbf{1}$ is a unit for the convolution. This also also clear from its frequency domain representation: indeed, we have $\mathbf{1}(z) = 1$ and $\mathbf{1}(f) = 1$, and therefore

$$(\mathbf{1} * x)(f) = \mathbf{1}(f) \times x(f) = x(f) = x(f) \times \mathbf{1}(f) = (x * \mathbf{1})(f).$$

**Definition – Filter, Impulse Response, Frequency Response, Transfer Function.** A *filter* is an operator mapping an input signal $x(t)$ to an output signal $y(t)$ related by the operation

$$y(t) = (h * u)(t)$$

where $h(t)$ is a signal called the filter *impulse response.* The filter *frequency response* is $h(f)$ and its *transfer function* is $h(z)$.

**Remark – Impulse Response.** The "impulse response" terminology is justified by the fact that if $u(t) = \mathbf{1}(t)$, then $y(t) = h(t)$: the impulse response is the filter output when the filter input is the unit impulse. For obvious reasons, the filters we have introduced so far are called finite impulse response (FIR) filters.

## Quickly Decreasing Signals

The assumption that $x(t)$ is finite simplifies the theory of frequency domain representation of signals, but it is also very restrictive. For example, in speech analysis, we routinely use auto-regressive filters; their impulse responses are are not finite, and yet their frequency representation is needed, for example to analyze the acoustic resonances of the vocal tract (or "formants").

Fortunately, the theory can be extended beyond finite signals. The extension is quite straightforward if $x(t)$ decrease quickly when $t \to \pm\infty$, where by "quickly decreasing" we mean that it has a sub-exponential decay:

**Definition – Quickly Decreasing Signal.** A signal $x(t)$ with sample period $\Delta t$ is *quickly decreasing* if

$$\exists\, \sigma > 0,\ \exists\, \kappa > 0,\ \forall\, t \in \mathbb{Z}\Delta t,\ |x(t)| \leq \kappa e^{-\sigma|t|}.$$

Given a quickly decreasing signal $x(t)$ in the time domain, as in the finite signal case, its representation in the frequency domain is

$$x(f) = \Delta t \sum_{t \in \mathbb{Z}\Delta t} x(t) e^{i2\pi f t}$$

and in the complex domain

$$x(z) = \Delta t \sum_{n \in \mathbb{Z}} x(t = n\Delta t) z^{-n}.$$

However, the sums are not finite anymore; we consider that the values of the functions $x(f)$ and $x(z)$ are well defined when the sums are absolutely summable.

**Theorem – Quickly Decreasing Signal.** Any quickly decreasing signal $x$ can be equivalently represented as:

1. a quickly decreasing function $x(t)$,
2. a holomorphic function $x(z)$ defined on some neighbourhood of $\mathbb{U}$,
3. a $\Delta f$-periodic and analytic function $x(f)$ on $\mathbb{R}$.

**Theorem – Inversion Formulas.** Let $x(t)$ be quickly decreasing signal and $x(z)$ be its representation in the complex domain, defined in the annulus $A_\rho$ for some $\rho \in [0, 1[$. For any $r > 0$ such that $r\partial\mathbb{U} \subset A_\rho$, we have

$$x(t = n\Delta t) = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{x(z)}{\Delta t} z^{n-1}\, dz.$$

As a special case, we have

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi f t}\, df.$$

**Proof.** If $x(t) \leq \kappa e^{-\sigma|t|}$ with $\kappa > 0$ and $\sigma > 0$, then

$$\limsup_{n \to +\infty} \sqrt[n]{|\Delta t \times x(t = -n\Delta t)|} \leq e^{-\sigma \Delta t},$$

hence $\Delta t \sum_{n\in -\mathbb{N}} x(t=n\Delta t) z^{-n}$ is defined and holomorphic in the disk $\{z \in \mathbb{C}\,,\, |z| < e^{\sigma \Delta t}\}$. Similarly,

$$\limsup_{n\to+\infty} \sqrt[n]{|\Delta t \times x(t=n\Delta t)|} \leq e^{-\sigma\Delta t},$$

hence $\Delta t \sum_{n\in\mathbb{N}^*} x(t=n\Delta t) z^{-n}$ is defined and holomorphic in the annulus $\{z\in\mathbb{C}\,,\, |z| > e^{-\sigma\Delta t}\}$. Finally $x(z) = \Delta t \sum_{n\in\mathbb{Z}} x(t=n\Delta t)z^{-n}$ is defined and holomorphic in $A_\rho$ with $\rho = e^{-\sigma\Delta t}$. The Cauchy formula for the computation of the coefficient of a Laurent series expansion yields for any $r \in \,]\rho, 1/\rho[$

$$\Delta t \times x(t=-n\Delta t) = \frac{1}{i2\pi}\int_{r[\circlearrowleft]} \frac{x(z)}{z^{n+1}}\, dz,$$

which is equivalent to the $z$-domain inversion formula. This formula yields

$$|x(t=n\Delta t)| \leq \kappa_r r^n$$

with

$$\kappa_r = \sup_{|z|=r} \frac{|x(z)|}{\Delta t}.$$

for any $r\in\,]\rho, 1/\rho[$ . Hence for any $r \in \,]\rho, 1[$

$$|x(t=n\Delta t)| \leq \kappa_r \exp\left(\frac{\log r}{\Delta t} n\Delta t\right),$$

and as $1/r \in \,]1, 1/\rho[$ , we also have

$$|x(t=n\Delta t)| \leq \kappa_{1/r} \exp\left(-\frac{\log r}{\Delta t} n\Delta t\right).$$

Consequently, for any $r\in\,]\rho,1[$ , $|x(t)| \leq \kappa e^{-\sigma|t|}$ with $\kappa = \max(\kappa_r, \kappa_{1/r})$ and $\sigma = -(\log r)/\Delta t$.

If $x(z)$ is holomorphic in a neighbourhood of the unit circle, then $x(f) = x(z = e^{i2\pi f \Delta t})$ is $\Delta f$-periodic and analytic. Conversely, if $x(f)$ analytic and $\Delta f$-periodic, it has a holomorphic extension, that we still denote $x(f)$, in some open neighbourhood $V$ of $\mathbb{R}$ in $\mathbb{C}$. We can always ensure that $x(f)$ is actually defined on as a tubular neighbourhood $V_\epsilon$ of $\mathbb{R}$ for some $\epsilon > 0$, where

$$V_\epsilon = \mathbb{R} + D_\epsilon = \{f \in \mathbb{C}\,,\, |\mathrm{Im}\, f| < \epsilon\}.$$

Indeed, we can select $\epsilon > 0$ such that $U_\epsilon = [-\Delta f/2, +\Delta f/2] + D_\epsilon$ is included in $V$, and define a new analytic extension $x'(f)$ on $V_\epsilon$ by $x'(f) = x(f - k\Delta f)$ where $k\in\mathbb{Z}$ is such that $f - k\Delta f \in U_\epsilon$ (by the isolated zeros theorem, this definition is unambigous). Let $\sigma = 2\pi\epsilon$; for any $z \in A_\rho$ with $\rho = e^{-\sigma\Delta t}$, $\phi(z) = x(f = (\log z)/i2\pi\Delta t)$ is independent of the determination of $\log z$ ; it is holomorphic and $x(f) = \phi(z = e^{i2\pi f\Delta t})$.

Finally, starting from

$$x(t = n\Delta t) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{x(z)}{\Delta t} z^{n-1}\, dz,$$

we may introduce the path $f \mapsto e^{i2\pi f \Delta t}$ for $f \in [-\Delta f/2, \Delta f/2]$, and we get

$$x(t = n\Delta t) = \frac{1}{i2\pi} \int_{-\Delta f/2}^{+\Delta f/2} \frac{x(z = e^{i2\pi f \Delta t})}{\Delta t} e^{i2\pi f (n-1)\Delta t} (i2\pi \Delta t e^{i2\pi f \Delta t}) df,$$

which after simplifications, yields for any $t \in \mathbb{Z}\Delta t$,

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi f t} df,$$

the expected result. ■

**Example – Auto-Regressive Filter.** The filter whose impulse response $h(t)$ is given by

$$h(t = n\Delta t) = (1/\Delta t) \times 2^{-n} \times \{n \geq 0\}$$

is an auto-regressive filter, ruled for finite inputs $u(t)$ by the dynamics

$$y(t) = 1/2 \times y(t - \Delta t) + u(t).$$

The transfer function $h(z)$ of this filter is

$$h(z) = \Delta t \sum_{n \in \mathbb{Z}} h(t = n\Delta t) z^{-n} = \sum_{n \in \mathbb{N}} (1/2z)^n.$$

This sum is absolutely convergent when $|1/2z| < 1$, that is $|z| > 1/2$, and

$$h(z) = \frac{1}{1 - 1/2z} = \frac{z}{z - 1/2}.$$

Consequently,

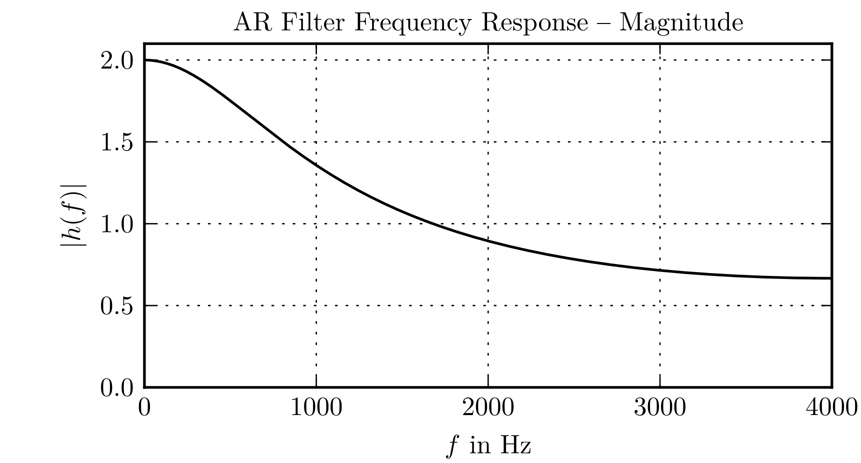
$$h(f) = \frac{e^{i2\pi f \Delta t}}{e^{i2\pi f \Delta t} - 1/2}.$$

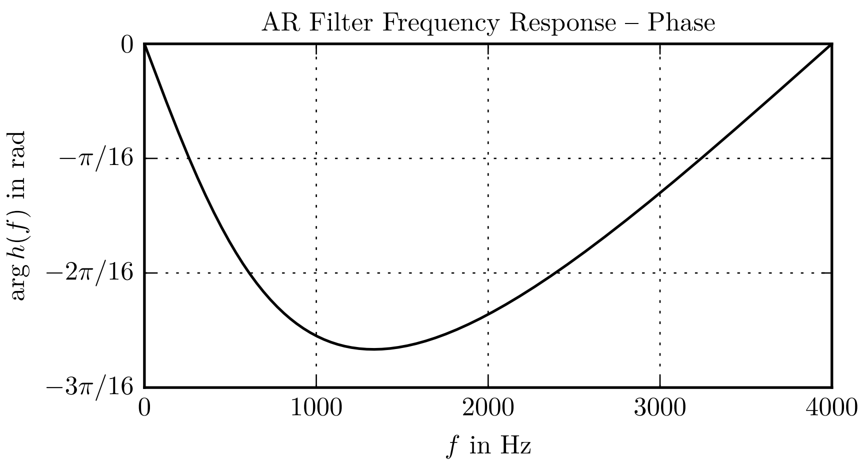
The modulus and argument of this complex-valued function are called the filter frequency response *magnitude* and *phase*. They are usually displayed on separate graphs. We know that $h(f)$ is $\Delta f$-periodic. Moreover, here $h(t)$ is real-valued, hence for any $f \in \mathbb{R}$, $h(-f) = \overline{h(f)}$. We can therefore plot the graphs for $f \in [0, +\Delta f/2]$ because all the information stored in the frequency response is available in this interval. The Python code below can be used to generate the graph data for $\Delta f = 8000$ Hz.

```
from numpy import *

df = 8000.0
```

```
dt = 1.0 / df
N = 1000
f = linspace(0.0, 0.5 * df, N)
z_f = exp(1j * 2 * pi * f * dt)
h_f = z_f / (z_f - 0.5)
```




## Slowly Increasing Signals

Once again the theory of representation of signals in the frequency domain can be extended, this time beyond quickly increasing signals. However, we will have to abandon the representation of $x(f)$ as a function, to adopt instead the representation of $x(f)$ as a *hyperfunction*.

The extension will be valid as long as $x(t)$ "increases slowly" when $t \to \pm\infty$, or more precisely, has an infra-exponential growth:

**Definition – Slowly Increasing Signal.** A signal $x(t)$ with sample period $\Delta t$ is *slowly increasing* if

$$\forall\, \sigma > 0,\ \exists\, \kappa > 0,\ \forall\, t \in \mathbb{Z}\Delta t,\ |x(t)| \leq \kappa e^{\sigma |t|}.$$

**Remark.** Quicky decreasing signals are obviously slowly increasing, but this class also include all bounded signals, and even all signals that are dominated by polynomials.

**Remark.** There is a way to get rid of the factor $\kappa$ in the definition of slowly increasing signal. Instead, we can check that the signal is *eventually* dominated by every increasing exponential function of $|t|$:

$$\forall\, \sigma > 0,\ \exists\, \tau \in \mathbb{N}\Delta t, \forall\, t \in \mathbb{Z}\Delta t,\ |t| > \tau \ \Rightarrow\ |x(t)| \leq e^{\sigma |t|}.$$

## Fourier Transform

**Definition – Abel-Poisson Windowing.** Let $r \in [0, 1[$. We denote $x_r(t)$ the signal derived from $x(t)$ by

$$x_r(t) = r^{|t/\Delta t|} x(t),$$

the application of the Abel-Poisson window $r^{|t/\Delta t|}$ to the original signal $x(t)$.

**Remark.** The family of signals $x_r(t)$ indexed by $r$, approximates $x(t)$: for any $t \in \mathbb{Z}\Delta t$, $x_r(t) \to x(t)$ when $r \uparrow 1$.

**Remark.** If $x(t)$ is only known to be slowly increasing, we cannot define its Fourier transform classically. However, for any $r \in [0, 1[$, the signal $x_r(t)$ is quickly decreasing and we may therefore compute its Fourier transform $x_r(f)$; we then leverage this property to define the Fourier transform $x(f)$ of $x(t)$ as the family of functions $x_r(f)$ indexed by $r$:

**Definition – Signal in the Frequency Domain, Fourier Transform.** The representation $x(f)$ in the frequency domain of a slowly increasing signal $x(t)$ is the $\Delta f$-periodic function with values in $[0, 1[ \to \mathbb{C}$ defined by:

$$x(f) = r \in [0, 1[ \mapsto x_r(f) \in \mathbb{C}.$$

The periodic hyperfunctions are then simply defined as the images of slowly increasing signals by the Fourier transform:

**Definition – Periodic Hyperfunction.** A *$\Delta f$-periodic hyperfunction* is a function

$$\phi : \mathbb{R} \to [0, 1[ \to \mathbb{C}$$

such that there is a slowly increasing signal $x(t)$ with sample rate $\Delta f$ satisfying

$$\phi(f)(r) = x_r(f).$$

**Remark – Multiple Representations in the Frequency Domain.** A signal $x(t)$ that is quickly decreasing is also slowly increasing; therefore it has two distincts representations in the frequency domain: a periodic function

$$f \in \mathbb{R} \mapsto x(f) \in \mathbb{C},$$

and a periodic hyperfunction

$$f \in \mathbb{R} \mapsto x(f) \in [0, 1[ \to \mathbb{C}.$$

Here, the Fourier-transform-as-a-function $x(f)$ is the uniform limit of the Fourier-transform-as-a-hyperfunction $x_r(f)$ when $r \uparrow 1$, hence we can easily recover the function representation of $x(f)$ from its hyperfunction representation.

**Remark – Hyperfunctions as Limits.** Is $x(f)$ the limit of $x_r(f)$ when $r \uparrow 1$? The short answer is "yes", but only when the question is framed appropriately, and we still lack of few tools to do it now. At this stage, it is probably more fruitful to think of $x(f)$ as the approximation process $r \mapsto x_r(f)$ itself than of its limit[2].

**Example – Fourier Transform of a Constant Signal.** Let $x(t) = 1$ for every $t \in \mathbb{Z}\Delta t$. This signal is not quickly decreasing, but it is slowly increasing, hence we may compute its Fourier transform as a periodic hyperfunction. By definition, $x_r(t) = r^{|t/\Delta t|} x(t) = r^{|t/\Delta t|}$, hence

$$x_r(f) = \Delta t \sum_{n \in \mathbb{Z}} r^{|n|} e^{-i2\pi f n \Delta t}.$$

We may split the sum in two:

$$x_r(f) = \Delta t \sum_{n \leq 0} (re^{i2\pi f \Delta t})^{-n} + \Delta t \sum_{n > 0} (re^{-i2\pi f \Delta t})^n.$$

Both terms in the right-hand side are sums of geometric series, which yields

$$\Delta t \sum_{n \leq 0} (re^{i2\pi f \Delta t})^{-n} = \frac{\Delta t}{1 - re^{i2\pi f \Delta t}},$$

$$\Delta t \sum_{n > 0} (re^{-i2\pi f \Delta t})^n = \frac{\Delta t \times re^{-i2\pi f \Delta t}}{1 - re^{-i2\pi f \Delta t}} = -\frac{\Delta t}{1 - r^{-1} e^{i2\pi f \Delta t}}.$$

[2] A similar situation happens in the construction of real numbers, at the stage where the rational numbers are available, but not yet the real numbers. You can then think of "$\pi$" as the sequence of decimal approximations 3, 31/10, 314/100, etc., but the question "Is $\pi$ the limit of this sequence?" is meaningless. It only starts to make sense when you have constructed the set of real numbers, embedded the rational numbers in it and defined a topology on the real numbers. Then, finally, the answer is "yes"!

Hence, if we define

$$x_\pm(z) = \frac{\Delta t}{1-z},$$

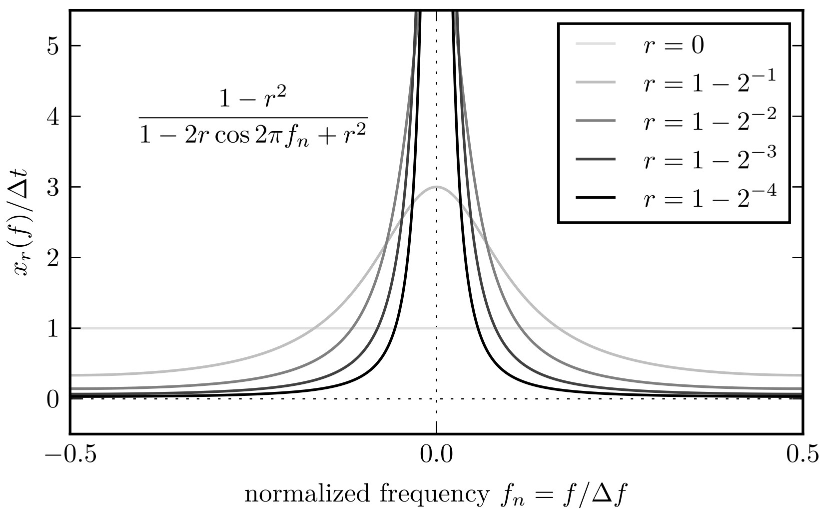
we can write $x_r(f)$ as

$$x_r(f) = x_\pm(z = re^{i2\pi f\Delta t}) - x_\pm(z = r^{-1}e^{i2\pi f\Delta t}).$$

We may compute another useful expression of $x_r(f)$

$$x_r(f) = \frac{\Delta t}{1 - re^{i2\pi f\Delta t}} - \frac{\Delta t \times re^{-i2\pi f\Delta t}}{1 - re^{-i2\pi f\Delta t}} = \frac{\Delta t(1-r^2)}{1 - 2r\cos 2\pi f\Delta t + r^2}$$

The representation of the functions $x_r(f)$ for several values of $r$ clearly demonstrates how the energy of the signals concentrates around $f = 0$ when $r \uparrow 1$.




## Standard Defining Function

The example of Fourier transform $x(f)$ computed in the previous section exhibited a very specific structure that is actually shared by all periodic hyperfunctions:

**Theorem & Definition – Standard Defining Function.** For every slowly increasing signal $x(t)$, there is a unique function $x_\pm(z)$ – called *standard defining function* of $x(f)$ – holomorphic in $\mathbb{C} \setminus \mathbb{U}$, with $x_\pm(z = \infty) = \lim_{|z| \to +\infty} x(z) = 0$, such that for any $r \in [0, 1[$:

$$x_r(f) = x_\pm(re^{i2\pi f\Delta t}) - x_\pm(r^{-1}e^{i2\pi f\Delta t}).$$

This function is defined by:

$$x_\pm(z) = \left| \begin{array}{ll} x_+(z) = +\Delta t \sum_{n \leq 0} x(t = n\Delta t) z^{-n} & \text{if } |z| < 1, \\ x_-(z) = -\Delta t \sum_{n > 0} x(t = n\Delta t) z^{-n} & \text{if } |z| > 1, \end{array} \right.$$

**Proof – Standard Defining Function.** The definition $x_r(t) = r^{|t/\Delta t|}x(t)$ yields

$$x_r(f) = \Delta t \sum_{n \in \mathbb{Z}} r^{|n|} x(t = n\Delta t) e^{-i2\pi f n \Delta t}.$$

We split the right-hand side in two sums:

$$\begin{aligned} x_r(f) = \;& \Delta t \sum_{n \leq 0} x(t = n\Delta t)(re^{i2\pi f \Delta t})^{-n} \\ & + \Delta t \sum_{n > 0} x(t = n\Delta t)(r^{-1}e^{i2\pi f \Delta t})^{-n} \end{aligned}$$

To prove the expansion of $x_\pm$ properly defines a (holomorphic) function on $\mathbb{C} \setminus \mathbb{U}$, we have to demonstrate that the power series in the right-hand side of this definition are absolutely convergent on the suitable domains. We only do it for the first expansion (the method for the second one is similar). Let $|z| < 1$; as $x(t)$ is slowly increasing, for any $\sigma > 0$, there is a $\kappa > 0$ such that for any $t \leq 0$, $|x(t)| \leq \kappa e^{\sigma |t|} = \kappa e^{-\sigma t}$. Hence, for any nonnegative integer $n$, we have $|x(t = n\Delta t) z^{-n}| \leq \kappa |e^{\sigma \Delta t} z|^{-n}$. We may select a $\sigma > 0$ such that $|e^{\sigma \Delta t} z| = r < 1$. The general term of the series is then dominated by $\Delta t \cdot \kappa r^{-n}$ which establishes the absolute convergence. Note that as only negative powers of $z$ are used in the expansion of $x_-(z)$, we have $x_\pm(\infty) = 0$.

If we set $z = re^{i2\pi \Delta t}$, we can write $x_r(f) = x_\pm(z) - x_\pm(1/\overline{z})$. Hence, if two functions $\phi$ and $\psi$ were suitable standard defining functions for $x(f)$, for any $z$ such that $|z| < 1$, we would have $\phi(z) - \phi(1/\overline{z}) = \psi(z) - \psi(1/\overline{z})$, or $\phi(z) - \psi(z) = \psi(1/\overline{z}) - \phi(1/\overline{z})$. The right-hand side $\chi(z)$ of this equation is antiholomorphic: $\overline{\chi}$ is holomorphic, hence $\phi - \psi$ is holomorphic and anti-holomorphic at the same time on the open unit disk, therefore it is constant. As $\phi(0) - \psi(0) = \psi(\infty) - \phi(\infty) = 0$, $\phi$ and $\psi$ are identical on the open unit disk and consequently on $\mathbb{C} \setminus \mathbb{U}$. ■

**Theorem – Inversion Formula.** Any holomorphic function defined on $\mathbb{C} \setminus \mathbb{U}$ and equal to 0 at $z = \infty$ is the standard defining function $x_\pm(z)$ of a unique slowly increasing signal $x(t)$, defined for any $r \in \,]0, 1[$ by

$$x(t = n\Delta t) = \frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{x_\pm(z)}{\Delta t} z^{n-1} \, dz.$$

**Proof – Inversion Formula.** As $x_\pm(z) = x_+(z) = \Delta t \sum_{n \leq 0} x(t = n\Delta t) z^{-n}$ inside the unit circle, when $n \leq 0$, $\Delta t \times x(t = n\Delta t)$ is the $(-n)$-th coefficient of the Taylor expansion of $x_+(z)$. Hence, when $t \leq 0$, for all $r \in \,]0, 1[$,

$$\Delta t \times x(t = n\Delta t) = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} x_\pm(z) z^{n-1} \, dz.$$

Outside the unit circle, the line integral

$$\int_{r^{-1}[\circlearrowleft]} x_\pm(z) z^{n-1} \, dz$$

is independent of $r \in \,]0,1[\,$, equal to its limit value when $r \uparrow 1$. When $n \leq 0$, as $x_{\pm}(z=\infty)=0$, $|z \times x_{\pm}(z)z^{n-1}| \to 0$ when $|z| \to \infty$. Consequently, by Jordan's lemma, this line integral is zero and the inversion formula holds for $t \leq 0$.

If $t = n\Delta t > 0$, $x_{\pm}(z)z^{n-1} = x_{+}(z)z^{n-1}$ inside the unit circle and has a Taylor series expansion. Hence it is holomorphic in the unit disk and for any $r \in \,]0,1[\,$,

$$\int_{r[\circlearrowleft]} x_{\pm}(z)z^{n-1}\,dz = 0.$$

Outside the unit circle $x_{\pm}(z) = x_{-}(z) = -\Delta t \sum_{n>0} x(t=n\Delta t)z^{-n}$. The $n$-th coefficient of this Laurent series expansion is $\Delta t \times x(t=n\Delta t)$, hence for any $r \in \,]0,1[\,$,

$$\Delta t \times x(t=n\Delta t) = \frac{1}{i2\pi}\int_{r^{-1}[\circlearrowleft]} x_{\pm}(z)z^{n-1}\,dz,$$

and the inversion formula holds for $t < 0$. ■

**Example – Inversion Formula.** Consider the signal whose Fourier transform has for standard defining function

$$x_{\pm}(z) = \frac{\Delta t}{1-z}.$$

The inversion formula provides

$$x(t=n\Delta t) = \frac{1}{i2\pi}\left[\int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]}\right]\frac{z^{n-1}}{1-z}\,dz.$$

The right-hand side is a line integral over the sequence of paths $\gamma$ made of $r[\circlearrowleft]$ (oriented counter-clockwise) and $r^{-1}[\circlearrowleft]$ (oriented clockwise). We have $\operatorname{ind}(\gamma,0)=0$ and $\operatorname{ind}(\gamma,1)=-1$, hence the residues theorem yields

$$x(t=n\Delta t) = \frac{1}{i2\pi}\int_{\gamma}\frac{z^{n-1}}{1-z}\,dz = -\operatorname{res}\left(\frac{z^{n-1}}{1-z}, z=1\right) = 1.$$

## Non-Standard Defining Functions

**Definition – Defining Function.** Let $x(f)$ be a $\Delta f$-periodic hyperfunction with standard defining function $x_{\pm}(z)$. A holomorphic function $\phi(z)$ defined on $V \setminus \mathbb{U}$ where $V$ is an open neighbourghood of $\mathbb{U}$ is a *defining function* of $x(f)$ if $\phi(z) - x_{\pm}(z)$ has an holomorphic extension to $V$. In the sequel, unless we use the "standard" qualifier, the notation $x_{\pm}(z)$ will be used to denote any of the defining function of a signal $x(t)$.

**Theorem – Inversion Formula.** Any holomorphic function defined on $A_{\rho} \setminus \mathbb{U}$ for some $\rho \in [0,1[$ is a defining function $x_{\pm}(z)$ of a unique slowly increasing signal $x(t)$, defined for any $r \in \,]\rho, 1[$ by

$$x(t) = \frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{x_\pm(z)}{\Delta t} z^{n-1} \, dz.$$

**Remark.** The domain of definition of a defining function $x_\pm(z)$ always contains a subset $A_\rho \setminus \mathbb{U}$ for some $\rho \in [0, 1[$ . As this restriction conveys enough information to described the signal $x(t)$, it is harmless and the assumption made in the theorem that the defining function is actually defined on such set is not overly restrictive.

**Proof – Inversion Formula.** The inversion formula holds for the standard defining function. Hence, if $x_\pm(z)$ is a defining function defined on $A_\rho$ for some $\rho \in [0, 1[$ and $\phi(z)$ denotes the difference between it and the standard one, the inversion formula is also valid for $x_\pm(z)$ if for any $r \in \,]\rho, 1[$ , we have

$$\frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{\phi(z)}{\Delta t} z^{n-1} \, dz = 0.$$

As $\phi(z)$ can be extended analytically to $A_\rho$, by the Cauchy integral theorem, this equality holds. ■

**Example – Quickly Decreasing Signals.** If $x(t)$ is a quickly decreasing signal, its $z$-transform $x(z)$ is defined in some open neighbourhood of $\mathbb{U}$ by

$$x(z) = \Delta t \sum_{n \in \mathbb{Z}} x(t = n\Delta t) z^{-n}$$

(see section Quickly Decreasing Signals); on the other hand its standard defining function $x_\pm(z)$ is given by

$$x_\pm(z) = \left| \begin{array}{ll} x_+(z) = +\Delta t \sum_{n \leq 0} x(t = n\Delta t) z^{-n} & \text{if } |z| < 1, \\ x_-(z) = -\Delta t \sum_{n > 0} x(t = n\Delta t) z^{-n} & \text{if } |z| > 1. \end{array} \right.$$

As $x(z)$ is defined and holomorphic on some open neighbourhood of $\mathbb{U}$, $x_-(z)$ and $x_+(z)$ can be extended as holomorphic functions to such a domain ; if we still denote $x_-(z)$ and $x_+(z)$ these extensions, we can write $x(z) = x_+(z) - x_-(z)$. Hence, the difference between

$$x_\pm(z) = +x(z) \times \{|z| < 1\},$$

and the standard defining function has an analytic extension – that is $x_-(z)$ – in a neighbourhood of $\mathbb{U}$ and $x_\pm(z)$ qualifies as a defining function. The function

$$x_\pm(z) = -x(z) \times \{|z| > 1\},$$

for similar reasons, also does.

## Ordinary Functions as Hyperfunctions

We still need to make our frequency-domain representations as hyperfunctions consistent with the classical framework. If a signal has a classical frequency-domain representation, as a complex-valued, locally integrable, $\Delta f$-periodic function $x(f)$ – or "ordinary function" representation – what is its frequency-domain representation as a hyperfunction?

The answer is – at least conceptually – pretty straightforward: if $x(f)$ is an ordinary function, the classic time-domain representation of $x(t)$ is given by

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi ft} \, df.$$

In particular, $x(t)$ is a bounded signal, hence it is slowly increasing signal, and we may define its frequency-domain representation as a hyperfunction: this is the representation of $x(f)$ as a hyperfunction.

**Theorem – Hyperfunction Representation of an Ordinary Function.** If $x(f)$ is an ordinary function, the standard defining function $x_\pm(z)$ of its representation as a hyperfunction is defined by:

$$x_\pm(z) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) \frac{\Delta t}{1 - ze^{-i2\pi f\Delta t}} \, df.$$

**Proof.** According to our construction of the representation of $x(f)$ as a hyperfunction, for any complex number $z$ such that $|z| < 1$, we have

$$x_+(z) = \Delta t \sum_{n \leq 0} \left[ \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi fn\Delta t} \, df \right] z^{-n}.$$

The series

$$\Delta t \sum_{n \leq 0} x(f) e^{i2\pi fn\Delta t} z^{-n} = x(f) \Delta t \sum_{m \geq 0} (ze^{-i2\pi fm\Delta t})^m$$

converges as a locally integrable function of $f$ to $x(f)\Delta t/(1 - ze^{-i2\pi f\Delta t})$, hence the formula for $x_\pm(z)$ of the theorem holds for $|z| < 1$.

If the complex number $z$ satisfies $|z| > 1$, we have

$$x_-(z) = -\Delta t \sum_{n > 0} \left[ \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi fn\Delta t} \, df \right] z^{-n},$$

and the series

$$-\Delta t \sum_{n > 0} x(f) e^{i2\pi fn\Delta t} z^{-n} = -x(f) \Delta t \sum_{n > 0} (z^{-1} e^{i2\pi fm\Delta t})^n$$

converges as a locally integrable function of $f$ to

$$-x(f)\Delta t(z^{-1}e^{i2\pi f\Delta t})/(1-z^{-1}e^{i2\pi f\Delta t}) = x(f)\Delta t/(1-ze^{-i2\pi f\Delta t}),$$

hence the formula for $x_{\pm}(z)$ of the theorem also holds for $|z|>1$. ■

**Example – Constant Frequency-Domain Representation.** The ordinary function $x(f)=1$ has a temporal representation given by

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} e^{i2\pi f t}\, df.$$

If $t=0$, $x(t)=\Delta f$ ; otherwise, $t=n\Delta t$ for some $n\neq 0$ and

$$x(t) = \left[\frac{e^{i2\pi f n\Delta t}}{i2\pi n\Delta t}\right]_{-\Delta f/2}^{+\Delta f/2} = \frac{(-1)^n-(-1)^{-n}}{i2\pi n\Delta t} = 0.$$

Hence, $x(t)=\mathbf{1}(t)$: the signal is the unit impulse. At this stage, it is easy to use the definition of the standard defining function to derive that $x_{\pm}(z)=\{|z|<1\}$. With the above theorem, we can also compute $x_{\pm}(z)$ directly from the definition $x(f)=1$. Indeed, we have

$$x_{\pm}(z) = \int_{-\Delta f/2}^{+\Delta f/2} \frac{\Delta t}{1-ze^{-i2\pi f\Delta t}}\, df = \int_{-\Delta f/2}^{+\Delta f/2} \frac{\Delta t}{1-ze^{-i2\pi f\Delta t}} \frac{de^{i2\pi f\Delta t}}{i2\pi\Delta t e^{i2\pi f\Delta t}},$$

hence

$$x_{\pm}(z) = \frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{\xi^{-1}}{1-z\xi^{-1}}\, d\xi = \frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{1}{\xi - z}\, d\xi,$$

which yields $x_{\pm}(z)=\{|z|<1\}$ as expected.

**Example – Defining Function of a Low-Pass Filter.** The impulse response $x(t)$ of a perfect low-pass filter whose cutoff frequency is $f_c=\Delta f/4$ – a filter whose passband and stopband have equal size – is defined in the frequency domain by

$$x(f) = \{|f|<\Delta f/4\},\ f\in[-\Delta f/2, \Delta f/2[\,.$$

The same kind of computations that we have made when we had $x(f)=1$ yield

$$x_{\pm}(z) = \frac{1}{i2\pi}\int_{\gamma} \frac{1}{\xi - z}\, d\xi$$

where $\gamma : f\in[-1/4,1/4]\to e^{i2\pi f}$. Inside or outside of the unit circle, if we differentiate under the integral sign and an integrate by parts the result, we end up with:

$$\frac{dx_{\pm}(z)}{dz} = \frac{1}{i2\pi}\left[\frac{1}{z-i}-\frac{1}{z+i}\right].$$

Let log denote the principal value of the logarithm. Inside the unit circle, the function

$$\left[z\mapsto \frac{1}{i2\pi}[\log(z-i)-\log(z+i)]\right].$$

is defined and holomorphic and its derivative matches the derivative of $x_\pm(z)$. As a direct computation shows that $x_\pm(z=0) = 1/2$, but $-1/2$ for this function, we have

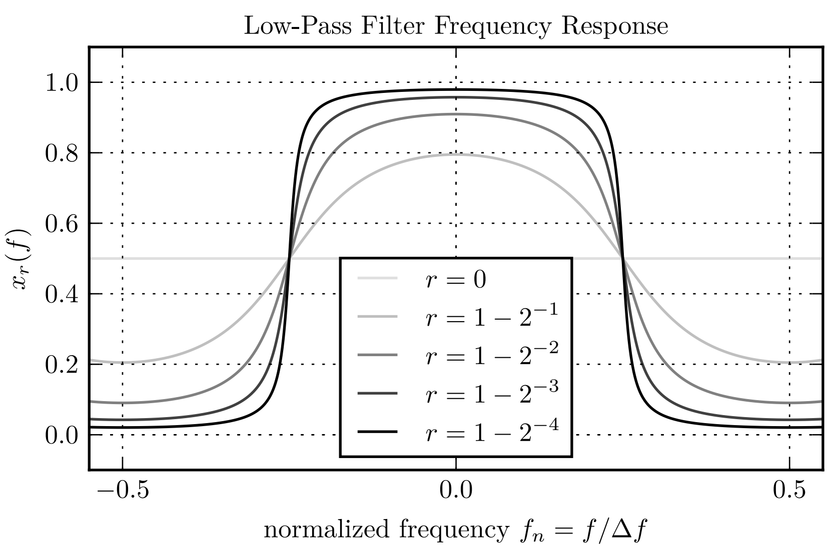
$$x_+(z) = \frac{1}{i2\pi}[\log(z-i) - \log(z+i)] + 1.$$

Outside the unit circle, the function

$$\left[z \mapsto \frac{1}{i2\pi} \log \frac{z-i}{z+i}\right]$$

is defined, holomorphic, and has the same derivative as $x_\pm(z)$. Moreover, it has the same limit when $|z| \to +\infty$, hence

$$x_-(z) = \frac{1}{i2\pi} \log \frac{z-i}{z+i}.$$




The time-domain representation of this filter is easy to determine: we have

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi f t}\, df = \int_{-\Delta f/4}^{+\Delta f/4} e^{i2\pi f t}\, df$$

hence

$$x(t = n\Delta t) = \left[\frac{e^{i2\pi f n \Delta t}}{i2\pi n \Delta t}\right]_{-\Delta f/4}^{+\Delta f/4} = \operatorname{sinc} \frac{\pi n}{2}.$$

## Calculus

The representation of signals in the frequency domain as hyperfunctions allows us to consider a large class of signals – the slowly increasing ones – but we now have to get familiar with the operations that we can perform with these mathematical objects. Some operations that are straightforward with functions cannot be carried to hyperfunctions – for example we cannot in general define the value of a hyperfunction $x(f)$ at a given frequency $f$ – some will be equally easy to perform and finally some – such as derivation with respect to $f$ – will be much easier to deal with in this new setting.

### Linear Combination

As the Fourier transform and the $z$-transforms are linear operators, the multiplication of signals by a complex scalar and sum of signals can be defined in the time domain, by

$$(\lambda x)(t) = \lambda x(t),\ (x+y)(t) = x(t) + y(t),$$

or equivalently in the frequency domain

$$(\lambda x)_r(f) = \lambda x_r(f),\ (x+y)_r(f) = x_r(f) + y_r(f),$$

as well as in the complex domain

$$(\lambda x)_\pm(z) = \lambda x_\pm(z),\ (x+y)_\pm(z) = x_\pm(z) + y_\pm(z).$$

### Modulation

Let $x(t)$ be a signal, $f_0 \in \mathbb{R}$ and

$$y(t) = x(t)e^{i2\pi f_0 t}.$$

Straighforward computations show that

$$y_r(f) = x_r(f - f_0)$$

and

$$y_\pm(z) = x_\pm(ze^{-i2\pi f_0 \Delta t})$$

**Example – Fourier Transform of Sine & Cosine.** Let $a > 0$, $\phi \in \mathbb{R}$, $f_0 > 0$ and let $x(t)$ be the signal defined by

$$x(t) = a\cos(2\pi f_0 t + \phi).$$

We can decompose $x(t)$ using complex exponentials:

$$x(t) = \frac{ae^{+i\phi}}{2} e^{i2\pi f_0 t} \times 1 + \frac{ae^{-i\phi}}{2} e^{-i2\pi f_0 t} \times 1$$

As we know the standard defining function of $t \mapsto 1$ is $\Delta t/(1-z)$, given the properties of linear combination and modulation in the complex domain, we have

$$x_\pm(z) = \frac{ae^{+i\phi}}{2} \frac{\Delta t}{1 - ze^{-i2\pi f_0 \Delta t}} + \frac{ae^{-i\phi}}{2} \frac{\Delta t}{1 - ze^{+i2\pi f_0 \Delta t}}.$$

## Integration (Frequency Domain)

Let $x(f)$ be a $\Delta f$-periodic hyperfunction. It would be natural to define the integral of $x(f)$ over one period as the limit when $r \uparrow 1$ of

$$\int_{-\Delta f/2}^{+\Delta f/2} x(f)\, df = \lim_{r \uparrow 1} \int_{-\Delta f/2}^{+\Delta f/2} x_r(f)\, df$$

but does this definition make sense? Are we sure that the limit always exists to begin with? Actually, it does and in a quite spectacular way: the integral under limit is eventually independent of $r$:

**Definition & Theorem – Integration in the Frequency Domain.** The integral over one period of a $\Delta f$-periodic hyperfunction $x(f)$ with defining function $x_\pm(z)$ is defined as

$$\int_{-\Delta f/2}^{+\Delta f/2} x(f)\, df = \int_{-\Delta f/2}^{+\Delta f/2} x_r(f)\, df = \frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{x_\pm(z)}{z\Delta t}\, dz$$

for any $r \in \,]\rho, 1[$ if the domain of definition of $x_\pm(z)$ contains $A_\rho \setminus \mathbb{U}$. This definition is sound: the right-hand sides of this formula are independent of the choice of $r$; they are also independent of the choice of the defining function.

**Proof.** Let $x_\pm(z)$ be the standard defining function of $x(f)$. For any $r \in \,]0, 1[$, we have

$$\int_{-\Delta f/2}^{+\Delta f/2} x_r(f)\, df = \int_{-\Delta f/2}^{+\Delta f/2} \left[ x_\pm(z = re^{i2\pi f \Delta t}) - x_\pm(z = r^{-1} e^{i2\pi f \Delta t}) \right] df.$$

If we rewrite the first term in the right-hand side as

$$\int_{-\Delta f/2}^{+\Delta f/2} x_\pm(z = re^{i2\pi f \Delta t})\, df = \frac{1}{i2\pi} \int_{-\Delta f/2}^{+\Delta f/2} \frac{x_\pm(z = re^{i2\pi f \Delta t})}{re^{i2\pi f \Delta t} \Delta t}\, d(re^{i2\pi f \Delta t}),$$

we see that it can be computed as a line integral:

$$\int_{-\Delta f/2}^{+\Delta f/2} x_\pm(z = re^{i2\pi f \Delta t})\, df = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{x_\pm(z)}{z\Delta t}\, dz.$$

The second integral can be computed similarly; we end up with

$$\int_{-\Delta f/2}^{\Delta f/2} x_r(f)\, df = \frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{x_\pm(z)}{z\Delta t}\, dz$$

By the Cauchy integral theorem, each integral in the right-hand side is independent of the choice of $r \in \left]0, 1\right[$.

Now, if $x_\pm(z)$ is any defining function whose domain contains $A_\rho \setminus \mathbb{U}$ and if $r \in \left]\rho, 1\right[$, let $\phi(z)$ be the extension to $A_\rho$ of the difference between $x_\pm(z)$ and the standard defining function. The difference of the integral formula based on $x_\pm(z)$ and the one based on the standard defining function is equal to

$$\frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{\phi(z)}{z\Delta t}\, dz.$$

The Cauchy integral theorem implies that this integral is equal to zero. ■

**Example – Constant Signal.** Let $x(t) = 1$ for every $t \in \mathbb{Z}\Delta t$. As the standard definition function of $x(f)$ is $x_\pm(z) = \Delta t/(1-z)$,

$$\int_{-\Delta f/2}^{\Delta f/2} x(f)\, df = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{1}{z(1-z)}\, dz - \frac{1}{i2\pi} \int_{r^{-1}[\circlearrowleft]} \frac{1}{z(1-z)}\, dz.$$

The pair of paths $\gamma$ made of $r\mathbb{U}$ (oriented counter-clockwise) and $r^{-1}\mathbb{U}$ (oriented clockwise) satisfies $\operatorname{ind}(\gamma, 0) = 0$ and $\operatorname{ind}(\gamma, 1) = -1$, hence

$$\int_{-\Delta f/2}^{\Delta f/2} x(f)\, df = (-1) \times \operatorname{res} \left[ \frac{1}{z(1-z)}, z = 1 \right] = 1.$$

## Differentiation (Frequency Domain)

Let $x(f)$ be a $\Delta f$-periodic hyperfunction. For every $r \in \left[0, 1\right[$, the function $x_r(f)$ is differentiable with respect to $f$. It would be natural to define the derivative of $x(f)$ with respect to $f$ by

$$\frac{dx(f)}{df} = \frac{\partial x_r(f)}{\partial f}.$$

and then every periodic hyperfunction would be differentiable. But does this definition make sense? Is $dx(f)/df$ well-defined as a hyperfunction?

**Definition & Theorem – Differentiation in the Frequency Domain.** Let $x(f)$ be a $\Delta f$-periodic hyperfunction with standard defining function $x_\pm(z)$. The derivative of $x(f)$ with respect to $f$ is the $\Delta f$-hyperfunction defined as

$$\frac{dx(f)}{df} = \frac{\partial x_r(f)}{\partial f}$$

and its standard defining function is

$$(i2\pi\Delta t)z\frac{dx_\pm(z)}{dz}.$$

**Proof.** We start from the equation

$$x_r(f) = x_\pm(z = re^{i2\pi f\Delta t}) - x_\pm(z = r^{-1}e^{i2\pi f\Delta t}).$$

The application of the chain rule to the right-hand side yields

$$\frac{\partial}{\partial f}x_\pm(z = re^{i2\pi f\Delta t}) = (i2\pi\Delta t)(re^{i2\pi f\Delta t})\frac{dx_\pm}{dz}(z = re^{i2\pi f\Delta t})$$

and

$$\frac{\partial}{\partial f}x_\pm(z = r^{-1}e^{i2\pi f\Delta t}) = (i2\pi\Delta t)(r^{-1}e^{i2\pi f\Delta t})\frac{dx_\pm}{dz}(z = r^{-1}e^{i2\pi f\Delta t}).$$

If we define

$$y_\pm(z) = (i2\pi\Delta t)z\frac{dx_\pm(z)}{dz},$$

we clearly have

$$\frac{dx_r(f)}{df} = y_\pm(re^{i2\pi f\Delta t}) - y_\pm(r^{-1}e^{i2\pi f\Delta t}).$$

The function $y_\pm(z)$ is defined and holomorphic on $\mathbb{C} \setminus \mathbb{U}$. Moreover, as $x_\pm(z = \infty) = 0$, the Laurent series expansion of $x_\pm(z)$ in a neighbourhood of $\infty$ has only negative powers of $z$ ; hence the expansion for $dx_\pm(z)/dz$ has only powers of $z$ less than $-2$, and the one of $y_\pm(z) = (i2\pi\Delta t)z dx_\pm(z)/dz$ only negative powers of $z$. Therefore, $y_\pm(z = \infty) = 0$ and $y_\pm(z)$ is an admissible standard defining function. ∎

**Example – Integral of a Derivative.** Let $x(f)$ be a periodic hyperfunction. We know that the standard defining function of $dx(f)/df$ is $(i2\pi\Delta t)z dx_\pm(z)/dz$. Hence, the integral

$$\int_{-\Delta f/2}^{+\Delta f/2} \frac{dx(f)}{df}\, df,$$

is equal to

$$\frac{1}{i2\pi}\left[\int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]}\right] \frac{(i2\pi\Delta t)z dx_\pm(z)/dz}{z\Delta t}\, dz$$

and after obvious simplifications, to

$$\int_{-\Delta f/2}^{+\Delta f/2} \frac{dx(f)}{df}\, df = \left[\int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]}\right] \frac{dx_\pm(z)}{dz}\, dz = 0.$$

## Convolution (Time Domain), Product (Frequency Domain)

**Theorem – Convolution.** The convolution $(x * y)(t)$ between a slowly increasing signal $x(t)$ and a quickly decreasing signal $y(t)$ is a slowly increasing signal.

**Proof – Convolution.** Assume that $\kappa > 0$, $\kappa' > 0$, $\sigma > 0$ and $\sigma' > 0$ are such that

$$|x(t)| \leq \kappa e^{\sigma |t|}$$

and

$$|y(t)| \leq \kappa' e^{-\sigma' |t|}.$$

We have

$$|(x * y)(t)| \leq \Delta t \sum_{\tau \in \mathbb{Z}\Delta t} |x(\tau)||y(t-\tau)| \leq \Delta t \kappa \kappa' \sum_{\tau \in \mathbb{Z}\Delta t} e^{\sigma|\tau|} e^{-\sigma'|t-\tau|}.$$

Using $|\tau| \leq |t| + |t-\tau|$, we get

$$e^{\sigma|\tau|} e^{\sigma'|t-\tau|} \leq e^{\sigma|t|} e^{-(\sigma'-\sigma)|t-\tau|}.$$

As long as $\sigma < \sigma'$,

$$\kappa'' = \sum_{\tau \in \mathbb{Z}\Delta t} e^{-(\sigma'-\sigma)|t-\tau|}$$

is finite and independent of $t$, hence

$$|(x * y)(t)| \leq \Delta t \kappa \kappa' \kappa'' e^{\sigma|t|}.$$

and $x * y$ is a slowly increasing signal. ■

**Definition – Product.** The product $w(f) = x(f) \times y(f)$ of a $\Delta f$-periodic hyperfunction $x(f)$ and a $\Delta f$-periodic analytic function $y(f)$ is the hyperfunction defined by

$$w_{\pm}(z) = x_{\pm}(z) \times y(z).$$

**Remark.** The product between arbitrary hyperfunctions is not defined in general.

**Remark – Product Soundness.** The definition of the product above is independent of the choice of the defining function for $x(f)$.

**Theorem.** The convolution $(x * y)(t)$ of a slowly increasing signal $x(t)$ and a quickly decreasing signal $y(t)$ is represented in the frequency domain as

$$(x * y)(f) = x(f) \times y(f).$$

**Proof.** Let $w(t) = (x * y)(t)$. For some $\rho \in ]0, 1[$ and any $z$ such that $\rho < |z| < 1$, we have

$$w_{+}(z) = \Delta t \sum_{n \leq 0} \left[ \Delta \sum_{m \in \mathbb{Z}} x(t = m\Delta t) y(t = (n-m)\Delta t) \right] z^{-n},$$

hence

$$w_+(z) = \sum_{n\leq 0}\sum_{m\in\mathbb{Z}} a_{mn} z^{-n}$$

with

$$a_{mn} = (\Delta t)^2 x(t = m\Delta t) y(t = (n-m)\Delta t),$$

and on the other hand

$$x_+(z)y(z) = \left[\Delta t \sum_{m\leq 0} x(t = m\Delta t) z^{-n}\right]\left[\Delta t \sum_{\ell\in\mathbb{Z}} y(t = \ell\Delta t) z^{-\ell}\right],$$

hence

$$x_+(z)y(z) = \sum_{m\leq 0}\sum_{\ell\in\mathbb{Z}} a_{m(m+\ell)} z^{-m-\ell} = \sum_{m\leq 0}\sum_{n\in\mathbb{Z}} a_{mn} z^{-n}.$$

Consequently,

$$\phi(z) = w_+(z) - x_+(z)y(z) = \sum_{n\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}} (\{n\leq 0\} - \{m\leq 0\}) a_{mn}\right] z^{-n}.$$

Similarly, for some $\rho \in ]0,1[$ and any $z$ such that $1 < |z| < 1/\rho$, we have

$$w_-(z) = -\sum_{n>0}\sum_{m\in\mathbb{Z}} a_{mn} z^{-n}$$

and

$$x_-(z)y(z) = -\sum_{m>0}\sum_{\ell\in\mathbb{Z}} a_{m(m+\ell)} z^{-m-\ell} = -\sum_{m>0}\sum_{n\in\mathbb{Z}} a_{mn} z^{-n},$$

hence

$$\psi(z) = w_-(z) - x_-(z)y(z) = \sum_{n\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}} (-\{n>0\} + \{m>0\}) a_{mn}\right] z^{-n}.$$

As $\{n > 0\} + \{n \leq 0\} = 1$ and $\{m > 0\} + \{m \leq 0\} = 1$, this expression can be rewritten as

$$\psi(z) = w_-(z) - x_-(z)y(z) = \sum_{n\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}} (\{n\leq 0\} - \{m\leq 0\}) a_{mn}\right] z^{-n}.$$

The functions $\phi(z)$ and $\psi(z)$ are defined in an non-empty annulus centered on the origin, inside and outside $\mathbb{U}$ respectively, and have the same Laurent series expansion. Consequently, they share a common holomorphic extension in a neighbourhood of $\mathbb{U}$. Hence, $x_\pm(z)y(z)$ is a defining function of $(x * y)(f)$. ■

**Example – Filtering a Pure Frequency.** Let $h(t)$ be a quickly decreasing signal and consider the filter that associates to the slowly increasing input $u(t)$ the slowly increasing output $y(t) = (h * u)(t)$. The transfer function $h(z)$ of this filter is holomorphic in a neighbourghood of $\mathbb{U}$. Let $f_0 > 0$; if $u(t) = e^{i2\pi f_0 t}$, we have

$$y_{\pm}(z) = h(z) \times \frac{\Delta t}{1 - ze^{-i2\pi f_0 \Delta t}}.$$

It is clear that the difference between this defining function and

$$\phi(z) = h(z = e^{i2\pi f_0 t}) \times \frac{\Delta t}{1 - ze^{-i2\pi f_0 \Delta t}}$$

can be extended to a holomorphic function in a neighbourghood of $\mathbb{U}$. Hence, $\phi(z)$ is also defining function for $y(f)$ (moreover, it is standard). From this defining function, the results of section Modulation show that

$$y(t) = h(f = f_0) \times e^{i2\pi f_0 t}.$$

## Fourier Inversion Formula

We already know enough about operational calculus of hyperfunctions to prove some interesting results. For example, we may now deal with the extension of the Fourier Inversion Formula to slowly increasing signals (in the time domain) or hyperfunctions (in the frequency domain).

**Theorem – Fourier Inversion Formula**. Let $x(t)$ be a slowly increasing signal and $x(f)$ its Fourier transform. We have

$$x(t) = \int_{-\Delta f/2}^{+\Delta f/2} x(f) e^{i2\pi f t} \, df.$$

**Remark.** The first step is obviously to check that the right-hand side means something, before that we prove that its is equal to $x(t)$. The Fourier transform $x(f)$ is defined as a $\Delta f$-periodic hyperfunction. For any time $t \in \mathbb{Z}\Delta t$, the function $f \mapsto e^{i2\pi f t}$ is analytic and $\Delta f$-periodic, hence $x(f)e^{i2\pi f t}$ is defined as a $\Delta f$-periodic hyperfunction. Therefore its integral over one period is well defined.

**Proof.** If $t = n\Delta t$, $e^{i2\pi f t} = (e^{i2\pi f \Delta t})^n$, hence the product $y(f) = x(f)e^{i2\pi f t}$ is defined by $y_{\pm}(z) = x_{\pm}(z) z^n$. The integral with respect to $f$ of $y(f)$ is then given for any $r \in ]0, 1[$ by

$$\frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{y_{\pm}(z)}{z\Delta t} \, dz$$

or after obvious simplifications

$$\frac{1}{i2\pi} \left[ \int_{r[\circlearrowleft]} - \int_{r^{-1}[\circlearrowleft]} \right] \frac{x_{\pm}(z)}{\Delta t} z^{n-1} \, dz$$

and we have already established in the "Inversion Formula" theorem of the Standard Defining Function section that this expression is equal to $x(t)$. ■

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# Chapter 14

# Exercises Answers

## Complex-Differentiability

### Antiholomorphic Functions

1. For any $\lambda \in \mathbb{R}$ and $w, z \in \mathbb{C}$, we have

$$\overline{w+z} = \overline{w} + \overline{z} \ \wedge \ \overline{\lambda z} = \lambda \overline{z},$$

therefore the function $c$ is real-linear. However, it is not complex-linear: for example $\overline{i} = -i \neq i \times \overline{1}$. The function $c$ is real-linear and continuous, hence it is real-differentiable and for any $z \in \mathbb{C}$, $dc_z = c$. This differential is not complex-linear, therefore the function is not complex-differentiable (or holomorphic). On the other hand, $\overline{c(z)} = z$, therefore it is antiholomorphic.

2. If the function $\overline{f} : \Omega \to \mathbb{R}$ is holomorphic, it is real-differentiable everywhere on its domain of definition. Hence $f = c \circ \overline{f}$ is real-differentiable as the composition of real-differentiable functions and

$$df_z = d(c \circ \overline{f})_z = c \circ d\overline{f}_z.$$

3. The complex-valued Cauchy-Riemann equation for $\overline{f}$ is

$$\frac{\partial \overline{f}}{\partial x}(z) = \frac{1}{i} \frac{\partial \overline{f}}{\partial y}(z), \ \text{ or } d\overline{f}_z(i) = i \times d\overline{f}_z(1)$$

On the other hand, we have

$$\frac{\partial \overline{f}}{\partial x}(z) = d(c \circ f)_z(1) = (c \circ df_z)(1) = \overline{\frac{\partial f}{\partial x}}$$

and

$$\frac{\partial \overline{f}}{\partial y}(z) = d(c \circ f)_z(i) = (c \circ df_z)(i) = \overline{\frac{\partial f}{\partial y}},$$

hence we can rewrite the Cauchy-Riemann equation for $\overline{f}$ as

$$\frac{\partial f}{\partial x}(z) = -\frac{1}{i}\frac{\partial f}{\partial y}(z), \quad \text{or } df_z(i) = -i \times df_z(1).$$

4. The composition of antiholomorphic functions is holomorphic. Indeed, if $f$ and $g$ are antiholomorphic, they are real-differentiable; their composition – assuming that it is defined – is $f \circ g = (c \circ \overline{f}) \circ (c \circ \overline{g})$; it satisfies

$$d(f \circ g)_z = d(c \circ \overline{f})_{c(\overline{g}(z))} \circ d(c \circ \overline{g})_z = c \circ d\overline{f}_{c(\overline{g}(z))} \circ c \circ d\overline{g}_z.$$

Since for any $h, w \in \mathbb{C}$,

$$c(hw) = \overline{h}c(w), \; d\overline{g}_z(hw) = hd\overline{g}(w), \; d\overline{f}_{c(\overline{g}(z))}(hw) = hd\overline{f}_{c(\overline{g}(z))}(w),$$

we have

$$d(f \circ g)_z(h) = hd(f \circ g)_z(1).$$

The differential of $f \circ g$ is complex-linear: the function is holomorphic.

5. If the point $w$ belongs to $\overline{\Omega}$, then $w = \overline{z}$ for some $z \in \Omega$, thus the complex number $\overline{f(\overline{z})}$ is defined. Additionally, the set

$$\overline{\Omega} = \{\overline{z} \mid z \in \Omega\} = \{w \in \mathbb{C} \mid \overline{w} \in \Omega\}$$

is an open set, as the inverse image of the open set $\Omega$ by the continous function $c$. The function $g$ satisfies

$$g = c \circ f \circ c = (c \circ f) \circ c$$

which is holomorphic as the composition of two antiholomorphic functions. We have

$$dg_z(h) = (c \circ df_{c(z)} \circ c)(h) = \overline{df_{\overline{z}}(\overline{h})} = \overline{f'(\overline{z})\overline{h}} = \overline{f'(\overline{z})}h,$$

hence $g'(z) = \overline{f'(\overline{z})}$.

## Principal Value of the Logarithm

1. Note that the definition of arg is non-ambiguous: for any nonzero real number $\epsilon$,

$$\arctan \epsilon + \arctan 1/\epsilon = \text{sgn}(\epsilon) \times \pi/2,$$

so if $x + iy$ belongs to two of the half-planes $x > 0$, $y < 0$ and $y > 0$, the two relevant expressions which may define $\arg(x + iy)$ are equal.

As $\arctan(\mathbb{R}) = ]-\pi/2, \pi/2[$, the three expressions that define arg have values in $]-\pi, \pi[$. Then, if for example $x > 0$, with $\theta = \arg(x + iy)$, we have

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \tan(\arctan y/x) = \frac{y}{x}.$$

Since $\cos\theta > 0$ and $x > 0$, there is a $\lambda > 0$ such that

$$x + iy = \lambda(\sin\theta + i\cos\theta) = \lambda e^{i\theta};$$

this equation yields

$$e^{i\arg(x+iy)} = \frac{x+iy}{|x+iy|}.$$

The proof for the half-planes $y > 0$ and $y < 0$ is similar.

2. The functions arg, ln and therefore log are continuously real-differentiable. If $x > 0$, for example, we have

$$\frac{\partial \arg(x+iy)}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$

and

$$\frac{\partial \arg(x+iy)}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$

On the other hand,

$$\frac{\partial}{\partial x}\ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

and

$$\frac{\partial}{\partial y}\ln\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$$

Finally

$$\frac{\partial \log}{\partial x}(x+iy) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{x+iy}$$

and

$$\frac{\partial \log}{\partial y}(x+iy) = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} = \frac{1}{i}\frac{1}{x+iy}.$$

The computations for $y > 0$ and $y < 0$ are similar. Conclusion: the function log is complex-differentiable and $\log'(z) = 1/z$.

## Conformal Mappings

1. If $L$ is $\mathbb{C}$-linear, then for any $\theta \in \mathbb{R}$, $L(e^{i\theta}) = e^{i\theta}L(1)$, hence it is angle-preserving. Reciprocally, if $L$ is angle-preserving, we have on one hand

$$L(e^{i\theta}) = \alpha_\theta \times e^{i\theta}L(1)$$

and on the other hand, as $e^{i\theta} = \cos\theta + i\sin\theta$,

$$L(e^{i\theta}) = \cos\theta \times L(1) + \sin\theta \times L(i) = (\cos\theta + \sin\theta \times \alpha_{\frac{\pi}{2}} i) L(1).$$

We know that $L(1) \neq 0$, hence these equations provide

$$\alpha_\theta = \cos\theta e^{-i\theta} + \sin\theta e^{-i\theta} \times \alpha_{\frac{\pi}{2}} i = \frac{1+\alpha_{\frac{\pi}{2}}}{2} + \frac{1-\alpha_{\frac{\pi}{2}}}{2} e^{-i2\theta}.$$

As $\alpha_\theta$ is real-valued, $\alpha_{\frac{\pi}{2}} = 1$. Consequently $\alpha_\theta = 1$ and $L$ is $\mathbb{C}$-linear.

2. A mapping $f : \Omega \to \mathbb{C}$ is conformal if it is $\mathbb{R}$-differentiable, $df_z$ is invertible everywhere and is $\mathbb{C}$-linear: this is exactly the class of holomorphic mappings $f$ on $\Omega$ such that $f'(z) \neq 0$ everywhere.

## Directional Derivative

1. The real-differentiability of $f$ at $z_0$ provides

$$f(z_0 + re^{i\alpha}) = f(z_0) + df_{z_0}(re^{i\alpha}) + \epsilon_{z_0}(re^{i\alpha})|r|$$

where $\lim_{h\to 0} \epsilon_{z_0}(h) = 0$. Therefore,

$$\frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0} = (re^{i\alpha})^{-1}(df_{z_0}(re^{i\alpha}) + \epsilon_{z_0}(re^{i\alpha})|r|).$$

Using the $\mathbb{R}$-linearity of $df_{z_0}$, we get

$$\frac{f(z_{r,\alpha}) - f(z_0)}{z_{r,\alpha} - z_0} = e^{-i\alpha} df_{z_0}(e^{i\alpha}) + \epsilon_{z_0,\alpha}(r)$$

for some function $\epsilon_{z_0,\alpha}$ such that $\lim_{r\to 0} \epsilon_{z_0,\alpha}(r) = 0$. Hence, the limit that defines $\ell_\alpha$ exists and

$$\ell_\alpha = e^{-i\alpha} df_{z_0}(e^{i\alpha}).$$

2. For every real number $\alpha$, we have

$$\ell_\alpha = e^{-i\alpha}\left(\frac{\partial f}{\partial x}(z_0)\cos\alpha + \frac{\partial f}{\partial y}(z_0)\sin\alpha\right).$$

Hence, if we use the equations

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2},\ \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i},$$

we obtain

$$\ell_\alpha = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) e^{-i2\alpha}.$$

Therefore, the set $A = \{\ell_\alpha \mid \alpha \in \mathbb{R}\}$ is a circle centered on

$$c = \frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

whose radius is

$$r = \left|\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right|.$$

3. The function $f$ is complex-differentiable at $z_0$ if and only if the Cauchy-Riemann equation

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

is met, which happens exactly when the radius of the circle $A$ is zero, that is, when $A$ is a single point in the complex plane.

# Line Integrals & Primitives

## Primitives of Power Functions

If $n \neq -1$, the function $z \mapsto z^{n+1}/(n+1)$ is a primitive of $z \mapsto z^n$. As $\mathbb{C}$ and $\mathbb{C}^*$ are path-connected, the other primitives differ from this one by a constant.

If $n = -1$, no primitive exist: the function $\gamma : t \in [0,1] \to e^{i2\pi t}$ is a closed rectifiable path of $\mathbb{C}^*$ and

$$\int_\gamma \frac{dz}{z} = \int_0^1 \frac{e^{i2\pi t} i2\pi}{e^{i2\pi t}}\, dt = i2\pi,$$

which is nonzero.

## Primitive of a Rational Function

We have

$$f(z) = -\frac{1}{z} + \frac{1}{z-1}.$$

The function $z \mapsto -1/z$ has no primitive on $D(0,1) \setminus \{0\}$: indeed if $\gamma(t) = 1/2 \times e^{i2\pi t}$, we have

$$\int_\gamma \frac{dz}{z} = i2\pi \neq 0.$$

On the other hand, on the same set, $z \mapsto \log(z-1)$ is a primitive of $z \mapsto 1/(z-1)$. Hence $f(z)$ has no primitive.

The function

$$g(z) = \log\frac{z-1}{z} = \log\left(1 - \frac{1}{z}\right)$$

is defined on $\mathbb{C} \setminus [0,1]$ and is a primitive of $f$. Indeed $g(z)$ is defined as long as neither of the conditions $z = 0$ and $1 - 1/z \in \mathbb{R}_-$ are met; they are equivalent to the condition $z \in [0,1]$, which is excluded. Moreover, $g$ satisfies

$$g'(z) = \frac{1/z^2}{1 - 1/z} = \frac{1}{z(z-1)}$$

hence it is a primitive of $f$.

## Reparametrization of Paths

1. The statement about the initial and terminal points is obvious. The one relative to the image holds because, under the assumptions that were made, the function $\phi$ is a bijection from $[0,1]$ on itself (and its inverse is also continuously differentiable).

2. We have

$$\int_\beta f(z)\,dz = \int_0^1 (f \circ \beta)(t)\beta'(t)\,dt = \int_0^1 (f \circ \alpha)(\phi(t))\alpha'(\phi(t))\,(\phi'(t)dt).$$

   The change of variable $s = \phi(t)$ leads to

$$\int_\beta f(z)\,dz = \int_0^1 (f \circ \alpha)(s)\alpha'(s)\,ds = \int_\alpha f(z)\,dz.$$

3. We have

$$\int_0^1 |\beta'(t)|\,dt = \int_0^1 |\alpha'(\phi(t))\phi'(t)|\,dt = \int_0^1 |\alpha'(\phi(t))|\,\phi'(t)dt$$

   The change of variable $s = \phi(t)$ leads to

$$\int_0^1 |\beta'(t)|\,dt = \int_0^1 |\alpha'(s)|\,ds,$$

   hence the lengths of $\alpha$ and $\beta$ are equal.

## The Logarithm: Alternate Choices

Let $\gamma$ be a closed rectifiable path of $\mathbb{C}_\alpha$. The path $\mu : [0,1] \mapsto e^{i(\pi-\alpha)}\gamma(t)$ is closed, rectifiable and its image is included in $\mathbb{C} \setminus \mathbb{R}_-$. Additionally

$$\int_\gamma \frac{dz}{z} = \int_\gamma \frac{d(e^{i(\pi-\alpha)}z)}{e^{i(\pi-\alpha)}z} = \int_\mu \frac{dz}{z}.$$

Since the principal value of the logarithm is a primitive if $z \mapsto 1/z$ on $\mathbb{C} \setminus \mathbb{R}_-$, the integral of $z \mapsto 1/z$ on $\mu$ is equal to zero. Therefore, there are primitives of $z \mapsto 1/z$ on $\mathbb{C}_\alpha$; since $\mathbb{C}_\alpha$ is connected, they all differ from an arbitrary constant.

Alternatively, we can build explicitly such a primitive: the function

$$f : z \mapsto \log(ze^{i(\pi-\alpha)});$$

it is defined and holomorphic on $\mathbb{C}_\alpha$ and for any $z \in \mathbb{C}_\alpha$,

$$f'(z) = \frac{1}{ze^{i(\pi-\alpha)}} \times e^{i(\pi-\alpha)} = \frac{1}{z}.$$

# Connected Sets

## Image of Path-Connected/Connected Sets

Suppose that $A$ is path-connected. Let $a, b \in f(A)$; there are some $c, d \in A$ such that $f(c) = a$ and $f(d) = b$. As $A$ is path-connected, there is a path $\gamma$ that joins $c$ and $d$ in $A$. By continuity of $f$, it is plain that its image $f \circ \gamma$ is a path of $f(A)$ that joins $a$ and $b$. Consequently, $f(A)$ is path-connected.

Now suppose that $A$ is connected. Let $g$ be a locally constant function defined on $f(A)$. The function $g \circ f$ is locally constant on $A$: if $a \in A$, there is a radius $r > 0$ such that $g$ is constant on $D(f(a), r) \cap f(A)$; by continuity of $f$, there is a $\epsilon > 0$ such that if $b \in D(a, \epsilon) \cap A$, $f(b) \in D(f(a), \epsilon) \cap f(A)$, thus $g \circ f$ is constant on $D(a, \epsilon) \cap A$ and finally, $g \circ f$ is locally constant. Since $A$ is connected, $g \circ f$ is actually constant and $g$ itself is constant: $f(A)$ is connected.

## Complement of a Compact Set

The compact set $K$ is closed, hence its complement is open. Therefore, the connected and path-connected components of $\mathbb{C} \setminus K$ are the same. The compact set $K$ is also bounded, hence there is a $r > 0$ such that the annulus

$$A = \{z \in \mathbb{C} \mid |z| > r\}$$

is included in $\mathbb{C} \setminus K$. The annulus $A$ is path-connected: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are in $A$, the path $\gamma = [r_1 \to r_2] e^{i[\theta_1 \to \theta_2]}$, which is defined by

$$\gamma(t) = ((1-t)r_1 + tr_2)e^{i((1-t)\theta_1 + t\theta_2)}$$

belongs to $A$ and joins $z_1$ and $z_2$. Hence, $A$ is included in some path-connected component of $\mathbb{C} \setminus K$. The collection of these path-connected components are a partition of $\mathbb{C} \setminus K$, hence every other component $C$ is a subset of $\mathbb{C} \setminus A = \overline{D}(0, r)$: it is bounded.

## Union of Separated Sets

1. No. For example, the sets $A = \{z \in \mathbb{C} \mid \Re(z) < 0\}$ and $B = \mathbb{C} \setminus A$ are disjoints, but their union is $\mathbb{C}$, which is connected.

2. Let $r = d(A, B)/2$. Under the assumption, the sets

$$A' = \cup_{a \in A} D(a, r), \; B' = \cup_{b \in B} D(b, r),$$

which are both open sets, are disjoints, hence their union is not path-connected. However $A' \cup B'$ is a dilation of $A \cup B$, hence $A \cup B$ is not connected.

Alternatively, consider the function $f$ equal to 1 on $A$ and 0 on $B$. It is locally constant – if $z \in A \cup B$, f is constant on $(A \cup B) \cap D(z, r)$ with $r = d(A, B)$ for example – but not constant, hence $A \cup B$ is not connected.

3. Consider again the function $f$ introduced in the previous answer. The assumption yields $A \cap B = \varnothing$; as $A$ and $B$ are non-empty, $f$ is not constant. If this function was not locally constant around some $a \in A$, we could find a sequence of $b_n \in (A \cup B) \setminus A = B$ such that $b_n \to a$. But that would imply that $a \in A \cap \overline{B}$ and would lead to a contradiction. Similarly, if it was not constant around some $b \in B$, that would lead to $b \in \overline{A} \cap B$, another contradiction. Hence, it is locally constant and $A \cup B$ is not connected.

## Anchor Set

1. Let $\mathcal{A}'$ be the collection of all the sets $A^* \cup A$ for $A \in \mathcal{A}$. For any $A \in \mathcal{A}$, the collection $\{A, A^*\}$ is composed of two path-connected/connected sets with a non-empty intersection; hence all the sets of $\mathcal{A}'$ in path-connected/connected. Moreover, the unions $\cup\mathcal{A}$ and $\cup\mathcal{A}'$ are identical. By assumption, unless $\mathcal{A}$ is empty, $A^*$ is not empty; hence the intersection $\cap\mathcal{A}'$ that contains $A^*$ is not empty. Therefore, $\cup\mathcal{A} = \cup\mathcal{A}'$ is path-connected/connected.

2. For any $a \in A$, $\gamma_a(0) = a$ and $\gamma_a([0, 1]) \subset A$, hence

$$A = \bigcup_{a \in A} \gamma_a([0, 1]).$$

For any $a \in A$, the set $\gamma_a([0, 1])$ is path-connected (as the image of a path-connected set by a continuous function), and $\gamma_a([0, 1]) \cap B$ is non-empty (it contains $\gamma_a(1)$). Consequently, the collection

$$\mathcal{A} = \{B\} \cup \{\gamma_a([0, 1]) \,|\, a \in A\}$$

satisfies the assumption of the previous question with $A^* = B$. Consequently, $A = \cup\mathcal{A}$ is path-connected/connected.

# Cauchy's Integral Theorem – Local Version

## A Fourier Transform

1. We may denote $x$ the extension to the complex plane of the original Gaussian function $x$, defined by:

$$\forall\, z \in \mathbb{C},\ x(z) = e^{-z^2/2}.$$

It is holomorphic as a composition of holomorphic functions. Let $\gamma = [-\tau \to \tau] + i\omega$. The line integral of $x$ along $\gamma$ satisfies

$$\int_\gamma x(z)\, dz = \int_0^1 x(-\tau(1-s) + \tau s + i\omega)\,(2\tau ds)$$

or, using the change of variable $t = -\tau(1-s) + \tau s$,

$$\int_\gamma x(z)\, dz = \int_{-\tau}^{\tau} x(t + i\omega)\, dt.$$

Since

$$x(t+i\omega) = e^{-(t+i\omega)^2/2} = e^{-t^2/2} e^{-i\omega t} e^{\omega^2/2},$$

we end up with

$$\int_\gamma x(z)\, dz = e^{\omega^2/2} \int_{-\tau}^{\tau} x(t) e^{-i2\pi f t}\, dt$$

2. Let $\nu = \tau + [0 \to i\omega]$; on the image of this path, we have

$$\forall\, s \in [0,1],\ |x(\nu(s))| = \left| e^{-(\tau + i\omega s)^2/2} \right| = e^{-\tau^2/2} e^{(\omega s)^2/2} \leq e^{-\tau^2/2} e^{\omega^2/2},$$

hence the M-L inequality provides

$$\left| \int_\mu x(z)\, dz \right| \leq (|\omega| e^{\omega^2/2}) e^{-\tau^2/2}$$

and thus,

$$\forall\, \omega \in \mathbb{R},\ \lim_{|\tau| \to +\infty} \int_\mu x(z)\, dz = 0.$$

We may apply Cauchy's integral theorem to the function $x$ on the closed polyline

$$[-\tau + i\omega \ \to\ \tau + i\omega \ \to\ \tau \ \to\ -\tau \ \to\ -\tau + i\omega].$$

It is the concatenation of $\gamma = [-\tau \to \tau] + i\omega$, the reverse of $\mu_+ = \tau + [0 \to i\omega]$, the reverse of $\gamma_0 = [-\tau \to \tau]$ and finally $\mu_- = -\tau + [0 \to i\omega]$.

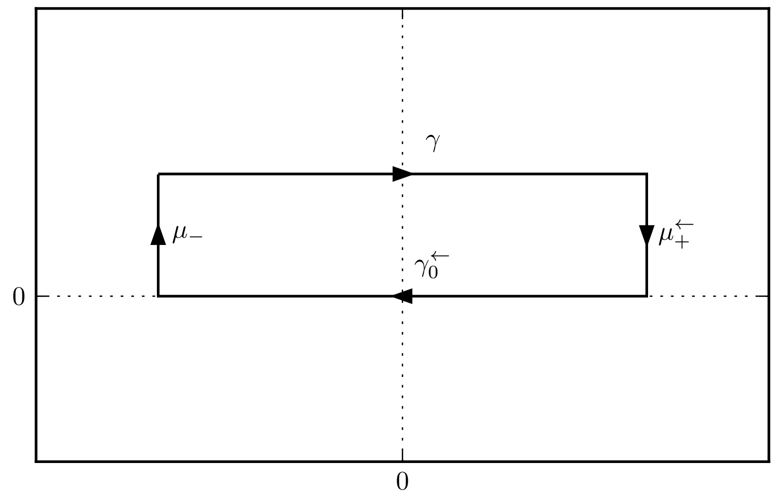


Figure 14.1: The closed path used in the application of Cauchy's integral theorem

The theorem provides

$$e^{\omega^2/2}\int_{-\tau}^{\tau} x(t)e^{-i\omega t}\,dt - \int_{\mu_+} x(z)\,dz$$
$$- \int_{-\tau}^{\tau} x(t)\,dt + \int_{\mu_-} x(z)\,dz = 0.$$

When $\tau \to +\infty$, this equality yields

$$\int_{-\infty}^{\infty} x(t)e^{-i2\pi f t}\,dt = \sqrt{2\pi}e^{-\omega^2/2}.$$

## Cauchy's Integral Formula for Disks

1. If $|z-c| > r$, the function $w \mapsto f(w)/(w-z)$ is defined and holomorphic in $\Omega \setminus \{z\}$. Let $\rho$ be the minimum between $|z-c|$ and the distance between $c$ and $\mathbb{C} \setminus \Omega$. By construction, the open disk $D(c,\rho)$ is a star-shaped subset of $\Omega \setminus \{z\}$ and it contains the image of $\gamma$. Thus, Cauchy's integral theorem provides

$$\frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw = 0.$$

2. When $z = c$, we have

$$\begin{aligned}\frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw &= \frac{1}{i2\pi}\int_0^1 \frac{f(c+re^{i2\pi t})}{c+re^{i2\pi t}-c}(i2\pi r e^{i2\pi t}dt) \\ &= \int_0^1 f(c+re^{i2\pi t})dt.\end{aligned}$$

By continuity of $f$ at $c$, the limit of this integral when $r \to 0$ is $f(c)$.

3. Assume for the sake of simplicity that $z = c + x$ for some real number $x \in [0, r[$. Let $\alpha = \arccos x/r$; define $\mu$ as the concatenation

$$\begin{array}{rcll}\mu &=& c + [x + i\epsilon \to re^{i\alpha}] & \mid \\ && c + re^{i[\alpha \to 2\pi - \alpha]} & \mid \\ && c + [re^{-i\alpha} \to x - i\epsilon] & \mid \\ && c + x + \epsilon e^{i[-\pi/2 \to -3\pi/2]} & .\end{array}$$

and $\nu$ as the concatenation

$$\begin{array}{rcll}\nu &=& c + [x - i\epsilon \to re^{-i\alpha}] & \mid \\ && c + re^{i[-\alpha \to \alpha]} & \mid \\ && c + [re^{i\alpha} \to x + i\epsilon] & \mid \\ && c + x + \epsilon e^{i[\pi/2 \to -\pi/2]} & .\end{array}$$

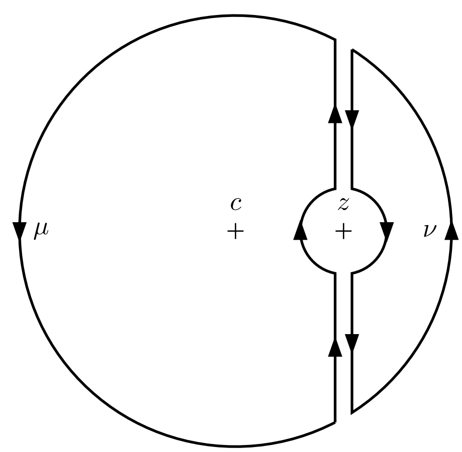


Figure 14.2: Cauchy's Integral Formula for Disks

Since the closure of $D(c, r)$ is included in $\Omega$, there is a $\rho > r$ such that $D(c, \rho) \subset \Omega$. The image of $\mu$ belongs to the set

$$D(c, \rho) \setminus \{z + t \mid t \geq 0\}$$

while the image of $\nu$ belongs to the set

$$D(c,\rho) \setminus \{z+t \mid t \leq 0\}.$$

Both sets are star-shaped and included in $\Omega$.

Additionally, for any continuous function $g : \Omega \setminus \{z\} \to \mathbb{C}$

- the integral of $g$ on $c + [x+i\epsilon \to re^{i\alpha}]$ and its reverse path on one hand, the integral of $g$ on $c + [re^{-i\alpha} \to x - i\epsilon]$ and its reverse path on the other hand are opposite numbers.
- the sum of the integral of $g$ on $c + re^{i[\alpha \to 2\pi - \alpha]}$ and $c + re^{i[-\alpha \to \alpha]}$ is equal to its integral on $\gamma = c + r[\circlearrowleft]$.
- the sum of the integral of $g$ on $c + x + \epsilon e^{i[-\pi/2 \to -3\pi/2]}$ and $c + x + \epsilon e^{i[\pi/2 \to -\pi/2]}$ is equal to the opposite of its integral on $\lambda = c + x + \epsilon[\circlearrowleft]$.

Therefore, the equality

$$\int_\gamma g(w)\,dw = \int_\lambda g(w)dw + \int_\mu g(w)\,dw + \int_\nu g(w)dw.$$

holds.

4. We may apply the result of the previous question to the function $w \mapsto f(w)/(w-z)$. As it is holomorphic on $\Omega \setminus \{z\}$, the star-shaped version of Cauchy's integral theorem provides

$$\int_\mu \frac{f(w)}{w-z}dw = \int_\nu \frac{f(w)}{w-z}dw = 0,$$

hence

$$\frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw = \frac{1}{i2\pi}\int_\lambda \frac{f(w)}{w-z}dw.$$

We proved in question 2. that the right-hand side of this equation tends to $f(z)$ when $\epsilon \to 0$, thus

$$\frac{1}{i2\pi}\int_\gamma \frac{f(w)}{w-z}dw = f(z),$$

which is Cauchy's integral formula for disks.

5. Let $z \in \Omega$. There are some $c \in \Omega$ and $r > 0$ such that $z \in D(c,r)$ and $\overline{D}(z,r) \subset \Omega$; let $\gamma = c + r[\circlearrowleft]$. For any complex number $h$ such that $|z+h-c| < r$, we have by Cauchy's formula

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{i2\pi}\int_\gamma \frac{1}{h}\left(\frac{1}{w-z-h} - \frac{1}{w-z}\right) f(w)\,dw \\ &= \frac{1}{i2\pi}\int_\gamma \frac{f(w)}{(w-z-h)(w-z)}\,dw\end{aligned}$$

To met the condition on $h$, assume that $|h| \leq \epsilon$ where

$$0 < \epsilon < \min_{t \in [0,1]} |z - \gamma(t)|.$$

Write the line integral above as an integral with respect to the real parameter $t \in [0,1]$; its integrand is dominated by a constant:

$$\forall\, t \in [0,1],\ \left| \frac{1}{i2\pi} \frac{f(\gamma(t))}{(\gamma(t) - z - h)(\gamma(t) - z)} \gamma'(t) \right| \leq \frac{1}{\epsilon^2} \max_{t \in [0,1]} |f(\gamma(t))|.$$

Thus, Lebesgue's dominated convergence theorem provides the existence of the derivative of $f$ at $z$ as well at its value:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{i2\pi} \int_\gamma \frac{f(w)}{(w-z)^2}\, dw.$$

Now it's pretty clear that we can iterate the previous argument: consider the right-hand side of the above equations as a function of $z$, build its difference quotient and pass to the limit. The process provides

$$f''(z) = \lim_{h \to 0} \frac{f'(z+h) - f'(z)}{h} = \frac{1}{i2\pi} \int_\gamma \frac{2f(w)}{(w-z)^3}\, dw.$$

The argument is valid for any $z \in \Omega$: the function $f'$ is also holomorphic.

## The Fundamental Theorem of Algebra

Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with no complex root. The function $f = 1/p$ is defined and holomorphic on $\mathbb{C}$. Additionally, as $|p(z)| \to +\infty$ when $|z| \to +\infty$, the modulus of $f$ is bounded. By Liouville's theorem, $f$ is constant, hence $p$ is constant too.

## Image of Entire Functions

Assume that the image of $f$ is not dense in $\mathbb{C}$: there is a $w \in \mathbb{C}$ and a $\epsilon > 0$ such that for any $z \in \mathbb{C}$, $|f(z) - w| \geq \epsilon$. Now consider the function $z \mapsto 1/(f(z) - w)$; it is defined and holomorphic in $\mathbb{C}$. Additionally,

$$\forall\, z \in \mathbb{C},\ \left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\epsilon}.$$

By Liouville's theorem, this function is constant, hence $f$ is constant too.

## The Winding Number

### Star-Shaped Sets

Let $\Omega$ be an open star-shaped subset of $\mathbb{C}$ with a center $c$.

For any $z \in \mathbb{C} \setminus \Omega$ and any $s \geq 0$, the point $w = z + s(z - c)$ belongs to $\mathbb{C} \setminus \Omega$. The ray of all such points $w$ is unbounded and connected, thus it is included in an unbounded component of $\mathbb{C} \setminus \Omega$. All components of $\mathbb{C} \setminus \Omega$ are therefore unbounded: $\Omega$ is simply connected.

Alternatively, let $\gamma$ be a closed path of $\Omega$ and let $z = c + re^{i\alpha} \in \mathbb{C} \setminus \Omega$. Since the ray $\{z + se^{i\alpha} \mid s \geq 0\}$ does not intersect $\Omega$, the function

$$\phi : t \in [0,1] \mapsto e^{-i(2\pi-\alpha)} \arg(e^{i(2\pi-\alpha)}(\gamma(t) - z))$$

is a continuous choice of the argument $w \mapsto \operatorname{Arg}(w - z)$ along $\gamma$, thus

$$\operatorname{ind}(\gamma, z) = \frac{1}{2\pi}[\phi(1) - \phi(0)] = 0.$$

Therefore, $\Omega$ is simply connected.

### The Argument Principle for Polynomials

1. Let $\gamma : t \in [0,1] \mapsto e^{i2\pi t}$; we have $(p \circ \gamma)(t) = (e^{i2\pi t})^3 + (e^{i2\pi t}) + 1$. The second figure shows that the graph of $t \mapsto |(p \circ \gamma)(t)|$ does not vanish on $[0,1]$, hence the image of $\gamma$ contains no root of $p$. The second figure shows that the variation of the argument of $z$ on the path $p \circ \gamma$ is $2\pi$ (a variation of $\pi$ between $t = 0$ and $t = 0.5$ and also a variation of $\pi$ between $t = 0.5$ and $t = 1.0$). Accordingly, we have

$$\operatorname{ind}(p \circ \gamma, 0) = 1.$$

   On the other hand, every zero $z$ of $p$ such that $|z| < 1$ satisfies $\operatorname{ind}(\gamma, z) = 1$ and every zero $z$ of $p$ such that $|z| > 1$ satisfies $\operatorname{ind}(\gamma, z) = 0$. Consequently, the expression

$$\sum_{k=1}^{m} \operatorname{ind}(\gamma, a_k) \times n_k$$

   provides the number of roots of $p$ – counted with their multiplicity – within the unit circle. By the argument principle, there is a unique root of $p$ within the unit circle.

2. If $\theta_0$ is an argument of $\lambda$, the sum

$$\theta : t \in [0,1] \mapsto \theta_0 + n_1\theta_1(t) \times \cdots + n_m\theta_m(t)$$

is continuous and

$$\begin{aligned} e^{i\theta(t)} &= e^{i\theta_0} \times e^{in_1\theta_1(t)} \times \cdots \times e^{in_m\theta_m(t)} \\ &= \frac{\lambda}{|\lambda|} \times \frac{(\gamma(t)-a_1)^{n_1}}{|\gamma(t)-a_1|^{n_1}} \times \cdots \times \frac{(\gamma(t)-a_m)^{n_m}}{|\gamma(t)-a_m|^{n_m}} \\ &= \frac{(p\circ\gamma)(t)}{|(p\circ\gamma)(t)|}, \end{aligned}$$

therefore $\theta$ is a choice of the argument of $z \mapsto z$ on $p \circ \gamma$. Consequently,

$$\begin{aligned} [z \mapsto \operatorname{Arg} z]_{p\circ\gamma} &= \theta(1) - \theta(0) \\ &= \theta_0 - \theta_0 + \sum_{k=1}^{m} n_k(\theta_k(1) - \theta_k(0)) \\ &= \sum_{k=1}^{m} n_k \times [z \mapsto \operatorname{Arg}(z - a_k)]_\gamma. \end{aligned}$$

A division of both sides of this equation by $2\pi$ concludes the proof.

3. The integral expression of the winding number is

$$\operatorname{ind}(p\circ\gamma, 0) = \frac{1}{i2\pi}\int_{p\circ\gamma} \frac{dz}{z}.$$

The polynomial $p$ is holomorphic on $\mathbb{C}$, hence we can perform the change of variable $z = p(w)$, which yields

$$\operatorname{ind}(p\circ\gamma, 0) = \frac{1}{i2\pi}\int_{\gamma} \frac{p'(w)}{p(w)} dw.$$

If we factor $p(w)$ as $(w-a_k)^{n_k} q(w)$, we see that

$$\frac{p'(w)}{p(w)} = \frac{n_k}{w-a_k} + \frac{q'(w)}{q(w)};$$

applying this process repeatedly for every $k \in \{1, \dots, m\}$, until $q$ is a constant, provides

$$\frac{p'(w)}{p(w)} = \sum_{k=1}^{m} \frac{n_k}{w-a_k}$$

and consequently

$$\begin{aligned} \operatorname{ind}(p\circ\gamma, 0) &= \frac{1}{i2\pi}\int_{\gamma} \left[\sum_{k=1}^{m} \frac{n_k}{w-a_k}\right] dw \\ &= \sum_{k=1}^{m} \left[\frac{1}{i2\pi}\int_{\gamma} \frac{dw}{w-a_k}\right] \times n_k \\ &= \sum_{k=1}^{m} \operatorname{ind}(\gamma, a_k) \times n_k. \end{aligned}$$

## Set Operations & Simply Connected Sets

1. **Intersection.** The statement holds true. Indeed, let $\gamma$ be a closed path of $A \cap B$; it is a path of $A$ and a path of $B$. As both sets are simply connected, the interior of $\gamma$ is included in $A$ and in $B$, that is in $A \cap B$: this intersection is simply connected.

   Alternatively, let $C$ be a component of

   $$\mathbb{C} \setminus (A \cap B) = (\mathbb{C} \setminus A) \cup (\mathbb{C} \setminus B),$$

   and let $z \in C$; we have $z \in \mathbb{C} \setminus A$ or $z \in \mathbb{C} \setminus B$. If $z \in \mathbb{C} \setminus A$, the component of $\mathbb{C} \setminus A$ that contains $z$ is unbounded; it is a connected set that contains $z$ and is included in $\mathbb{C} \setminus (A \cap B)$, hence, it is also included in $C$. Consequently, $C$ is unbounded. If instead $z \in \mathbb{C} \setminus B$, a similar argument provides the same result. Consequently, all components of $\mathbb{C} \setminus (A \cap B)$ are unbounded: $A \cap B$ is simply connected.

2. **Complement.** The statement does not hold: consider $A = D(0, 3)$ and $C = \overline{D(0, 1)}$. The set $A$ is open and simply connected and the set $C$ is closed and connected. The set $C$ is actually a component of $A \setminus C$: it is included in $A \setminus C$, connected and maximal.

   However, the statement holds if additionally the set $C \setminus A$ is not empty. Let $\gamma$ be a closed path of $A \setminus C$ and let $z \in \mathbb{C} \setminus (A \setminus C)$. If $z \in \mathbb{C} \setminus A$, as $A$ is simply connected, $z$ belongs to the exterior of $\gamma$. Otherwise, $z \in A \cap C$; as $C$ is a connected subset that does not intersect the image of $\gamma$, the function $w \in C \mapsto \operatorname{ind}(\gamma, w)$ is constant. There is a $w \in C \setminus A$ and $\operatorname{ind}(\gamma, z) = \operatorname{ind}(\gamma, w) = 0$. Therefore $z$ also belongs to the exterior of $\gamma$: $A \setminus C$ is simply connected.

   Alternatively, let $D$ be a component of

   $$\mathbb{C} \setminus (A \setminus C) = (\mathbb{C} \setminus A) \cup C.$$

   Some of its elements are in $\mathbb{C} \setminus A$: otherwise, $C$ would be a connected superset of $D$ that is included in $\mathbb{C} \setminus (A \setminus C)$; we would have $C = D$ and therefore $C \setminus A$ would be empty. Now, as $D$ contains at least a point $z$ of $\mathbb{C} \setminus A$, it contains the component of $\mathbb{C} \setminus A$ that contains $z$; therefore $D$ is unbounded. Consequently, $A \setminus C$ is simply connected.

3. **Union.** The statement doesn't hold: consider

   $$A_s = \{e^{i2\pi t} \mid t \in [0, 1/2]\}, \; B_s = \{e^{i2\pi t} \mid t \in [1/2, 1]\}.$$

   and the associated dilations

   $$A = \{z \in \mathbb{C} \mid d(z, A_s) < 1\}, \; B = \{z \in \mathbb{C} \mid d(z, B_s) < 1\}.$$

   They are both open, connected and simply connected (their complement in the plane has a single path-connected component and it is unbounded) but their union $A \cup B$ is the annulus $D(0,3) \setminus D(0,1)$. We already considered this set in question 2: it is not simply connected.

However, the statement holds if additionally, the intersection $A \cap B$ is connected. Let $\gamma$ be a closed path of $A \cup B$ and let $z \in \mathbb{C} \setminus (A \cap B)$. We have to prove that $\operatorname{ind}(\gamma, z) = 0$.

There exist[1] a sequence $(\gamma_1, \dots, \gamma_n)$ of consecutive paths of $A \cup B$ whose concatenation is $\gamma$ and such that for any $k \in \{1, \dots, n\}$, $\gamma_k([0,1]) \subset A$ or $\gamma_k([0,1]) \subset B$.

Let $a_k$ be the initial point of $\gamma_k$ and let $w \in A \cap B$. As $A$, $B$ and $A \cap B$ are connected, for any $k \in \{1, \dots, n\}$, there is a path $\beta_k$ from $w$ to $a_k$ such that $\beta_k([0,1]) \subset A$ if $a_k \in A$ and $\beta_k([0,1]) \subset B$ if $a_k \in B$. We denote $\beta_{n+1} = \beta_1$ for convenience; define the paths $\alpha_k$ as the concatenations

$$\alpha_k = \beta_k \,|\, \gamma_k \,|\, \beta_{k+1}.$$

By construction

$$[x \mapsto \operatorname{Arg}(x - z)]_\gamma = \sum_{k=1}^{n} [x \mapsto \operatorname{Arg}(x - z)]_{\alpha_k}.$$

Every path $\alpha_k$ is closed, hence this is equivalent to

$$\operatorname{ind}(\gamma, z) = \sum_{k=1}^{n} \operatorname{ind}(\alpha_k, z),$$

but every $\alpha_k$ belongs either to $A$ or $B$, which are simply connected, hence the right-hand-side is equal to zero. (This proof was adapted from Ronnie Brown's argument on Math Stack Exchange)

# Cauchy's Integral Theorem – Global Version

## Cauchy's Converse Integral Theorem

For any $w \in \mathbb{C} \setminus \Omega$, the function $f : z \in \Omega \mapsto 1/(z - w)$ is defined and holomorphic, thus

$$\operatorname{ind}(\gamma, w) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z - w} = 0$$

[1] The collection $\{A, B\}$ is an open cover of $\gamma([0,1])$ which is compact. Now, for any positive integer $n$, consider the sequence $(\gamma_1^n, \dots, \gamma_n^n)$ where

$$\gamma_k^n(t) = \gamma((k - 1 + t)/n).$$

By uniform continuity of $\gamma$, the diameters of the $\gamma_k^n$ tends uniformly to zero when $n$ tends to $+\infty$. The conclusion follows from Lebesgue's Number Lemma.

and therefore $\text{Int}\,\gamma \subset \Omega$. Now, suppose that this conclusion holds for any sequence $\gamma$ of closed rectifiable paths of $\Omega$. Since the winding number is locally constant and since for any closed path $\mu$ of $\Omega$ and any $\epsilon > 0$, there is a closed rectifiable path $\gamma$ of $\Omega$ such that

$$\forall\, t \in [0, 1],\ |\gamma(t) - \mu(t)| < \epsilon,$$

we also have $\text{ind}(\mu, w) = \text{ind}(\gamma, w) = 0$. Therefore $\text{Int}\,\mu \subset \Omega$: the set $\Omega$ is simply connected.

## Cauchy Transform of Power Functions

For any $z \in \mathbb{C}$ such that $|z| \neq 1$, the function

$$\psi_z : w \mapsto \frac{w^n}{w - z}$$

is defined and holomorphic on $\Omega = \mathbb{C} \setminus \{z\}$ if $n \geq 0$; it is defined and holomorphic on $\Omega = \mathbb{C} \setminus \{0, z\}$ if $n < 0$. The interior of $[\circlearrowleft]$ is the open unit disk.

We now study separately four configurations.

1. Assume that $n \geq 0$ and $|z| > 1$. The interior of $[\circlearrowleft]$ is included in $\Omega$, hence by Cauchy's integral theorem, $\phi(z) = 0$.

   Alternatively, Cauchy's formula was also applicable.

2. Assume that $n \geq 0$ and $|z| < 1$. The unique singularity of $\psi_z$ is $w = z$; it satisfies $\text{ind}([\circlearrowleft], z) = 1$. Let $\gamma(r) = z + r[\circlearrowleft]$; we have

   $$\text{res}(\psi_z, z) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{w - z} dw = \lim_{r \to 0} \int_0^1 (z + re^{i2\pi t})^n dt = z^n,$$

   hence by the residues theorem, $\phi(z) = z^n$.

   Alternatively, Cauchy's formula was also applicable.

3. Assume that $n < 0$ and $|z| > 1$. We have

   $$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{1}{w^{|n|}(w - z)}\, dw.$$

   If $n = -1$, Cauchy's formula provides the answer:

   $$\phi(z) = \frac{1}{0 - z} = -z^{-1}.$$

Otherwise $n < -1$, we may use integration by parts (several times):

$$\begin{aligned}\phi(z) &= \frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{1}{w^{|n|}(w-z)}\,dw \\ &= -\frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{-1}{|n|-1}\frac{1}{w^{|n|-1}}\frac{-1}{(w-z)^2}\,dw \\ &= \dots \\ &= (-1)^{|n|-1}\frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{1}{(|n|-1)!}\frac{1}{w}\frac{(|n|-1)!}{(w-z)^{|n|}}\,dw \\ &= -\frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{1}{w}\frac{1}{(z-w)^{|n|}}\,dw.\end{aligned}$$

At this point, Cauchy's formula may be used again and we obtain

$$\phi(z) = -z^n.$$

Alternatively, we may perform the change of variable $w = 1/\xi$ :

$$\begin{aligned}\phi(z) &= \frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{w^n}{w-z}dw \\ &= -\frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{\xi^{-n}}{\xi^{-1}-z}\left(-\frac{d\xi}{\xi^2}\right) \\ &= -\frac{1}{z}\frac{1}{i2\pi}\int_{[\circlearrowleft]} \frac{\xi^{-n-1}}{\xi - z^{-1}}d\xi.\end{aligned}$$

As $-n-1 \geq 0$ and $|z^{-1}| < 1$, we may invoke the result obtained for the first configuration: it provides $\phi(z) = -z^n$.

Alternatively, we may perform a partial fraction decomposition of $w \mapsto 1/(w^{|n|}(w-z))$. Since

$$1 - \left(\frac{w}{z}\right)^{|n|} = \left(1 - \frac{w}{z}\right)\left(1 + \frac{w}{z} + \dots + \left(\frac{w}{z}\right)^{|n|-1}\right),$$

we have

$$\frac{1}{w-z} = -\frac{1}{z}\left(1 + \frac{w}{z} + \dots + \left(\frac{w}{z}\right)^{|n|-1}\right) + \frac{w^{|n|}/z^{|n|}}{w-z}$$

and therefore

$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \dots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w-z}.$$

The integral along $\gamma$ of $w \in \mathbb{C} \mapsto 1/w^p$ is zero for $p > 1$ since this function has a primitive. The integral of $w \mapsto 1/(w-z)$ is also zero since $|z| > 1$. Finally,

$$\phi(z) = \frac{1}{i2\pi}\int_\gamma -\frac{1/z^{|n|}}{w^{-1}}dw = -z^n.$$

4. Assume that $n < 0$ and $|z| < 1$. There are two singularities of $\psi_z$ in the interior of $[\circlearrowleft]$, $w = 0$ and $w = z$, unless of course if $z = 0$.

   If $z = 0$, we have
$$\phi(z) = \frac{1}{i2\pi} \int_{[\circlearrowleft]} w^{n-1} dw = 0$$
   because $w \in \mathbb{C}^* \mapsto w^n/n$ is a primitive of $w \in \mathbb{C}^* \mapsto w^{n-1}$.

   We now assume that $z \neq 0$. The residue associated to $w = z$ can be computed directly with Cauchy's formula; with $\gamma(r) = z + r[\circlearrowleft]$, we have
$$\text{res}(\psi_z, z) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{\gamma(r)} \frac{w^n}{(w-z)} dw = z^n.$$
   On the other hand, using computations similar to those of the previous question, we can derive
$$\text{res}(\psi_z, 0) = \lim_{r \to 0} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{w^n}{(w-z)} dw = -z^n.$$
   Consequently, $\phi(z) = 0$.

   In the case $z \neq 0$, we may perform again the change of variable $w = 1/\xi$ that provides
$$\phi(z) = -\frac{1}{z} \frac{1}{i2\pi} \int_{[\circlearrowleft]} \frac{\xi^{-n-1}}{\xi - z^{-1}} d\xi.$$
   As $-n - 1 \geq 0$ and $|z^{-1}| > 1$, we may invoke the result obtained for the first configuration: it yields $\phi(z) = 0$.

   There is yet another method: we can notice that for $r > 1$, the interior of the path sequence $(r[\circlearrowleft], [\circlearrowleft]^{\leftarrow})$, which is the annulus $\{z \in \mathbb{C} \mid 1 < |z| < r\}$, is included in $\Omega$. Cauchy's integral theorem provides
$$\forall\, r > 1,\ \phi(z) = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{w^n}{w - z} dw.$$
   and the M-L estimation lemma
$$\forall\, r > 1,\ |\phi(z)| \leq \frac{1}{r^{|n|-1}(r - |z|)}.$$
   The limit of the right-hand side when $r \to +\infty$ yields $\phi(z) = 0$.

   Finally, we may use again the partial fraction decomposition of $w \mapsto 1/(w^{|n|}(w - z))$:
$$\frac{1}{w^{|n|}(w-z)} = -\left(\frac{z}{w^{|n|}} + \frac{1/z^2}{w^{|n|-1}} + \cdots + \frac{1/z^{|n|}}{w^{-1}}\right) + \frac{1/z^{|n|}}{w - z}.$$
   The integral along $\gamma$ of $w \in \mathbb{C} \mapsto 1/w^p$ is zero for $p > 1$ since this function has a primitive. Therefore
$$\phi(z) = \frac{1}{i2\pi} \int_\gamma -\frac{1/z^{|n|}}{w^{-1}} dw + \frac{1}{i2\pi} \int_\gamma \frac{1/z^{|n|}}{w - z} dw = 0.$$

# Power Series

## The Fibonacci Sequence

1. The discriminant $\Delta$ of the quadratic equation $x^2 - x - 1 = 0$ is
$$\Delta = (-1)^2 - 4 \times 1 \times (-1) = 5,$$
therefore the solutions are
$$x = \frac{1 \pm \sqrt{5}}{2}.$$
The golden ratio $\phi$, equal to $(1+\sqrt{5})/2$, is the largest of the two. The fact that the other root $\psi$ of the equation is equal to $-1/\phi$ can be demonstrated directly; we have indeed
$$\psi = \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{1+\sqrt{5}}\frac{1-\sqrt{5}}{2} = \frac{1^2 - \sqrt{5}^2}{2(1+\sqrt{5})} = -\frac{2}{1+\sqrt{5}}.$$
Alternatively, we know that
$$x^2 - x - 1 = (x-\phi)(x-\psi) = x^2 - (\phi+\psi)x + \phi\psi,$$
hence $\phi\psi = -1$.

2. It is clear that $a_0 = 0 \leq 1 = \phi^0$ and $a_1 = 1 \leq \phi = \phi^1$. If we assume that the inequality $a_n \leq \phi^n$ holds for $n = 0, 1, \ldots, m+1$, the recursive definition of the Fibonacci sequence yields
$$a_{m+2} = a_m + a_{m+1} \leq \phi^m + \phi^{m+1} = \phi^m(1+\phi) = \phi^{m+2}.$$
Hence, by induction, the inequality holds for every $n \in \mathbb{N}$.

3. The inequality $a_n \leq \phi^n$ provides
$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} \leq \phi,$$
and hence, by the Cauchy-Hadamard formula, the radius of convergence of the series $\sum_{n\geq 0} a_n z^n$ is at least $1/\phi$.

4. If $|z| < 1/\phi$, we can write the expansion of $f(z)$ as
$$f(z) = \sum_{n=0}^{+\infty} a_n z^n = a_0 + a_1 z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2} = z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2}.$$
Using $a_{n+2} = a_n + a_{n+1}$, we deduce that
$$f(z) = z + z^2 \sum_{n=0}^{+\infty} a_n z^n + z \sum_{n=0}^{+\infty} a_{n+1} z^{n+1} = z + z^2 f(z) + z f(z),$$
hence
$$f(z) = \frac{z}{1 - z - z^2}.$$

5. The roots of the polynomial $1 - z - z^2$ are $-\phi$ and $-\psi$, hence

$$-z^2 - z + 1 = -(z + \phi)(z + \psi).$$

Thus, for any $|z| < 1/\phi$, we have

$$f(z) = \frac{-z}{(z+\phi)(z+\psi)} = \frac{1}{\phi - \psi}\left[\frac{-\phi}{z+\phi} + \frac{\psi}{z+\psi}\right],$$

or equivalently, using $\psi = -1/\phi$,

$$f(z) = \frac{1}{\phi - \psi}\left[\frac{-1}{1-\psi z} + \frac{1}{1-\phi z}\right].$$

If $|z| < 1/\phi$, then $|\phi z| < 1$ and $|\psi z| < 1$ and consequently

$$\frac{1}{1-\phi z} = \sum_{n=0}^{+\infty} \phi^n z^n, \quad \frac{1}{1-\psi z} = \sum_{n=0}^{+\infty} \psi^n z^n.$$

Thus, $f(z)$ can be expanded as

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{\phi - \psi}\left[\phi^n - \psi^n\right] z^n.$$

The power series expansion of $f(z)$ in the disk centered on the origin with radius $1/\phi$ is unique, therefore

$$a_n = \frac{1}{\phi - \psi}\left[\phi^n - \psi^n\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for every $n \in \mathbb{N}$.

## Entire Functions Dominated By Polynomials

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the power series expansion of $f$ in $\mathbb{C}$. For any $r > 0$, we have

$$a_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

hence by the M-L estimation lemma,

$$|a_n| \leq \frac{\sup\left\{|P(re^{i2\pi t})| \mid t \in [0,1]\right\}}{r^n}.$$

For any $n > p$, letting $r \to +\infty$ provides $a_n = 0$. Hence, the function $f$ is a polynomial of degree at most $p$.

## Existence of Primitives

The function $f$ is defined and holomorphic in $\mathbb{C} \setminus [-1, 1]$ (the zeros of $\sin \pi/z$ are $z = 1/k$ for $k \in \mathbb{N}^*$).

We first consider the restriction of $f$ to the annulus $A(0, 1, +\infty)$. For any $z$ in this annulus, $-z$ also belong to it and $f(-z) = f(z)$. Hence, if $\sum_{n=-\infty}^{+\infty} a_n z^n$ is a Laurent series expansion of $f$, $\sum_{n=-\infty}^{+\infty} (-1)^n a_n z^n$ is another valid one. The uniqueness of the expansion yields that $a_n = 0$ if $n$ is odd; in particular, $a_{-1} = 0$ and the sum

$$\sum_{p=-\infty}^{+\infty} \frac{a_{2p}}{2p+1} z^{2p+1}$$

provide a primitive of $f$ on the annulus.

Now, let $\gamma$ be an arbitrary closed rectifiable path of $\mathbb{C} \setminus [-1, 1]$. Let $n = \operatorname{ind}(\gamma, 0)$; define the path $\mu : t \in [0, 1] \mapsto 2e^{i2\pi nt}$ and the sequence of paths $\nu = (\gamma, \mu^{\leftarrow})$. As $[-1, 1]$ is a connected subset of $\mathbb{C} \setminus \nu([0, 1])$, for any $z \in [-1, 1]$, $\operatorname{ind}(\nu, z) = \operatorname{ind}(\nu, 0) = 0$. Consequently, $\operatorname{Int} \nu \subset \mathbb{C} \setminus [-1, 1]$ and Cauchy's integral theorem provides

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

As $f$ has a primitive on the annulus $A(0, 1, +\infty)$, the integral in the right-hand side of this equation is equal to zero. The classic criteria therefore proves that primitives of $f$ exist in $\mathbb{C} \setminus [-1, 1]$.

## A Removable Set

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the Taylor series expansion of $f$ in $D(0, 1)$; we are going to prove that this expansion is actually a valid expansion of $f$ in $\mathbb{C}$. Consider the Laurent expansion $\sum_{n=-\infty}^{+\infty} b_n z^n$ of $f$ in $A(0, 1, +\infty)$. For any $n \in \mathbb{Z}$ and any $r > 1$, we have

$$b_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

thus, by continuity of $f$

$$\begin{aligned} b_n &= \lim_{r \to 1^+} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \\ &= \lim_{r \to 1^-} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \end{aligned}$$

and consequently, $b_n = a_n$ if $n$ is non-negative and zero otherwise. The sum $\sum_{n=0}^{+\infty} a_n z^n$ is defined for any $|z| > 1$, thus its open disk of convergence is the full complex plane. It is equal to $f$ on $\mathbb{C} \setminus \mathbb{U}$ and both functions are continuous on $\mathbb{C}$, hence they are equal on $\mathbb{C}$: the function $f$ is entire.

### Derivative of Power Series

Let $f_m(z) = \sum_{n=0}^{m} a_n (z-c)^n$. Every polynomial $f_m$ is holomorphic and the sequence converges locally uniformly to $f(z) = \sum_{n=0}^{+\infty} a_n (z-c)^n$ in the open disk of convergence $D(c,r)$ of the series, thus $f$ is holomorphic.

For any holomorphic function $\phi$ in $D(c,r)$ and any $\rho \in \,]0, r[$

$$\phi'(z) = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{\phi(w)}{(w-z)^2}.$$

Thus, for any $m \in \mathbb{N}$,

$$f_m'(z) = \sum_{n=1}^{m} n a_n (z-c)^{n-1} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2}.$$

The integrand above converges locally uniformly in $D(c,r)$, hence

$$\lim_{m \to +\infty} \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f(w)}{(w-z)^2} = f'(z).$$

Finally,

$$\sum_{n=1}^{+\infty} n a_n (z-c)^{n-1} = \lim_{m \to +\infty} \sum_{n=1}^{m} n a_n (z-c)^{n-1} = f'(z).$$

## Zeros & Poles

### The Weierstrass-Casorati Theorem

Assume that the image of $f$ is not dense in $\mathbb{C}$; let then $w \in \mathbb{C}$ be such that

$$\exists\, \epsilon > 0,\ \forall\, z \in \Omega,\ |f(z) - w| \geq \epsilon.$$

The function $z \in \Omega \mapsto 1/(f(z) - w)$ is defined and holomorphic. As it satisfies

$$\forall\, z \in \Omega,\ \left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\epsilon},$$

it is also bounded. Thus, the point $a$ is a removable singularity of the function, that can be extended as a holomorphic function $g$ on $\Omega \cup \{a\}$:

$$\forall\, z \in \Omega,\ g(z) = \frac{1}{f(z) - w}$$

By construction, $g$ has no zero in $\Omega$, thus $a$ is either not a zero of $g$, or a zero of finite multiplicity. Since

$$\forall\, z \in \Omega,\ f(z) = w + \frac{1}{g(z)}$$

in the first case $f(z) \to w + 1/g(a)$ when $z \to a$ thus $a$ is a removable singularity of $a$; in the second one, $|f(z)| \to +\infty$ when $z \to a$ thus $a$ is a pole of $f$.

Note that either way, there is a non-negative integer $p$ and a holomorphic function $h : \Omega \cup \{a\} \to \mathbb{C}$ such that $h(a) \neq 0$ and

$$\forall\, z \in \Omega \cup \{a\},\ g(z) = h(z)(z-a)^p.$$

As the function $g$ has no zero on $\Omega$, the function $h$ has no zero on $\Omega \cup \{a\}$; the function $1/h$ is defined and holomorphic on $\Omega \cup \{a\}$, $1/h(a) \neq 0$ and

$$\forall\, z \in \Omega,\ f(z) = w + \frac{1}{g(z)} = w + \frac{1}{h(z)} \frac{1}{(z-a)^p}.$$

Therefore, the point $a$ is either a removable singularity of $f$ (if $p = 0$), or a pole of order $p$ (if $p \geq 1$).

## The Maximum Principle

For any holomorphic function $f : \Omega \to \mathbb{C}$ and $a \in \Omega$, the point $a$ is a zero of the holomorphic function $z \mapsto f(z) - f(a)$. We will prove shortly that if $a$ is a zero of finite multiplicity of this function, $|f|$ does not have a local maximum at $a$. The conclusion of the proof follows by the Isolated Zeros Theorem.

Suppose that there is a positive integer $p$ such that

$$f(z) = f(a) + g(z)(z-a)^p$$

for some holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(a) \neq 0$; there is a function $\epsilon_a : \Omega \to \mathbb{C}$ such that $\epsilon_a(z) \to 0$ when $z \to a$ and

$$f(z) = f(a) + g(a)(z-a)^p + \epsilon_a(z)(z-a)^p$$

Assume that $f(a) \neq 0$ (if $f(a) = 0$, it is plain that $|f(a)| = 0$ cannot be a local maximum of $|f|$ at $a$). Let $\alpha$, $\beta$ and $\gamma$ be some real numbers such that

$$f(a) = |f(a)|e^{i\alpha},\ g(a) = |g(a)|e^{i\beta},\ \gamma = \frac{\theta - \alpha}{p}.$$

For small enough values $r > 0$, we have

$$|f(a + re^{i\gamma}) - (|f(a)| + |g(a)|r^p)e^{i\alpha}| \leq |\epsilon_a(a + re^{i\gamma})|r^p \leq \frac{|g(a)|}{2} r^p,$$

which yields

$$|f(a + re^{i\gamma})| \geq |f(a)| + |g(a)|r^p - \frac{|g(a)|}{2} r^p > |f(a)|.$$

Therefore $f$ has no maximum at $a$.

## The $\Pi$ Function

1. The function $t \in \mathbb{R}_+^* \mapsto t^z e^{-t}$ is continuous and thus measurable. Additionally, for any $t > 0$,

$$|t^z e^{-t}| = |e^{z \ln t} e^{-t}| = e^{(\operatorname{Re} z) \ln t} e^{-t} = t^{\operatorname{Re} z} e^{-t},$$

hence it is integrable if and only if $\operatorname{Re} z > -1$: the domain of $\Pi$ is

$$\{z \in \mathbb{C} \mid \operatorname{Re} z > -1\}$$

and it is open. Now, let $z$ and $h$ be complex numbers in this domain; the associated difference quotient satisfies

$$\begin{aligned}
\frac{\Pi(z+h) - \Pi(z)}{h} &= \int_0^{+\infty} \frac{t^{z+h} - t^z}{h} e^{-t} dt \\
&= \int_0^{+\infty} \frac{t^h - 1}{h} t^z e^{-t} dt \\
&= \int_0^{+\infty} \frac{e^{h \ln t} - 1}{h} t^z e^{-t} dt \\
&= \int_0^{+\infty} \left[ \frac{e^{h \ln t} - 1}{h \ln t} \right] t^z \ln t \, e^{-t} dt
\end{aligned}$$

The integrand converges pointwise when $h \to 0$:

$$\forall\, t > 0,\ \lim_{h \to 0} \left[ \frac{e^{h \ln t} - 1}{h \ln t} \right] t^z \ln t \, e^{-t} = t^z \ln t \, e^{-t}.$$

Additionally, we have

$$\forall\, z \in \mathbb{C}^*,\ \left| \frac{e^z - 1}{z} \right| \leq e^{|z|};$$

indeed, for any nonzero complex number $z$, the Taylor expansion of $e^z$ at the origin provides

$$\left| \frac{e^z - 1}{z} \right| = \left| \sum_{n=0}^{+\infty} \frac{1}{(n+1)!} z^n \right| = \sum_{n=0}^{+\infty} \frac{1}{(n+1)!} |z|^n \leq \sum_{n=0}^{+\infty} \frac{1}{n!} |z|^n.$$

Hence,

$$\left| \frac{e^{h \ln t} - 1}{h \ln t} \right| \leq e^{|h| |\ln t|} \leq \max(t^{|h|}, t^{-|h|})$$

and our integrand is dominated by

$$\max(t^{z+|h|}, t^{z-|h|}) \ln t \, e^{-t}$$

which is integrable whenever $\operatorname{Re}(z - |h|) > -1$. Finally, Lebesgue's dominated convergence theorem applies and $\Pi$ is holomorphic.

2. If $\operatorname{Re} z > -1$, then $\operatorname{Re}(z+1) > -1$ and

$$\Pi(z+1) = \int_0^{+\infty} t^{z+1} e^{-t}\, dt.$$

By integration by parts,

$$\begin{aligned}\Pi(z+1) &= [t^{z+1}(-e^{-t})]_0^{+\infty} - \int_0^{+\infty} (z+1)t^z(-e^{-t})\, dt \\ &= (z+1)\Pi(z).\end{aligned}$$

We have

$$\Pi(0) = \int_0^{+\infty} e^{-t}\, dt = [-e^{-t}]_0^{+\infty} = 1$$

and hence, by induction, $\Pi(n) = n!$ for any $n \in \mathbb{N}$.

3. There is at most one holomorphic extension $\Pi$ of the original function to the connected open set $\Omega$ by the isolated zeros theorem (two extensions would be identical on the original domain of $\Pi$, which is a non-empty open set: the set of zeros of their difference would not be isolated).

   It is plain that the function $z \mapsto \Pi(z+1) - (z+1)\Pi(z)$ is defined and holomorphic on $\Omega$, a connected open set of the plane. Similarly, by the isolated zeros theorem, it is identically zero and hence the functional equation $\Pi(z+1) = (z+1)\Pi(z)$ holds on $\Omega$.

4. We may define the extension $\Pi(z)$ as

$$\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}$$

for any natural number $n$ such $\operatorname{Re}(z+n) > -1$. This definition does not depend on the choice of $n$: if $m > n$, we have $\operatorname{Re}(z+m) > -1$ and

$$\Pi(z+m) = \Pi(z+n) \times (z+n+1)\cdots(z+m),$$

hence

$$\frac{\Pi(z+m)}{(z+1)(z+2)\cdots(z+m)} = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}.$$

It is plain that this extension of the original function $\Pi$ is holomorphic.

5. Let $n$ be a positive integer. Let $z$ be a complex number such that $|z - (-n)| < 1$; it satisfies $\operatorname{Re}(z+n) > -1$ and thus

$$\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n)}.$$

Consequently,

$$(z-(-n))\Pi(z) = \frac{\Pi(z+n)}{(z+1)(z+2)\cdots(z+n-1)}$$

and

$$\lim_{z\to -n}(z-(-n))\Pi(z) = \frac{\Pi(0)}{(-n-1)(-n-2)\cdots(-1)} = \frac{(-1)^{n-1}}{(n-1)!}.$$

As this number differ from zero, $z=-n$ is a simple pole of $\Pi$ and

$$\mathrm{res}(\Pi,-n) = \frac{(-1)^{n-1}}{(n-1)!}.$$

## Singularities and Residues

The function $z \mapsto \sin \pi z$ is defined and holomorphic in $\mathbb{C}$. Its Taylor expansion, valid for any $z \in \mathbb{C}$, is

$$\sin \pi z = \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} z^{2n+1}.$$

The function $z \mapsto \frac{\sin \pi z}{\pi z}$ is therefore defined and holomorphic in $\mathbb{C}^*$ where its Laurent expansion is

$$\frac{\sin \pi z}{\pi z} = \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n}}{(2n+1)!} z^{2n}.$$

The series on the right-hand side of this equation has no negative power of $z$: it is a power series that converges for any $z \in \mathbb{C}^*$, thus its open disk of convergence is actually $\mathbb{C}$. Its limit is a holomorphic function that extends $z \mapsto \frac{\sin \pi z}{\pi z}$ to $\mathbb{C}$, hence 0 is a removable singularity of this function (and its residue is 0).

The singularities of $z \mapsto 1/(\sin \pi z)^2$ are the zeros of $z \in \mathbb{C} \mapsto \sin \pi z$: the integers. The function is invariant if we substitute $z+k$ to $z$ for any $k \in \mathbb{Z}$, hence we may limit our analysis of the singularities to the origin. If $z$ is not an integer, we have

$$\frac{1}{(\sin \pi z)^2} = \frac{1}{\pi^2 z^2}\left(\frac{\pi z}{\sin \pi z}\right)^2.$$

The function $z \mapsto (\pi z / \sin \pi z)^2$ has a removable singularity at the origin and the value of its holomorphic extension at the origin is nonzero (it is 1), thus the origin is a double pole of the function. We have therefore

$$\mathrm{res}\left(z \mapsto \frac{1}{(\sin \pi z)^2}, 0\right) = \lim_{z\to 0}\left[\frac{z^2}{2}\frac{1}{(\sin \pi z)^2}\right]'.$$

We have

$$\left[\frac{z^2}{2}\frac{1}{(\sin \pi z)^2}\right]' = \frac{1}{\pi}\left(\frac{(\pi z)\sin \pi z - (\pi z)^2 \cos \pi z}{(\sin \pi z)^3}\right).$$

The Taylor expansions of the functions sin and cos on $\mathbb{C}$ provide

$$\sin w = w\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}w^{2n}\right) = w - \frac{w^3}{6} + w^5\left(\sum_{n=2}^{+\infty}\frac{(-1)^n}{(2n+1)!}w^{2n-4}\right)$$

and

$$\cos w = 1 - \frac{w^2}{2} + w^4\left(\sum_{n=2}^{+\infty}\frac{(-1)^n}{(2n)!}w^{2n-4}\right),$$

thus there are entire functions $f$ and $g$ such that

$$w \sin w - w^2 \cos w = \left(w^2 - \frac{1}{6}w^4\right) - \left(w^2 - \frac{1}{2}w^4\right) + w^6 f(w)$$

and

$$(\sin w)^3 = w^3 g(w), \; g(0) = 1.$$

Consequently,

$$\text{res}\left(z \mapsto \frac{1}{(\sin \pi z)^2}, 0\right) = \lim_{w \to 0}\frac{1}{\pi}\frac{w/3 + w^3 f(w)}{g(w)} = 0.$$

Alternatively, to compute the residue, we may notice that if $z$ is not an integer

$$\frac{1}{(\sin \pi z)^2} = \frac{1}{(\sin \pi(-z))^2},$$

thus if $\sum_{n=-\infty}^{+\infty} a_n z^n$ is the Laurent expansion of the right-hand side in $D(0,1) \setminus \{0\}$, the Laurent expansion $\sum_{n=-\infty}^{+\infty}(-1)^n a_n z^n$ is also valid in the same annulus. The uniqueness of the Laurent expansion yields $a_n = 0$ for every odd $n$, thus the residue of the function at the origin – which is $a_{-1}$ – is zero.

The function $z \mapsto \sin \frac{\pi}{z}$ is defined and holomorphic on $\mathbb{C}^*$. It has a Laurent expansion in this annulus, which is

$$\sin \frac{\pi}{z} = \sum_{n=0}^{+\infty}\frac{(-1)^n \pi^{2n+1}}{(2n+1)!} z^{-(2n+1)}.$$

There are an infinite number of nonzero coefficients associated with negative powers of $z$, thus 0 is an essential singularity of this function. Its residue at 0 is the coefficient of $z^{-1}$, which is $\pi$.

The zeros of $z \in \mathbb{C} \mapsto \sin \pi z$ are the integers, thus $z \mapsto 1/\sin \frac{\pi}{z}$ is defined and holomorphic on the open set $\Omega = \mathbb{C}^* \setminus \{1/k \mid k \in \mathbb{Z}^*\}$. We can write the function as the quotient of $f(z) = 1$ and $g(z) = \sin \frac{\pi}{z}$. The functions $f$ and $g$ are defined and holomorphic in $\mathbb{C}^*$ and

$$g'(z) = \left(\cos \frac{\pi}{z}\right)\left(-\frac{\pi}{z^2}\right).$$

Thus, for any $k \in \mathbb{Z}^*$, $1/k$ is a simple pole of $z \mapsto 1/\sin \frac{\pi}{z}$ and

$$\operatorname{res}\left(z \mapsto \frac{1}{\sin \frac{\pi}{z}}, \frac{1}{k}\right) = \frac{1}{(\cos \frac{\pi}{k^{-1}})(-\frac{\pi}{(k^{-1})^2})} = \frac{(-1)^{k+1}}{\pi k^2}.$$

The origin $z = 0$ is also singularity of $z \mapsto 1/\sin \frac{\pi}{z}$, but it is not isolated, thus its residue is not defined.

## Integrals of Functions of a Real Variable

1. Let $f$ be the function $z \mapsto 1/(1+z^n)$, defined and holomorphic on

$$\Omega = \mathbb{C} \setminus \left\{ e^{\frac{i(2k+1)\pi}{n}} \;\middle|\; k \in \{0, \dots, n-1\} \right\}.$$

Let $r > 1$ and define the rectifiable paths $\gamma_1$, $\gamma_2$ and $\gamma_3$ as

$$\gamma_1 = [0 \to r],\; \gamma_2 = re^{i[0 \to 2\pi/n]},\; \gamma_3 = [re^{i2\pi/n} \to 0],$$

then set $\gamma = \gamma_1 \mid \gamma_2 \mid \gamma_3$. It is plain that

$$\lim_{r \to 0} \int_{\gamma_1} \frac{dz}{1+z^n} = \int_0^{+\infty} \frac{dx}{1+x^n}.$$

Similarly,

$$\int_{\overleftarrow{\gamma_3}} \frac{dz}{1+z^n} = \int_0^1 \frac{re^{i\frac{2\pi}{n}}\, dt}{1+(rt)^n (e^{i\frac{2\pi}{n}})^n} = e^{i\frac{2\pi}{n}} \int_0^r \frac{dx}{1+x^n},$$

thus

$$\lim_{r \to 0} \int_{\gamma_3} \frac{dz}{1+z^n} = -e^{i\frac{2\pi}{n}} \int_0^{+\infty} \frac{dx}{1+x^n}.$$

Finally, by the M-L inequality,

$$\left| \int_{\gamma_2} \frac{dz}{1+z^n} \right| \leq \frac{1}{r^n - 1} \times \left(\frac{2\pi}{n} r\right),$$

hence

$$\lim_{r \to +\infty} \int_{\gamma_2} \frac{dz}{1+z^n} = 0.$$

On the other hand, the complex number $e^{i\frac{\pi}{n}}$ is the unique singularity of $f$ in the interior of $\gamma$; more precisely, we have $\operatorname{ind}(\gamma, e^{i\frac{\pi}{n}}) = 1$. The function $f$ is the quotient of the holomorphic functions $p : z \in \mathbb{C} \mapsto 1$ and $q : z \in \mathbb{C} \mapsto 1 + z^n$; the derivative of $q$ at this singularity is

$$q'(e^{i\frac{\pi}{n}}) = n(e^{i\frac{\pi}{n}})^{n-1} = n(e^{i\frac{\pi}{n}})^n e^{-i\frac{\pi}{n}} = -ne^{-i\frac{\pi}{n}},$$

thus

$$\operatorname{res}(f, e^{i\frac{\pi}{n}}) = \frac{p(e^{i\frac{\pi}{n}})}{q'(e^{i\frac{\pi}{n}})} = -\frac{e^{i\frac{\pi}{n}}}{n}$$

Given these results, the residue theorem provides

$$\left(1 - e^{i\frac{2\pi}{n}}\right) \int_0^{+\infty} \frac{dx}{1+x^n} = (i2\pi) \times \left(-\frac{e^{i\frac{\pi}{n}}}{n}\right)$$

or equivalently,

$$\int_0^{+\infty} \frac{dx}{1+x^n} = \frac{\pi}{n} \frac{2i}{e^{i\frac{\pi}{n}} - e^{-i\frac{\pi}{n}}} = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

2. Let $\log_0$ be the function defined on $\mathbb{C} \setminus \mathbb{R}_+$ by

$$\log_0 z = \log(-z) + i\pi.$$

This function is an analytic choice of the logarithm on $\mathbb{C} \setminus \mathbb{R}_+$: it is holomorphic and $\exp \circ \log_0$ is the identity. It also satisfies

$$\log_0 re^{i\theta} = (\ln r) + i\theta,\ r > 0,\ \theta \in \left]0, 2\pi\right[.$$

We use this function to define

$$f : z \mapsto \frac{e^{\frac{1}{2}\log_0 z}}{1 + z + z^2}.$$

The roots of the polynomial $z \mapsto 1 + z + z^2$ are $j$ and $j^2$, where $j = e^{i\frac{2\pi}{3}}$, thus $f$ is defined and holomorphic in $\Omega = \mathbb{C} \setminus \mathbb{R}_+ \setminus \{j, j^2\}$.

Now, let $r > 1$ and $0 < \alpha < 2\pi/3$; we define four rectifiable paths that depend on $r$ and $\alpha$:

$$\begin{aligned}
\gamma_1 &= [r^{-1}e^{i\alpha} \to re^{i\alpha}],\\
\gamma_2 &= re^{i[\alpha \to 2\pi - \alpha]},\\
\gamma_3 &= [re^{i(2\pi-\alpha)} \to r^{-1}e^{i(2\pi-\alpha)}],\\
\gamma_4 &= r^{-1}e^{i[2\pi-\alpha \to \alpha]}.
\end{aligned}$$

We also consider their concatenation

$$\gamma = \gamma_1 \mid \gamma_2 \mid \gamma_3 \mid \gamma_4.$$

We have

$$\int_{\gamma_1} f(z)\,dz = \int_{r^{-1}}^{r} \frac{e^{\frac{1}{2}((\ln x)+i\alpha)}}{1+xe^{i\alpha}+x^2e^{i2\alpha}}\, e^{i\alpha}dx$$
$$= e^{i3\alpha/2}\int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+xe^{i\alpha}+x^2e^{i2\alpha}}\, dx$$

and thus by the dominated convergence theorem[2]

$$\lim_{\alpha\to 0}\int_{\gamma_1} f(z)\,dz = \int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+x+x^2}\, dx.$$

Similarly,

$$\int_{\gamma_3^{\leftarrow}} f(z)\,dz = \int_{r^{-1}}^{r} \frac{e^{\frac{1}{2}((\ln x)+i(2\pi-\alpha))}}{1+xe^{-i\alpha}+x^2e^{-i2\alpha}}\, e^{-i\alpha}dx$$
$$= -e^{-i3\alpha/2}\int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+xe^{-i\alpha}+x^2e^{-i2\alpha}}\, dx$$

and thus by the dominated convergence theorem

$$\lim_{\alpha\to 0}\int_{\gamma_3} f(z)\,dz = \int_{r^{-1}}^{r} \frac{\sqrt{x}}{1+x+x^2}\, dx$$

On the other hand,

$$\left|e^{\frac{1}{2}\log_0 z}\right| = e^{\mathrm{Re}(\frac{1}{2}\log_0 z)} = e^{\frac{1}{2}\ln|z|} = |z|^{\frac{1}{2}};$$

by the M-L inequality, this equality provides

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{r^{\frac{1}{2}}}{-1-r+r^2}\times 2(\pi-\alpha)r$$

and

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{r^{-\frac{1}{2}}}{1-r^{-1}-r^{-2}}\times 2(\pi-\alpha)r^{-1},$$

hence

$$\lim_{r\to+\infty}\left(\lim_{\alpha\to 0}\int_{\gamma_2} f(z)\,dz\right) = \lim_{r\to+\infty}\left(\lim_{\alpha\to 0}\int_{\gamma_4} f(z)\,dz\right) = 0.$$

Now the function $f$ is the quotient of the two functions $z\mapsto e^{\frac{1}{2}\log_0 z}$ and $z\mapsto 1+z+z^2$, defined and holomorphic in a neighbourhood of the

[2] the function $(\alpha,x)\mapsto\left|\sqrt{x}/(1+xe^{i\alpha}+x^2e^{i2\alpha})\right|$ is defined and continuous in the compact set $[0,\pi/2]\times[r^{-1},r]$, thus it has a finite upper bound.

singularities $j$ and $j^2$. The derivative of $z \mapsto 1+z+z^2$ is $z \mapsto 1+2z$, it is nonzero at $j$ and $j^2$. Thus,

$$\operatorname{res}(f,j) = \frac{e^{\frac{1}{2}\log_0 j}}{1+2j} = \frac{e^{i\frac{\pi}{3}}}{i\sqrt{3}}$$

and

$$\operatorname{res}(f,j^2) = \frac{e^{\frac{1}{2}\log_0 j^2}}{1+2j^2} = \frac{e^{i\frac{2\pi}{3}}}{-i\sqrt{3}}.$$

The winding number of $\gamma$ around $j$ and $j^2$ is 1; by the residue theorem,

$$2\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\,dx = (i2\pi)(\operatorname{res}(f,j)+\operatorname{res}(f,j^2))$$

or equivalently

$$\int_0^{+\infty} \frac{\sqrt{x}}{1+x+x^2}\,dx = \frac{\pi}{\sqrt{3}}(e^{i\frac{\pi}{3}} - e^{i\frac{2\pi}{3}}) = \frac{\pi}{\sqrt{3}}.$$

# Analytic Functions

## Taylor Series of a Rational Function

1. The function $f$ is the restriction of the holomorphic function

$$f^* : z \in \mathbb{C} \setminus \{i,-i\} \mapsto \frac{1}{1+z^2}.$$

2. For any $x_0 \in \mathbb{R}$, the disk

$$D(x_0) = D(x_0, \sqrt{1+x_0^2})$$

is included in $\mathbb{C} \setminus \{i,-i\}$. The radius of the disk of convergence of the Taylor expansion of $f^*$ at $x_0$ is therefore at least $\sqrt{1+x_0^2}$; it cannot exceed this threshold: otherwise, the sum $g(z)$ of its Taylor series would be defined and holomorphic in an open set that contains $\overline{D_0}$ and therefore bounded on $D_0$; but $g$ and $f^*$ are identical on $D_0$ where $f^*$ is unbounded. Finally, as the Taylor expansion of $f$ at $x_0$ has the same coefficient as the Taylor expansion of $f^*$ at $x_0$, the open interval of convergence of $f^*$ at $x_0$ is

$$\left]x_0 - \sqrt{1+x_0^2}, x_0 + \sqrt{1+x_0^2}\right[.$$

## Analytic Functions of a Real Variable

1. By induction, for any $x > 0$, $f^{(n)}(x) = g_n(x)e^{-1/x}$ where $g_n$ is a rational function, defined for $x > 0$ by

   $$g_0(x) = 1 \ \wedge \ \forall\, n \in \mathbb{N},\ g_{n+1}(x) = g_n'(x) + \frac{g_n(x)}{x^2}.$$

   On the other hand, for $x \leq 0$, the $n$-th order derivative (left-derivative at $x = 0$) of $f$ at $x$ is defined and equal to 0. To prove that $f$ is smooth, we now have to prove that the right, $n$-th order derivative of $f$ at 0 exists and is equal to its left derivative, that is zero. We may proceed by induction: suppose that $f^{(n)}(0)$ exists and is zero; then

   $$\frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \frac{1}{h}\int_0^h f^{(n+1)}(x)dx.$$

   But for any $n$, we have

   $$\lim_{x \to 0^+} f^{(n+1)}(x) = \lim_{x \to 0^+} g_{n+1}(x)e^{-1/x} = 0,$$

   hence the right-hand side of the equation tends to zero when $h \to 0^+$: the $n+1$-th order right derivative of $f$ exists at $x = 0$ and is equal to zero.

   Now, the function $f$ cannot be analytic: given that its derivatives at $x = 0$ are zero at all order, its Taylor series expansion at the origin is zero. The function $f$ would be zero in a neighbourhood of the origin, and this property does not hold.

2. If the function $f$ is smooth, for any real numbers $c$ and $y$ in $K$, the Taylor formula with integral remainder is applicable at any order $n$:

   $$f(y) = \sum_{p=0}^{n} \frac{f^{(p)}(c)}{p!}(y-c)^p + \int_c^y \frac{f^{(n+1)}(x)}{n!}(x-c)^n dx.$$

   If there exist $\alpha > 0$ and $r > 0$ such that the inequality

   $$\forall\, x \in K,\ |f^{(n)}(x)| \leq \alpha\, r^n n!$$

   holds, the remainder satisfies

   $$\begin{aligned}\left|\int_c^y \frac{f^{(n+1)}(x)}{n!}(x-c)^n dx\right| &\leq \alpha\, r^{n+1}\frac{(n+1)!}{n!}\left|\int_c^y (x-c)^n\, dx\right|. \\ &= \alpha\,(r|y-c|)^{n+1}\end{aligned}$$

   Thus, if $|y - c| < 1/r$, the Taylor expansion of $f$ at $y$ centered on $c$ is convergent. As $c$ is an arbitrary point of $I$, the function $f$ is analytic.

   Conversely, if $f$ is analytic, it has a holomorphic extension – that we may still denote $f$ – to some open neighbourhood $U$ of $K$. The distance $d$

between $K$ and $\mathbb{C}\setminus U$ is positive: for any $c \in K$, the disk $D(c,d)$ is included in $U$. Let $r$ be a positive radius smaller than $d$ and $\alpha$ be an upper bound of $|f|$ on $K + \overline{D}(0,r)$; for any natural number $n$, we have

$$\left|\frac{f^{(n)}(c)}{n!}\right| = \left|\frac{1}{i2\pi}\int_{r[\circlearrowleft]+c} \frac{f(z)}{(z-c)^{n+1}}dz\right| \leq \alpha\left(r^{-1}\right)^n$$

which concludes the proof.

## Periodic Analytic Functions

1. The set $\mathbb{U}$ is compact and the set $\mathbb{C}\setminus U$ is closed; their intersection is empty, thus the distance $d = d(\mathbb{U}, \mathbb{C}\setminus U)$ is positive (it may be $+\infty$ if $U = \mathbb{C}$). On the other hand, for any $r < 1$,

$$d(\mathbb{U}, \mathbb{C}\setminus A_r) = \min(1-r, 1/r-1) = 1-r.$$

Thus, for any $r$ such that $1 - r < d$, the annulus $A_r$ is a subset of $U$.

2. The $2\pi$-periodicity of $g$ is clear: for any $t \in \mathbb{R}$,

$$g(t+2\pi) = f(e^{i(t+2\pi)}) = f(e^{it}e^{i2\pi}) = f(e^{it}) = g(t).$$

The assumption on $f$ and the result of the previous question provide a holomorphic extension $f^* : A_r \to \mathbb{C}$ to $f : \mathbb{U} \to \mathbb{C}$ for some $r < 1$. Now,

$$|e^{iz}| = e^{\mathrm{Re}\ iz} = e^{-\mathrm{Im}\ z},$$

thus if $|\mathrm{Im}\ z| < \ln 1/r$, then $\ln r < -\mathrm{Im}\ z < \ln 1/r$ which yields $r < |e^{iz}| < 1/r$. Therefore, if we set $\epsilon = \ln 1/r > 0$, we have

$$\forall\, z \in \mathbb{C},\ (z \in \Omega_\epsilon \Rightarrow e^{iz} \in A_r).$$

Consequently, setting $g^*(z) = f^*(e^{iz})$ defines a function $g^*$ on $\Omega_\epsilon$; it is an extension of $g : \mathbb{R} \to \mathbb{C}$ and it is holomorphic as the composition of holomorphic functions. Therefore, the function $g : \mathbb{R} \to \mathbb{C}$ is analytic.

**Alternate proof.** Consider the Laurent series expansion of $f^*$ in $A_r$:

$$f^*(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

For any real numbers $t_0$ and $t$, we have

$$(e^{it})^n = e^{int} = e^{int_0}e^{in(t-t_0)} = e^{int_0}\sum_{m=0}^{+\infty}\frac{1}{m!}i^m n^m (t-t_0)^m,$$

therefore

$$g(t) = f(e^{it}) = \sum_{n=-\infty}^{+\infty} a_n e^{int_0} \left[ \sum_{m=0}^{+\infty} \frac{1}{m!} i^m n^m (t-t_0)^m \right].$$

We can change the order of the summation in this double series if

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{Z}} \left| a_n e^{int_0} \frac{1}{m!} i^m n^m (t-t_0)^m \right| < +\infty$$

The general term of this double series satisfies

$$\left| a_n e^{int_0} \frac{1}{m!} i^m n^m (t-t_0)^m \right| \le |a_n| \frac{1}{m!} |n|^m |t-t_0|^m$$

hence the sum is bounded by

$$\sum_{n=-\infty}^{+\infty} |a_n| (e^{|t-t_0|})^{|n|} \le \sum_{n=-\infty}^{+\infty} |a_n| (e^{t-t_0})^n + \sum_{n=-\infty}^{+\infty} |a_n| (e^{-(t-t_0)})^n.$$

The Laurent series expansion of $f^*$ is absolutely convergent in $A_r$, hence the sums in the right-hand side of this inequality are finite if

$$r < e^{t-t_0} < 1/r \text{ and } r < e^{-(t-t_0)} < 1/r$$

that is if $|t-t_0| < \epsilon = \ln 1/r$. After the change in the order of the summation, we end up with:

$$\forall\, t \in \mathbb{R},\ |t-t_0| < \epsilon \Rightarrow g(t) = \sum_{m=0}^{+\infty} b_m (t-t_0)^m$$

where

$$b_m = \left[ \sum_{n=-\infty}^{+\infty} a_n e^{int_0} \frac{1}{m!} i^m n^m \right],$$

hence the function $g$ is analytic.

3. The function $g$ is analytic; let $g^0$ be an analytic extension of $g$ in some open neighbourhood $\Omega$ of $\mathbb{R}$. However, if the distance between $\mathbb{R}$ and $\mathbb{C} \setminus \Omega$ is equal to zero – it may happen as both sets but neither of them is compact – then $\Omega$ contains no strip $\Omega_\epsilon$.

   Let's build a new analytic extension $g^*$ on such a strip from $g^0$. First, the set $\Omega$ contains some open tubular neighbourhood $V_\epsilon$ of $[0, 2\pi]$ for any $\epsilon > 0$ small enough:
   $$V_\epsilon = \{z \in \mathbb{C} \mid d(z, [0, 2\pi]) < \epsilon\} \subset \Omega.$$
   Indeed, $[0, 2\pi]$ is compact, $\mathbb{C} \setminus \Omega$ is closed and their intersection is empty, hence $d(\mathbb{C} \setminus \Omega, [0, 2\pi]) > 0$; any $\epsilon$ smaller than (or equal to) this distance is admissible.

Consider the function $g^*$ defined on $\Omega_\epsilon$ by

$$g^*(z) = g^0(z + 2\pi k) \ \text{ if } \ k \in \mathbb{Z} \ \text{ and } \ z + 2\pi k \in V_\epsilon.$$

It is plain that $g^*$ is analytic and extends $g$ to $\Omega_\epsilon$; by construction it also satisfies the property

$$\forall\, z \in \Omega_\epsilon,\ g^*(z + 2\pi) = g^*(z).$$

The only point to check is that this definition is unambiguous, as we may have for some $z$ several integers $k$ and $\ell$ such that $z_k = z + 2\pi k \in V_\epsilon$ and $z_\ell = z + 2\pi\ell \in V_\epsilon$. Assume for example that $k < \ell$; in this case $z_k \in D(0,\epsilon)$ and $\ell = k + 1$, *i.e.* $z_\ell = z_k + 2\pi$. The functions

$$w \in D(0,\epsilon) \mapsto g^0(w) \ \text{ and } \ w \in D(0,\epsilon) \mapsto g^0(w + 2\pi)$$

are holomorphic and identical on $]-\epsilon, \epsilon[$; by the isolated zeros theorem, they are identical on $D(0,\epsilon)$ (which is connected) and in particular $g(z_k) = g(z_\ell)$. The definition of $g^*$ is actually unambiguous.

4. To answer the question, we exhibit an analytic function $f^* : A_r \to \mathbb{C}$ with $\epsilon = \ln 1/r$ (or equivalently $r = e^{-\epsilon}$) such that

$$\forall\, z \in A_r,\ f^*(e^{iz}) = g^*(z).$$

For any $w \in \mathbb{C}^*$, there is a solution $z_0$ to the equation

$$e^{iz} = w,\ z \in \mathbb{C}$$

and the other solutionss are $z_0 + 2\pi k$, for $k \in \mathbb{Z}$. Additionally, if $w \in A_r$, then $z \in \Omega_\epsilon$ with $\epsilon = \ln 1/r$. We may define $f^* : A_r \to \mathbb{C}$ by

$$f^*(w) = g^*(z),\ e^{iz} = w.$$

This definition is unambiguous: two $z$ that correspond to the same $w$ differ from a multiple of $2\pi$, but $g^*$ is $2\pi$-periodic hence the right-hand sides of this definition are equal.

Let's prove that $f^*$ is analytic. Let $w_0$ in $A_r$ and $z_0$ such that $e^{iz_0} = w_0$, the expression

$$\phi(w) = -i \log \frac{w}{w_0} + z_0$$

defines an analytic function $\phi$ in an neighbourhood of $w_0$. It satisfies $e^{i(\phi(w) - z_0)} = w/w_0$, thus

$$e^{i\phi(w)} = w.$$

Consequently, in a neighbourhood of $w_0$,

$$f^*(w) = g^*(\phi(w))$$

and $f^*$ is holomorphic – locally everywhere – as a composition of holomorphic functions.

# Integral Representations

## Functions of Several Complex Variables

We may define the embedding functions $e_{k,z} : \mathbb{C} \to \mathbb{C}^n$ by

$$e_{k,z}(w) = (z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n).$$

It is plain that the $e_{k,z}$ are continuous. A function $f_{k,z}$ is defined on the preimage of the open set $\Omega$ by $e_{k,z}$ which is therefore an open set.

Assume that $f$ is complex-differentiable; it is continuous. Additionally, $f_{k,z} = f \circ e_{k,z}$; as the function $e_{k,z}$ is complex-linear, it is complex-differentiable and $f_{k,z}$ is complex-differentiable (or holomorphic) as the composition of complex-differentiable functions.

Conversely, if $f$ if continuous and every partial function $f_{k,z}$ is complex-differentiable, the function $f$ itself is complex-differentiable as every partial derivative $z \in \Omega \mapsto (\partial f / \partial z_k)(z)$ is continuous – not merely as a function of its $k$-th variable which is plain, but as a function of all its variables.

Let $z = (z_1, \ldots, z_n) \in \Omega$, let $c \in \mathbb{C}$ and $r > 0$ such that

$$\forall\, w \in \mathbb{C},\ |w - c| \leq r \ \to\ (z_1, \ldots, z_{k-1}, w, z_k, \ldots, z_n) \in \Omega.$$

Cauchy's formula, applied to the partial function $f_{k,z}$ for the path $\gamma = c + r[\circlearrowleft]$, provides

$$f(z_1, \ldots, z_n) = \frac{1}{i2\pi} \int_\gamma \frac{f(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{w - z_k} dw$$

The integrand is continuous with respect to the pair $(z_1, w_1)$ and complex-differentiable with respect to $z_1$, thus we may compute the partial derivative of $f$ with respect to $z_k$ satisfies by differentiation under the integral sign:

$$\frac{\partial f}{\partial z_k}(z_1, \ldots, z_n) = \frac{1}{i2\pi} \int_\gamma \frac{f(z_1, \ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{(w - z_k)^2} dw.$$

As the function $f$ is continuous, the partial derivative is also continous.