Complex Analysis

R. ROOPKUMAR

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Complex Analysis

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Complex Analysis

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Preface

There are many good text books on complex analysis, some of which are listed in the bibliography. In my opinion, most of the books are written with the assumption that the reader can understand the intricacies of the proofs by filling the gaps in the arguments. However, based on my personal experience of offering several courses on complex analysis to the present generation of students, I sensed the necessity to write a book on complex analysis by explaining each and every argument in any proof in a lucid manner so that the book would be an ideal self study material for the students. The present book has been written to address this need. Since many concepts in complex analysis are geometrical in nature, more geometrical arguments are given, without any compromise in rigor. While the detailed proofs presented in the book may appear to be self-evident to the experts in complex analysis, beginner students who try to learn the subject with rigor and without any assumptions will find such treatment helpful. At the same time, this book may also be used as a hand book by a young teacher who needs explanations for some tedious theorems in complex analysis.

This text book is intended for both under graduate and post graduate courses in complex analysis. The first chapter consists of the basic concepts of complex numbers, operations on the complex numbers, topological properties and limiting concepts with more details. All the results given in this chapter are used at least once in the later part of this book. For an undergraduate course, the chapters 2,3,4,5,6 may be prescribed by omitting big theorems. For a postgraduate course, chapters 1 to 6 may be prescribed as a first course on complex analysis. For those who have the desired level of exposure to complex analysis in an under graduate course, directly chapters 4,5,6,7,8 may be prescribed in the post graduate course. In the last chapter, the properties of Jacobian elliptic functions are having very long proofs. Indeed, these proofs have been presented to motivate the young students and urge them persevere instead of being bewildered by the subtle nuances and complications involved. However, my personal opinion is that it is not advisable to ask the proofs of those results in any examination.

The content of the book is subdivided into 8 chapters with two objectives. The first objective is to club similar topics under a chapter title and the second one is to maintain the size of the chapters as uniform as possible. I have made sure that each chapter has a sufficient number of results and problems of varied lengths, so that it becomes suitable for setting questions of all types. For doing this, I realized that I either had to compromise on at least one of the above said objectives or I had to use two results from the later chapters in earlier chapters. I have chosen the latter option. However, I have confirmed that there is no begging of question by using a result before it is proved. There are some concepts such as the Jordan curve Theorem, Hardamard's Theorem regarding the estimate of genus of an infinite product, introduction to univalent and star-like functions that are not discussed here, since these are to be taken up at an advanced level beyond the scope of this book. I have taken stringent efforts to make this book error-free. Nonetheless, if any error is found, please write to me so that they may be eliminated in future editions. My e-mail address is roopkumar_r@yahoo.co.in

I am grateful to Dr R. Vembu for his constant encouragement, fruitful discussions and valuable suggestions while preparing the content for this book. My thanks are also due to the editors at Pearson Education who worked on this book, for their professionalism and diligence.

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Complex Analysis

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1

Preliminaries

1.1 INTRODUCTION AND BRIEF PREREQUISITES

Every mathematical structure has some good properties as well as some deficiencies. To overcome a particular deficiency on a mathematical structure, it is, therefore, customary to generalize the mathematical structure. However, it should be noted that any generalized mathematical structure having a good property that is not in the earlier structure will not satisfy another good property of an earlier structure. Consider the following examples:

1. $\mathbb{Z}$ is an ordered abelian group, whereas $\mathbb{N}$ is an ordered commutative semi-group but not a group. *However, every non-empty subset of* $\mathbb{N}$ *has a minimum in* $\mathbb{N}$*, which is not true in* $\mathbb{Z}$*.*

2. $\mathbb{Q}$ is an ordered field, whereas $\mathbb{Z}$ is an ordered integral domain but not a field. *However, for every element of* $\mathbb{Z}$*, we can find previous element and next element in* $\mathbb{Z}$*, which is not true in* $\mathbb{Q}$*.*

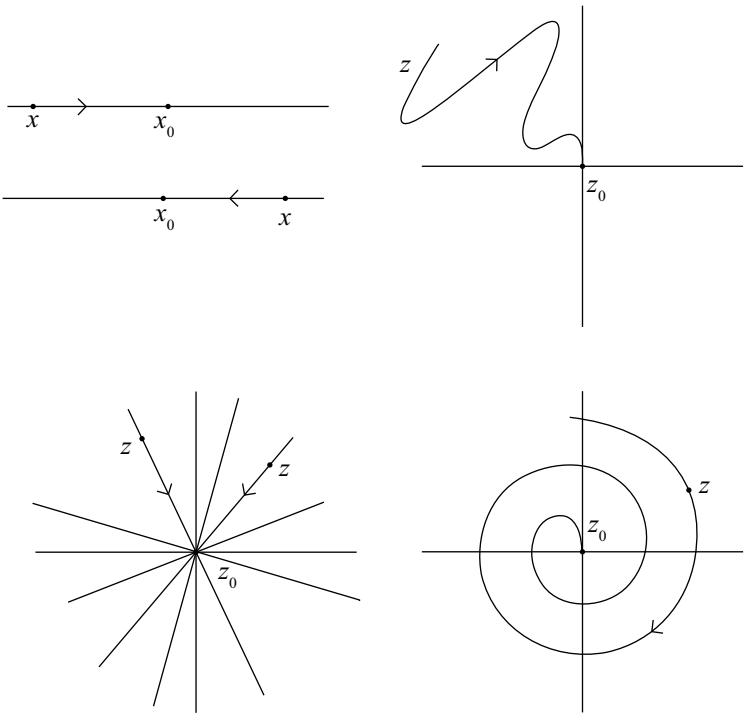
3. $\mathbb{R}$ is an ordered field with least upper bound property, whereas $\mathbb{Q}$ is not having least upper bound property. *However,* $\mathbb{Q}$ *is countable, whereas* $\mathbb{R}$ *is not countable.*

The set $\mathbb{R}$ of all real numbers is a very good setup in which we have lot of mathematical structures such as Archimedian field with least upper bound property, complete metric space having Heine–Borel property and Banach space (a complete normed linear space), and so on. In fact, it is identified with the set of all points on a straight line. However, in the algebraic point of view, not all polynomials of degree n (for some $n \in \mathbb{N}$) over $\mathbb{R}$ have n roots in $\mathbb{R}$. To be specific, $x^2 + 1$ is a polynomial of degree 2 over $\mathbb{R}$, which has no solution in $\mathbb{R}$. This is the motivation for introducing the complex number system in which every polynomial of degree n over $\mathbb{C}$ has exactly n roots in $\mathbb{C}$. In fact, $\mathbb{C}$ is the splitting field of $x^2 + 1$, considering

$x^2 + 1$ as a polynomial over $\mathbb{R}$. As in the earlier generalization of number systems (just listed above), $\mathbb{R}$ is an ordered field, whereas there is no order relation on $\mathbb{C}$, which makes it as an ordered field. However, complex analysis has more good and different properties that cannot be expected in real analysis.

As $\mathbb{R}$ and $\mathbb{C}$ are metric spaces, we can say all the topological properties that are common in a metric space such as the properties of open sets, closed sets, limit point of a set, convergent sequences and series, limit of a function, and continuous functions etc are similar in $\mathbb{R}$ and $\mathbb{C}$. However, it is interesting to note some of the good differences between real and complex analysis.

1. If $x \to x_0$ in $\mathbb{R}$, then x can approach only in two directions along the real axis. However, if $z \to z_0$ in $\mathbb{C}$, then z gets closer to z_0 through uncountable number of paths. See the following diagrams for some examples.



2. We know that interior of A is connected for every connected subset A of $\mathbb{R}$. However, there is a connected subset A of $\mathbb{C}$ whose interior is not connected.

3. If $f = (u, v) : \mathbb{C} \to \mathbb{C}$ is differentiable, then for the given u, we can find v and for the given v, we can find u (uniquely up to a constant). However, if $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is differentiable, then neither f_2 can be found from f_1 nor f_1 can be found from f_2.

4. If $f : \Omega \to \mathbb{C}$ is (once) differentiable (where Ω is an open connected subset of $\mathbb{C}$), then f is infinitely many times differentiable. However, if $f : I \to \mathbb{R}$ (where I is an interval) is differentiable, then f' need not be even continuous.

5. If $f : \mathbb{C} \to \mathbb{C}$ is a bounded differentiable function, then f is a constant function. However, it is not true for a bounded differentiable real-valued function on $\mathbb{R}$.

6. If f is a complex-valued differentiable function on $\Omega \subseteq \mathbb{C}$ such that $f = 0$ on a set having a limit point must be identically zero. However, it is not true for a differentiable real-valued function on an interval of $\mathbb{R}$.

7. Every non-constant differentiable function $f : \Omega \to \mathbb{C}$ is an open map. However, this is not true for a differentiable real-valued function on an interval of $\mathbb{R}$.

8. Although the above points are some positive properties of complex analysis that are not in real analysis, graph of a complex-valued function of a complex variable cannot be visualized, whereas the graph of a real-valued function of a real variable can be plotted. However, the geometric properties of a complex-valued function of a complex variable can be studied by seeing the image of a curve or a region under the given function.

To discuss any branch of Mathematics, it is necessary to know the required basic definitions and results. Therefore, to make this this book self-contained, we carefully selected the most useful definitions and results from set theory, real analysis, and algebra, and presented them in this section.

1. We call a collection of '*well defined*' objects a set.

2. A set containing no elements is called an empty set. To denote this special set, we use the notation $\varnothing$.

3. To say that x is an element of a set, we write $x \in A$.

4. We say that B is a subset of A if $x \in A$ whenever $x \in B$. We write this symbolically as $B \subseteq A$.

5. We say that A and B are equal if $A \subseteq B$ and $B \subseteq A$.

6. Let X be a set, A, B and $A_\alpha \subseteq X$, $\forall \alpha \in I$ for some index set I, then

   (a) the union $\underset{\alpha \in I}{\cup} A_\alpha$ of $\{A_\alpha\}_{\alpha \in I}$ is defined by $\{x : x \in A_\alpha$, for some $\alpha \in I\}$.

   (b) the intersection $\underset{\alpha \in I}{\cap} A_\alpha$ of $\{A_\alpha\}_{\alpha \in I}$ is defined by $\{x : x \in A_\alpha$, $\forall \alpha \in I\}$.
   When $I = \{1, 2, 3, \ldots, n\}$, we also use the following equivalent notations:

   $$\underset{\alpha \in I}{\cup} A_\alpha = \overset{n}{\underset{\alpha=1}{\cup}} A_\alpha = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$

   and

   $$\underset{\alpha \in I}{\cap} A_\alpha = \overset{n}{\underset{\alpha=1}{\cap}} A_\alpha = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n,$$

   (c) The complement $A^c = X \setminus A$ of A in X is defined by the set $\{x \in X : x \notin A\}$.

   (d) A and B are said to be disjoint if $A \cap B = \varnothing$. We also note that $A \cap B = \varnothing$ iff $A \subseteq B^c$ iff $B \subseteq A^c$.

   (e) **De-Morgan's laws**
   (i) $\left(\underset{\alpha \in I}{\cup} A_\alpha\right)^c = \underset{\alpha \in I}{\cap} A_\alpha^c$ and (ii) $\left(\underset{\alpha \in I}{\cap} A_\alpha\right)^c = \underset{\alpha \in I}{\cup} A_\alpha^c$.

7. Let $\{X_i\}_{i=1}^n$ be a finite collection of sets, then the Cartesian product

   $$X_1 \times X_2 \times \cdots \times X_n \text{ (or) } \prod_{i=1}^{n} X_i$$

   consists of all elements of the form $(x_1, x_2, x_3, \ldots, x_n)$ called ordered n-tuples, where $x_i \in X_i$, $\forall i = 1, 2, 3, \ldots, n$. We say that two elements $(x_1, x_2, x_3, \ldots, x_n)$ and $(y_1, y_2, y_3, \ldots, y_n)$ are said to be equal in $\prod_{i=1}^{n} X_i$ if $x_i = y_i$ and $\forall i = 1, 2, 3, \ldots, n$. The members of $\prod_{i=1}^{n} X_i$ are called ordered pairs when $n = 2$ and ordered triplets when $n = 3$.

One can say that mathematical analysis cannot be discussed without inequalities. Therefore, now we recall the definitions of order relation, lower bound, upper bound, infimum, and supremum and state some useful inequalities.

1. A relation R on a non-empty set X is a subset of $X \times X$.

2. A relation R on X is said to be an order relation if

   (a) $(x,y),(y,z) \in R \Rightarrow (x,z) \in R$;

   (b) for every $x, y \in X$, one of the following is true:
   (i) $x = y$ (ii) $(x,y) \in R$ (iii) $(y,x) \in R$

   If R is an ordered relation on X, then it is customary to write $(x,y) \in R$ by $x < y$ and $(X, <)$ is called an ordered set. In an ordered set $(X, <)$, the notations $x \leq y$ and $a \geq b$ mean that ($x < y$ or $x = y$) and ($b < a$ or $a = b$), respectively.

3. Let $(X, <)$ be an ordered set and $\varnothing \neq A \subseteq X$, then

   (a) an element $\alpha \in X$ is said to be an upper bound of A, if

   $$x \leq \alpha, \ \forall x \in A.$$

   (b) A is said to be bounded above in X if A has an upper bound $\alpha \in X$.

   (c) least upper bound of A is an upper bound β of A with the property that

   $$\beta \leq \gamma, \text{ for every upper bound } \gamma \text{ of } A.$$

   least upper bound of A is also called supremum of A and is denoted by $\sup A$ in X or simply by $\sup A$.

   (d) an element $\alpha \in X$ is said to be a lower bound of A, if

   $$\alpha \leq x, \ \forall x \in A.$$

   (e) A is said to be bounded below in X if A has a lower bound $\alpha \in X$.

   (f) greatest lower bound of A is a lower bound β of A with the property that

   $$\gamma \leq \beta, \text{ for every lower bound } \gamma \text{ of } A.$$

   greatest lower bound of A is also called infimum of A and is denoted by $\inf A$ in X or simply by $\inf A$.

(g)
- $\sup A$ is unique if it exists.
- $\inf A$ is unique if it exists.
- If $\alpha = \sup A$ and $\alpha \in A$, then α is called the maximum of A.
- If $\alpha = \inf A$ and $\alpha \in A$, then α is called the minimum of A.

(h) X is said to have least upper bound property, if for every bounded above subset A of X, $\sup A$ exists in X.

(i) X is said to have greatest lower bound property, if for every bounded below subset A of X, $\inf A$ exists in X.

(j) X has greatest lower bound property iff it has least upper bound property.

(k) $\mathbb{R}$ is an ordered set with least upper bound property and Archimedian property[1].

(l) For given $x, y \in \mathbb{R}$ such that $x < y$, there exists $p \in \mathbb{Q}$ such that $x < p < y$.

(m) For a given $x > 0$ there exists unique $y > 0$ such that $y^n = x$. This y is called the n^{th} root of x and is denoted by $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Although the content of this book includes some topological properties of subsets of the complex plane, mainly we concentrate on functions of a complex variable. Therefore, now we recall some basic ideas from function theory.

1. Let A and B be non-empty sets. We say that f is a function from A into B (denoted by $f : A \to B$) if f is a subset of $A \times B$ with the following properties:

   (a) for every $a \in A$, there exists $b \in B$ such that $(a, b) \in f$.

   (b) if $(a, b_1), (a, b_2) \in f$, then $b_1 = b_2$.

2. Let $f : A \to B$ be a function.

   (a) Then A and B are called the domain and codomain of f, respectively.

   (b) $(a, b) \in f$ is equivalently denoted by $f(a) = b$.

   (c) If $f : A \to A$ is defined by $f(x) = x$, $\forall x \in A$, then f is called the identity function on A.

[1] **Archimedian property:** Given $x > 0$ and $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.

(d) If $A_1 \subseteq A$, then the image $f(A_1)$ of A_1 under f is defined by $f(A_1) = \{f(a) : a \in A_1\}$.

(e) If $B_1 \subseteq B$, then the pre-image or inverse image $f^{-1}(B_1)$ of B_1 is defined by $f^{-1}(B_1) = \{a \in A : f(a) \in B_1\}$.

(f) If $g : B \to C$, then we define the composition $g \circ f : A \to C$ of f and g by $(g \circ f)(x) = g(f(x)),\ \forall x \in A$.

(g) f is invertible if there exists a function $g : B \to A$ such that $g \circ f$ is the identity function on A and $f \circ g$ is the identity function on B. Furthermore, g is called the inverse of f and is denoted by f^{-1}.

3. A function $f : A \to B$ is said to be

   (a) one-to-one (or injective) if $f(x) \neq f(y)$ whenever $x, y \in A$ with $x \neq y$.

   (b) onto (or surjective) if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.

   (c) bijective if it is injective and surjective.

4. Let $f : A \to B$. Then, f is invertible iff f is a bijection.
   Furthermore, for a bijective function $f : A \to B$, the inverse $f^{-1} : B \to A$ of f is defined by $f^{-1}(y) = x$, where $x \in A$ is unique such that $f(x) = y$.

5. Let $A, B \subseteq \mathbb{R}$. A function $f : A \to B$ is said to be

   (a) an increasing function if $x, y \in A$ such that $x < y$, then $f(x) \leq f(y)$.

   (b) a strictly increasing function if $x, y \in A$ such that $x < y$, then $f(x) < f(y)$.

   (c) a decreasing function if $x, y \in A$ such that $x < y$, then $f(x) \geq f(y)$.

   (d) a strictly decreasing function if $x, y \in A$ such that $x < y$, then $f(x) > f(y)$.

As algebra is inevitable to discuss analysis, we briefly recall some concepts like group, ring, field, and so on.
Let A be a non-empty set.

1. **Binary operator.** A function $f : A \times A \to A$ is called a binary operator on A. We also denote a binary operator f by $\star$ and the value $f(a, b)$ by $a \star b$.

2. **Unary operator.** A function $f : A \to A$ is called a unary operator on A.

3. Let A be a non-empty set and $\star$ be a binary operator on A, then
   (a) $\star$ is said to be associative if $a \star (b \star c) = (a \star b) \star c$, $\forall a, b, c \in A$.
   (b) $\star$ is said to have an identity element in A if there exists $e_\star \in A$ such that $a \star e_\star = e_\star \star a = a$, $\forall a \in A$.
   (c) an element a of A is said to have an inverse $b \in A$ with respect to $\star$ if $a \star b = b \star a = e_\star$. *We know that inverse of a is unique.*
   (d) $\star$ is said to be commutative if $a \star b = b \star a$, $\forall a, b \in A$.
4. $(A, \star)$ is said to be an abelian group if $\star$ satisfies the above four properties (a), (b), (c) for every element of A, and (d).
5. $(A, +, \cdot)$ is said to be a field if $(A, +)$ and $(A \setminus \{0\}, \cdot)$ are abelian groups, where 0 is the additive identity and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$, $\forall$ $a, b, c \in A$.
6. A four tuple $(A, +, \cdot, <)$ is said to be an ordered field if $(A, <)$ is an ordered set and $(A, +, \cdot)$ is a field with the following properties:
   (a) If $a, b \in A$ such that $a < b$, then $a + c < b + c$, $\forall c \in A$.
   (b) If $a, b \in A$ such that $a > 0$ and $b > 0$, then $a \cdot b > 0$.
7. If $(A, +, \cdot, <)$ is an ordered field, then
   (a) $a \cdot a > 0$, $\forall a \in A \setminus \{0\}$.
   (b) $a > 0$ iff $-a < 0$, where $-a$ is the additive inverse of a.
   (c) $a > 0$ iff $a^{-1} > 0$, where a^{-1} is the multiplicative inverse of a.
   (d) if $a < b$ and $c > 0$, then $ac < bc$.
   (e) if $a < b$ and $c < 0$, then $ac > bc$.
   (f) if $0 < a < b$, then $b^{-1} < a^{-1}$.
8. The set $\mathbb{R}$ of all real numbers is an ordered field with respect to usual $+$, $\cdot$, and $<$.
   (a) The modulus function on $\mathbb{R}$ is defined by $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0, \end{cases}$ $\forall x \in \mathbb{R}$.
   (b) The modulus function on $\mathbb{R}$ satisfies the following properties: (i) $|x| \geq 0$, $\forall x \in \mathbb{R}$, (ii) $|x| = 0$ iff $x = 0$, (iii) $|xy| = |x|\ |y|$, (iv) $|x + y| \leq |x| + |y|$, and (v) $x^2 \leq y^2$ iff $|x| < |y|$.

1.2 COMPLEX NUMBERS AND GEOMETRICAL REPRESENTATIONS

The terminology *Complex numbers* was due to C. F. Gauss in 1831. The rigorous definition of complex number as an ordered pair of real numbers (a_1, a_2) was introduced by W. R. Hamilton in 1837. The set of all complex numbers is denoted by $\mathbb{C}$. Then immediately it follows that $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. In the set of all complex numbers, there is a special element $(0, 1)$, which is denoted by i. Euler named this number i by the imaginary number. Note that this number i was used as a number satisfying $i^2 = -1$ before the rigorous construction of complex numbers, and hence, it was called an imaginary number.

Since there is no element in $\mathbb{R}$ whose square is a negative number and $i^2 = -1$, this number is not in $\mathbb{R}$ and hence, it might be called an imaginary number. As the numbers on a straight line are not imaginary, the name set of real numbers is given to $\mathbb{R}$.

We define addition and multiplication on $\mathbb{C}$ as follows:

$$\begin{aligned}(a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2),\\ (a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1), \text{ and}\end{aligned}$$

$(a_1, a_2), (b_1, b_2) \in \mathbb{C}$.

THEOREM 1.2.1 The set $\mathbb{C}$ of all complex numbers is a field with respect to addition and multiplication as defined above.

Proof: Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, and $c = (c_1, c_2) \in \mathbb{C}$. Using the fact that $\mathbb{R}$ is a field, we get the following:

1. Clearly,
$$\begin{aligned}a + b &= (a_1, a_2) + (b_1, b_2)\\ &= (a_1 + b_1, a_2 + b_2)\\ &= (b_1 + a_1, b_2 + a_2)\\ &= (b_1, b_2) + (a_1, a_2)\\ &= b + a\end{aligned}$$

2.
$$\begin{aligned}a + (b + c) &= (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]\\ &= (a_1, a_2) + (b_1 + c_1, b_2 + c_2)\\ &= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2))\\ &= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2)\\ &= (a_1 + b_1, a_2 + b_2) + (c_1, c_2)\\ &= [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)\\ &= (a + b) + c.\end{aligned}$$

3. $(0,0) \in \mathbb{C}$ is the identity element with respect to $+$ because $(a_1, a_2) + (0,0) = (a_1 + 0, a_2 + 0) = (a_1, a_2)$ and $(0,0) + (a_1, a_2) = (0 + a_1, 0 + a_2) = (a_1, a_2)$.

4. If $(a_1, a_2) \in \mathbb{C}$, then $(-a_1, -a_2) \in \mathbb{C}$ and $(a_1, a_2) + (-a_1, -a_2) = (a_1 + (-a_1), a_2 + (-a_2)) = (0,0)$.

5. $$\begin{aligned}(a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \\ &= (b_1 a_1 - b_2 a_2, b_2 a_1 + b_1 a_2) \\ &= (b_1 a_1 - b_2 a_2, b_1 a_2 + b_2 a_1) \\ &= (b_1, b_2) \cdot (a_1, a_2).\end{aligned}$$

6. $$\begin{aligned}& a \cdot (b \cdot c) \\ &= (a_1, a_2) \cdot [(b_1, b_2) \cdot (c_1, c_2)] \\ &= (a_1, a_2) \cdot [(b_1 c_1 - b_2 c_2, b_1 c_2 + b_2 c_1)] \\ &= (a_1(b_1 c_1 - b_2 c_2) - a_2(b_1 c_2 + b_2 c_1), a_1(b_1 c_2 + b_2 c_1) \\ &\quad + a_2(b_1 c_1 - b_2 c_2)) \\ &= (a_1 b_1 c_1 - a_1 b_2 c_2 - a_2 b_1 c_2 - a_2 b_2 c_1, a_1 b_1 c_2 + a_1 b_2 c_1 \\ &\quad + a_2 b_1 c_1 - a_2 b_2 c_2) \\ &= ((a_1 b_1 - a_2 b_2) c_1 - (a_1 b_2 + a_2 b_1) c_2, (a_1 b_1 - a_2 b_2) c_2 \\ &\quad + (a_1 b_2 + a_2 b_1) c_1) \\ &= ((a_1 b_1 - a_2 b_2), (a_1 b_2 + a_2 b_1)) \cdot (c_1, c_2) \\ &= ((a_1, b_1) \cdot (a_2, b_2)) \cdot (c_1, c_2) \\ &= (a \cdot b) \cdot c.\end{aligned}$$

7. Clearly, $(1,0) \in \mathbb{C}$ and $(a_1, a_2) \cdot (1,0) = (a_1 \cdot 1 - a_2 \cdot 0, a_1 \cdot 0 + a_2 \cdot 1) = (a_1, a_2)$.

8. If $(a_1, a_2) \in \mathbb{C} \neq (0,0)$, then at least one of a_1 and a_2 is non-zero. Therefore, $a_1^2 + a_2^2 \neq 0$, and hence, $\left(\dfrac{a_1}{a_1^2 + a_2^2}, \dfrac{-a_2}{a_1^2 + a_2^2} \right) \in \mathbb{C}$. Now,
$$\begin{aligned}(a_1, a_2) \cdot \left(\frac{a_1}{a_1^2 + a_2^2}, \frac{-a_2}{a_1^2 + a_2^2} \right) &= \left(\frac{a_1^2 + a_2^2}{a_1^2 + a_2^2}, \frac{-a_1 a_2 + a_2 a_1}{a_1^2 + a_2^2} \right) \\ &= (1,0).\end{aligned}$$

9. $a \cdot (b + c)$
$$\begin{aligned}
&= (a_1, a_2) \cdot ((b_1, b_2) + (c_1, c_2)) \\
&= (a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2) \\
&= ((a_1(b_1 + c_1) - a_2(b_2 + c_2)), a_1(b_2 + c_2) + a_2(b_1 + c_1)) \\
&= (a_1b_1 + a_1c_1 - a_2b_2 - a_2c_2, a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1) \\
&= ((a_1b_1 - a_2b_2) + (a_1c_1 - a_2c_2), (a_1b_2 + a_2b_1) + (a_1c_2 + a_2c_1)) \\
&= (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1) + (a_1c_1 - a_2c_2, a_1c_2 + a_2c_1) \\
&= ((a_1, a_2) \cdot (b_1, b_2)) + ((a_1, a_2) \cdot (c_1, c_2)) \\
&= (a \cdot b) + (a \cdot c).
\end{aligned}$$

Hence, $(\mathbb{C}, +, \cdot)$ is a field. □

We identify every real number x with the element $(x, 0)$ of $\mathbb{C}$. By this identification, the set of all real numbers is regarded as a subset of $\mathbb{C}$. It is also customary to write a complex number $z = (x, y)$ as $x + iy$, which is the simplified form of the expression $(x, 0) + (0, 1) \cdot (y, 0)$. We call x, y the real and imaginary parts of z and denote them by $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$, respectively.

LEMMA 1.2.2 $i \cdot i = (-1, 0)$.

Proof: By direct computation, we get $i \cdot i = (0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0)$. □

Remark 1.2.3: $\mathbb{C}$ is not an ordered field with respect to any order relation.

Reason: By the observation 7(a) in page 8, if $\mathbb{C}$ were an ordered field, then we would get

$$-1 = i \cdot i > 0 \text{ and } 1 = 1 \cdot 1 > 0,$$

which contradicts the observation 7(b) in page 8.

Example 1.2.4 Write the following numbers in $a + ib$ form.

1. $(3 + i5)(4 - i2)$.

2. $\dfrac{(3 + i)^2}{1 - i2}$.

3. $\sin(1 + i)$.

Solution:

1. $(3+i5)(4-i2) = (12+10)+i(-6+20) = 22+i14$.
2. $\dfrac{(3+i)^2}{1-i2} = \dfrac{(9-1+i6)(1+i2)}{5} = \dfrac{(8-12)+i(16+6)}{5} = \dfrac{-4+i22}{5}$.
3. $\sin(1+i) = \sin(1)\cos(i) + \cos(1)\sin(i) = \sin(1)\cosh(1) + i\cos(1)\sinh(1)$.

Exercise 1.2.5 Write the following numbers in $a+ib$ form.

1. $\dfrac{(2+i)(3-i2)}{1+i4}$.
2. $(4+i3)^4$.
3. $\cos(1-i2)$.
4. $\exp(i(1+i))$.
5. $\cos(i)\exp(i3+2)$.

Answers:

1. $\dfrac{4-i33}{17}$.
2. $-527-i336$.
3. $\cos(1)\cosh(2)+i\sin(1)\sinh(2)$.
4. $\exp(-1)\cos(1)+i\exp(-1)\sin(1)$.
5. $\cosh(1)\exp(2)\cos(3)+i\cosh(1)\exp(2)\sin(3)$.

Definition 1.2.6 We define the complex conjugate $\overline{z}$ of $z \in \mathbb{C}$ by $\overline{z} = (x,-y)$ whenever $z = (x,y)$.

RESULT 1.2.7 (Properties of complex conjugate)
Let $z, z_1, z_2 \in \mathbb{C}$ be arbitrary.

1. $z + \overline{z} = 2\mathrm{Re}\, z$ and $z - \overline{z} = i2\mathrm{Im}\, z$.
2. $\overline{\overline{z}} = z$.

3. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

4. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

5. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, when $z_2 \neq (0,0)$.

6. If $z = x + iy$ then $z \cdot \bar{z} = x^2 + y^2$.

7. $z = \bar{z}$ if and only if z is a real number.

Proof: Let $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$.

1. $z + \bar{z} = (x+iy) + (x-iy) = 2x = 2\text{Re } z$ and $z - \bar{z} = (x+iy) - (x-iy) = i2y = i2 \text{ Im } z$.

2. $\bar{\bar{z}} = \overline{\overline{x + iy}} = \overline{x - iy} = x + iy = z$.

3. $$\begin{aligned} \overline{(z_1 + z_2)} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= \overline{z_1} + \overline{z_2}. \end{aligned}$$

4. $$\begin{aligned} \overline{z_1 \cdot z_2} &= \overline{(x_1 + iy_1) \cdot (x_2 + iy_2)} \\ &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} \\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= (x_1x_2 - (-y_1)(-y_2)) + i(x_1(-y_2) + x_2(-y_1)) \\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) \\ &= \overline{z_1} \cdot \overline{z_2}. \end{aligned}$$

5. First, we prove that $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{\bar{z}}$, for $z \neq (0,0)$.

$$\begin{aligned} \overline{\left(\frac{1}{z}\right)} &= \overline{\left(\frac{1}{x + iy}\right)} \\ &= \overline{\left(\frac{x - iy}{(x - iy)(x + iy)}\right)} \\ &= \overline{\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right)} \end{aligned}$$

$$\begin{aligned} &= \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} \\ &= \frac{x+iy}{x^2+y^2} \\ &= \frac{x+iy}{(x+iy)(x-iy)} \\ &= \frac{1}{x-iy} \\ &= \frac{1}{\overline{z}}. \end{aligned}$$

Then, applying (4), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(z_1 \cdot \frac{1}{z_2}\right)} = \overline{z_1} \cdot \overline{\left(\frac{1}{z_2}\right)} = \overline{z_1} \cdot \frac{1}{\overline{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}.$$

6. By a direct computation, we get $z \cdot \overline{z} = (x+iy) \cdot (x-iy) = x^2 + y^2$.
7. $z = \overline{z}$ iff $x + iy = x - iy$ iff $y = -y$ iff $y = 0$ iff z is a real number. □

We also have one more unary operator called the modulus or absolute value of a complex number, which is defined by

$$|z| = \sqrt{z \cdot \overline{z}} \text{ or equivalently } |(x,y)| = \sqrt{x^2+y^2}$$

here, we take the positive square root. This unary operator satisfies the following properties:

RESULT 1.2.8 Let $z, z_1, z_2 \in \mathbb{C}$ be arbitrary.

1. $|\text{Re } z| \leq |z|$ and $|\text{Im } z| \leq |z|$.
2. $|z| = |\overline{z}|$.
3. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.
4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

Proof: Let $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$.

1. $x^2 \leq x^2 + y^2$ implies that $|x| \leq \sqrt{x^2+y^2} = |z|$. As $\sqrt{x^2+y^2}$ is symmetric in x and y, by interchanging x and y, we get $|y| \leq |z|$.

2. $|z| = \sqrt{x^2+y^2} = \sqrt{x^2+(-y)^2} = |\overline{z}|$.

3. $$\begin{aligned} |z_1 \cdot z_2| &= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)| \\ &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\ &= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} \\ &= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} \\ &= |z_1| \cdot |z_2|. \end{aligned}$$

4. As
$$\left|\frac{1}{z}\right| = \left|\frac{x-iy}{x^2+y^2}\right| = \frac{\sqrt{x^2+y^2}}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{|z|},$$
we get
$$\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = |z_1|\left|\frac{1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$ □

RESULT 1.2.9 (Triangle inequality)
If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$. Furthermore, $|z_1 + z_2| = |z_1| + |z_2|$ iff $z_1 \cdot \overline{z_2}$ is real and is non-negative.

Proof: Using Results 1.2.7 and 1.2.8, we get

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)\cdot\overline{(z_1+z_2)} \\ &= (z_1+z_2)\cdot(\overline{z_1}+\overline{z_2}) \\ &= (z_1\cdot\overline{z_1} + z_1\cdot\overline{z_2} + z_2\cdot\overline{z_1} + \overline{z_2}\cdot\overline{z_2} \\ &= |z_1|^2 + |z_2|^2 + z_1\cdot\overline{z_2} + \overline{z_1\cdot\overline{z_2}} \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\cdot\overline{z_2}) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1\cdot\overline{z_2}| \\ &= |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2| \\ &= (|z_1| + |z_2|)^2. \end{aligned}$$

Hence, we get $|z_1 + z_2| \leq |z_1| + |z_2|$.

Next, we find the necessary and sufficient condition to get equality in the above inequality.

From the proof of the triangle inequality, we get

$$\begin{aligned} |z_1+z_2| = |z_1| + |z_2| &\Leftrightarrow |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1\cdot\overline{z_2}) \\ &= |z_1|^2 + |z_2|^2 + 2|z_1\cdot\overline{z_2}| \\ &\Leftrightarrow \mathrm{Re}\,(z_1\cdot\overline{z_2}) = |z_1\cdot\overline{z_2}| \\ &\Leftrightarrow z_1\cdot\overline{z_2} \text{ is real and non-negative.} \end{aligned}$$

Hence, the result follows. □

As a corollary of the above result, we have another version of triangle inequality, which follows.

COROLLARY 1.2.10 If $z_1, z_2 \in \mathbb{C}$, then $| \, |z_1| - |z_2| \, | \leq |z_1 - z_2|$.

Proof: Now, $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$, which implies that

$$|z_1| - |z_2| \leq |z_1 - z_2|.$$

Interchanging z_1, z_2 and using $|-z| = |-1||z| = |z|$, we get

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|.$$

Hence, by the definition of the modulus function on $\mathbb{R}$, the corollary follows. □

RESULT 1.2.11 If $z_j \in \mathbb{C}$, $1 \leq j \leq n$, then $\left| \sum_{j=1}^{n} z_j \right| \leq \sum_{j=1}^{n} |z_j|$.

Proof: We prove this by induction on n. For $n = 1$, this result holds obviously. Assume that this result is true for some n. Now, using triangle inequality (Result 1.2.9) and by induction hypothesis, we get

$$\left| \sum_{j=1}^{n+1} z_j \right| = \left| \sum_{j=1}^{n} z_j + z_{n+1} \right| \leq \left| \sum_{j=1}^{n} z_j \right| + |z_{n+1}| \leq \sum_{j=1}^{n} |z_j| + |z_{n+1}| = \sum_{j=1}^{n+1} |z_j|.$$

This completes the proof of this result. □

Exercise 1.2.12 Prove that $\left| \sum_{j=1}^{n} z_j \right| = \sum_{j=1}^{n} |z_j|$ iff $z_j \overline{z_k}$ is real and non-negative, $\forall j, k \in \{1, 2, \ldots, n\}$.

RESULT 1.2.13 (Cauchy–Schwarz inequality)
If $x_i, y_i \in \mathbb{C}$, $\forall i = 1, 2, 3, \ldots, n$, then

$$\left| \sum_{k=1}^{n} x_k \overline{y_k} \right|^2 \leq \sum_{k=1}^{n} |x_k|^2 \cdot \sum_{k=1}^{n} |y_k|^2. \tag{1.1}$$

Proof: Let $c = \sum_{k=1}^{n} x_k\overline{y_k}$, $\|\mathbf{x}\| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}$ and $\|\mathbf{y}\| = \left(\sum_{k=1}^{n} |y_k|^2\right)^{\frac{1}{2}}$. Consider

$$\begin{aligned}
0 &\leq \sum_{k=1}^{n} \left|(\|\mathbf{y}\|^2 x_k - c y_k)\right|^2 \\
&= \sum_{k=1}^{n} (\|\mathbf{y}\|^2 x_k - c y_k)(\|\mathbf{y}\|^2 \overline{x_k} - \overline{c}\,\overline{y_k}) \\
&= \sum_{k=1}^{n} (\|\mathbf{y}\|^4 |x_k|^2 - \|\mathbf{y}\|^2 \overline{c} x_k \overline{y_k} - \|\mathbf{y}\|^2 c y_k \overline{x}_k + c\overline{c} y_k \overline{y}_k) \\
&= \|\mathbf{y}\|^4 \cdot \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 \cdot |c|^2 - \|\mathbf{y}\|^2 \cdot |c|^2 + \|\mathbf{y}\|^2 \cdot |c|^2 \\
&= \|\mathbf{y}\|^2 (\|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - |c|^2)
\end{aligned}$$

If $\|\mathbf{y}\|^2 = 0$, then equality holds in (1.1); otherwise, we get $|c|^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$. This implies (1.1). □

RESULT 1.2.14 (Schwarz's inequaltiy)
If $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$, then $\|x + y\| \leq \|x\| + \|y\|$, where $\|\mathbf{z}\| = \left(\sum_{k=1}^{n} |z_k|^2\right)^{\frac{1}{2}}$, $\forall \mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$.

Proof:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= \sum_{k=1}^{n} |x_k + y_k|^2 \\
&= \sum_{k=1}^{n} (x_k + y_k)\overline{(x_k + y_k)} \\
&= \sum_{k=1}^{n} (x_k \overline{x_k} + x_k \overline{y_k} + y_k \overline{x_k} + y_k \overline{y_k}) \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \sum_{k=1}^{n} x_k \overline{y_k} + \sum_{k=1}^{n} y_k \overline{x_k} \\
&\quad \text{(note that } \sum_{k=1}^{n} x_k \overline{y_k} + \sum_{k=1}^{n} y_k \overline{x_k} \text{ is real)}
\end{aligned}$$

$$\begin{aligned}
&\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \left|\sum_{k=1}^{n} x_k\overline{y_k} + \sum_{k=1}^{n} y_k\overline{x_k}\right| \\
&\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \left|\sum_{k=1}^{n} x_k\overline{y_k}\right| + \left|\sum_{k=1}^{n} y_k\overline{x_k}\right| \\
&\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{x}\|\|\mathbf{y}\| \\
&\qquad \text{(by Cauchy–Schwarz inequality)} \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

Thus, the result follows. □

Geometrically, the set of all complex numbers can be viewed as the set of all points in a plane. We fix a point arbitrarily and name it the origin and denote it by $(0,0)$. We draw two lines on the plane that pass through $(0,0)$ horizontally and vertically. We call the horizontal line and vertical line, respectively, the real axis and the imaginary axis. Now, every element $(x,y) \in \mathbb{C}$ is represented by the point of intersection of the horizontal line passing through $(0,y)$ and the vertical line passing through $(x,0)$. This representation is also called the Cartesian form of a complex number.

Every non-zero complex number $z = (x,y)$ can also be represented in another form $r\exp(i\theta)$ (or $re^{i\theta}$) called polar form[2], where $r = |z|$ is the length of the line segment joining $(0,0)$ and z (called the modulus of z), and θ is the angle between the positive side of the x-axis and the line segment joining $(0,0)$ and z, measured in the anti-clock wise sense (called an argument θ of z). Using the Pythagoras theorem, one can get the length of the line segment as $\sqrt{x^2+y^2} = |(x,y)| = |z|$.

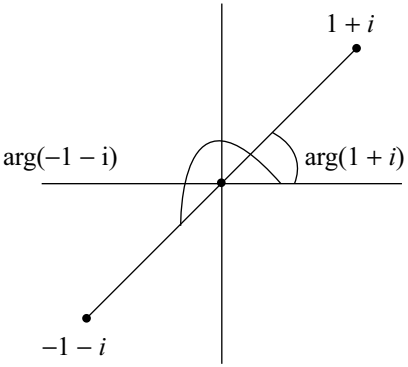
Note that the modulus of a complex number is unique but the argument is not. The reason for the second statement is that if θ is one of the values of argument of (a,b), then for every $k \in \mathbb{Z}$, $\theta + 2k\pi$ is also an argument of the same (a,b). Hence, we define the principal argument of a non-zero complex number (a,b) as the argument θ of (a,b) satisfying the condition $-\pi < \theta \leq \pi$.

The subsets $\{(a,b) : a > 0, b > 0\}$, $\{(a,b) : a < 0, b > 0\}$, $\{(a,b) : a < 0, b < 0\}$ and $\{(a,b) : a > 0, b < 0\}$ are called first quadrant, second quadrant, third quadrant, and fourth quadrant, respectively.

[2] At present, treat $r\exp(i\theta)$ as just a symbol. The reason for using this notation is explained in Corollary 2.4.25.

Remark 1.2.15: At this juncture, the reader is advised not to use arg $(a + ib) = \arctan\left(\frac{b}{a}\right)$, for all non-zero $a + ib \in \mathbb{C}$.

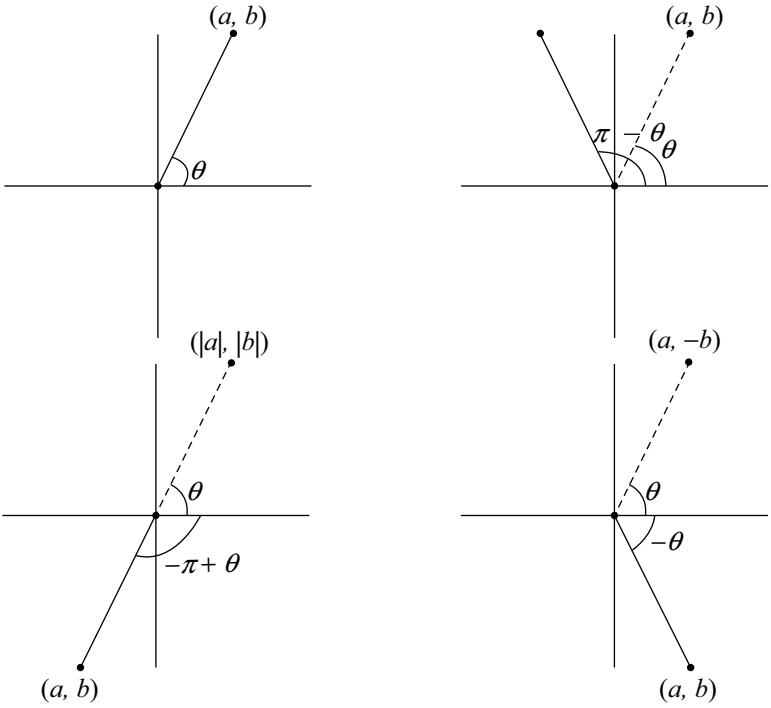
To justify this remark, if we use the above formula, then we will get arg $(1 + i)$ $= \arg(-1 - i) = \frac{\pi}{4}$, but the first point is in the first quadrant and the second point is in the third quadrant, and hence obviously, their arguments should be different. See the following diagram.



Definition 1.2.16 (Principal argument)[3] Let $a + ib \neq 0$.

- Case (i): If $a = 0, b > 0$, then $\theta = \frac{\pi}{2}$; If $a = 0, b < 0$, then $\theta = \frac{-\pi}{2}$; If $a < 0, b = 0$, then $\theta = \pi$.
- Case (ii): If $a > 0, b \in \mathbb{R}$, then $\theta = \arctan\left(\frac{b}{a}\right)$.
- Case (iii): If $a < 0, b > 0$, then $\theta = \pi - \arctan\left(\frac{b}{|a|}\right)$.
- Case (iv): If $a < 0, b < 0$, then $\theta = -\pi + \arctan\left(\frac{|b|}{|a|}\right)$.

[3] For arctan and its properties, we refer the reader to Section 2.4.



Example 1.2.17 Write the given numbers in the polar form with principal arguments: (1) $2 + i3$ and (2) $\frac{1}{2}(1 - \sqrt{3})$.

1. $|3 + i4| = \sqrt{9 + 16} = \sqrt{25} = 5$ and $\arg 3 + i4 = \arctan\left(\frac{4}{3}\right)$
   $\Rightarrow 3 + i4 = \sqrt{25}\, e^{i\arctan\left(\frac{4}{3}\right)}$.

2. $\left|\frac{1}{2}(1 - \sqrt{3})\right| = \frac{1}{2}\sqrt{1+3} = 1$ and $\arg \frac{1}{2}(1 - \sqrt{3}) = -\arctan\left(\sqrt{3}\right) = -\frac{\pi}{3} \Rightarrow \frac{1}{2}(1 - \sqrt{3}) = e^{\frac{-\pi}{3}}$.

Definition 1.2.18 Given a non-zero complex number $z = re^{i\theta}$, we define $\sqrt[n]{z} = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2\pi k}{n}\right)}$, where $k = 0, 1, 2, \ldots, n-1$.

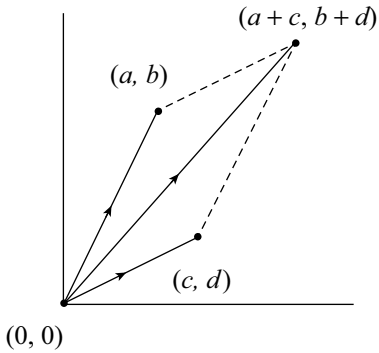
Example 1.2.19 Find the cube roots of -8.
First, we write $-8 = 8e^{i\pi}$, and hence, $(-8)^{1/3} = 2e^{i(\pi+2k\pi)/3}$, $k = 0, 1, 2$. Hence, the three values of $(-8)^{1/3}$ are as follows:

1. $2e^{i\pi/3} = 2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + i\sqrt{3}$.

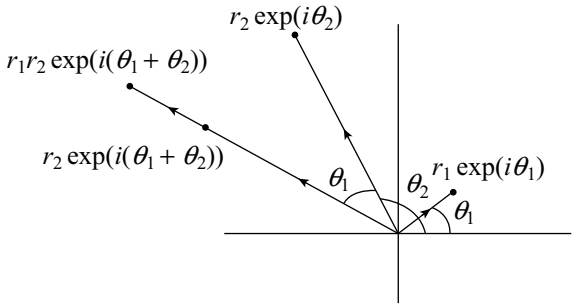
2. $2e^{i\pi} = 2(\cos(\pi) + i\sin(\pi)) = -2$.

3. $2e^{i4\pi/3} = 2(\cos(4\pi/3) + i\sin(4\pi/3)) = 1 - i\sqrt{3}$.

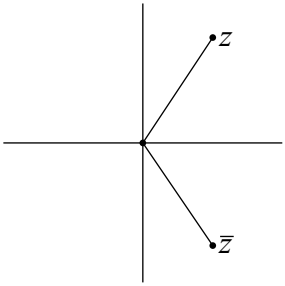
The sum of two complex numbers (a, b) and (c, d) can be viewed geometrically as follows. Construct a parallelogram using the line segment joining $(0, 0)$, (a, b) and the line segment joining $(0, 0)$, (c, d), as the adjacent sides. Then the fourth vertex of the parallelogram is $(a, b)+(c, d)$. See the following figure.



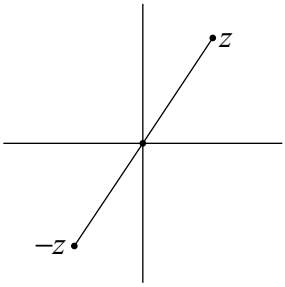
The product of two complex numbers z_1 and z_2 can be understood easily if we represent them in polar coordinates rather than in Cartesian coordinates. Let $r_1 \exp(\theta_1), r_2 \exp(\theta_2)$ be the given two non-zero complex numbers. (If at least one of them is zero, then the product is obvious.) Then the product of these two complex numbers is the complex number $r_1 \cdot r_2 \exp(\theta_1 + \theta_2)$. Geometrically, $r_1 \cdot r_2 \exp(\theta_1 + \theta_2)$ is obtained by rotating the line joining the origin and $r_1 \exp(\theta_1)$ through an angle θ_2 and then multiplying it by r_2. Multiplying $r_1 \exp(\theta_1 + \theta_2)$ by r_2 expands (or shrinks) the length of line segment joining $(0, 0)$ and $r_1 \exp(\theta_1 + \theta_2)$ by the scale r_2, when $r_2 > 1$ (or $0 < r_2 < 1$). At this stage, take one of the end points of the resulting line segment, other than origin, as the product of the given complex numbers. See the following figure.



The complex conjugate of a complex number (a, b) can be viewed as the point of reflection with respect to the real axis.

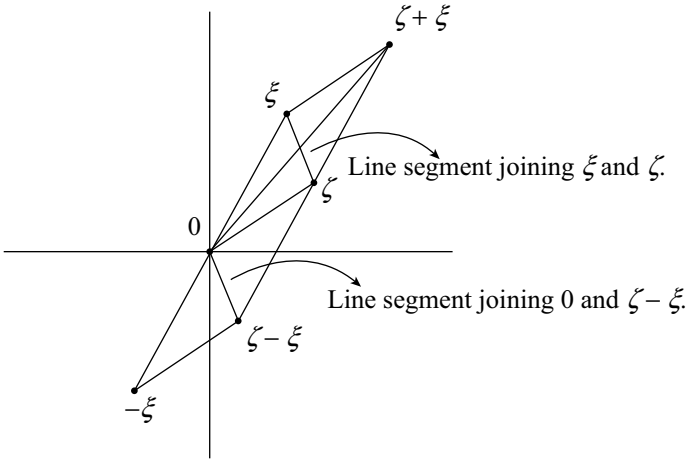


Geometrically, the additive inverse of a complex number can be viewed as the rotation of the given non-zero complex number through an angle π or equivalently, it is the reflection of z with respect to the real axis followed by another reflection with respect to the imaginary axis.



Exercise 1.2.20 Geometrically explain the multiplicative inverse of a complex number.

Now, we find the equation of the straight line passing through the given two points ζ and ξ. Geometrically, the slope of the line segment joining ξ and ζ is same as the slope of the line segment joining 0 and $\zeta - \xi$.



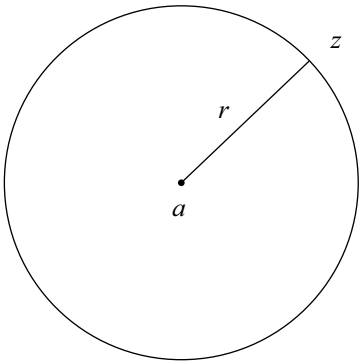
Therefore, the straight line passing through ξ and ζ is same as the straight line passing through ξ with the slope as that of the line segment joining 0 and $\zeta - \xi$. Note that if t moves from 0 towards ∞, then $t(\zeta - \xi)$ moves from 0 to ∞, and it passes through $\zeta - \xi$ when $t = 1$. If t moves from 0 to $-\infty$, then $t(\zeta - \xi)$ moves from 0 to ∞ through $-(\zeta - \xi)$ when $t = -1$. Hence, the equation of the straight line passing through 0 and $\zeta - \xi$ is $t(\zeta - \xi)$, $t \in \mathbb{R}$. If the straight line passing through 0 and $\zeta - \xi$ is translated by ξ, then the resulting straight line passes through ξ and ζ. Hence, its equation becomes $\xi + t(\zeta - \xi)$, $t \in \mathbb{R}$. Therefore, the equation of a straight line passing through a and having slope as the slope of the line segment joining 0 and b, for some $a, b \in \mathbb{C}$ with $b \neq 0$, is given by

$$z = a + tb, \ t \in \mathbb{R}. \tag{1.2}$$

Now, we find an implicit equation of the same straight line.

$$\begin{aligned}
z \text{ lies on the line (1.2)} \quad &\text{iff} \quad \frac{z-a}{b} \in \mathbb{R} \\
&\text{iff} \quad \frac{z-a}{b} = \overline{\left(\frac{z-a}{b}\right)} \\
&\text{iff} \quad z\bar{b} - \bar{z}b + \bar{a}b - a\bar{b} = 0 \\
&\text{iff} \quad z(\overline{ib}) + \bar{z}ib + 2\text{Im}\, a\bar{b} = 0.
\end{aligned}$$

The circle with centre $a \in \mathbb{C}$ and radius $r > 0$ can be defined as the set of all points that are at distance r units from a. More explicitly $\{z : |z - a| = r\}$. We shall prove latter (in Corollary 2.4.25) that $z - a = r(\exp(i\theta))$ for some $\theta \in [0, 2\pi]$. Thus, an equation of a circle can be given by $z = a + r\exp(i\theta)$, $\theta \in [0, 2\pi]$.



The implicit equation of the circle $|z - a| = r$ is given as follows.

z belongs to the circle iff $|z - a|^2 = r^2$

iff $|z|^2 - z\overline{a} - \overline{z}a + |a|^2 - r^2 = 0$

iff $|z|^2 + z\overline{(-a)} - \overline{z}(-a) + |a|^2 - r^2 = 0$.

The interesting fact is that there is a combined version of an equation for straight lines and circles. This is simply

$$\alpha|z|^2 + \beta\overline{z} + \overline{\beta}z + \gamma = 0, \quad \text{where } \alpha, \gamma \in \mathbb{R} \text{ and } \beta \in \mathbb{C} \text{ with not } \alpha = 0 = \beta.$$

The above equation represents a $\begin{cases} \text{straight line} & \text{if } \alpha = 0. \\ \text{circle} & \text{if } \alpha \neq 0. \end{cases}$

1.3 SEQUENCES AND SERIES OF COMPLEX NUMBERS

In this section, first we recall some definitions and results on sequences and series of complex numbers.

Definition 1.3.1 A sequence (a_n) of complex numbers is said to be

1. convergent if there exists $a \in \mathbb{C}$ such that given $\epsilon > 0$, then there exists $m \in \mathbb{N}$ such that $|a_n - a| < \epsilon$, $\forall n \geq m$. In this case, we call 'a' the limit of (a_n) and we write $a_n \to a$ as $n \to \infty$ or $a = \lim_{n\to\infty} a_n$ or (a_n) converges to a.

2. a Cauchy sequence if given $\epsilon > 0$, then there exists $m \in \mathbb{N}$ such that $|a_j - a_k| < \epsilon, \forall j, k \geq m$.

3. bounded if there exists $M > 0$ such that $|a_n| \leq M, \forall n \in \mathbb{N}$.

RESULT 1.3.2 If (a_n) is a convergent sequence, then (a_n) is a Cauchy sequence.

Proof: Let (a_n) be a convergent sequence. Then, there exists $a \in \mathbb{C}$ such that given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2}, \ \forall n \geq m.$$

For the same $m \in \mathbb{N}$, if $j, k \geq m$, then

$$|a_j - a_k| \leq |a_j - a| + |a_k - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, (a_n) is a Cauchy sequence. □

RESULT 1.3.3 If (a_n) is a Cauchy sequence, then (a_n) is a bounded sequence.

Proof: For $\epsilon = 1$, there exists $m \in \mathbb{N}$ such that

$$|a_j - a_k| < 1, \ \forall j, k \geq m.$$

Then, put

$$K = \max\{1 + |a_m|, |a_n| : n = 1, 2, 3, \ldots, m - 1\}.$$

Then, $K > 0$ and for every $n \leq m - 1$, obviously we have $|a_n| \leq K$ and for $n \geq m$,

$$|a_n| \leq |a_n - a_m| + |a_m| < 1 + |a_m| \leq K,$$

which implies that (a_n) is a bounded sequence. □

From the above two results, we obtain the following corollary.

COROLLARY 1.3.4 If (a_n) is a convergent sequence, then (a_n) is a bounded sequence.

Example 1.3.5 Every constant sequence is obviously a convergent sequence.

Example 1.3.6 $\left(\dfrac{n}{n+1}\right)$ converges to 1.

Given $\epsilon > 0$, applying Archimedian property, we choose $N > \dfrac{1}{\epsilon}$. If $n \geq N$, then

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon.$$

Thus, $\dfrac{n}{n+1} \to 1$ as $n \to \infty$.

Definition 1.3.7 Given a sequence (a_n), we mean (a_{n_k}) a subsequence of (a_n) if (n_k) is a strictly increasing sequence of natural numbers. That is, $n_k < n_j$ if $k < j$.

LEMMA 1.3.8 If (x_n) is a Cauchy sequence and it has a subsequence (x_{n_k}) converging to x, then (x_n) also converges to the same x.

Proof: Let $\epsilon > 0$ be given. Then, there exist N, $j \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\epsilon}{2},\ \forall m, n \geq N \text{ and } |x_{n_k} - x| < \frac{\epsilon}{2},\ \forall k \geq j.$$

If $p = \max\{N, n_j\}$, then for every $n \geq p$, as $n_p \geq p \geq N$ and $p \geq n_j \geq j$, we have,

$$|x_n - x| \leq |x_n - x_{n_p}| + |x_{n_p} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, the lemma follows. □

THEOREM 1.3.9 Let (a_n) and (b_n) be complex sequences and $a, b \in \mathbb{C}$.

(1) $a_n \to a$ as $n \to \infty$ iff $Re\ a_n \to Re\ a$ and $Im\ a_n \to Im\ a$ as $n \to \infty$.

Let $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then,

(2) $ca_n \to ca$ as $n \to \infty$.

(3) Every subsequence of (a_n) converges to a.

(4) $a_n + b_n \to a + b$ as $n \to \infty$.

(5) $a_n \cdot b_n \to a \cdot b$ as $n \to \infty$.

(6) $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ as $n \to \infty$ if $b \neq 0,\ b_n \neq 0,\ \forall n \in \mathbb{N}$.

Proof:

1. Assume that $a_n \to a$ as $n \to \infty$. Then, for a given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|a_n - a| < \epsilon$, $\forall n \geq m$. For the same $m \in \mathbb{N}$, if $n \geq m$, we have

$$|\text{Re } a_n - \text{Re } a| \leq |a_n - a| < \epsilon \text{ and } |\text{Im } a_n - \text{Im } a| \leq |a_n - a| < \epsilon$$

Conversely, assume that $\text{Re } a_n \to \text{Re } a$ and $\text{Im } a_n \to \text{Im } a$ as $n \to \infty$. Given $\epsilon > 0$, there exist $m_1, m_2 \in \mathbb{N}$ such that

$$|\text{Re } a_n - \text{Re } a| < \frac{\epsilon}{\sqrt{2}}, \ \forall n \geq m_1$$

and

$$|\text{Im } a_n - \text{Im } a| < \frac{\epsilon}{\sqrt{2}}, \ \forall n \geq m_2.$$

For $n \geq \max\{m_1, m_2\}$, we have

$$|a_n - a| = \sqrt{|\text{Re } a_n - \text{Re } a|^2 + |\text{Im } a_n - \text{Im } a|^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon.$$

Hence, $a_n \to a$ as $n \to \infty$.

2. Using $a_n \to a$ as $n \to \infty$ for a given $\epsilon > 0$, we choose $m \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{1 + |c|}, \ \forall n \geq m.$$

For $n \geq m$,

$$|ca_n - ca| = |c||a_n - a| \leq |c|\frac{\epsilon}{1 + |c|} \leq \epsilon.$$

Therefore, $ca_n \to ca$ as $n \to \infty$.

3. Let (a_{n_k}) be a subsequence of the given sequence (a_n). Then, (n_k) is a strictly increasing sequence of natural numbers. Given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|a_n - a| < \epsilon$, $\forall n \geq m$. Choose $j \in \mathbb{N}$ such that $n_j > m$. If $k \geq j$, then $n_k \geq n_j > m$, and hence, $|a_{n_k} - a| < \epsilon$.

4. Given $\epsilon > 0$, choose $m_1, m_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2}, \ \forall n \geq m_1 \text{ and } |b_n - b| < \frac{\epsilon}{2}, \ \forall n \geq m_2.$$

If $n \geq \max\{m_1, m_2\}$, then

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $a_n + b_n \to a + b$ as $n \to \infty$.

5. Using Corollary 1.3.4, we find an $M > 0$ such that $|a_n| \leq M$, $\forall n \in \mathbb{N}$. Given $\epsilon > 0$, choose $m_1, m_2 \in \mathbb{N}$, such that

$$|a_n - a| < \frac{\epsilon}{2(1 + |b|)}, \ \forall n \geq m_1 \text{ and } |b_n - b| < \frac{\epsilon}{2M}, \ \forall n \geq m_2.$$

If $n \geq \max\{m_1, m_2\}$, then

$$\begin{aligned} |a_n \cdot b_n - a \cdot b| &= |a_n \cdot b_n - a_n \cdot b + a_n \cdot b - a \cdot b| \\ &\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \\ &< M\frac{\epsilon}{2M} + |b|\frac{\epsilon}{2(1 + |b|)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

6. It is left as an exercise to the reader. For a hint, see Theorem 1.6.3. □

LEMMA 1.3.10 Let (a_n) and (b_n) be real sequences such that $a_n \leq b_n \ \forall n \in \mathbb{N}$. If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $a \leq b$.

Proof: If $a > b$, then $\frac{a-b}{2} > 0$. Then, for $\epsilon = \frac{a-b}{2}$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{a - b}{2}, \ \forall n \geq N_1 \text{ and } |b_n - b| < \frac{a - b}{2}, \ \forall n \geq N_2.$$

If $m \geq \max\{N_1, N_2\}$, then

$$|a_m - a| < \frac{a - b}{2} \text{ and } |b_m - b| < \frac{a - b}{2}.$$

Therefore,

$$a - b = a - a_m + a_m - b \leq (a - a_m) + (b_m - b) < \frac{a - b}{2} + \frac{a - b}{2} = a - b.$$

which is a contradiction. Thus, $a \leq b$. □

THEOREM 1.3.11 (Cauchy criterion)
Let (a_n) be a sequence of complex numbers. Then, (a_n) is a Cauchy sequence iff (a_n) is a convergent sequence.

The half of the proof of the above theorem is already proved and the remaining half follows from the fact that $\mathbb{C}$ is a complete metric space. (See Theorem 1.4.46.)

Definition 1.3.12 Let (a_n) be a sequence of complex numbers. We write $a_n \to \infty$ as $n \to \infty$ if for every $K > 0$, there exists $m \in \mathbb{N}$ such that $|a_n| > K, \forall n \geq m$.

Definition 1.3.13 Let (x_n) be a sequence of real numbers.

1. We say that $x_n \to +\infty$ as $n \to \infty$ if for every $K > 0$, there exists $m \in \mathbb{N}$ such that $x_n > K,\ \forall n \geq m$.
2. We say that $x_n \to -\infty$ as $n \to \infty$ if for every $K > 0$, there exists $m \in \mathbb{N}$ such that $x_n < -K,\ \forall n \geq m$.

Definition 1.3.14 Let (x_n) be a sequence of real numbers, then an extended real number α is said to be the limit superior or upper limit, which is denoted by $\limsup_{n\to\infty} x_n$, if it satisfies the following conditions:

1. There exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to \alpha$ as $k \to \infty$.
2. Given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $a_n < \alpha + \epsilon,\ \forall n \geq m$.

Definition 1.3.15 Let (x_n) be a sequence of real numbers, then an extended real number α is said to be the limit inferior or lower limit, which is denoted by $\liminf_{n\to\infty} x_n$, if it satisfies the following conditions:

1. There exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to \alpha$ as $k \to \infty$.
2. Given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $a_n > \alpha - \epsilon,\ \forall n \geq m$.

LEMMA 1.3.16 For a real sequence (x_n), $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Proof: Let $\epsilon > 0$ be given. If $\alpha = \liminf_{n\to\infty} x_n$ and $\beta = \limsup_{n\to\infty} x_n$, then there exist $N_1,\ N_2 \in \mathbb{N}$ such that

$$x_n > \alpha - \epsilon,\ \forall n \geq N_1 \text{ and } x_n < \beta + \epsilon,\ \forall n \geq N_2.$$

If $m \geq \max\{N_1, N_2\}$, then $\alpha - \epsilon < x_m$ and $x_m < \beta + \epsilon$ imply that $\alpha - \epsilon < \beta + \epsilon$. As $\epsilon > 0$ is arbitrary, we get $\alpha \leq \beta$. □

RESULT 1.3.17 Let (x_n) be a sequence of real numbers. (x_n) converges to x iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

The proof of the above result is straightforward from the definitions.

LEMMA 1.3.18 If (x_n) and (y_n) be sequences of real numbers such that $x_n \leq y_n, \forall n \in \mathbb{N}$, then $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$ and $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$.

Proof: Let x and y be the limit superior of (x_n) and (y_n), respectively, then by definition, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $y_n < y + \epsilon,\ \forall n \geq N$. From the hypothesis, we get

$$x_n \leq y_n < y + \epsilon,\ \forall n \geq N.$$

Let (x_{n_k}) be a subsequence of (x_n) such that

$$n_1 \geq N \text{ and } x_{n_k} \to x \text{ as } k \to \infty.$$

As the constant sequence converges to the same constant, in view of Lemma 1.3.10, from $x_{n_k} < y + \epsilon,\ \forall k \in \mathbb{N}$, we infer that $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} x \leq y + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty} x_n = x \leq y = \limsup_{n\to\infty} y_n.$$

Similarly, we can prove the other inequality. □

THEOREM 1.3.19 Let (x_n) and (y_n) be sequences of positive real numbers. If (x_n) converges to a positive real number, then $\limsup_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n$.

Proof: Let $x = \lim_{n\to\infty} x_n$ and $y = \limsup_{n\to\infty} y_n$. First, we assume that $y \in \mathbb{R}$. Given $\epsilon \in (0, 1)$, we find $m_1, m_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{1 + x + y} \Leftrightarrow x - \frac{\epsilon}{1 + x + y} < x_n < x + \frac{\epsilon}{1 + x + y}\ \forall n \geq m_1$$

and

$$y_n < y + \frac{\epsilon}{1 + x + y},\ \forall n \geq m_2.$$

For $n \geq \max\{m_1, m_2\}$, we get

$$\begin{aligned} x_n \cdot y_n &< x \cdot y + \frac{\epsilon(x+y)}{1+x+y} + \left(\frac{\epsilon}{1+x+y}\right)^2 \\ &< x \cdot y + \frac{\epsilon(x+y)}{1+x+y} + \frac{\epsilon}{1+x+y} \quad \left(\text{as } \frac{\epsilon}{1+x+y} < 1\right) \\ &< x \cdot y + \frac{\epsilon(1+x+y)}{1+x+y} \\ &= x \cdot y + \epsilon. \end{aligned}$$

If (y_{n_k}) is a subsequence of (y_n) converges to y, then using Theorem 1.3.9, we get $(x_{n_k} \cdot y_{n_k})$ converges to $x \cdot y$. Thus, $x \cdot y = \limsup_{n\to\infty}(x_n \cdot y_n)$.

If $y = +\infty$, then there exists a subsequence (y_{n_k}) of y such that $y_{n_k} \to \infty$ as $k \to \infty$. Let $K > 0$ be given, then by assumption, there exists $j \in \mathbb{N}$ such that $y_{n_k} > \frac{2K}{x}$, $\forall k \geq j$. For $\epsilon = \frac{x}{2} > 0$, we choose m such that $|x_n - x| < \frac{x}{2}$, $\forall n \geq m$. Hence,

$$x - x_n \leq |x_n - x| < \frac{x}{2} \Rightarrow x_n > \frac{x}{2},\ \forall n \geq m.$$

Next, choose $p \in \mathbb{N}$ such that $n_p \geq m$. Now, for $k \geq \max\{j, p\}$, we get

$$x_{n_k} \cdot y_{n_k} > \frac{x}{2} \cdot \frac{2K}{x} = K.$$

Hence, $x_{n_k} \cdot y_{n_k} \to \infty$ as $k \to \infty$. As we have

$$x_n \cdot y_n < \infty + \epsilon,\ \forall \epsilon > 0,\ \forall n \in \mathbb{N}.$$

we get $\limsup_{n\to\infty}(x_n \cdot y_n) = \infty = x \cdot y$. □

THEOREM 1.3.20 Let (x_n) be a sequence of positive real numbers, then

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$$

Proof: Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{x_n}$ and $\beta = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$. If $\beta = +\infty$, then obviously we have $\alpha \leq \beta$. Therefore, we assume that $\beta < +\infty$. Then by definition of limit superior, for a given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\frac{x_{n+1}}{x_n} < \beta + \epsilon$, $\forall n \geq m$. For $n \geq m$, we have

$$x_n = \frac{x_n}{x_{n-1}} \times \frac{x_{n-1}}{x_{n-2}} \times \cdots \times \frac{x_{m+1}}{x_m} \times x_m < x_m \times (\beta + \epsilon)^{n-m}.$$

which implies that $\sqrt[n]{x_n} < \left(\frac{x_m}{(\beta+\epsilon)^m}\right)^{1/n}(\beta+\epsilon)$. If $c = \frac{x_m}{(\beta+\epsilon)^m}$, then we get $c^{1/n} \to 1$ as $n \to \infty$. (Because if $c > 1$, then $c^{1/n} = 1 + \delta_n$ for some $\delta_n > 0 \Rightarrow c = (1+\delta_n)^n > n\delta_n \Rightarrow \delta_n < \frac{c}{n} \to 0$ as $n \to \infty \Rightarrow c^{1/n} \to 1$ as $n \to \infty$. If $c < 1$, then $c^{-1} > 1 \Rightarrow c^{-1/n} \to 1 \Rightarrow c^{1/n} \to 1$ as $n \to \infty$). Therefore, using Lemma 1.3.18, we get

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \beta + \epsilon.$$

As $\epsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty} \sqrt[n]{x_n} = \alpha \leq \beta = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Similarly, we can prove that

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n}.$$

Using Lemma 1.3.16, we get

$$\liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n}.$$

Thus, the theorem follows. □

Definition 1.3.21 Let $f_n : A \to \mathbb{C}$, $\forall n \in \mathbb{N}$, where $A \subseteq \mathbb{C}$. We say that the sequence (f_n) of functions converges

1. point-wise to f on A, if for a given $z \in A$ and a given $\epsilon > 0$, there exists $m_z \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$, $\forall n \geq m_z$.

2. uniformly to f on A, if given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$, $\forall n \geq m$ and $\forall z \in A$.

Definition 1.3.22 Let $x_0 \in [a,b]$ and $f : (a,b) \times (c,d) \to \mathbb{C}$. We say that $f(x,y) \to \ell$ as $x \to x_0$ uniformly in y on (c,d). If given $\epsilon > 0$, then there exists $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x,y) - f(x_0,y)| < \epsilon$, $\forall y \in (c,d)$.

It is customary to define series of complex numbers by a formal sum $\sum_{n=1}^{\infty} a_n$. Before talking about its convergence, the sum has no meaning other than in the expression $\sum_{n=1}^{\infty} a_n$, first term is a_1, second term is a_2, third term is a_3, and so on. In the sequence (a_n) of complex numbers also, we say that first term is a_1, second term is a_2, third term is a_3, and so on. So what is the difference between (a_n) and $\sum_{n=1}^{\infty} a_n$? To find the difference, we can redefine the series of complex numbers as follows.

Definition 1.3.23 (Rigorous Definition)
Let (a_n) be a sequence of complex numbers. We say that the sequence (a_n) is summable if (S_n) converges, where $S_n = \sum_{k=1}^{n} a_k$, $\forall n \in \mathbb{N}$. In this case, we call $\lim_{n\to\infty} S_n$ the sum of (a_n) and is denoted by $\sum_{n=1}^{\infty} a_n$. The sum $\sum_{n=1}^{\infty} a_n$ is also called a series (or) an infinite series.

However, the old definition of a series (formal sum) is in usage for a long time and most of the readers are accustomed with using the notation $\sum_{n=1}^{\infty} a_n$ before discussing its convergence, a few similar less rigorous statements on series are also used in this book, only to facilitate the reader

Definition 1.3.24 A series $\sum_{n=0}^{\infty} a_n$ of complex numbers is said to be convergent if $s_m = \sum_{k=0}^{m} a_k$, $\forall m \in \mathbb{N}$ and $s_m \to s$ as $m \to \infty$ for some complex number s. In this case, we call 's' the sum of the series $\sum_{n=0}^{\infty} a_n$ and is also denoted by the series itself.

Definition 1.3.25 A series $\sum_{n=0}^{\infty} a_n$ of complex numbers is said to be absolutely convergent if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

RESULT 1.3.26 A series $\sum_{n=0}^{\infty} a_n$ converges iff $\lim_{m\to\infty} \sum_{n=m}^{\infty} a_n = 0$.

Proof: $\sum_{n=0}^{\infty} a_n$ converges to s iff $s_m \to s$ as $m \to \infty$, where $s_m = a_0 + a_1 + a_2 + \cdots + a_m$, $\forall m \in \mathbb{N}$ and $s = \sum_{n=0}^{\infty} a_n$ iff $s_{m-1} - s \to 0$ as $m \to \infty$ iff $\lim_{m\to\infty} \sum_{n=m}^{\infty} a_n = 0$. □

The following result gives a necessary condition for a convergent series of complex numbers.

RESULT 1.3.27 If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $\sum_{n=0}^{\infty} a_n$ converge. If

$$s_m = a_0 + a_1 + a_2 + \cdots + a_m, \ \forall m \in \mathbb{N},$$

then (s_m) is a convergent sequence and by Result 1.3.2, (s_m) is a Cauchy sequence. Let $\epsilon > 0$ be given. Then, there exists $m \in \mathbb{N}$ such that

$$|s_j - s_k| < \epsilon, \ \forall j, k \geq m.$$

If $n \geq m + 1$, then $n, n - 1 \geq m$, and hence, $|a_n| = |s_n - s_{n-1}| < \epsilon$. Hence, (a_n) converges to 0. □

Definition 1.3.28 (Rearrangement of a series)
Let $\sum_{n=1}^{\infty} a_n$ be a given series. If $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\sum_{n=1}^{\infty} a_{f(n)}$ is called a rearrangement of $\sum_{n=1}^{\infty} a_n$.

RESULT 1.3.29 If $\sum_{n=1}^{\infty} a_n$ converges to S absolutely, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to the same S.

Proof: Let $\sum_{n=1}^{\infty} a_{f(n)}$ be a rearrangement of $\sum_{n=1} a_n$, where $f : \mathbb{N} \to \mathbb{N}$ is a bijection. Let $S_m = \sum_{n=1}^{m} |a_n|$ and $T_m = \sum_{n=1}^{m} a_{f(n)}$, $\forall m \in \mathbb{N}$. Since (S_m) converges, by Cauchy criterion (Theorem 1.3.11), we have (S_m) is a Cauchy

sequence and hence for a given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$|S_p - S_q| = \sum_{n=p}^{q} |a_n| < \frac{\epsilon}{2} \ \forall q \geq p \geq N_0.$$

Since $\sum_{n=1}^{\infty} a_n$ convergs to s, there exists $N \in \mathbb{N}$ such that $N \geq N_0$ and

$$\left| \sum_{n=1}^{p} a_n - S \right| \leq \frac{\epsilon}{2} \ \forall p \geq N. \tag{1.3}$$

Since $f(\mathbb{N}) = \mathbb{N}$ we can choose $M \in \mathbb{N}$ such that $\{1, 2, \ldots, N\} \subseteq \{f(n) : 1 \leq n \leq M\}$. Now for all $p \in \mathbb{N}$ with $p \geq M$, we have

$$\begin{aligned}
|T_p - S| &= \left| \sum_{n=1}^{p} a_{f(n)} - \sum_{n=1}^{q} a_n \right| + \left| \sum_{n=1}^{q} a_n - S \right| \text{ (where } q \geq p.) \\
&\leq \sum_{n \epsilon A} |a_n| + \frac{\epsilon}{2} \quad \text{(using } q \geq p \geq M \geq N \text{ and (1.3))} \\
&\quad \text{where } A \subseteq \mathbb{N} \text{ such that } K \leq n \leq J, \forall n \, \epsilon \, A \text{ for some } K, J \, \epsilon \, \mathbb{N} \\
&\quad \text{with } N \leq K \leq J. \\
&\leq \sum_{K}^{J} |a_n| + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

because by the choice of M, we have $N \leq K \leq J$. Thus $\sum_{n=1}^{\infty} a_{f(n)}$ converges to S. □

THEOREM 1.3.30 (Comparison test)
If (a_n) is a sequence of complex numbers and (b_n) is a sequence of non-negative real numbers such that $|a_n| \leq b_n$, $\forall n \geq m$, for some $m \in \mathbb{N}$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Proof: For each $n \in \mathbb{N}$, let

$$s_n = |a_0| + |a_1| + \cdots + |a_n| \text{ and } t_n = b_0 + b_1 + \cdots + b_n.$$

By assumption, (t_n) is a convergent sequence, and hence, (t_n) is a Cauchy sequence. Then, for a given $\epsilon > 0$, there exists $p \in \mathbb{N}$ such that

$$|t_j - t_k| < \epsilon, \ \forall j, k \geq p.$$

Now for the same p, if $j > k \geq p$, then we have

$$|s_j - s_k| = \sum_{n=k+1}^{j} |a_n| \leq \sum_{n=k+1}^{j} b_n = |t_j - t_k| < \epsilon.$$

Hence, it follows that $|s_j - s_k| < \epsilon$ whenever $j, k \geq p$. Therefore, (s_n) converges, by Cauchy's criterion, and hence, $\sum_{n=0}^{\infty} a_n$ converges absolutely. □

Example 1.3.31 If $z \in \mathbb{C}$ with $|z| < 1$, then the geometric series $\sum_{n=0}^{\infty} z^n$ converges.

If $z = 0$, then obviously the series converges to 0. For $z \neq 0$, then we have $0 < |z| < 1$. Hence, we write $|z| = \dfrac{1}{1+r}$, where $r = \dfrac{1}{|z|} - 1 > 0$. For every $n > 2$, since

$$(1+r)^n > nC_1 r = nr$$

we get

$$0 < |z|^n = \frac{1}{(1+r)^n} < \frac{1}{nr} \to 0 \text{ as } n \to \infty.$$

If $s_n = 1 + z + z^2 + \cdots + z^n$, then

$$s_n(1-z) = (1 + z + z^2 + \cdots + z^n)(1-z) = 1 - z^{n+1}, \forall n \in \mathbb{N}.$$

Therefore, it follows that

$$s_n = \frac{1 - z^{n+1}}{1-z} \to \frac{1}{1-z} \text{ as } n \to \infty$$

and hence, $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$.

1.4 SOME TOPOLOGICAL PROPERTIES OF THE COMPLEX PLANE

Although we assume that the reader is familiar with mathematical analysis, we briefly recall some important definitions and results on topology of complex plane, which will be required in the following sequel.

Definition 1.4.1 The standard metric $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is defined by $d(z, w) = |z - w|$, $\forall z, w \in \mathbb{C}$.

From the properties of $|\cdot|$, we obtain the following:

1. $|z - w| \geq 0$, $\forall z, w \in \mathbb{C}$,
2. $|z - w| = |w - z|$, $\forall z, w \in \mathbb{C}$,
3. $|z - w| = 0$ iff $z = w$,
4. $|z - w| \leq |z - \zeta| + |\zeta - w|$, $\forall z, \zeta, w \in \mathbb{C}$.

Definition 1.4.2 For a given $z_0 \in \mathbb{C}$ and $r > 0$, by a neighbourhood of z_0 or by the open ball with center z_0 and radius $r > 0$, we mean the set $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Definition 1.4.3 Let $U \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. We say that a is

1. a limit point of U if for every $r > 0$, we have $(B(a, r) \setminus \{a\}) \cap U \neq \varnothing$.
2. an interior point of U if there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$.

Definition 1.4.4 A subset U of $\mathbb{C}$ is said to be

1. an open set if every point of U is an interior point of U.
2. a closed set if every limit point of U is a point of U.

RESULT 1.4.5 Let $U \subseteq \mathbb{C}$. U is an open set iff U^c is a closed set.

Proof:

U is open	iff	every point of U is an interior point of U
	iff	for every $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subseteq U$
	iff	for every $x \notin U^c$, there exists $r_x > 0$ such that $B(x, r_x) \cap U^c = \varnothing$
	iff	for every $x \notin U^c$, there exists $r_x > 0$ such that $(B(x, r_x) \setminus \{x\}) \cap U^c = \varnothing$
	iff	every $x \notin U^c$ is not a limit point of U^c
	iff	every limit point of U^c is a point of U^c
	iff	U^c is closed.

Hence, the result. □

THEOREM 1.4.6 Let $A_\alpha, A_j \subseteq \mathbb{C}$ for all $\alpha \in \Lambda$ and for all $j \in \{1, 2, \ldots, n\}$, where Λ is an arbitrary index set.

(i) If A_α is open $\forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is open.

(ii) If A_j is open $\forall j \in \{1, 2, \ldots, n\}$, then $\bigcap_{j=1}^{n} A_j$ is open.

Proof: To prove $\bigcup_{\alpha \in \Lambda} A_\alpha$ is open, let $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$ be arbitrary. Then $x \in A_\beta$ for some $\beta \in \Lambda$. As A_β is open, there exists $r > 0$ such that $B(x, r) \subseteq A_\beta \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$. Thus, $\bigcup_{\alpha \in \Lambda} A_\alpha$ is open.

Next, if $x \in \bigcap_{j=1}^{n} A_j$ is arbitrary, then $x \in A_j$, $\forall j \in \{1, 2, \ldots, n\}$. Since each A_j is open, there exists $r_j > 0$ such that $B(x, r_j) \subseteq A_j$, $\forall j \in \{1, 2, \ldots, n\}$. If $s = \min\{r_j : 1 \leq j \leq n\}$, then $s > 0$, and hence, $B(x, s) \subseteq B(x, r_j) \subseteq A_j$, $\forall j \in \{1, 2, \ldots, n\} \Rightarrow B(x, s) \subseteq \bigcap_{j=1}^{n} A_j$. Thus, $\bigcap_{j=1}^{n} A_j$ is open. □

THEOREM 1.4.7 Let $B_\alpha, B_j \subseteq \mathbb{C}$ for all $\alpha \in \Lambda$ and for all $j \in \{1, 2, \ldots, n\}$, where Λ is an arbitrary index set.

(i) If B_α is closed $\forall \alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} B_\alpha$ is closed.

(ii) If B_j is closed $\forall j \in \{1, 2, \ldots, n\}$, then $\bigcup_{j=1}^{n} B_j$ is closed.

Proof: Proof of this theorem follows from Theorem 1.4.5, De-Morgan's laws, and Theorem 1.4.6. □

Definition 1.4.8 Let $A \subset \mathbb{C}$. Define closure $\mathrm{Cl}\, A$ of A by $A \cup A'$, where A' is the set of all limit points of A.

LEMMA 1.4.9 $x \in \mathrm{Cl}\, A$ iff $B(x, r) \cap A \neq \varnothing$ for every $r > 0$.

Proof of this lemma follows directly from the definition of $\mathrm{Cl}\, A$.

RESULT 1.4.10 The closure operator satisfies the following properties:

1. $A \subseteq B \Rightarrow \text{Cl}\, A \subseteq \text{Cl}\, B$.
2. A is closed iff $A = \text{Cl}\, A$.
3. $\text{Cl}\, A$ is the smallest closed set containing A.
4. $\text{Cl Cl}\, A = \text{Cl}\, A$.
5. $\text{Cl}\,(A \cup B) = (\text{Cl}\, A) \cup (\text{Cl}\, B)$.
6. $\text{Cl}\,(A \cap B) \subseteq (\text{Cl}\, A) \cap (\text{Cl}\, B)$.

Proof:

1. If $A \subset B$, then using the previous lemma, we get

$$\begin{aligned} x \in \text{Cl}\, A &\Rightarrow B(x, r) \cap A \neq \varnothing,\ \forall r > 0 \\ &\Rightarrow B(x, r) \cap B \neq \varnothing,\ \forall r > 0 \\ &\Rightarrow x \in \text{Cl}\, B. \end{aligned}$$

2. $A = \text{Cl}\, A \Leftrightarrow A = A \cup A' \Leftrightarrow A' \subseteq A \Leftrightarrow A$ is closed.

3. Let $x \notin \text{Cl}\, A$. Then by previous lemma, there exists $r > 0$ such that $B(x, r) \cap A = \varnothing$. For every $y \in B(x, r)$, there exists $s_y = r - |x - y| > 0$ such that $B(y, s_y) \subseteq B(x, r)$, and hence, $B(y, s_y) \cap A = \varnothing$. Hence, it follows that no point of $B(x, r)$ is a point of $\text{Cl}\, A$ by the same lemma. Therefore, $B(x, r) \subseteq (\text{Cl}\, A)^c$. Thus, we have proved that $(\text{Cl}\, A)^c$ is an open set, and hence, $\text{Cl}\, A$ is a closed set.
   Next if B is a closed subset of $\mathbb{C}$ such that $A \subset B$, then using the properties (1) and (2), we have $\text{Cl}\, A \subseteq \text{Cl}\, B$. Therefore, $\text{Cl}\, A \subseteq \text{Cl}\, B = B$. Thus, $\text{Cl}\, A$ is the smallest closed set containing A.

4. Using the property (3), $\text{Cl}\, A$ is a closed set, and by using the property (2), we get $\text{Cl Cl}\, A = \text{Cl}\, A$.

5. Clearly, $A \subseteq \text{Cl A}$ and $B \subseteq \text{Cl B}$ imply that $A \cup B \subseteq (\text{Cl}\, A) \cup (\text{Cl}\, B)$. As $(\text{Cl}\, A) \cup (\text{Cl}\, B)$ is a closed set (being a union of two closed sets) containing $A \cup B$ and $\text{Cl}\,(A \cup B)$ is the smallest closed set containing $A \cup B$, we get $\text{Cl}\,(A \cup B) \subseteq (\text{Cl}\, A) \cup (\text{Cl}\, B)$. On the other hand, using the property (1), $A \subseteq A \cup B$ and $B \subseteq A \cup B$ imply that $\text{Cl}\, A \subseteq \text{Cl}\,(A \cup B)$ and $\text{Cl}\, B \subseteq \text{Cl}\,(A \cup B)$. Therefore, $(\text{Cl}\, A) \cup (\text{Cl}\, B) = \text{Cl}\,(A \cup B)$. Thus, the property (5) holds.

6. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, using the property (1), we have $\mathrm{Cl}\,(A \cap B) \subseteq \mathrm{Cl}\,A$ and $\mathrm{Cl}\,(A \cap B) \subseteq \mathrm{Cl}\,B$. Thus, $\mathrm{Cl}\,(A \cap B) \subseteq (\mathrm{Cl}\,A) \cap (\mathrm{Cl}\,B)$.

□

Example 1.4.11 For every $a \in \mathbb{C}$ and $r > 0$, $\mathrm{Cl}\,B(a,r) = \{x \in \mathbb{C} : |x-a| \leq r\}$.
Suppose $y \notin \{x \in \mathbb{C} : |x-a| \leq r\}$. Then, $|y-a| > r$. If $s \in \mathbb{R}$ is such that $0 < s < |y-a| - r$, then we claim that $B(y,s) \cap B(a,r) = \varnothing$. If $z \in B(y,s) \cap B(a,r)$, then $|z-y| < s$ and $|z-a| < r$ which implies that

$$|y-a| \leq |y-z| + |z-a| < s + r < |y-a|.$$

This is not possible. Hence, $B(y,s) \cap B(a,r) = \varnothing$. Therefore, $y \notin \mathrm{Cl}\,B(a,r)$ by Lemma 1.4.9. Conversely, if $y \notin \mathrm{Cl}\,B(a,r)$, then there exists $t > 0$ such that $B(y,t) \cap B(a,r) = \varnothing$. As $y - \frac{t}{2}\frac{y-a}{|y-a|} \in B(y,t)$, we have $y - \frac{t}{2}\frac{y-a}{|y-a|} \notin B(a,r)$, and hence by applying $|y-a| > t \Rightarrow 0 < \frac{t}{2|y-a|} < 1$, we get

$$\begin{aligned} r &\leq \left| y - \frac{t}{2}\frac{y-a}{|y-a|} - a \right| \\ &= |y-a| \left| 1 - \frac{t}{2|y-a|} \right| \\ &= |y-a| \left(1 - \frac{t}{2|y-a|} \right) \\ &< |y-a|. \end{aligned}$$

Therefore, $y \notin \{x \in \mathbb{C} : |x-a| \leq r\}$, and hence, $\mathrm{Cl}\,B(a,r) = \{x \in \mathbb{C} : |x-a| \leq r\}$.

Example 1.4.12 There exists $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$ such that $\mathrm{Cl}\,(A \cap B) \subsetneqq (\mathrm{Cl}\,A) \cap (\mathrm{Cl}\,A)$.
We consider $A = B(1,1)$ and $B = B(2,1)$. Clearly, $\mathrm{Cl}\,(A \cap B) = \mathrm{Cl}\,\varnothing = \varnothing$ but $(\mathrm{Cl}\,A) \cap (\mathrm{Cl}\,B) = \{x \in \mathbb{C} : |x-1| \leq 1\} \cap \{x \in \mathbb{C} : |x-2| \leq 1\} = \{1\}$.

Exercise 1.4.13 Prove that $x \in \mathrm{Cl}\,A$ iff there exists a sequence (x_n) in A such that $x_n \to x$ as $n \to \infty$.

Exercise 1.4.14 If A is a bounded subset of $\mathbb{R}$, then prove that $\sup A, \inf A \in \mathrm{Cl}\,A$.

Definition 1.4.15 A subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if $\text{Cl}\, A = \mathbb{R}$. Similarly, if $A \subseteq \mathbb{C}$ with $\text{Cl}\, A = \mathbb{C}$, then A is called a dense subset of $\mathbb{C}$.

Example 1.4.16 The set of all rational numbers is dense in $\mathbb{R}$.

To justify this example, we shall show that every point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$. Let $r > 0$ be given, then using the observation 3(l) in page 6, we choose $q \in \mathbb{Q}$ such that $x < q < x + r$. Therefore, $q \in [B(x, r) \setminus \{x\}] \cap \mathbb{Q}$. Thus, $\mathbb{R} \subseteq \mathbb{Q}' \subseteq \text{Cl}\, \mathbb{Q}$.

Definition 1.4.17 Let $A \subset \mathbb{C}$. The interior $\text{Int}\, A$ of A is defined by the set of all interior points of A.

Exercise 1.4.18 Let $A, B \subseteq \mathbb{C}$. Then prove that

(i) $A \subseteq B \Rightarrow \text{Int}\, A \subseteq \text{Int}\, B$.

(ii) A is open $\Leftrightarrow A = \text{Int}\, A$.

(iii) $\text{Int}\, A$ is the largest open set contained in A.

(iv) $\text{Int Int}\, A = A$.

(v) $\text{Int}\, (A \cap B) = \text{Int}\, A \cap \text{Int}\, B$.

(vi) $\text{Int}\, (A \cup B) \supseteq \text{Int}\, A \cup \text{Int}\, B$.

Definition 1.4.19 Let $A, B \subset \mathbb{C}$. A and B are said to be separated if $(\text{Cl}\, A) \cap B = \varnothing = A \cap (\text{Cl}\, B)$.

Definition 1.4.20 Let S be a subset of $\mathbb{C}$. S is said to be connected if S cannot be a subset of union of two separated sets A and B such that $A \cap S \neq \varnothing$ and $B \cap S \neq \varnothing$.

Definition 1.4.21 (Interval) A subset I of $\mathbb{R}$ is said to be an interval if $z \in I$ whenever $x, y \in I$ and $x < z < y$.

There are four types of intervals, which are as follows:

1. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
2. $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
3. $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.
4. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

THEOREM 1.4.22 A subset S of $\mathbb{R}$ is connected iff S is an interval.

Proof: Let S be a connected subset of $\mathbb{R}$. Suppose S is not an interval, then there exist $x, y \in S$ and $z \in \mathbb{R}$ such that $x < z < y$ but $z \notin S$.
Put $A = (-\infty, z)$ and $B = (z, \infty)$. Clearly, it follows that

$$S \subseteq A \cup B, A \cap S \neq \varnothing \text{ and } B \cap S \neq \varnothing.$$

As

$$A \cap (\mathrm{Cl}\, B) = (-\infty, z) \cap [z, \infty) = \varnothing = (-\infty, z] \cap (z, \infty) = (\mathrm{Cl}\, A) \cap B,$$

we get that S is disconnected, which is a contradiction. Therefore, S is an interval.
Conversely, suppose S is an interval and if there exist subsets A, B of $\mathbb{R}$ such that

$$S \subseteq A \cup B,\ S \cap A \neq \varnothing \neq S \cap B \text{ and } (\mathrm{Cl}\, A) \cap B = \varnothing = (\mathrm{Cl}\, B) \cap A.$$

As $S \cap A \neq \varnothing \neq S \cap B$, we choose $\alpha \in S \cap A$ and $\gamma \in S \cap B$. Without loss of generality, we assume that $\alpha < \gamma$, since $\alpha \neq \gamma$. If $\beta = \sup([\alpha, \gamma] \cap A)$, then $\alpha \leq \beta \leq \gamma$ and $\alpha \in \mathrm{Cl}\, A$. As $\alpha, \gamma \in S$ and S is an interval, we get $\beta \in S$ and hence,

$$\beta \in A \text{ or } \beta \in B \tag{1.4}$$

Using $\beta \in \mathrm{Cl}\, A$ and $(\mathrm{Cl}\, A) \cap B = \varnothing$, we obtain $\beta \notin B$. Then, $\beta \in A$, and hence, $\beta \notin \mathrm{Cl}\, B$. Hence, $(\beta, \gamma) \nsubseteq B$; otherwise, $\beta \in [\beta, \gamma] = \mathrm{Cl}\,(\beta, \gamma) \subseteq \mathrm{Cl}\, B$, which is not possible. Therefore, there exists $\delta \in (\beta, \gamma)$ such that $\delta \notin B$. From the definition of β and from the inequality $\beta < \delta$, it follows that $\delta \notin A$. Thus, $\delta \notin S$, as $S \subseteq A \cup B$. This implies that S is not an interval, as $\alpha < \delta < \gamma$ and $\alpha, \gamma \in S$. This is a contradiction, and hence S is connected. □

COROLLARY 1.4.23 The set $\mathbb{R}$ of all real numbers is a connected set.

Proof: As $\mathbb{R}$ itself is obviously an interval, it is connected by the previous theorem. □

Definition 1.4.24 (Line segment) Let a, b be two points in $\mathbb{C}$. We call the set $L_{a,b} = \{(1-t)a + tb : t \in [0, 1]\}$ the line segment joining a and b. The point a is called the initial point of the line segment and b is called the end point of the line segment.

Definition 1.4.25 (Polygon) Let $\{a_i : i = 0, 1, 2, \ldots, n\}$ be a finite set of complex numbers. The union $\bigcup_{i=1}^{n} L_{a_{i-1},a_i}$ of the given finite number of line segments of the form $L_{a_{i-1},a_i}, i = 1, 2, \ldots, n$ is called a polygon.

THEOREM 1.4.26 If E is a subset of $\mathbb{C}$ such that every pair of points a, b can be joined by a polygon $P_{a,b} \subseteq E$, then E is a connected subset of $\mathbb{C}$.

Proof: Suppose E is disconnected, then there exists a pair of sets A, B in $\mathbb{C}$ such that $E \subseteq A \cup B$ and $A \cap E \neq \varnothing \neq E \cap B$ and $(\mathrm{Cl}\, A) \cap B = A \cap (\mathrm{Cl}\, B) = \varnothing$.

Choose $a \in E \cap A$ and $b \in E \cap B$. By hypothesis, there exists $P_{a,b} \subseteq E$. On examining the line segments from a to b, we can find a line segment $l = L_{c,d} \subseteq P_{a,b}$ such that $c \in A$ and $d \in B$. Put

$$A_1 = \{t \in [0, 1] : (1 - t)c + td \in A \cap E\}$$

$$A_2 = \{t \in [0, 1] : (1 - t)c + td \in B \cap E\}$$

As the line segment $l \subseteq E \subseteq A \cup B$, we get $[0, 1] = A_1 \cup A_2$ by an easy verification. We note that

$$c \in A \cap E \Rightarrow 0 \in A_1 \Rightarrow A_1 \neq \varnothing$$

and

$$d \in B \cap E \Rightarrow 1 \in A_2 \Rightarrow A_2 \neq \varnothing$$

Claim. $(\mathrm{Cl}\, A_1) \cap A_2 = A_1 \cap (\mathrm{Cl}\, A_2) = \varnothing$.

Suppose $t \in (\mathrm{Cl}\, A_1) \cap A_2$. Then, there exists a sequence $\{t_n\}$ from A_1 such that $t_n \to t$ as $n \to \infty$. Therefore, $(1 - t_n)c + t_n d \in A$, $\forall n \in \mathbb{N}$, and so,

$$(1 - t)c + td = \lim_{n \to \infty} (1 - t_n)c + t_n d \in \mathrm{Cl}\, A$$

Clearly, $(1 - t)c + td \in B$ because $t \in A_2$. Similarly, if $s \in A_1 \cap (\mathrm{Cl}\, A_2)$, then we can show that $(1 - s)c + sd \in A \cap (\mathrm{Cl}\, B)$. Therefore, $(\mathrm{Cl}\, A_1) \cap A_2 = A_1 \cap (\mathrm{Cl}\, A_2) = \varnothing$. This gives a separation for the interval $[0, 1]$, which is a contradiction. □

Immediately, we get the following corollaries.

COROLLARY 1.4.27 Every line segment is connected in $\mathbb{C}$.

COROLLARY 1.4.28 Every polygon is connected.

COROLLARY 1.4.29 In $\mathbb{C}$, every open disc $B(a, r) = \{x \in \mathbb{C} : |x - a| < r\}$ is connected.

Proof: Let $x, y \in B(a,r)$. We shall show that $L_{x,y} = \{(1-t)x + ty : t \in [0,1]\} \subseteq B(a,r)$.
Now, if $t \in [0,1]$, then

$$\begin{aligned} |(1-t)x + ty - a| &= |(1-t)x + ty - (1-t)a + ta| \\ &\leq (1-t)|x-a| + t|y-a| \\ &< (1-t)r + tr = r \end{aligned}$$

and hence, we get $(1-t)x + ty \in B(a,r)$. Hence, the corollary follows from the above theorem. □

Definition 1.4.30 An open connected subset of $\mathbb{C}$ is called a region.

THEOREM 1.4.31 If E is a region, then given a pair of points x, y of E, there exists a polygon $Q_{x,y}$ joining x and y such that $Q_{x,y} \subseteq E$ and whose line segments are parallel to the coordinate axes.

Proof: Throughout this proof, by $Q_{s,t}$ we mean a polygon joining s and t such that $Q_{s,t} \subseteq E$ and whose line segments are parallel to the coordinate axes. Assume that E is connected. Fix $a \in E$ arbitrarily. Define

$$A = \{x \in E : \text{ there exists a polygon } Q_{x,a} \subseteq E\} \text{ and } B = E \setminus A.$$

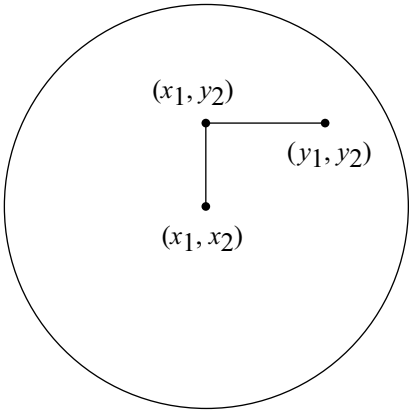
Clearly, $E = A \cup B$ and $A \neq \varnothing$. To prove this theorem, we shall show that $E = A$. (Because if it is proved and if $x, y \in E$ is arbitrary, then there exist $Q_{x,a}, Q_{y,a} \subseteq E$, and hence, their union is a polygon in E joining x and y.)

Claim. $(\mathrm{Cl}\, A) \cap B = \varnothing$.

If $x \in (\mathrm{Cl}\, A) \cap B$, then there exists $r > 0$ such that $B(x,r) \subseteq E$ (as $x \in E$ and E is open). Then, there exists $y \in A \cap B(x,r)$, using $x \in \mathrm{Cl}\, A$. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then let $z = (x_1, y_2)$. From

$$|x - z| = |(x_1, x_2) - (x_1, y_2)| = |x_2 - y_2| \leq |x - y| < r$$

we get $z \in B(x,r)$.



By applying the proof of Corollary 1.4.29 twice, we get $L_{x,z}, L_{z,y} \subseteq B(x,r) \subseteq E$, and hence, $P_{x,y} = L_{x,z} \cup L_{z,y}$ is completely contained in E. Since $y \in A$, there exists $P_{y,a} \subset E$ and hence $P_{x,a} = P_{x,y} \cup P_{y,a} \subseteq E$. Thus, $x \in A$. This contradicts the fact that $A \cap B = \varnothing$. Therefore, $(\mathrm{Cl}\, A) \cap B = \varnothing$.

Claim. $A \cap (\mathrm{Cl}\, B) = \varnothing$.

If $a \in A \cap (\mathrm{Cl}\, B)$, then choose $s > 0$ such that $B(a,s) \subseteq E$. As $a \in \mathrm{Cl}\, B$ there exists $b \in B \cap B(a,s)$. By a similar argument used in the justification of previous claim, we can find a polygon $P_{a,b} \subseteq E$. This implies that $b \in A$, which is a contradiction.

Hence, $B = \varnothing$, and hence, $A = E$. This completes the proof of the theorem. □

Example 1.4.32 There exists a connected subset E of $\mathbb{C}$ such that it contains a pair of points that cannot be joint by a polygon inside E.

If $E = \{z \in \mathbb{C} : |z| = 1\}$, then E is connected because $E = f([0, 2\pi])$ where $f(t) = e^{it}$, $t \in [0, 2\pi]$; at this level, assume that the equality holds and the fact that f is continuous. However, for any two distinct points z and w of E, no point of $L_{z,w}$ other than z and w belongs to E.

To justify the last statement, let $z, w \in E$ be such that $z \neq w$. Then, there exist $\theta_1, \theta_2 \in [0, 2\pi)$ such that $z = e^{i\theta_1}$ and $w = e^{i\theta_2}$ so that $0 < |\theta_1 - \theta_2| < 2\pi$. Hence, for any $t \in (0,1)$, $(1-t)z\overline{tw} = (1-t)te^{i(\theta_1 - \theta_2)}$, which is not real and non-negative, and hence, $|(1-t)z + tw| < |(1-t)z| + |tw| = (1-t)|z| + t|w| \leq (1-t) + t = 1$. Thus, $(1-t)z + tw \notin E$. □

Definition 1.4.33 Let $S \subseteq \mathbb{C}$. A subset A of S is said to be a component of S if A is a largest connected subset of $\mathbb{C}$ contained in S.

That is, there is no connected subset B of $\mathbb{C}$ such that $A \subsetneqq B \subseteq S$.

RESULT 1.4.34 The collection of all components of a set S is a partition of S. That is, the union of all components of S is S, and components are pairwise disjoint.

Proof: For each, $x \in S$, let $C_S[x]$ be the union of all connected subsets of S containing x. Clearly, this union is non-empty as $\{x\}$ is a connected subset of S containing x. We claim that $C_S[x]$ is connected. Suppose, $C_S[x] \subseteq A \cup B$, with $C_S[x] \cap A \neq \varnothing$, $C_S[x] \cap B \neq \varnothing$ and $A \cap \mathrm{Cl}\, B = \varnothing$ and $B \cap \mathrm{Cl}\, A = \varnothing$. Let $a \in A$ and $b \in B$. Then, there exist connected subsets C_a and C_b of S such that $a, x \in C_a \subseteq S$ and $b, x \in C_b \subseteq S$. Therefore, $C_a \subseteq A \cup B$ and $C_b \subseteq A \cup B$. Since C_a is a connected set either $C_a \subseteq A$ or $C_a \subseteq B$, otherwise $C_a \cap A \neq \varnothing$ and $C_a \cap B \neq \varnothing$, and hence, C_a becomes a disconnected set, which is a contradiction. As $a \in A \cap C_a$, we conclude that $C_a \subseteq A$. Similarly, we conclude that $C_b \subseteq B$. Therefore, $x \in C_a \cap C_b \subseteq A \cap B \subseteq A \cap \mathrm{Cl}\, B = \varnothing$, which is a contradiction. Hence, $C_S[x]$ is a connected set. Next we observe that $C_S[x]$ cannot be properly contained in a connected subset of S because if K is a connected subset of S containing $C_S[x]$, then $x \in K$, and hence by definition of $C_S[x]$, we have $K \subseteq C_S[x]$ which implies $C_S[x] = K$. Thus, $C_S[x]$ is a component of S.

As for every $x \in S$, there exists a component $C_S[x]$ of S containing x; to conclude this theorem, we show that any two component of S are either identical or disjoint. If $C_S[x]$ and $C_S[y]$ are two components of S such that $C_S[x] \cap C_S[y] \neq \varnothing$, then let $z \in C_S[x] \cap C_S[y]$. If $C_S[z]$ is the component of S containing z, then $C_S[x] \subseteq C_S[z]$, since $C_S[x]$ is a connected subset of S containing z. Now, $x \in C_S[x] \subseteq C_S[z]$ imply that $C_S[z]$ is a connected subset of S containing x, and hence, $C_S[z] \subseteq C_S[x] \Rightarrow C_S[x] = C_S[z]$. Similarly, we show that $C_S[y] = C_S[z]$. Thus, $C_S[x] = C_S[z] = C_S[y]$. □

RESULT 1.4.35 Every non-empty connected subset A of a set S is contained in one and only one component of S.

Proof: Let A be a non-empty connected subset S. Then, there exists $x \in A \subseteq S$. If $C_S[x]$ is the component of S containing x, then obviously $C_S[x]$ is a component containing A, as $C_S[x]$ is the union of all connected subsets of S containing x. Since the components of S are pairwise disjoint (by previous result), $C_S[x]$ is the unique component of S containing A. □

Exercise 1.4.36 Let $E \subseteq \mathbb{C}$. Then prove the following statements:

1. If E is written as a disjoint union of two non-empty open subsets of $\mathbb{C}$, then E is not connected.

2. If E is written as a disjoint union of two non-empty closed subsets of $\mathbb{C}$, then E is not connected.

Definition 1.4.37 A subset K of $\mathbb{C}$ is called a compact set if for a given collection $\{E_\alpha : \alpha \in I\}$ of open subsets of $\mathbb{C}$ such that $K \subset \bigcup_{\alpha \in I} E_\alpha$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $K \subset \bigcup_{j=1}^{n} E_{\alpha_j}$.

THEOREM 1.4.38 Let K be a compact subset of $\mathbb{C}$. Then K is closed and bounded.

Proof: Let $x \notin K$. Then, for every $y \in K$, we have $y \neq x$. If $r_y = \dfrac{|x-y|}{2}$, then $r_y > 0$ and $B(x, r_y) \cap B(y, r_y) = \varnothing$, $\forall y \in K$. As $\{B(y, r_y) : y \in K\}$ is a collection of open sets satisfying $K \subset \bigcup_{y \in K} B(y, r_y)$. Hence, there exist $y_1, y_2, \ldots, y_m \in K$ such that $K \subset \bigcup_{k=1}^{m} B(y_k, r_{y_k})$. If

$$r = \min\{r_{y_k} : 1 \leq k \leq m\}$$

then

$$B(x, r) \cap K \subseteq B(x, r) \cap \bigcup_{k=1}^{m} B(y_k, r_{y_k}) \subseteq \bigcup_{k=1}^{m} B(x, r_{y_k}) \cap B(y_k, r_{y_k}) = \varnothing$$

and hence we get $B(x, r) \subseteq K^c$. Thus, K^c is an open set, and hence, K is closed.

Consider the collection of open sets $\{B(x, 1) : x \in K\}$ with $K \subset \bigcup_{x \in K} B(x, 1)$. By definition, there exist $x_1, x_2, \ldots, x_n \in K$ such that $K \subset \bigcup_{j=1}^{n} B(x_j, 1)$. If $M = \max\{|x_j| : j \in \{1, 2, \ldots, n\}\}$, then we claim that $|x| \leq M + 1$, $\forall x \in K$. If $x \in K$, then $x \in B(x_j, 1)$ for some $1 \leq j \leq n$. Therefore,

$$|x| \leq |x - x_j| + |x_j| \leq 1 + M.$$

Hence, K is bounded. □

THEOREM 1.4.39 Every infinite subset of a compact set K has a limit point in K.

Proof: Let S be an infinite subset of K. If S has no limit point in K, then for each $z \in K$, there exists $r_z > 0$ such that $B(z, r_z) \cap S \subseteq \{z\}$. Obviously, $\{B(z, r_z) : z \in K\}$ is a collection of open sets such that $\underset{z\in K}{\cup} B(z, r_z) = K$. Since K is compact, there exists $z_1, z_2, \ldots, z_n \in K$ such that $\underset{j=1}{\overset{n}{\cup}} B(z_n, r_{z_n}) = K$, and hence, $S = S \cap K = S \cap \underset{j=1}{\overset{n}{\cup}} B(z_n, r_{z_n}) = \underset{j=1}{\overset{n}{\cup}} S \cap B(z_n, r_{z_n}) \subseteq \{z_1, z_2, \ldots, z_n\}$, which is a contradiction to the assumption on S. □

THEOREM 1.4.40 Closed subset of a compact set is compact.

Proof: Let K be a compact subset of $\mathbb{C}$ and $F \subset K$ be closed. If $\{E_\alpha : \alpha \in I\}$ is a collection of open sets such that $F \subseteq \underset{\alpha\in I}{\cup} E_\alpha$, then

$$K \subseteq \mathbb{C} = F \cup F^c \subseteq \underset{\alpha\in I}{\cup} E_\alpha \cup F^c.$$

As F is closed, $\{E_\alpha, F^c : \alpha \in I\}$ is a collection of open sets; therefore, using the compactness of K, we get $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $K \subseteq E_{\alpha_1} \cup E_{\alpha_2} \cup \cdots \cup E_{\alpha_n} \cup F^c$. Thus, $F \subseteq E_{\alpha_1} \cup E_{\alpha_2} \cup \cdots \cup E_{\alpha_n}$, and hence, F is compact. □

THEOREM 1.4.41 Let A be a closed subset of $\mathbb{C}$ and B be a compact subset of $\mathbb{C}$. If $A \cap B = \varnothing$, then $\inf\{|z - w| : z \in A, w \in B\} > 0$.

Proof: Since $A \cap B = \varnothing$, w is not a limit point of A, $\forall w \in B$. Therefore, there exists $r_w > 0$ such that $B(w, r_w) \cap A = \varnothing$, $\forall w \in B$. As $\left\{B\left(w, \frac{r_w}{2}\right) : w \in B\right\}$ is a collection of open sets such that $B \subseteq \underset{w\in B}{\cup} B\left(w, \frac{r_w}{2}\right)$, there exist $w_k \in B$, $k = 1, 2, \ldots, n$ such that $B \subseteq \underset{k=1}{\overset{n}{\cup}} B\left(w_k, \frac{r_{w_k}}{2}\right)$. Let $\mu = \min\left\{\frac{r_{w_k}}{2} : k = 1, 2, \ldots, n\right\}$, then $\mu > 0$. We claim that $\inf\{|z - w| : z \in A, w \in B\} \geq \mu$. Otherwise, there exist $z \in A$ and $w \in B$ such that $|z - w| < \mu$. Using $w \in B$, we find $j \in \{1, 2, \ldots, n\}$ such that $w \in B\left(w_j, \frac{r_{w_j}}{2}\right)$. Then we have $|w - w_j| < \frac{r_{w_j}}{2}$. Now,

$$|z - w_j| \leq |z - w| + |w - w_j| < \mu + \frac{r_{w_j}}{2} \leq r_{w_j}$$

which implies that $z \in B\left(w_j, r_{w_j}\right)$. This is a contradiction to $B\left(w_j, r_{w_j}\right) \cap A = \varnothing$. Hence, the theorem follows. □

THEOREM 1.4.42 (Cantor's intersection theorem)
If (K_n) is a sequence of non-empty compact subsets of $\mathbb{C}$ such that $K_n \supseteq K_{n+1}$, $\forall n \in \mathbb{N}$, and $D(K_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} K_n$ is a singleton set, where $D(K_n) = \sup\{|z - w| : z, w \in K_n\}$, $\forall n \in \mathbb{N}$.

Proof: First, we show that $\bigcap_{n=1}^{\infty} K_n$ has at most one point. If $x, y \in \bigcap_{n=1}^{\infty} K_n$, then $x, y \in K_n$, $\forall n \in \mathbb{N}$. Then, $0 \leq |x - y| \leq D(K_n) \to 0$ as $n \to \infty$, and hence, $x = y$.

Suppose that $\bigcap_{n=1}^{\infty} K_n = \varnothing$. Then $K_1 \cap \bigcap_{n=2}^{\infty} K_n = \varnothing \Rightarrow K_1 \subseteq \left(\bigcap_{n=2}^{\infty} K_n\right)^c = \bigcup_{n=2}^{\infty} K_n^c$. As each K_n is a closed set (Theorem 1.4.38), K_n^c is open, $\forall n \geq 2$. Hence, by the compactness of K_1, we have $K_1 \subseteq \left(\bigcup_{j=1}^{m} K_{n_j}^c\right) = \left(\bigcap_{j=1}^{m} K_{n_j}\right)^c$ for some $2 \leq n_1 < n_2 < \cdots < n_m$. Thus, $K_{n_m} = K_1 \cap \left(\bigcap_{j=1}^{m} K_{n_j}\right) = \varnothing$, which is a contradiction. Hence, the theorem follows. □

THEOREM 1.4.43 (Heine-Borel theorem)
Every closed and bounded interval in $\mathbb{R}$ is compact.

Proof: Let $I = [a, b]$ be the given interval, where $a, b \in \mathbb{R}$ with $a < b$. Suppose there exists a collection $\mathscr{C}$ of open sets in $\mathbb{R}$ such that $[a, b] \subseteq \bigcup_{E \in \mathscr{C}} E$ and I is not contained in the union of any finite sub-collection of $\mathscr{C}$. Let $c_1 = \frac{a+b}{2}$ be the mid-point of a and b. Then at least one of the two intervals $[a, c_1]$ and $[c_1, b]$ is not contained in the union of any finite sub-collection of $\mathscr{C}$. Choose such an interval and denote it by $I_1 = [a_1, b_1]$. If $D(I)$ is the diameter of I, then $D(I) = b - a$ and $D(I_1) = b_1 - a_1 = \frac{b-a}{2}$. Proceeding like this, we can find a sequence $\{I_n\}$ of intervals with the following properties:

1. $I_{n+1} \subseteq I_n$, $\forall n \in \mathbb{N}$.

2. I_n is not contained in the union of any finite sub-collection of $\mathscr{C}$.

3. $D(I_{n+1}) = \frac{1}{2} D(I_n) = \frac{1}{2^n} D(I)$, $\forall n \in \mathbb{N}$.

If $I_n = [a_n, b_n],\ \forall n \in \mathbb{N}$, then we have

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_n, \forall m, n \in \mathbb{N}.$$

Hence, it follows that for each $m \in \mathbb{N}$, b_m is an upper bound of $\{a_n : n \in \mathbb{N}\}$. Using the fact that $\mathbb{R}$ has least upper bound property, there exists $\alpha \in \mathbb{R}$ such that $\alpha = \sup\{a_n : n \in \mathbb{N}\}$. Then immediately, we have $a_n \leq \alpha \leq b_m,\ \forall m, n \in \mathbb{N}$. Thus, $\alpha \in \bigcap_{n=1}^{\infty} I_n$. As $\alpha \in I$, there exists $E \in \mathscr{C}$ such that $\alpha \in E$. Since E is open, there exists $r > 0$ such that $B(\alpha, r) \subseteq E$. As $\left\{\frac{1}{2^n} D(I)\right\}$ converges to 0, there exists $m \in \mathbb{N}$ such that $\frac{1}{2^m} D(I) < r$. We now claim that $I_m \subseteq E$. If $x \in I_m$, then $|x - \alpha| \leq \frac{1}{2^m} D(I) < r$, because $x, \alpha \in I_m$. Thus, $x \in B(\alpha, r) \subseteq E$. Hence, our claim follows. However, this is contradiction to a property (2) of I_m. Hence, $[a, b]$ is compact. □

COROLLARY 1.4.44 Every closed and bounded subset of $\mathbb{R}$ is compact.

Proof: Let K be a closed and bounded subset of $\mathbb{R}$. Since K is bounded, there exists $M > 0$ such that $|x| \leq M,\ \forall x \in K$. Therefore, $K \subseteq [-M, M]$. By previous theorem, $[-M, M]$ is compact. As K is a closed subset of the compact set $[-M, M]$, K is compact. □

Exercise 1.4.45 Every closed and bounded subset of $\mathbb{C}$ is compact.

Hint: First prove that $[a, b] \times [c, d]$ is compact as in the proof of Heine–Borel theorem. Next find positive real numbers M_1 and M_2 such that $K \subset [-M_1, M_1] \times [-M_2, M_2]$ and complete the remaining proof as in the proof of previous corollary.

THEOREM 1.4.46 Every Cauchy sequence in $\mathbb{C}$ is a convergent sequence.

Proof: Let (z_n) be a Cauchy sequence in $\mathbb{C}$. If $z_n = x_n + iy_n,\ \forall n \in \mathbb{N}$, then

$$|x_n - x_m| \leq |z_n - z_m| \text{ and } |y_n - y_m| \leq |z_n - z_m|,\ \forall n \in \mathbb{N}.$$

Therefore, (x_n) and (y_n) are Cauchy sequences of real numbers. Then by Result 1.3.3, (x_n) and (y_n) are bounded sequences. Therefore, we can find $M > 0$ such that $x_n \in [-M, M],\ \forall n \in \mathbb{N}$. If (x_n) has a subsequence as a constant sequence, then obviously the subsequence is convergent (Example 1.3.5), and by Lemma 1.3.8, (x_n) itself is convergent. If (x_n) has no such subsequence, then $\{x_n : n \in \mathbb{N}\}$ is an infinite set of $[-M, M]$, which is a

compact set by Heine–Borel theorem. Now applying Theorem 1.4.39, we get $\{x_n : n \in \mathbb{N}\}$ has a limit point x in $[-M, M]$. Hence, for each $k \in \mathbb{N}$, there exists $x_{n_k} \in B(x, 1/k)$, with $n_k > n_j$ whenever $k > j$. Therefore, $(x_{n,k})$ is a convergent subsequence of (x_n). Again by using Theorem 1.4.39, we get (x_n) is a convergent sequence. By the same argument, we can prove that (y_n) is a convergent sequence. Thus, by Theorem 1.3.9 (2), we get that (z_n) is a convergent sequence. □

The above theorem can be rephrased as that $\mathbb{C}$ is a complete metric space.

THEOREM 1.4.47 (Bolzano–Weierstrass property) Every bounded infinite subset of $\mathbb{C}$ has a limit point.

Proof of this theorem is a consequence of Exercise 1.4.45 and Theorem 1.4.39.

THEOREM 1.4.48 The set $\mathbb{C}$ of all complex numbers is second countable. That is, there exists a countable collection $\mathscr{B}$ of open subsets of $\mathbb{C}$ such that for every open subset G of $\mathbb{C}$ and $z \in G$, there exists $B \in \mathscr{B}$ such that $z \in B \subseteq G$.

Proof: We know that the set of all rational numbers is countable and dense in $\mathbb{R}$. Therefore, the collection

$$\mathscr{B} = \left\{B(w_j, q) : w_j \in \mathbb{Q} \times \mathbb{Q}, q \in \mathbb{Q} \cap (0, \infty)\right\}$$

is also a countable collection of open subsets of $\mathbb{C}$. For $z = x + iy \in \Omega$, there exists a positive real number R such that $B(z, R) \subset \Omega$. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose a rational number r such that $0 < r < R$ so that $B(z, r) \subseteq B(z, R) \subseteq G$. Again using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we can choose $u, v \in \mathbb{Q}$ such that

$$x < u < x + \frac{r}{4} \text{ and } y < v < y + \frac{r}{4}.$$

If $w = u + iv$, then

$$|w - z| = |(u + iv) - (x + iy)| \leq |u - x| + |v - y| < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}.$$

We also have

$$\begin{aligned} \zeta \in B\left(w, \frac{r}{2}\right) &\Rightarrow |w - \zeta| < \frac{r}{2} \\ &\Rightarrow |z - \zeta| \leq |z - w| + |w - \zeta| < \frac{r}{2} + \frac{r}{2} \end{aligned}$$

$$\Rightarrow \quad \zeta \in B(z, r).$$

Therefore, $z \in B\left(w, \frac{r}{2}\right) \subseteq B(z, r) \subseteq G$. We note that $B\left(w, \frac{r}{2}\right) \in \mathscr{B}$. Hence, the theorem follows. □

1.5 EXTENDED COMPLEX NUMBERS AND STEREOGRAPHIC PROJECTION

The set $\mathbb{C}$ of all complex numbers along with a symbol ∞, denoted by $\mathbb{C}_\infty$, is called the set of all extended complex numbers (or the extended complex plane) if the symbol is assigned with the following properties:

1. $a \pm \infty = \infty \pm a = \infty$, for all $a \in \mathbb{C}$,
2. $c \cdot \infty = \infty \cdot c = \infty$, for all $c \in \mathbb{C} \setminus \{0\}$,
3. $\overline{\infty} = \infty$,
4. $\dfrac{a}{0} = \infty$, for all $a \in \mathbb{C}_\infty \setminus \{0\}$,
5. $\dfrac{a}{\infty} = 0, \forall a \in \mathbb{C}$.

$$\infty - \infty, \frac{\infty}{\infty} \text{ and } 0 \cdot \infty \text{ are not defined.}$$

Definition 1.5.1 (Topology on $\mathbb{C}_\infty$)
A set $A \subseteq \mathbb{C}_\infty$ is said to be an open subset of $\mathbb{C}_\infty$ if

1. $A = \mathbb{C}_\infty \setminus K$ for some compact subset K of $\mathbb{C}$ or
2. A is an open subset of $\mathbb{C}$.

THEOREM 1.5.2 (Stereographic projection)
There exists a bijection ϕ from the unit ball U in $\mathbb{R}^3$ onto $\mathbb{C}_\infty$.

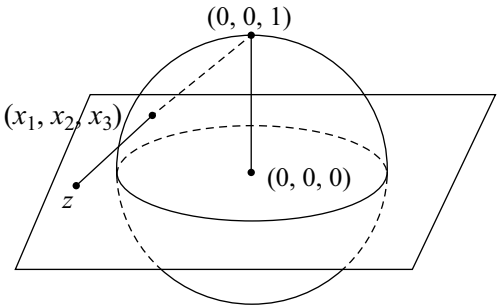
Proof: Given a point $(x_1, x_2, x_3) \in U$ other than the north pole $(0, 0, 1)$, we define the image of $\phi((x_1, x_2, x_3)))$ by the intersection of the line passing through (x_1, x_2, x_3) and $(0, 0, 1)$ with the complex plane $\mathbb{C}$.

To find $\phi((x_1, x_2, x_3)))$ explicitly, first we consider the equation of the line passing through (x_1, x_2, x_3) and $(0, 0, 1)$, which is given by

$$(1-t)(0,0,1) + t(x_1, x_2, x_3) = (tx_1, tx_2, (1-t) + tx_3),\ t \in \mathbb{R}.$$

This line meets the x_1x_2 -plane if $1 - t + tx_3 = 0 \Rightarrow t = \frac{1}{1-x_3}$. Therefore,

$$\phi((x_1, x_2, x_3)) = \begin{cases} \left(\dfrac{x_1}{1-x_3}, \dfrac{x_2}{1-x_3}, 0\right) & \text{if } x_3 \neq 1 \\ \infty & \text{if } x_3 = 1 \end{cases}.$$



To prove $\phi : U \to \mathbb{C}_\infty$ is onto let $z \in \mathbb{C}$. If $(x_1, x_2, x_3) \in U$ such that $z = \phi(x_1, x_2, x_3)$, then using $x_1^2 + x_2^2 + x_3^2 = 1$, we get

$$\begin{aligned} |z|^2 &= \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1 - x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3} \\ &\Rightarrow (1-x_3)|z|^2 = (1+x_3) \\ &\Rightarrow x_3(1+|z|^2) = |z|^2 - 1 \\ &\Rightarrow x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \end{aligned}$$

Next, $\dfrac{x_1}{1-x_3} = \operatorname{Re} z = \dfrac{z+\bar{z}}{2} \Rightarrow x_1 = \dfrac{z+\bar{z}}{2}\left(1 - \dfrac{|z|^2-1}{|z|^2+1}\right) = \dfrac{z+\bar{z}}{|z|^2+1}$.
Similarly, we obtain that

$$\frac{x_2}{1-x_3} = \operatorname{Im} z = \frac{z-\bar{z}}{i2} \Rightarrow x_2 = \frac{z-\bar{z}}{i2}\left(1 - \frac{|z|^2-1}{|z|^2+1}\right) = \frac{z-\bar{z}}{i(|z|^2+1)}.$$

Next we show, that (x_1, x_2, x_3) obtained as above belongs to U and it satisfies $\phi(x_1, x_2, x_3) = z$. Now,

$$x_1^2 + x_2^2 + x_3^2 = \frac{(z+\bar{z})^2 + (z-\bar{z})^2 + (|z|^2-1)^2}{(|z|^2+1)^2}$$

$$\begin{aligned} &= \frac{2(z^2+\bar{z}^2)+|z|^4+1-2|z|^2}{(|z|^2+1)^2} \\ &= \frac{|z|^4+1+2|z|^2}{(|z|^2+1)^2} \\ &= 1. \end{aligned}$$

and hence, $(x_1,x_2,x_3) \in U$. Clearly, we have $\phi((x_1,x_2,x_3)) = z$. Moreover, we have $\phi(\infty) = (0,0,1)$. Thus, $\phi : U \to \mathbb{C}_\infty$ is onto. To prove $\phi : U \to \mathbb{C}_\infty$ is one-to-one, let $(x_1,x_2,x_3),(y_1,y_2,y_3) \in U$ such that $\phi((x_1,x_2,x_3)) = \phi((y_1,y_2,y_3)) = z$. If $z = \infty$, then by definition, $(x_1,x_2,x_3) = (0,0,1) = (y_1,y_2,y_3)$. Hence, assume that $z \in \mathbb{C}$. Then, from the previous argument,

$$(x_1,x_2,x_3) = \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{z-\bar{z}}{i(|z|^2+1)}, \frac{|z|^2-1}{|z|^2+1}\right) = (y_1,y_2,y_3).$$

Thus, $\phi : U \to \mathbb{C}_\infty$ is one-to-one. □

RESULT 1.5.3 The image of a circle on the sphere U under the map ϕ is either a circle or a straight line in $\mathbb{C}_\infty$.

Proof: Any circle on the sphere can be interpreted as the intersection of U by a plane (say) $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$. Using the values of x_1, x_2, and x_3 in terms of z, we get

$$\alpha_1(z+\bar{z}) + \frac{\alpha_2}{i}(z-\bar{z}) + \alpha_3(|z|^2-1) = \alpha_0(|z|^2+1).$$

Putting $z = x+iy$ in the last equation, we have $2\alpha_1 x+2\alpha_2 y+\alpha_3(x^2+y^2-1) = \alpha_0(x^2+y^2+1) \Rightarrow (\alpha_0-\alpha_3)(x^2+y^2)-2\alpha_1 x-2\alpha_2 y+\alpha_0+\alpha_3 = 0$, which is the equation of a circle if $(\alpha_0-\alpha_3) \neq 0$ and is the equation of a straight line if $(\alpha_0-\alpha_3) = 0$. □

Definition 1.5.4 Given any two points $z, w \in \mathbb{C}_\infty$, define the distance $D(z,w)$ by the actual distance between $\phi^{-1}(z)$ and $\phi^{-1}(w)$ in $\mathbb{R}^3$.

PROBLEM 1.5.5 Find $D(z,w)$ explicitly.

Proof:
Case 1: $z, w \in \mathbb{C}$.
Let $\phi^{-1}(z) = (x_1,x_2,x_3)$ and $\phi^{-1}(w) = (y_1,y_2,y_3)$. Then,

$$(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$

$$
\begin{aligned}
&= x_1^2 + y_1^2 - 2x_1y_1 + x_2^2 + y_2^2 - 2x_2y_2 + x_3^2 + y_3^2 - 2x_3y_3 \\
&= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) - 2(x_1y_1 + x_2y_2 + x_3y_3) \\
&= 2 - 2(x_1y_1 + x_2y_2 + x_3y_3) \\
&\quad (\text{as } x_1^2 + x_2^2 + x_3^2 = 1 = y_1^2 + y_2^2 + y_3^2) \\
&= 2 - 2\frac{(z+\bar{z})(w+\overline{w}) - (z-\bar{z})(w-\overline{w}) + (|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)} \\
&= 2 - 2\left(\frac{2(z\overline{w} + \bar{z}w) + (|z|^2|w|^2 - |z|^2 - |w|^2 + 1)}{(|z|^2+1)(|w|^2+1)}\right) \\
&= 2 - 2\left(\frac{2(-|z|^2 - |w|^2 + z\overline{w} + \bar{z}w) + (|z|^2|w|^2 + |z|^2 + |w|^2 + 1)}{(|z|^2+1)(|w|^2+1)}\right) \\
&= 2 - 2\left(\frac{-2|z-w|^2 + (|z|^2+1)(|w|^2+1)}{(|z|^2+1)(|w|^2+1)}\right) \\
&= \frac{4|z-w|^2}{(|z|^2+1)(|w|^2+1)}.
\end{aligned}
$$

Case 2: $z \in \mathbb{C}$ and $w = \infty$.
In this case, $\phi^{-1}(w) = (0, 0, 1)$ and let $\phi^{-1}(z) = (x_1, x_2, x_3)$. Therefore,

$$
\begin{aligned}
(x_1 - 0)^2 + (x_2 - 0)^2 + (x_3 - 1)^2 &= 2 - 2x_3 \\
&= 2 - 2\frac{|z|^2 - 1}{|z|^2 + 1} \\
&= \frac{4}{|z|^2 + 1}
\end{aligned}
$$

Hence, $D(z, w) = \begin{cases} \dfrac{2|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}} & \text{if } z, w \in \mathbb{C} \\ \dfrac{2}{\sqrt{|z|^2+1}} & \text{if } z \in \mathbb{C} \text{ and } w = \infty. \end{cases}$ □

1.6 LIMIT AND CONTINUITY

Definition 1.6.1 Let Ω be a region, $z_0 \in \Omega$, $l \in \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. We say that $f(z) \to l$ as $z \to z_0$ if given $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.
In this case, we write $\lim_{z \to z_0} f(z) = l$.

THEOREM 1.6.2 Let $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega$, $l, c \in \mathbb{C}$ and $\lim_{z \to z_0} f(z) = l$. Then,

1. $\lim_{z\to z_0} \text{Re}\,(f(z)) = \text{Re}\, l$,
2. $\lim_{z\to z_0} \text{Im}\,(f(z)) = \text{Im}\, l$,
3. $\lim_{z\to z_0} \overline{f(z)} = \bar{l}$,
4. $\lim_{z\to z_0} |f(z)| = |l|$,
5. $\lim_{z\to z_0} (f + c)(z) = l + c$,
6. $\lim_{z\to z_0} (cf)(z) = cl$.

Proof: Let $\epsilon > 0$. Then, there exists $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$. We choose the same δ for proving (1), (2), (3), (4), (5). If $0 < |z - z_0| < \delta$, then

1. $|\text{Re}\,(f(z)) - \text{Re}\, l| = |\text{Re}\,(f(z) - l)| \leq |f(z) - l| < \epsilon$
2. $|\text{Im}\,(f(z)) - \text{Im}\, l| = |\text{Im}\,(f(z) - l)| \leq |f(z) - l| < \epsilon$
3. $|\overline{f(z)} - \bar{l}| = |\overline{f(z) - l}| \leq |f(z) - l| < \epsilon$
4. $|\,|f(z)| - |l|\,| \leq |f(z) - l| < \epsilon$, by using Corollary 1.2.10
5. $|(f(z) + c) - (l + c)| = |f(z) - l| < \epsilon$
6. For proving the last assertion, we first choose $\delta > 0$ such that

$$|f(z) - l| < \frac{\epsilon}{|c| + 1} \text{ whenever } 0 < |z - z_0| < \delta$$

If $0 < |z - z_0| < \delta$, then $|(cf(z)) - (cl)| = |c||f(z) - l| \leq \dfrac{|c|\epsilon}{|c| + 1} < \epsilon$. □

THEOREM 1.6.3 Let f and g be functions on Ω, $z_0 \in \Omega$ and $\lim_{z\to z_0} f(z) = l_1$, $\lim_{z\to z_0} g(z) = l_2$, where $l_1, l_2 \in \mathbb{C}$. Then,

1. $\lim_{z\to z_0} (f + g)(z) = l_1 + l_2$,
2. $\lim_{z\to z_0} (fg)(z) = l_1 l_2$,

3. $\lim_{z\to z_0}\left(\frac{f}{g}\right)(z) = \frac{l_1}{l_2}$, provided $l_2 \neq 0$.

Proof: Let $\epsilon > 0$ be given.

1. For a given $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - l_1| < \frac{\epsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_1$$

and

$$|g(z) - l_2| < \frac{\epsilon}{2} \text{ whenever } 0 < |z - z_0| < \delta_2.$$

Let $0 < \delta < \min\{\delta_1, \delta_2\}$. If $0 < |z - z_0| < \delta$, then $0 < |z - z_0| < \delta_1$ and $0 < |z - z_0| < \delta_2$. Thus, we get

$$|(f + g)(z) - (l_1 + l_2)| \leq |f(z) - l_1| + |g(z) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2. Consider

$$\begin{aligned} |(fg)(z) - (l_1 l_2)| &\leq |f(z)g(z) - f(z)l_2)| + |f(z)l_2 - l_1 l_2| \\ &= |f(z)||g(z) - l_2| + |l_2||f(z) - l_1|. \end{aligned} \tag{1.5}$$

Given $\epsilon > 0$ choose $\delta_1 > 0$ such that

$$|f(z) - l_1| < 1 \text{ if } 0 < |z - z_0| < \delta_1.$$

Hence, $|f(z)| < |l_1| + 1$, whenever $0 < |z - z_0| < \delta_1$. For the same ϵ, find $\delta_2 > 0$ such that

$$|g(z) - l_2| < \frac{\epsilon}{2(|l_1| + 1)} \text{ whenever } 0 < |z - z_0| < \delta_2.$$

Next choose $\delta_3 > 0$ such that

$$|f(z) - l_1| < \frac{\epsilon}{2(|l_2| + 1)} \text{ whenever } 0 < |z - z_0| < \delta_3.$$

If $0 < \delta < \min\{\delta_1, \delta_2, \delta_3\}$, then from (1.5), we get

$$\begin{aligned} |(fg)(z) - (l_1 l_2)| &\leq |f(z)||g(z) - l_2| + |l_2||f(z) - l_1| \\ &\leq (|l_1| + 1)\frac{\epsilon}{2(|l_1| + 1)} + |l_2|\frac{\epsilon}{2(|l_2| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

3. First we prove that $\lim_{z\to z_0} \frac{1}{g(z)} = \frac{1}{l_2}$ if $l_2 \neq 0$. Given $\epsilon > 0$, choose $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |g(z) - l_2| < \min\left\{\frac{|l_2|}{2}, \frac{|l_2|^2\epsilon}{2}\right\}.$$

Thus, if $0 < |z - z_0| < \delta$, then

$$||l_2| - |g(z)|| \leq |g(z) - l_2| < \frac{|l_2|}{2} \Rightarrow |g(z)| > \frac{|l_2|}{2} \Rightarrow \frac{1}{|g(z)|} < \frac{2}{|l_2|}.$$

We also have

$$\left|\frac{1}{g(z)} - \frac{1}{l_2}\right| = \frac{|g(z) - l_2|}{|g(z)l_2|} < \frac{|l_2|^2\epsilon}{2} \cdot \frac{2}{|l_2|^2} = \epsilon.$$

Hence, our claim follows. Next by using (2) of this theorem, we get $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} f(z)\frac{1}{g(z)} = \lim_{z\to z_0} f(z) \lim_{z\to z_0} \frac{1}{g(z)} = \frac{l_1}{l_2}$. □

Definition 1.6.4 Let $f : \mathbb{C} \to \mathbb{C}$. We say that $f(z) \to \infty$ as $z \to \infty$ if given $M > 0$, then there exists $K > 0$ such that $|f(z)| > M$ whenever $|z| > K$. In this case, we write $\lim_{z\to\infty} f(z) = \infty$.

Definition 1.6.5 Let $f : \mathbb{C} \to \mathbb{C}$ and $a \in \mathbb{C}$. We say that $f(z) \to \infty$ as $z \to a$ if given $M > 0$, then there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies that $|f(z)| > M$. We write this by $\lim_{z\to a} f(z) = \infty$.

Similarly, we define $\lim_{z\to\infty} f(z) = a$.

THEOREM 1.6.6 Let $f : \Omega \to \mathbb{C}$, $g : \Omega \to \mathbb{C}$, $A \in \mathbb{C}$ and $a \in \Omega$. If $\lim_{z\to a} f(z) = \infty$ and $\lim_{z\to a} g(z) = A$, then

1. $\lim_{z\to a}(cf)(z) = \infty$ for every non-zero complex number c,
2. $\lim_{z\to a}(f + g)(z) = \infty$,
3. $\lim_{z\to a}(f \cdot g)(z) = \infty$ if $A \neq 0$,
4. $\lim_{z\to a} \frac{f(z)}{g(z)} = \infty$,

5. $\lim_{z\to a} \dfrac{g(z)}{f(z)} = 0.$

Proof: Let $M > 0$ be given.

1. We choose $\delta > 0$ such that

$$0 < |z-a| < \delta \Rightarrow |f(z)| > \frac{M}{|c|}.$$

If $0 < |z-a| < \delta$, then $|cf(z)| = |c||f(z)| > |c|\dfrac{M}{|c|} = M.$

2. We choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |z-a| < \delta_2 \Rightarrow |f(z)| > M + 1 + |A|$$

and

$$0 < |z-a| < \delta_2 \Rightarrow |g(z)-A| < 1 \Rightarrow |g(z)| \le |g(z)-A|+|A| < 1+|A|.$$

If $0 < |z-a| < \min\{\delta_1, \delta_2\}$, then

$$|f(z)+g(z)| \ge |f(z)| - |g(z)| > M + 1 + |A| - 1 - |A| = M.$$

Hence, $\lim_{z\to a}(f+g)(z) = \infty.$

3. We choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |z-a| < \delta_2 \Rightarrow |f(z)| > \frac{2M}{|A|}$$

and

$$0 < |z-a| < \delta_2 \Rightarrow |g(z)-A| < \frac{|A|}{2} \Rightarrow |g(z)| \ge \frac{|A|}{2}.$$

If $0 < |z-a| < \min\{\delta_1, \delta_2\}$, then

$$|f(z)g(z)| > \frac{2M}{|A|}\frac{|A|}{2} = M.$$

Hence, $\lim_{z\to a} f(z)g(z) = \infty.$

4. We choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |z-a| < \delta_2 \Rightarrow |f(z)| > M(1+|A|)$$

and

$$0 < |z-a| < \delta_2 \Rightarrow |g(z)-A| < 1 \Rightarrow |g(z)| \leq |g(z)-A|+|A| < 1+|A|.$$

If $0 < |z-a| < \min\{\delta_1, \delta_2\}$, then

$$\left|\frac{f(z)}{g(z)}\right| > \frac{M(1+|A|)}{1+|A|} = M.$$

Hence, $\lim\limits_{z\to a} \dfrac{f(z)}{g(z)} = \infty$.

5. Given $\epsilon > 0$ we choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |z-a| < \delta_1 \Rightarrow |f(z)| > \frac{1+|A|}{\epsilon}$$

and

$$0 < |z-a| < \delta_2 \Rightarrow |g(z)-A| < 1 \Rightarrow |g(z)| \leq 1+|A|.$$

If $0 < |z-a| < \min\{\delta_1, \delta_2\}$, then

$$\left|\frac{g(z)}{f(z)}\right| < \frac{\epsilon}{1+|A|}(1+|A|) = \epsilon.$$

Hence, $\lim\limits_{z\to a} \dfrac{g(z)}{f(z)} = 0$. □

Example 1.6.7 Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \dfrac{xy(x+iy)}{x^3+y^3} & z = x+iy \neq 0 \\ 0 & z = 0 \end{cases}, \ \forall z \in \mathbb{C}.$$

Prove that $\lim\limits_{z\to 0} f(z)$ does not exist.

If we allow $z \to 0$ along the line $y = cx$, then we get

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x(cx)(x+i(cx))}{x^3+(cx)^3} = \lim_{x\to 0} \frac{c(1+ic)}{1+c^3} = \frac{c(1+ic)}{1+c^3}.$$

which is depending on c. That is, if we let $z \to 0$ along different lines in the family $y = cx$ of straight lines, then $f(z)$ approaches different values, and hence, $\lim\limits_{z\to 0} f(z)$ does not exist.

LEMMA 1.6.8 Let $f : \Omega \to \mathbb{C}$ and let $(a, b) \in \Omega$, where $\Omega \subseteq \mathbb{C}$ is open. If $\lim\limits_{(x,y)\to(a,b)} f(x, y) = A$ for some $A \in \mathbb{C}$, then $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y) = A = \lim\limits_{y\to b}\lim\limits_{x\to a} f(x, y)$.

Proof: First we choose $r > 0$ such that $B((a, b), r) \subseteq \Omega$. For the given $\epsilon > 0$, there exists $0 < \delta < r$ such that

$$0 < |(x, y) - (a, b)| < \delta \Rightarrow |f(x, y) - A| < \frac{\epsilon}{2},$$

which implies that

$$|f(x,y) - f(u,v)| \leq |f(x,y) - A| + |A - f(u,v)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad (1.6)$$

whenever $(x, y), (u, v) \in B((a, b), \delta) \setminus \{(a, b)\}$. Now for a fixed $x \in \mathbb{R}$ such that $|x - a| < \delta$,

$$\text{if } 0 < |y - b| < \delta, \text{ then } |(x, y) - (x, b)| = |y - b| < \delta$$

and hence, $|f(x, y) - f(x, b)| < \epsilon$. In other words, we have proved that $\lim\limits_{y\to b} f(x, y) = f(x, b)$ for every $x \in \mathbb{R}$ with $|x - a| < \delta$. Using the definition of $\lim\limits_{(x,y)\to(a,b)} f(x, b) = A$, we get

$$0 < |x - a| < \delta \Rightarrow |(x, b) - (a, b)| = |x - a| < \delta \Rightarrow |f(x, b) - A| < \epsilon.$$

Thus, $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y) = \lim\limits_{x\to a} f(x, b) = A$. Similarly, we can prove that $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x, y) = A$. □

Exercise 1.6.9 Let $f : \Omega \to \mathbb{C}$, $a \in \Omega$, and $A \in \mathbb{C}$. Prove that $\lim\limits_{z\to a} f(z) = A$ iff $f(z_n) \to A$ as $n \to \infty$ whenever $z_n \to a$ as $n \to \infty$ in Ω with $z_n \neq a$, $\forall n \in \Omega$.

Now we discuss left limit and right limit of a real-valued function on an interval I of $\mathbb{R}$.

Definition 1.6.10 Let $f : I \to \mathbb{R}$ and $x \in I$, where I is an interval of $\mathbb{R}$. We write that

1. $\lim\limits_{y\to x^-} f(y) = l$ exists for some $l \in \mathbb{R}$ if given $\epsilon > 0$, then there exists $\delta > 0$ such that $(x - \delta, x) \subset I$ and $x - \delta < y < x \Rightarrow |f(y) - l| < \epsilon$.

2. $\lim_{y\to x^+} f(y) = l$ exists for some $l \in \mathbb{R}$ if given $\epsilon > 0$, then there exists $\delta > 0$ such that $(x, x+\delta) \subset I$ and $x < y < x+\delta \Rightarrow |f(y) - l| < \epsilon$.

3. $\lim_{y\to x^-} f(y) = +\infty$ exists if given $M > 0$, then there exists $\delta > 0$ such that $(x-\delta, x) \subset I$ and $x - \delta < y < x \Rightarrow f(y) > M$.

4. $\lim_{y\to x^+} f(y) = +\infty$ exists if given $K > 0$, then there exists $\delta > 0$ such that $(x, x+\delta) \subset I$ and $x < y < x+\delta \Rightarrow f(y) > K$.

Similarly, we can define $\lim_{y\to x^-} f(y) = -\infty$ and $\lim_{y\to x^+} f(y) = -\infty$.

Exercise 1.6.11 Let $f : \Omega \to \mathbb{C}$, $g : \Omega \to \mathbb{C}$, and $a \in \Omega$. If $\lim_{z\to a} f(z) = \infty$ and g is bounded in a neighbourhood of a, then prove that $\lim_{z\to a}(fg)(z) = \infty$.

Remark 1.6.12: Let $f : \Omega \to \mathbb{C}$, $g : \Omega \to \mathbb{C}$, and $a \in \Omega$. If $\lim_{z\to a} f(z) = \infty$ and $\lim_{z\to a} g(z) = \infty$, then $\lim_{z\to a}(f+g)(z)$ need not be ∞.

For example, for a fixed $c \in \mathbb{C}$, if $f(z) = \begin{cases} \frac{1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$ and $g(z) = \begin{cases} -\frac{1}{z} + c & z \neq 0 \\ 0 & z = 0 \end{cases}$, then clearly, $\lim_{z\to 0} f(z) = \infty$ and $\lim_{z\to 0} g(z) = \infty$ but $\lim_{z\to 0}(f+g)(z) = c \neq \infty$.

Definition 1.6.13 Let $f : \Omega \to \mathbb{C}$.

1. f is said to be continuous at $z_0 \in \Omega$ if $\lim_{z\to z_0} f(z) = f(z_0)$.

2. f is said to be continuous on Ω if f is continuous at every point $z_0 \in \Omega$.

THEOREM 1.6.14 If f and g are continuous at $z_0 \in \Omega$, then $f+g$, fg, and $\frac{f}{g}$ are continuous at z_0.

Proof: Proof of this theorem follows from Theorem 1.6.3. □

Example 1.6.15 Every constant function on $\mathbb{C}$ is continuous.
Let $z_0 \in \mathbb{C}$ be arbitrary. Given $\epsilon > 0$, we choose $\delta > 0$ arbitrarily. If $|z - z_0| < \delta$, then $|f(z) - f(z_0)| = 0 < \epsilon$. Hence, f is continuous at z_0.

Example 1.6.16 If $f(z) = z$, $\forall z \in \mathbb{C}$, then f is continuous on $\mathbb{C}$.
For a given $\epsilon > 0$, we choose $\delta = \epsilon$. Then obviously, we get $|z - z_0| < \epsilon$ when $|z - z_0| < \delta$.

Example 1.6.17 Every polynomial is continuous on $\mathbb{C}$.
Using Theorem 1.6.14 and the above two examples, we get every polynomial is a continuous function on $\mathbb{C}$.

Example 1.6.18 $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, and $|z|$ are continuous on $\mathbb{C}$.
For justification, see Example 1.6.16 and Theorem 1.6.2.

LEMMA 1.6.19 f is continuous iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous iff $\overline{f}$ is continuous.

Proof: Let f be continuous at z_0. For a given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, whenever $|z - z_0| < \delta$. For the same δ,

$$\text{if } |z - z_0| < \delta, \text{ then } |\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| \leq |f(z) - f(z_0)| < \epsilon.$$

Similarly, for the same δ,

$$\text{if } |z - z_0| < \delta, \text{ then } |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| \leq |f(z) - f(z_0)| < \epsilon.$$

Conversely, assume that $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0. Then, for a given $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| < \dfrac{\epsilon}{\sqrt{2}}$, whenever $|z - z_0| < \delta_1$ and $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| < \dfrac{\epsilon}{\sqrt{2}}$, whenever $|z - z_0| < \delta_2$. If $\delta = \min\{\delta_1, \delta_2\}$, then $\delta > 0$, and if $|z - z_0| < \delta$, then

$$\begin{aligned} |f(z) - f(z_0)| &= \sqrt{|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)|^2 + |\operatorname{Im} f(z) - \operatorname{Im} f(z_0)|} \\ &< \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon. \end{aligned}$$

Thus, f is continuous at z_0.
Using $|z| = |\bar{z}|$, one can prove that f is continuous iff $\overline{f}$ is continuous. □

THEOREM 1.6.20 Composition of two continuous functions is continuous.

Proof: Let $f : \Omega \to \mathbb{C}$ and $g : f(\Omega) \to \mathbb{C}$ be continuous. If $h = g \circ f$, then we show that h is continuous on Ω. Let $z_0 \in \Omega$ be arbitrary. Then $f(z_0) \in$

$f(\Omega)$. Using the continuity of g at $f(z_0)$ for a given $\epsilon > 0$, choose $r > 0$ such that

$$|g(w) - g(f(z_0))| < \epsilon \text{ if } |w - f(z_0)| < r.$$

Next, using the continuity of f at z_0, for this $r > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < r \text{ if } |z - z_0| < \delta.$$

Therefore, combining these two statements, we get

$$\text{if } |z - z_0| < \delta, \text{ then } |h(z) - h(z_0)| = |g(f(z)) - g(f(z_0))| < \epsilon.$$

Hence, h is continuous at z_0. □

RESULT 1.6.21 Let $f : A \to \mathbb{C}$ and $x \in A$, then f is continuous at x iff $f(x_n) \to f(x)$ as $n \to \infty$, whenever $x_n \to x$ as $n \to \infty$.

Proof: Assume that f is continuous at x. For a given $\epsilon > 0$, we can find $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

If $x_n \to x_0$ as $n \to \infty$, for the $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \delta, \ \forall n \geq N.$$

Therefore, if $n \geq N$, then

$$|x_n - x| < \delta \Rightarrow |f(x_n) - f(x)| < \epsilon.$$

Conversely, assume that f is not continuous at x. Then, there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $x_n \in A$ with $|x_n - x| < \frac{1}{n}$ and $|f(x) - f(x_n)| \geq \epsilon$. Hence, there exists a sequence (x_n) such that $x_n \to x$ as $n \to \infty$ but $f(x_n) \not\to f(x)$ as $n \to \infty$. Hence, the result follows. □

Definition 1.6.22 Let $m, n \in \mathbb{N}$, $E \subseteq \mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in E$, and $f = (f_1, f_2, \ldots, f_m) : E \to \mathbb{R}^m$ be continuous at $(x_1, x_2, \ldots, x_n)$ if given $\epsilon > 0$, then there exists $\delta > 0$ such that

$$\sqrt{\sum_{k=1}^{m} |f_k(t_1, t_2, \ldots, t_n) - f_k(x_1, x_2, \ldots, x_n)|^2} < \epsilon$$

whenever

$$(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \text{ and } \sqrt{\sum_{j=1}^{n} |t_j - x_j|^2} < \delta.$$

Limit of a function $f = (f_1, f_2, \ldots, f_m) : E \to \mathbb{R}^m$ at a point $(x_1, x_2, \ldots, x_n)$ can also be defined in an analogous way. As in the earlier discussions, one can prove that $f = (f_1, f_2, \ldots, f_m) : E \to \mathbb{R}^m$ is continuous at $(x_1, x_2, \ldots, x_n)$ iff $f_k(x_1(\nu), x_2(\nu), \ldots, x_n(\nu)) \to f_k(x_1, x_2, \ldots, x_n)$ as $\nu \to \infty$, $\forall k = 1, 2, \ldots, m$, whenever $x_j(\nu) \to x_j$ as $\nu \to \infty$, $\forall j = 1, 2, \ldots, n$.

THEOREM 1.6.23 Let $\phi : U \to \mathbb{C}_\infty$ be the stereographic projection. Then

- ϕ is a continuous map on $U \setminus \{(0, 0, 1)\}$ and

$$\phi(x_1, x_2, x_3) \to \infty \text{ as } (x_1, x_2, x_3) \to (0, 0, 1),$$

- ϕ^{-1} is a continuous map on $\mathbb{C}$ and

$$\phi^{-1}(z) \to (0, 0, 1) \text{ as } z \to \infty.$$

Proof: If $\phi_1(x_1, x_2, x_3) = \dfrac{x_1}{1 - x_3}$ and $\phi_2(x_1, x_2, x_3) = \dfrac{x_2}{1 - x_3}$ $\forall (x_1, x_2, x_3) \in U \setminus \{(0, 0, 1)\}$, then $\phi = (\phi_1, \phi_2)$. (Theorem 1.5.2.)

1. (a) As $x_3 \neq 1$, $\forall (x_1, x_2, x_3) \in U \setminus \{(0, 0, 1)\}$, whenever $x_j(\nu) \to x_j$ as $\nu \to \infty$ for all $j = 1, 2, 3$, we have

$$\frac{x_1(\nu)}{1 - x_3(\nu)} \to \frac{x_1}{1 - x_3} \text{ as } \nu \to \infty.$$

and

$$\frac{x_2(\nu)}{1 - x_3(\nu)} \to \frac{x_2}{1 - x_3} \text{ as } \nu \to \infty.$$

Therefore, ϕ is continuous on $U \setminus \{(0, 0, 1)\}$.

(b) If $x_j(\nu) \to 0$ for $j = 1, 2$ and $x_3(\nu) \to 1$ as $\nu \to \infty$, then

$$\phi_k(x_1(\nu), x_2(\nu), x_3(\nu)) \to \pm\infty \text{ as } \nu \to \infty \text{ for } k = 1, 2$$

in the extended real number system, and hence,

$$\phi(x_1(\nu), x_2(\nu), x_3(\nu)) \to \infty \text{ in } \mathbb{C}_\infty \text{ as } \nu \to \infty.$$

2. As the remaining part of the proof of this theorem is similar, we leave it as an exercise to the reader.

Therefore, ϕ and ϕ^{-1} are continuous functions. □

THEOREM 1.6.24 Let $f : A \to \mathbb{C}$, where A is open in $\mathbb{C}$. Then, f is continuous on A iff $f^{-1}(E)$ is open in A whenever E is open in $\mathbb{C}$.

Proof: Assume that f is continuous on A. Let $x \in f^{-1}(E)$. Then $f(x) \in E$. As E is open, there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq E$. Since f is continuous at x, for this $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - x| < \delta \Rightarrow |f(z) - f(x)| < \epsilon.$$

Therefore, if $z \in B(x, \delta) \Rightarrow f(z) \in B(f(x), \epsilon)$. Hence,

$$f(B(x, \delta)) \subseteq B(f(x), \epsilon)) \subseteq E \Rightarrow B(x, \delta) \subseteq f^{-1}(E).$$

Thus, $f^{-1}(E)$ is open in A.

Conversely, assume that $f^{-1}(E)$ is open in A, whenever E is open in $\mathbb{C}$. Let $x \in A$ and $\epsilon > 0$ be given. Then $B(f(x), \epsilon)$ is an open subset of $\mathbb{C}$. Then by hypothesis, $f^{-1}(B(f(x), \epsilon))$ is open in A and $x \in f^{-1}(B(f(x), \epsilon))$. Then, there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) \Rightarrow f(B(x, \delta)) \subseteq B(f(x), \epsilon)$. Thus, we have proved that

$$|z - x| < \delta \Rightarrow |f(z) - f(x)| < \epsilon$$

and hence, f is continuous on A. □

Definition 1.6.25 Let $f : E \to \mathbb{C}$, where $E \subset \mathbb{C}$. f is said to be uniformly continuous on E if given $\epsilon > 0$, there exits $\delta > 0$ such that $x, y \in E$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

THEOREM 1.6.26 If $f : K \to \mathbb{C}$ is continuous and K is a compact subset of $\mathbb{C}$, then f is uniformly continuous.

Proof: Let $\epsilon > 0$ be given. For every $w \in K$, there exists $\delta_w > 0$ such that

$$|z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\epsilon}{3}. \tag{1.7}$$

Since K is compact and $\left\{B\left(w, \frac{\delta_w}{2}\right) : w \in K\right\}$ is a collection of open sets such that $K \subset \bigcup_{w \in K} B\left(w, \frac{\delta_w}{2}\right)$, there exist

$$w_1, w_2, w_3, \ldots, w_n \in K \text{ such that } K \subset \bigcup_{j=1}^{n} B\left(w_j, \frac{\delta_{w_j}}{2}\right).$$

If $\delta = \min\left\{\frac{\delta_{w_j}}{2} : j = 1, 2, \ldots, n\right\}$, then $\delta > 0$. If $x, y \in K$ with $|x - y| < \delta$, then $x \in B\left(w_j, \frac{\delta_{w_j}}{2}\right)$ for some $1 \leq j \leq n$. We note that $y \in B\left(w_j, \delta_{w_j}\right)$ for the same j, because

$$|y - w_j| \leq |y - x| + |x - w_j| \leq \delta + \frac{\delta_{w_j}}{2} < \frac{\delta_{w_j}}{2} + \frac{\delta_{w_j}}{2} = \delta_{w_j}.$$

Hence, using (1.7), we get

$$|f(x) - f(y)| \leq |f(x) - f(w_j)| + |f(w_j) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.$$

Thus, f is uniformly continuous on K. □

THEOREM 1.6.27 Continuous image of a compact set is compact.

Proof: Let $\{E_\alpha : \alpha \in I\}$ be a collection of open sets such that $f(K) \subseteq \bigcup_\alpha E_\alpha$. By Theorem 1.6.24, $f^{-1}(E_\alpha)$ is open for every $\alpha \in I$. Now, if $x \in K$, then $f(x) \in f(K)$. Then $f(x) \in E_\alpha$ for some α. This implies $x \in f^{-1}(E_\alpha)$ for this α. Therefore,

$$K \subseteq \bigcup_\alpha f^{-1}(E_\alpha).$$

As K is compact, we have $K \subseteq \bigcup_{i=1}^{n} f^{-1}(E_{\alpha_i})$ for some suitable $\alpha_i \in I$, $i = 1, 2, \ldots, n$. Then

$$f(K) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(E_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} E_{\alpha_i}.$$

Therefore, $f(K)$ is compact. □

THEOREM 1.6.28 If f is a real-valued continuous function on a compact subset K of $\mathbb{C}$, then f is bounded and there exist $x, y \in K$ such that $f(x) = \sup f(K)$ and $f(y) = \inf f(K)$.

Proof: By previous theorem, $f(K)$ is a compact subset of $\mathbb{R}$, and hence, it is closed and bounded. Thus, f is a bounded function on K. As $\sup f(K) \in \mathrm{Cl}\, f(K) = f(K)$ (cf. Exercise 1.4.14), then there exists $x \in K$ such that $f(x) = \sup f(K)$. Similarly, we can prove that there exists $y \in K$ such that $f(y) = \inf f(K)$. □

THEOREM 1.6.29 Continuous image of a connected set is connected.

Proof: Let C be a connected subset of $\mathbb{C}$. Suppose if $f(C)$ is not connected in $\mathbb{C}$, then there exist non-empty separated sets A and B in $\mathbb{C}$ such that

$$f(C) \subseteq A \cup B \text{ and } f(C) \cap A \neq \varnothing \neq f(C) \cap B.$$

Therefore, clearly we have

$$C \subseteq f^{-1}(A) \cup f^{-1}(B) \text{ and } C \cap f^{-1}(A) \neq \varnothing \neq C \cap f^{-1}(B).$$

Now we claim that $f^{-1}(A)$ and $f^{-1}(B)$ are separated sets in $\mathbb{C}$. Suppose $f^{-1}(A) \cap (\mathrm{Cl} f^{-1}(B)) \neq \varnothing$, then there exists a point $x \in f^{-1}(A)$ and a sequence (x_n) in $f^{-1}(B)$ such that $x_n \to x$ as $n \to \infty$, indeed choose

$$x_n \in B\left(x, \frac{1}{n}\right) \cap f^{-1}(B), \ \forall n \in \mathbb{N}.$$

As f is continuous, using Result 1.6.21, we have

$$f(x_n) \to f(x) \text{ as } n \to \infty.$$

Note that $f(x) \in A$ and $f(x_n) \in B \ \forall n \in \mathbb{N}$. Therefore, $f(x) \in \mathrm{Cl}\, B$ as for each $r > 0$,

$$f(x_n) \in B(f(x), r) \cap B \text{ for all but finitely many } n,$$

and hence, $A \cap \mathrm{Cl}\, B \neq \varnothing$. This contradicts the fact that A and B are separated sets. By a similar argument, we can show that

$$(\mathrm{Cl} f^{-1}(A)) \cap f^{-1}(B) \neq \varnothing.$$

This implies that C is not connected, which is a contradiction. Therefore, $f(C)$ is connected. □

THEOREM 1.6.30 (Intermediate value theorem)
If $f : [a, b] \to \mathbb{R}$ is a continuous function and $\lambda \in \mathbb{R}$ lies between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = \lambda$.

Proof: As $[a, b]$ is a connected subset of $\mathbb{R}$ (as well as connected subset of $\mathbb{C}$), by previous theorem, $f([a, b])$ is a connected subset of $\mathbb{R}$. Hence, $f([a, b])$ is an interval by Theorem 1.4.22. Thus, by definition of an interval, if λ lies between $f(a)$ and $f(b)$, then $\lambda \in f([a, b])$. That is, there exists $c \in [a, b]$ such that $f(c) = \lambda$. As $f(a) < f(c) < f(b)$, we have $a \neq c \neq b$, and hence, $c \in (a, b)$. □

THEOREM 1.6.31 Let (f_n) be a sequence of complex-valued continuous functions on $E \subseteq \mathbb{C}$. If $f_n \to f$ as $n \to \infty$ uniformly on E for some $f : E \to \mathbb{C}$, then f is also continuous on $[a, b]$.

Proof: Let $z_0 \in E$ and $\epsilon > 0$ be arbitrary. By hypothesis, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}, \ \forall z \in E, \ \forall n \geq N. \tag{1.8}$$

As f_N is continuous at z_0, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f_N(z) - f_N(z_0)| < \frac{\epsilon}{3}. \tag{1.9}$$

For the same $\delta > 0$, if $|z - z_0| < \delta$, then using (1.8) and (1.9), we get

$$|f(z)-f(z_0)| \leq |f(z)-f_N(z)|+|f_N(z)-f_N(z_0)|+|f_N(z_0)-f(z_0)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

Therefore, f is continuous on E. □

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2

Analytic Functions

2.1 DIFFERENTIABILITY

Definition 2.1.1 Let Ω be a region, $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$. f is said to be differentiable at z_0 if $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists in $\mathbb{C}$ or equivalently, $\lim\limits_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists in $\mathbb{C}$, and this limit is called the derivative of f at z_0, which is denoted by $f'(z_0)$.

Definition 2.1.2 A function f is said to be differentiable on a region Ω if f is differentiable at every point of Ω.

Definition 2.1.3 Let Ω be a region, $z_0 \in \Omega$ and $f: \Omega \to \mathbb{C}$.

1. f is said to be analytic at z_0 if f is differentiable on some neighbourhood of z_0.
2. f is said to be analytic on Ω if f is analytic at every point of Ω.

Remark 2.1.4: Suppose f is a function on a region Ω. Then, f is differentiable on Ω iff f is analytic on Ω.

RESULT 2.1.5 If f is differentiable at z_0, then f is continuous at z_0.

Proof: Using Theorem 1.6.3, we get

$$\begin{aligned}
\lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{(f(z) - f(z_0))}{z - z_0} \cdot (z - z_0) \\
&= \lim_{z \to z_0} \frac{(f(z) - f(z_0))}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) \\
&= f'(z_0) \cdot 0 = 0.
\end{aligned}$$

Thus, f is continuous at z_0. □

THEOREM 2.1.6 If f and g are differentiable at z_0, then $f+g$ and fg are differentiable at z_0. If $g'(z_0) \neq 0$, then $\frac{f}{g}$ is differentiable at z_0.

Proof: By assumption, we have

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \quad \text{and} \quad \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = g'(z_0).$$

Using Theorems 1.6.2 and 1.6.3 and Result 2.1.5, we get

1. $$\begin{aligned} &\lim_{z\to z_0} \frac{(f+g)(z)-(f+g)(z_0)}{z-z_0} \\ &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} + \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} \\ &= f'(z_0)+g'(z_0). \end{aligned}$$

2. $$\begin{aligned} &\lim_{z\to z_0} \frac{(fg)(z)-(fg)(z_0)}{z-z_0} \\ &= \lim_{z\to z_0} \frac{f(z)g(z)-f(z_0)g(z)+f(z_0)g(z)-f(z_0)g(z_0)}{z-z_0} \\ &= \lim_{z\to z_0} \frac{g(z)(f(z)-f(z_0))}{z-z_0} + \lim_{z\to z_0} \frac{f(z_0)(g(z)-g(z_0))}{z-z_0} \\ &= \lim_{z\to z_0} g(z) \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} + f(z_0) \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} \\ &= f'(z_0)g(z_0)+f(z_0)g'(z_0). \end{aligned}$$

3. $$\begin{aligned} &\lim_{z\to z_0} \frac{\left(\frac{f}{g}\right)(z)-\left(\frac{f}{g}\right)(z_0)}{z-z_0} \\ &= \lim_{z\to z_0} \frac{f(z)g(z_0)-f(z_0)g(z)}{g(z)g(z_0)(z-z_0)} \\ &= \lim_{z\to z_0} \frac{f(z)g(z_0)-f(z_0)g(z_0)+f(z_0)g(z_0)-f(z_0)g(z)}{g(z)g(z_0)(z-z_0)} \\ &= \frac{\lim_{z\to z_0} \left(\frac{g(z_0)(f(z)-f(z_0))}{z-z_0} - \frac{f(z_0)(g(z)-g(z_0))}{z-z_0} \right)}{\lim_{z\to z_0} g(z)g(z_0)} \\ &= \frac{g(z_0)f'(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2}. \end{aligned}$$

□

Example 2.1.7 Every constant function is differentiable on $\mathbb{C}$ and its derivative is 0.
Let $z_0 \in \mathbb{C}$ be arbitrary. If f is a constant function, then obviously $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{0}{z-z_0} = 0$.

Example 2.1.8 If $f(z) = z^n,\ \forall z \in \mathbb{C}$, where $n \in \mathbb{N}$, then f is differentiable on $\mathbb{C}$ and $f'(z) = nz^{n-1}, \forall z \in \mathbb{C}$.
For an arbitrary $z_0 \in \mathbb{C}$, consider

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} &= \lim_{z\to z_0} \frac{z^n - z_0^n}{z-z_0} \\ &= \lim_{z\to z_0} \frac{(z-z_0)(z^{n-1}+z^{n-2}z_0+\cdots+zz_0^{n-2}+z_0^{n-1})}{z-z_0} \\ &= \lim_{z\to z_0} \left(z^{n-1}+z^{n-2}z_0+\cdots+zz_0^{n-2}+z_0^{n-1}\right) \\ &= nz_0^{n-1}.\end{aligned}$$

Thus, $f'(z) = nz^{n-1},\ \forall z \in \mathbb{C}$.

Example 2.1.9 Every polynomial is differentiable on $\mathbb{C}$.
Applying Theorem 2.1.6 repeatedly and using the above two examples, we get every polynomial to be differentiable on $\mathbb{C}$.

THEOREM 2.1.10 (Chain rule)
If f is differentiable on Ω, g is differentiable on $f(\Omega)$, and $h = g \circ f$, then h is differentiable on Ω and $h'(z) = g'(f(z))f'(z),\ \forall z \in \Omega$.

Proof: Let $z_0 \in \Omega$ be arbitrary, then we have

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \text{ and } \lim_{w\to f(z_0)} \frac{g(w)-g(f(z_0))}{w-f(z_0)} = g'(f(z_0)).$$

Thus, if we set

$$R(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \text{ in } B(z_0, r) \setminus \{z_0\}$$

and

$$S(w) = \frac{g(w)-g(f(z_0))}{w-f(z_0)} - g'(f(z_0)) \text{ in } B(f(z_0), r) \setminus \{f(z_0)\}$$

for a small $r > 0$, then

$$\begin{aligned} f(z) - f(z_0) &= (z - z_0)(f'(z_0) + R(z)), \\ g(w) - g(f(z_0)) &= (w - f(z_0))[g'(f(z_0)) + S(w)], \end{aligned}$$

and $R(z) \to 0$ as $z \to z_0$, $S(w) \to 0$ as $w \to f(z_0)$. As f is continuous at z_0 (Result 2.1.5), we have $f(z) \to f(z_0)$ as $z \to z_0$. Therefore,

$$\begin{aligned} \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\ &= \lim_{z \to z_0} \frac{[g'(f(z_0)) + S(f(z))](f(z) - f(z_0))}{z - z_0} \\ &= \lim_{z \to z_0} \frac{[g'(f(z_0)) + S(f(z))][f'(z_0) + R(z)](z - z_0)}{z - z_0} \\ &= \lim_{z \to z_0} [g'(f(z_0)) + S(f(z))](f'(z_0) + R(z)) \\ &= g'(f(z_0))f'(z_0). \end{aligned}$$

Hence, h is differentiable on Ω and $h'(z) = g'(f(z))f'(z)$, $\forall z \in \Omega$. □

Definition 2.1.11 (Higher order derivatives)
Let $f : \Omega \to \mathbb{C}$, where Ω be a region.

1. The second derivative of f is defined by the derivative of f' and is denoted by $f^{(2)}$ or f''.
2. The third derivative of f is defined by the derivative of $f^{(2)}$ and is denoted by $f^{(3)}$ or f'''.
3. Proceeding further inductively, we define the k^{th} derivative of f by the derivative of $f^{(k-1)}$ and is denoted by $f^{(k)}$, $\forall k > 1$.

We shall also use the notation $f^{(0)}$ to denote the function f. We also mean that f is k times differentiable, by writing that $f^{(k)}$ exists, for $k \in \{1, 2, 3, \ldots\}$.

Remark 2.1.12: In the above definition, the k^{th} derivative of f could be defined if $f^{(k-1)}$ is differentiable. However, we shall prove that if f is analytic, then $f^{(k)}$ exists, for all $k \in \mathbb{N}$. (See Corollary 4.3.3.)

RESULT 2.1.13 (Leibniz rule)

Let f and g be analytic functions, then $(f \cdot g)^{(k)} = \sum_{j=0}^{k} {}^{k}C_j f^{(j)} g^{(k-j)}$, for all

$k \in \mathbb{N}$, where $kC_j = \dfrac{k \times (k-1) \times \cdots \times (k-j+1)}{1 \times 2 \times \cdots \times j}$

Proof: We prove this theorem by induction on k. For $k = 1$, the result follows from Theorem 2.1.6 immediately. Assume that this result holds for some $k \in \mathbb{N}$. First, we note that

$$\begin{aligned}
&kC_j + kC_{j-1} \\
&\quad = \frac{k \times (k-1) \times \cdots \times (k-j+1)}{1 \times 2 \times \cdots \times j} \\
&\qquad \times \frac{k \times (k-1) \times \cdots \times (k-j+2)}{1 \times 2 \times \cdots \times j-1} \\
&\quad = \frac{k \times (k-1) \times \cdots \times (k-j+2)}{1 \times 2 \times \cdots \times j-1} \times \left(\frac{(k-j+1)}{j} + 1 \right) \\
&\quad = \frac{k \times (k-1) \times \cdots \times (k-j+2)}{1 \times 2 \times \cdots \times j-1} \times \frac{(k+1)}{j} \\
&\quad = (k+1)C_j.
\end{aligned}$$

Again applying Theorem 2.1.6 and by using $nC_0 = nC_n = 1$, $\forall n \in \mathbb{N}$, we get

$$\begin{aligned}
(f \cdot g)^{(k+1)} &= ((f \cdot g)^{(k)})' \\
&= \left(\sum_{j=0}^{k} kC_j f^{(j)} \cdot g^{(k-j)} \right)' \\
&= \sum_{j=0}^{k} kC_j \left(f^{(j+1)} \cdot g^{(k-j)} + f^{(j)} \cdot g^{(k-j+1)} \right) \\
&= \sum_{j=1}^{k+1} kC_{j-1} f^{(j)} \cdot g^{(k-j+1)} + \sum_{j=0}^{k} kC_j f^{(j)} \cdot g^{(k-j+1)} \\
&= kC_k f^{(k+1)} \cdot g + kC_0 f \cdot g^{(k+1)} \\
&\quad + \sum_{j=1}^{k} \left(kC_j + kC_{j-1} \right) f^{(j)} \cdot g^{(k-j+1)}
\end{aligned}$$

$$
\begin{aligned}
&= (k+1)C_0 f \cdot g^{(k+1)} + \sum_{j=1}^{k} (k+1)C_j f^{(j)} \cdot g^{(k+1-j)} \\
&\quad + (k+1)C_{k+1} f^{(k+1)} \cdot g \\
&= \sum_{j=0}^{(k+1)} (k+1)C_j f^{(j)} \cdot g^{(k+1-j)}.
\end{aligned}
$$

Thus, the result follows. □

A function f of a complex variable z can also be viewed as a function of two real variables x and y. Therefore, we can discuss about the partial derivatives of f with respect to x and y, and the relation between the differentiability of f and partial differentiabilities of f.

Definition 2.1.14 Let $f : \Omega \to \mathbb{C}$ and $(x_0, y_0) \in \Omega$, where Ω is a region.

1. f is said to be partially differentiable with respect to x at (x_0, y_0) if $\lim_{s \to 0} \dfrac{f(x_0 + s, y_0) - f(x_0, y_0)}{s}$ exists and is denoted by $\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$.
2. f is said to be partially differentiable with respect to y at (x_0, y_0) if $\lim_{t \to 0} \dfrac{f(x_0, y_0 + t) - f(x_0, y_0)}{t}$ exists and is denoted by $\dfrac{\partial f}{\partial y}(x_0, y_0)$ or $f_y(x_0, y_0)$.

In the above two limits, s and t approach 0 through reals.

THEOREM 2.1.15 If f and g are partially differentiable with respect to x (with respect to y), then

1. $f + g$ is partially differentiable with respect to x (with respect to y) and $(f+g)_x = f_x + g_x$ ($(f+g)_y = f_y + g_y$).
2. cf is partially differentiable with respect to x (with respect to y) and $(cf)_x = cf_x$ ($(cf)_y = cf_y$), where $c \in \mathbb{C}$.

Proof: Proof of this theorem is analogous to that of Theorem 2.1.6. □

As an immediate consequence, we have the following corollary.

COROLLARY 2.1.16 Let $f = u + iv$, then f is partially differentiable with respect to x (with respect to y) iff u and v are partially differentiable with respect to x (with respect to y). In this case, $f_x = u_x + iv_x$ ($f_y = u_y + iv_y$).

Example 2.1.17 In general, though f_{xy} and f_{yx} exist at a point, they need not be equal to each other.

$$\text{Consider} \quad f(x,y) = \begin{cases} \dfrac{x^3y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = 0 \end{cases}.$$

Now for $s, t \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} f_x(0,0) &= \lim_{s\to 0} \frac{f(0+s,0)-f(0,0)}{s} = \lim_{s\to 0} \frac{0-0}{s} = 0 \\ f_x(0,t) &= \lim_{s\to 0} \frac{f(0+s,t)-f(0,t)}{s} = \lim_{s\to 0} \frac{s^2t}{s^2+t^2} = 0 \\ f_y(0,0) &= \lim_{t\to 0} \frac{f(0,0+t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0 \\ f_y(s,0) &= \lim_{t\to 0} \frac{f(s,0+t)-f(s,0)}{t} = \lim_{t\to 0} \frac{s^3}{s^2+t^2} = s. \end{aligned}$$

Therefore, we have

$$f_{xy}(0,0) = \lim_{s\to 0} \frac{f_y(s,0)-f_y(0,0)}{s} = \lim_{s\to 0} \frac{s}{s} = 1$$

and

$$f_{yx}(0,0) = \lim_{t\to 0} \frac{f_x(0,t)-f_x(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Next we recall mean-value theorem from real analysis, which will be applied in the following sequel.

THEOREM 2.1.18 (Mean-value theorem)
Let $f : [a,b] \to \mathbb{R}$ be a continuous function. If f is differentiable on (a,b), then there exists $x \in (a,b)$ such that $f(b) - f(a) = f'(x)(b-a)$.

THEOREM 2.1.19 (Complex version of mean-value theorem)
Let $f : [a,b] \to \mathbb{C}$ be a continuous function. If f is differentiable on (a,b), then there exists $x \in (a,b)$ such that $|f(b) - f(a)| \leq |f'(x)|(b-a)$.

Proof: If $f = (u,v)$, then u and v are real-valued continuous functions on $[a,b]$, and they are differentiable on (a,b). Furthermore,

$$f'(t) = (u'(t), v'(t)), \ \forall\, t \in [a,b]$$

Define $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(t) = xu(t) + yv(t),\ \forall\, t \in [a,b].$$

where $(x,y) = f(b) - f(a) = (u(b) - u(a), v(b) - v(a))$

Therefore, ϕ is a real-valued continuous function on $[a,b]$ and $\phi'(t) = xu'(t) + yv'(t), \forall t \in (a,b)$.

Now applying mean-value theorem (Theorem 2.1.18), we choose $t \in (a,b)$ such that

$$\phi(b) - \phi(a) = \phi'(t)(b-a).$$

Therefore,

$$\begin{aligned}
|(x,y)|^2 &= x^2 + y^2 \\
&= x(u(b) - u(a)) + y(v(b) - v(a)) \\
&= xu(b) + yv(b) - (xu(a) + yv(a)) \\
&= \phi(b) - \phi(a) \\
&= \phi'(t)(b-a) \\
&= (xu'(t) + yv'(t))(b-a) \\
&\leq \sqrt{x^2+y^2}\sqrt{(u'(t))^2 + (v'(t))^2}\,(b-a) \\
&\quad \text{(by Cauchy–Schwarz inequality.)} \\
&= |(x,y)||f'(t)|(b-a).
\end{aligned}$$

If $|(x,y)| \neq 0$, then we have $|(x,y)| \leq |f'(t)|(b-a)$. If $|(x,y)| = 0$, then $|(x,y)| = 0 \leq |f'(t)|(b-a)$. □

THEOREM 2.1.20 (Young's theorem)
Let $f : \Omega \to \mathbb{R}$ and $(a,b) \in \Omega$. If f_x exists in a neighbourhood of (a,b) and f_{xy} is continuous at (a,b), then f_{yx} exists at (a,b) and $f_{yx}(a,b) = f_{xy}(a,b)$.

Proof: Let f_x, f_y, and f_{xy} exist on $B((a,b),r)$. Fix $(s,t) \in \mathbb{C}$ such that $(a+s, b+t) \in B((a,b),r)$. If $F : [b, b+t] \to \mathbb{R}$ is defined by

$$F(y) = f(a+s,y) - f(a,y),\ \forall y \in [b, b+t],$$

then F is differentiable on $[b, b+t]$, and hence, by applying mean-value theorem (Theorem 2.1.18) twice, we get

$$f(a+s,b+t)-f(a,b+t)-[f(a+s,b)-f(a,b)]$$

$$\begin{aligned}
&= F(b+t)-F(b)\\
&= tF'(b+\lambda t),\ \text{for some } \lambda \in (0,1)\\
&= t\left[f_y(a+s,b+\lambda t)-f_y(a,b+\lambda t)\right]\\
&= stf_{xy}(a+\mu s,b+\lambda t),\ \text{for some } \mu \in (0,1).
\end{aligned}$$

Using the continuity of f_{xy} at (a,b), we obtain

$$\begin{aligned}
f_{xy}(a,b) &= \lim_{(s,t)\to(0,0)} f_{xy}(a+\mu s,b+\lambda t)\\
&= \lim_{t\to 0}\lim_{s\to 0}\frac{1}{t}\left[\frac{f(a+s,b+t)-f(a,b+t)}{s}-\frac{f(a+s,b)-f(a,b)}{s}\right]\\
&\quad \text{(using Lemma 1.6.8)}\\
&= \lim_{t\to 0}\frac{f_x(a,b+t)-f_x(a,b)}{t}\\
&= f_{yx}(a,b).
\end{aligned}$$

Thus, the theorem follows. □

2.2 CAUCHY–RIEMANN EQUATIONS

Definition 2.2.1 (Cauchy–Riemann equations)
Let u and v be real-valued functions of two real variables x and y, then we say that u and v satisfy Cauchy–Riemann equations (or simply C–R equations) if $u_x = v_y$ and $v_x = -u_y$.

THEOREM 2.2.2 Let Ω be a region, $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$. If $f = u + iv$ is differentiable at z_0, then f is partially differentiable at $z_0 = x_0 + iy_0 = (x_0, y_0)$, and u and v satisfy C–R equations.

Proof: As f is differentiable at z_0, we have $f'(z_0) = \lim_{h\to 0}\dfrac{f(z_0+h)-f(z_0)}{h}$ exists or equivalently,

$$f'(x_0+iy_0) = \lim_{s+it\to 0}\frac{f((x_0+s)+i(y_0+t))-f(x_0+iy_0)}{s+it} \text{ exists.} \qquad (2.1)$$

By letting $t = 0$ and $s \to 0$ in equation (2.1), we get

$$\begin{aligned} f'(x_0 + iy_0) &= \lim_{s\to 0} \frac{f((x_0 + s) + iy_0) - f(x_0 + iy_0)}{s} \\ &= \lim_{s\to 0} \frac{f(x_0 + s, y_0) - f(x_0, y_0)}{s} \\ &= f_x(x_0, y_0). \end{aligned}$$

If $s = 0$ and $t \to 0$ in equation (2.1), then we get

$$\begin{aligned} f'(x_0 + iy_0) &= \lim_{t\to 0} \frac{f(x_0 + i(y_0 + t)) - f(x_0 + iy_0)}{it} \\ &= \frac{1}{i} \lim_{t\to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t} \\ &= -if_y(x_0, y_0). \end{aligned}$$

Hence, f_x and f_y exist at (x_0, y_0), and we also have $f_x = f' = -if_y$. As $f = u + iv$, using Corollary 2.1.16, we have

$$u_x + iv_x = f_x = -if_y = -i(u_y + iv_y) = -iu_y + v_y.$$

Equating the real and imaginary parts on both sides, we get $u_x = v_y$ and $v_x = -u_y$. Thus, u and v satisfy C–R equations. □

Remark 2.2.3: From the above theorem, we conclude that a necessary condition on f has to be differentiable at (x, y) such that f should satisfy the C–R equations at (x, y) (i.e., $f_x = -if_y$).

Example 2.2.4 If $f(z) = x^2 - iy^3$ and $g(z) = \exp(2x + i3y)$, $\forall z \in \mathbb{C}$, where $x = \text{Re } z$ and $y = \text{Im z}$, then justify that f and g are not differentiable on $\mathbb{C}$.
We now check the C–R equations for f.
If $u(x, y) = x^2$, $v(x, y) = -y^3$, $\forall (x, y) \in \mathbb{C}$, then $f = u + iv$ and we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x, & \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} &= 0, & \frac{\partial v}{\partial y} &= -3y^2. \end{aligned}$$

Therefore, $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$. Thus, f does not satisfy C–R equations at any $(x, y) \in \mathbb{C}\backslash\{(0, 0)\}$, and hence, it is not differentiable on $\mathbb{C}$.

If $u(x,y) = \exp(2x)\cos(3y)$, $v(x,y) = \exp(2x)\sin(3y)$, $\forall (x,y) \in \mathbb{C}$, then $g = u + iv$ and we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2\exp(2x)\cos(3y), & \frac{\partial u}{\partial y} &= -3\exp(2x)\sin(3y) \\ \frac{\partial v}{\partial x} &= 2\exp(2x)\sin(3y), & \frac{\partial v}{\partial y} &= 3\exp(2x)\cos(3y). \end{aligned}$$

Therefore, f does not satisfy C–R equations at any $(x,y) \in \mathbb{C}$, and hence, it is not differentiable on $\mathbb{C}$.

Definition 2.2.5 Let Ω be a region, then a function u on Ω is called a harmonic function if u satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ on Ω.

RESULT 2.2.6 If $f = u + iv$ and f is differentiable on Ω, then u and v are harmonic functions on Ω.

Proof: (At present, assume the fact that partial derivatives of u and v of all orders exist and they are continuous. This will be obtained as a consequence of Theorem 4.3.3.) By Theorem 2.2.2, if f is differentiable on Ω and u and v satisfy C–R equations, then $u_x = v_y$ and $u_y = -v_x$. Now applying Young's theorem (Theorem 2.1.20), we get

1. $u_{xx} + u_{yy} = \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} = \dfrac{\partial v_y}{\partial x} + \dfrac{-\partial v_x}{\partial y} = v_{yx} - v_{xy} = 0$. Hence, u is harmonic.
2. $v_{xx} + v_{yy} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = \dfrac{-\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y} = u_{xy} - u_{yx} = 0$. Hence, v is harmonic. □

Definition 2.2.7 Given a harmonic function u on a region Ω, another harmonic function v on Ω is called a harmonic conjugate of u if $u + iv$ is a differentiable function on Ω.

RESULT 2.2.8 v is a harmonic conjugate of u iff $-u$ is a harmonic conjugate of v.

Proof: v is a harmonic conjugate of u iff $u + iv$ is an analytic function iff $-i(u + iv) = v - iu$ is an analytic function iff $-u$ is a harmonic conjugate of v. □

Algorithm 2.2.9 (To find harmonic conjugate.)

Step 1. Let the given function be u and find u_x.

Step 2. Put $v_y = u_x$ and integrate v_y partially with respect to y and write $v = \int v_y\, dy + \phi$, where ϕ is a function of x.

Step 3. Using this v, find v_x.

Step 4. From the given u, find u_y.

Step 5. Using $u_y = v_x$, find the value of ϕ' and then ϕ.

Now, we present two more algorithms (by Milne-Thompson) to find an analytic function f whose real part u (or imaginary part v) is the given harmonic function. Replacing z by $x + iy$ and finding the real and imaginary parts of $f(x + iy)$, we can also find the harmonic conjugate of u (or v).

Algorithm 2.2.10 (Finding an analytic function $f = u + iv$ from the given u)

Step 1. Check whether the given function u is a harmonic function. If yes, then go to next step.

Step 2. Find u_x and u_y.

Step 3. Put $f'(z) = u_x(z, 0) - iu_y(z, 0)$.

Step 4. Find $f = \int f'(z)\, dz + c$, where c is a constant.

Step 5. If you want to find the harmonic conjugate of u, then put $z = x + iy$ and find real and imaginary parts of f.

Step 6. Write the harmonic conjugate of u by $\operatorname{Im} f$.

Algorithm 2.2.11 (Finding an analytic function $f = u + iv$ from the given v)

Step 1. Check whether given function v is a harmonic function.

Step 2. Find v_x and v_y.

Step 3. Put $f'(z) = v_y(z, 0) + iv_x(z, 0)$.

Step 4. Find $f = \int f'(z)\, dz + c$, where c is a constant.

Step 5. If you want to find the harmonic conjugate of v, then put $z = x + iy$ and find real and imaginary parts of f.

Step 6. Write the harmonic conjugate of v by $-\operatorname{Re} f$.

Example 2.2.12 If $u(x,y) = x^3 - 3xy^2 + 2x^2 - 2y^2 + x$, then prove that u is a harmonic function and find its harmonic conjugate.
As

$$\begin{aligned} u_x &= 3x^2 - 3y^2 + 4x + 1,\ u_{xx} = 6x + 4 \\ u_y &= -6xy - 4y \quad \text{and} \quad u_{yy} = -6x - 4 \end{aligned}$$

we have

$$u_{xx} + u_{yy} = 6x + 4 - 6x - 4 = 0.$$

Hence, u is a harmonic function. If v is a harmonic conjugate of u, then u and v satisfy C–R equations (i.e., $u_x = v_y,\ u_y = -v_x$). The first equation

$$v_y = u_x = 3x^2 - 3y^2 + 4x + 1$$

implies

$$v = \int v_y\, dy = \int (3x^2 - 3y^2 + 4x + 1)\, dy = 3x^2y - y^3 + 4xy + y + \phi(x)$$

for some function ϕ of x. Using $u_y = -v_x$, we get

$$-6xy - 4y = -6xy - 4y - \phi'(x).$$

Therefore, $\phi'(x) = 0$, and hence, ϕ is a constant. Thus,

$$v = 3x^2y - y^3 + 4xy + y + c$$

for some real constant c.

Example 2.2.13 If $u(x,y) = \dfrac{\sin(2x)}{\cosh(2y) - \cos(2x)}$, then find the analytic function whose real part is u.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(\cosh(2y) - \cos(2x))2\cos(2x) - \sin(2x)2\sin(2x)}{(\cosh(2y) - \cos(2x))^2} \\ &= \frac{2(\cos(2x)\cosh(2y) - 1)}{(\cosh(2y) - \cos(2x))^2} \\ \frac{\partial u}{\partial y} &= -\frac{2\sin(2x)\sinh(2y)}{(\cosh(2y) - \cos(2x))^2}. \end{aligned}$$

If $f = u + iv$, then

$$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)$$

$$\begin{aligned} &= \frac{2(\cos(2z)-1)}{(1-\cos(2z))^2} \\ &= \frac{2}{\cos(2z)-1} \\ &= \frac{2}{\cos^2(z)-\sin^2(z)-\cos^2(z)-\sin^2(z)} \\ &= \frac{2}{-2\sin^2(z)} \\ &= -\csc^2(z) \end{aligned}$$

and hence, we get $f(z) = \cot(z) + c$ for some complex constant c.

Example 2.2.14 If $u(x,y) = \cosh(x)\sin(y),\ \forall (x,y) \in \mathbb{C}$, then prove that u is a harmonic function and find the analytic function f whose real part is u. Find also its harmonic conjugate.

$$\begin{aligned} u_x &= \sinh(x)\sin(y), & u_{xx} &= \cosh(x)\sin(y) \\ u_y &= \cosh(x)\cos(y) & u_{yy} &= -\cosh(x)\sin(y). \end{aligned}$$

Therefore, $u_{xx} + u_{yy} = \cosh(x)\sin(y) - \cosh(x)\sin(y) = 0$, and hence, u is a harmonic function.

As $u_x = \sinh(x)\sin(y)$ and $u_y = \cosh(x)\cos(y)$, we put

$$f'(z) = u_x(z,0) - iu_y(z,0) = -i\cosh(z).$$

Therefore, $f(z) = \int f'(z)\,dz = \int -i\cosh(z)\,dz + c = -i\sinh(z) + c$.
Writing $z = x + iy$, we get

$$\begin{aligned} f(x+iy) &= -i\sinh(x+iy) + c \\ &= -\sin(i(x+iy)) + c \\ &= -\sin(ix - y) + c \\ &= -\sin(ix)\cos(y) + \sin(y)\cos(ix) + c \\ &= -i\sinh(x)\cos(y) + \sin(y)\cosh(x) + c. \end{aligned}$$

Hence, the harmonic conjugate of $\cosh(x)\sin(y)$ is $-\sinh(x)\cos(y) + r$ for some real constant r.

Example 2.2.15 Prove that $v(x,y) = \exp(x)(x\sin(y) + y\cos(y)),\ \forall (x,y) \in \mathbb{C}$ is a harmonic function. Find an analytic function whose imaginary part is v and also find its harmonic conjugate.

$$\begin{aligned} v_x &= \exp(x)(x\sin(y) + y\cos(y)) + \exp(x)\sin(y) \\ &= \exp(x)((x+1)\sin(y) + y\cos(y)) \\ v_{xx} &= \exp(x)((x+1)\sin(y) + y\cos(y)) + \exp(x)\sin(y) \\ &= \exp(x)((x+2)\sin(y) + y\cos(y)) \\ v_y &= \exp(x)(x\cos(y) - y\sin(y) + \cos(y)) \\ &= \exp(x)((x+1)\cos(y) - y\sin(y)) \\ v_{yy} &= \exp(x)(-(x+1)\sin(y) - y\cos(y) - \sin(y)) \\ &= \exp(x)(-(x+2)\sin(y) - y\cos(y)). \end{aligned}$$

As

$$\begin{aligned} v_{xx} + v_{yy} &= \exp(x)((x+2)\sin(y) + y\cos(y) - (x+2)\sin(y) - y\cos(y)) \\ &= 0 \end{aligned}$$

v is a harmonic function. We know that $f' = f_x = u_x + iv_x = v_y + iv_x$; therefore,

$$\begin{aligned} f' &= \exp(x)((x+1)\cos(y) - y\sin(y)) \\ &\quad + i(\exp(x)((x+1)\sin(y) + y\cos(y))) \\ &= \exp(z)(z+1) \text{ (by replacing } x \text{ and } y \text{ by } z \text{ and 0 respectively).} \end{aligned}$$

Now,

$$\begin{aligned} f(z) &= \int \exp(z)(z+1)\,dz \\ &= (z+1)\exp(z) - \int \exp(z)\,dz \\ &= (z+1)\exp(z) - \exp(z) = z\exp(z). \end{aligned}$$

Therefore, $f(z) = z\exp(z) + c, \forall z \in \mathbb{C}$ for some constant c. Writing $z = x + iy$ and finding the real part of $f(z)$, we get

$$\begin{aligned} u(x,y) &= \text{Re } [(x+iy)\exp(x+iy)] \\ &= \text{Re } [(x+iy)\exp(x)(\cos(y) + i\sin(y))] \\ &= \exp(x)(x\cos(y) - y\sin(y)). \end{aligned}$$

Therefore, the harmonic conjugate of $v(x,y)$ is

$$-u(x,y) = \exp(x)(y\sin(y) - x\cos(y))$$

Example 2.2.16 Find an analytic function f whose real and imaginary parts are u and v, respectively, if $u + v = \exp(-y)\,(\cos(x) - \sin(x))$.
We find

$$\begin{aligned} u_x + v_x &= \exp(-y)\,(-\sin(x) - \cos(x))\,, \qquad (2.2)\\ u_y + v_y &= -\exp(-y)\,(\cos(x) - \sin(x))\,,\\ -v_x + u_x &= \exp(-y)\,(\sin(x) - \cos(x))\,. \qquad (2.3) \end{aligned}$$

Now, equations (2.2) + (2.3) and (2.2) − (2.3) imply that

$$\begin{aligned} u_x &= -\exp(-y)\cos(x),\\ v_x &= -\exp(-y)\sin(x). \end{aligned}$$

Therefore,

$$\begin{aligned} f'(z) &= u_x(z,0) + iv_x(z,0)\\ &= -(\cos(z) + i\sin(z)) = -\exp(iz) \end{aligned}$$

and hence,

$$f(z) = i\exp(iz) + c = \exp(iz) + c$$

for some $c \in \mathbb{C}$.

Exercise 2.2.17

1. Prove that the following functions are harmonic and find the harmonic conjugates.
   (a) $u(x,y) = x^3 - 3xy^2$.
   (b) $u(x,y) = x^4 + y^4 - 6x^2y^2 + 2xy - x$.
   (c) $u(x,y) = \exp(-x)(x\sin(y) - y\cos(y))$.
   (d) $u(x,y) = \cosh(x)\sin(y)$.
   (e) $u(x,y) = \exp(-x)(x\cos(y) + y\sin(y))$.
2. Verify that $u(x,y) = \dfrac{y}{x^2+y^2}$, $\forall (x,y) \in \mathbb{C} \setminus \{(0,0)\}$ is a harmonic function and find an analytic function whose real part is u.
3. Verify that $v(x,y) = -3\exp(2x)\sin(2y)$, $\forall (x,y) \in \mathbb{C}$ is a harmonic function and find an anlytic function whose imaginary part is v.
4. Verify that $u(x,y) = -\exp(x^2 - y^2)\sin(2xy)$, $\forall (x,y) \in \mathbb{C}$ and find an analytic function whose real part is u. Find also the harmonic conjugate of u.

5. Verify that v is a harmonic function and find the analytic function whose imaginary part is v, where $v(x,y) = x^2 - y^2 + 2y$, $\forall (x,y) \in \mathbb{C}$. Find also its harmonic conjugate.

6. Determine the analytic function $f = u + iv$ if

   (a) $(u - v)(x,y) = x^3 - y^3 + 3x(x - y)$.

   (b) $(u + v)(x,y) = \dfrac{\sin(2x) + \sinh(2y)}{\cosh(2y) + \cos(2x)}$.

Answers: (1) (a) $v(x,y) = 3x^2y - y^3 + c$.
(b) $4x^3y - 4xy^3 - (x^2 - y^2) - y$.
(c) $v(x,y) = \exp(-x)(y\sin(y) + x\cos(y)) + c$.
(d) $-\sinh(x)\cos(y) + c$.
(e) $\exp(-x)(y\cos(y) - x\sin(y))$.
(2) $\dfrac{i}{z}$.
(3) $-3\exp(2z)$.
(4) $-\exp(z^2)$, $\exp(x^2 - y^2)\cos(2xy)$.
(5) $iz^2 + 2z$, $2xy - 2x$.
(6) (a) $f(z) = -iz^3 + c$.
(b) $\tan z + c$.

LEMMA 2.2.18 Let $\phi : (a,b) \to \mathbb{R}$ be a differentiable function.

1. If $\phi'(t) = 0$, $\forall t \in (a,b)$, then ϕ is constant on (a,b).
2. If $\phi'(t) \geq 0$, $\forall t \in (a,b)$, then ϕ is increasing on (a,b).
3. If $\phi'(t) \leq 0$, $\forall t \in (a,b)$, then ϕ is decreasing on (a,b).
4. If $\phi'(t) > 0$, $\forall t \in (a,b)$, then ϕ is strictly increasing on (a,b).
5. If $\phi'(t) < 0$, $\forall t \in (a,b)$, then ϕ is strictly decreasing on (a,b).

Proof: Let $x, y \in (a,b)$ be arbitrary such that $x \neq y$. We assume that $x < y$ without loss of generality. Then, ϕ is a differentiable function on $[x,y]$, and then by using mean-value theorem (Theorem 2.1.18), there exists $t \in (x,y)$ such that

$$\phi(y) - \phi(x) = \phi'(t)(x - y) = 0.$$

1. If $\phi' = 0$, then $\phi(y) = \phi(x)$. Thus, ϕ is a constant.

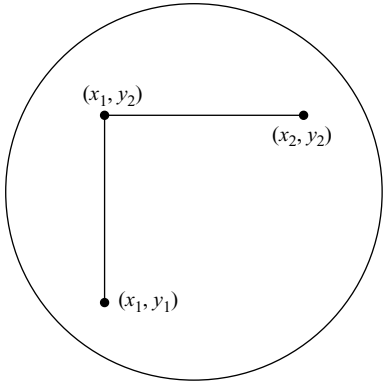
2. If $\phi' \geq 0$, then $\phi(y) \geq \phi(x)$. Thus, ϕ is increasing.

The proof of the remaining assertions are similar. □

THEOREM 2.2.19 If f is analytic on a region Ω such that $f' = 0$, then f is constant.

Proof: Let $z_0 \in \Omega$ be arbitrary. Choose $r > 0$ such that $B(z_0, r) \subseteq \Omega$. If $f = u + iv$, then $f' = u_x + iv_x = v_y - iu_y$ implies $u_x = v_x = u_y = v_y = 0$. From $u_x = 0$, we get that u is a constant c_y on $H_z \cap B(z_0, r)$, where H_z is the horizontal line passing through $z = (x, y)$ for every $z \in B(z_0, r)$. Similarly, from $u_y = 0$, we get u, which is a constant c_x on $V_z \cap B(z_0, r)$, where V_z is the vertical line passing through $z = (x, y)$ for every $z \in B(z_0, r)$.

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2) \in B(z_0, r)$, then the polygon passing through $(x_1, y_1), (x_1, y_2)$, and (x_2, y_2) is completely lying inside $B(z_0, r)$. See Corollary 1.4.29 and the following diagram.



As (x_1, y_1) and (x_1, y_2) lie on a vertical line segment in $B(z_0, r)$, we have $u(x_1, y_1) = u(x_1, y_2)$. As (x_1, y_2) and (x_2, y_2) lie on a horizontal line segment in $B(z_0, r)$, we have $u(x_1, y_2) = u(x_2, y_2)$. Thus, we get $u(z_1) = u(z_2)$, and hence, u is constant on $B(z_0, r)$.

By the same argument, from $v_x = v_y = 0$, we can prove that v is constant on $B(z_0, r)$, and so f is a constant function on $B(z_0, r)$.

If a, b are any two points in Ω, then by Theorem 1.4.31, we can find a polygon such that

1. it passes through the vertices $a = a_0, a_1, a_2, \ldots a_n = b$,

2. for each $i = 1, 2, \ldots, n$, the line segment joining a_{i-1} and a_i is either vertical or horizontal,

3. for each $i = 1, 2, \ldots, n$, the line segment joining a_{i-1} and a_i is contained in $B(z, r) \subseteq \Omega$ for some $z \in \Omega$ and $r > 0$.

Therefore, $f(a) = f(a_0) = f(a_1) = f(a_2) = \cdots = f(a_n) = f(b)$.
Hence, f is constant on Ω. □

RESULT 2.2.20 Let $f = u + iv$ be an analytic function on a region Ω, then f is a constant if any one of the following statements holds:

1. u is constant.
2. v is constant.
3. $|f|$ is constant.
4. $\arg f$ is constant.

Proof: Assume that u is constant. Then, $u_x = u_y = 0$. As $v_x = -u_y$, we have $f'(z) = f_x = u_x + iv_x = 0$ on Ω. Therefore, by using Theorem 2.2.19, we get that f is constant.
By a similar argument, using the C–R equations, one can prove that if v is constant, then u is constant, and hence, f is constant.
If $|f| = 0$, then obviously $f = 0$, and hence, f is constant. If $|f|$ is constant and non-zero, then we have $u^2 + v^2$ as constant and non-zero. Partially differentiating with respect to x and y, we get

$$2uu_x + 2vv_x = 0,\ 2uu_y + 2vv_y = 0.$$

Using C–R equations, we get

$$uu_x - vu_y = 0 \text{ and } vu_x + uu_y = 0.$$

As determinant of the coefficient matrix $\begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2 \neq 0$, this system of homogeneous linear equations in the two variables u_x and u_y has a unique solution and is $u_x = 0 = u_y$. Hence, u is constant. Therefore, f is constant.
If $\arg f$ is constant, then $\dfrac{v}{u}$ is constant, say c. It follows that $v - cu$ is constant, but $v - cu = \text{Im}\,(1 - ci)f$, and hence, $(1 - ci)f$ is constant. Thus, f is constant. □

RESULT 2.2.21 If $f = u + iv$ is differentiable, then we have $u_x v_x + u_y v_y = 0$. Geometrically, this result is rephrased by the family $u(x, y) = c$ of curves is orthogonal with the family $v(x, y) = d$ of curves.

Proof: As u and v satisfy C–R equations, we have $u_x = v_y$ and $v_x = -u_y$. Therefore, $u_x v_x + u_y v_y = -u_x u_y + u_x u_y = 0$. □

RESULT 2.2.22 If $f = u + iv$ is analytic, then $\dfrac{\partial^2 u^2}{\partial x^2} + \dfrac{\partial^2 u^2}{\partial y^2} = 2|f'|^2$.

Proof: First we note that u and v satisfy the C–R equations, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.
Now,

$$\frac{\partial^2 u^2}{\partial x^2} = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2}.$$

Similarly, we get $\dfrac{\partial^2 u^2}{\partial y^2} = 2\left(\dfrac{\partial u}{\partial y}\right)^2 + 2u\dfrac{\partial^2 u}{\partial y^2}$. Therefore, using the fact that u is harmonic, we get

$$\begin{aligned}
\frac{\partial^2 u^2}{\partial^2 x^2} + \frac{\partial^2 u^2}{\partial y^2} &= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \\
&= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{(using C–R equations)} \\
&= 2|f'|^2 \quad \left(\text{as } f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).
\end{aligned}$$

□

Exercise 2.2.23 If f is an analytic function, then

1. $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$
2. $\left[\left(\dfrac{\partial}{\partial x}\right)^2 + \left(\dfrac{\partial}{\partial y}\right)^2\right]|f(z)|^2 = |f'(z)|^2$.

RESULT 2.2.24 (Cauchy–Riemann equations in polar form)
If u and v satisfy C–R equations, then $r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta}$ and $r\dfrac{\partial v}{\partial r} = -\dfrac{\partial u}{\partial \theta}$.

Proof: We know that C–R equations in Cartesian form are given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. From the relation $x = r\cos(\theta)$ and $y = r\sin(\theta)$ between the Cartesian coordinates and polar coordinates, we get,

$$\begin{aligned}
r\frac{\partial u}{\partial r} &= r\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) \\
&= -\frac{\partial v}{\partial x} r\sin(\theta) + \frac{\partial v}{\partial y} r\cos(\theta) \\
&= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial v}{\partial \theta}, \\
r\frac{\partial v}{\partial r} &= r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) \\
&= \frac{\partial u}{\partial x} r\sin(\theta) - \frac{\partial u}{\partial y} r\cos(\theta) \\
&= -\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\frac{\partial u}{\partial \theta}.
\end{aligned}$$

Hence, C–R equations in polar form are obtained. □

RESULT 2.2.25 If f is a complex-valued function on a region Ω satisfying the C–R equation $\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$, then $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof: If $z = x + iy$, then we have $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{i2}(z - \bar{z})$. Therefore,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Hence, the result follows. □

Example 2.2.26 There exists a function f on $\mathbb{C}$ such that f_x and f_y exist and satisfy C–R equation at a point, but it is not differentiable at the same point.

Define

$$f(x,y)=\begin{cases}\dfrac{xy(x+iy)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases},\ \forall (x,y)\in\mathbb{C}$$

As $f(x,0)=0,\ \forall x\in\mathbb{R}$, we obtain $f_x(0,0)=0$, and as $f(0,y)=0,\ \forall y\in\mathbb{R}$, we get $f_y(0,0)=0$. Hence, the C–R equation $f_x=-if_y$ is satisfied at $(0,0)$. However,

$$\begin{aligned}\lim_{h\to 0}\frac{f(2h,h)-f(0,0)}{2h+ih} &= \lim_{h\to 0}\frac{2h^2(2h+ih)}{5h^2(2h+ih)}\\ &= \frac{2}{5}\\ &\neq \frac{1}{2}\\ &= \lim_{h\to 0}\frac{h^2(h+ih)}{2h^2(h+ih)}\\ &= \lim_{h\to 0}\frac{f(h,h)-f(0,0)}{h+ih}\end{aligned}$$

Hence, f is not differentiable at $(0,0)$.

Example 2.2.27 There exists a function f on $\mathbb{C}$ such that f_x and f_y exist and are continuous at a point, but it is not differentiable at the same point. Consider the function

$$f(x,y)=x-iy,\ \forall (x,y)\in\mathbb{C}$$

Now, $f_x=1$ and $f_y=-i,\ \forall (x,y)\in\mathbb{C}$. Clearly, f_x and f_y are continuous functions. However, it is not differentiable at $(0,0)$ as f does not satisfy C–R equations at $(0,0)$.

THEOREM 2.2.28 If $f=u+iv$ has continuous partial derivatives with respect to x, y on a region Ω and u, v satisfy C–R equations, then f is differentiable on Ω.

Proof: Let $z_0=(x_0,y_0)\in\Omega$ be arbitrary, then by applying mean-value theorem, we get

$$\begin{aligned}&u(x_0+s,y_0+t)-u(x_0,y_0)\\ &\quad=u(x_0+s,y_0+t)-u(x_0,y_0+t)+u(x_0,y_0+t)-u(x_0,y_0)\\ &\quad=su_x(x_0+ps,y_0+t)+tu_y(x_0,y_0+qt)\\ &\qquad\qquad(\text{for some } p,q\in(0,1))\end{aligned}$$

$$= s(u_x(x_0, y_0) + r_1) + t(u_y(x_0, y_0) + r_2), \text{ where } r_1 \to 0,\ r_2 \to 0$$
$$\text{as } (s, t) \to (0, 0), \text{(by using the continuity of } u_x \text{ and } u_y)$$
$$= s(u_x + r_1) + t(u_y + r_2)$$
$$\text{(after writing } u_x(x_0, y_0) \text{ and } u_y(x_0, y_0) \text{ simply by } u_x \text{ and } u_y).$$

Similarly, we get $v(x_0 + s, y_0 + t) - v(x_0, y_0) = s(v_x + r_3) + t(v_y + r_4)$, where $r_3 \to 0,\ r_4 \to 0$ as $(s, t) \to (0, 0)$.
Therefore,

$$\begin{aligned} & f(x_0 + s, y_0 + t) - f(x_0, y_0) \\ &= s(u_x + r_1) + t(u_y + r_2) + is(v_x + r_3) + it(v_y + r_4) \\ &= s(u_x + r_1) + t(-v_x + r_2) + is(v_x + r_3) + it(u_x + r_4) \\ &= (s + it)u_x + i(s + it)v_x + s(r_1 + ir_3) + t(r_2 + ir_4) \\ &= (s + it)(u_x + iv_x) + s(r_1 + ir_3) + t(r_2 + ir_4) \\ &= (s + it)f_x + s(r_1 + ir_3) + t(r_2 + ir_4). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\leq \left| \frac{f(x_0 + s, y_0 + t) - f(x_0, y_0)}{s + it} - f_x \right| \\ &\leq \left| \frac{s}{s + it}(r_1 + ir_3) + \frac{t}{s + it}(r_2 + ir_4) \right| \\ &\leq |r_1| + |r_3| + |r_2| + |r_4| \to 0 \text{ as } (s, t) \to (0, 0). \end{aligned}$$

Thus, $f'(z_0) = \lim_{s+it \to 0} \dfrac{f((x_0 + iy_0) + (s + it)) - f(x_0 + iy_0)}{s + it}$ exists and is equal to $f_x(z_0)$. □

RESULT 2.2.29 Let $f : \Omega \to \mathbb{C}$, where Ω is a region such that $z \in \Omega \Rightarrow \bar{z} \in \Omega$. If $g(z) = \overline{f(\bar{z})}$, $\forall z \in \Omega$, then f is differentiable iff g is differentiable.

Proof: Let $f = u + iv$ and $g = \varphi + i\psi$. As

$$\varphi(x, y) = u(x, -y) \text{ and } \psi(x, y) = -v(x, -y),\ \forall (x, y) \in \Omega,$$

we get

$$\varphi_x(x, y) = u_x(x, -y),\ \varphi_y(x, y) = -u_y(x, -y),\ \forall (x, y) \in \Omega$$

and

$$\psi_x(x, y) = -v_x(x, -y),\ \psi_y(x, y) = v_y(x, -y),\ \forall (x, y) \in \Omega.$$

Hence, u_x, u_y, v_x and v_y exist and are continuous iff φ_x, φ_y, ψ_x and ψ_y exist and are continuous. Furthermore, $u_x = v_y$ and $u_y = -v_x$ imply

$$\varphi_x(x, y) = u_x(x, -y) = v_y(x, -y) = \psi_y(x, y)$$

and

$$\varphi_y(x, y) = -u_y(x, -y) = v_x(x, -y) = -\psi_x(x, y).$$

Similarly, we can prove that $\varphi_x = \psi_y$ and $\varphi_y = -\psi_x$ imply $u_x = v_y$ and $u_y = -v_x$. Hence, using Theorems 2.2.2 and 2.2.28, we get that f is differentiable iff u_x, u_y, v_x and v_y are continuous and satisfy C–R equations iff φ_x, φ_y, ψ_x and ψ_y are continuous and satisfy C–R equations iff g is differentiable. □

2.3 POWER SERIES AND ABEL'S THEOREMS

Definition 2.3.1 The series of functions $\sum_{n=0}^{\infty} a_n(z-a)^n$ is called a power series, where $a \in \mathbb{C}$ and (a_n) is a sequence of complex numbers.

The convergence of a power series depends on the sequence (a_n) and the point z.

THEOREM 2.3.2 (Abel's theorem on convergence of a power series)
For a given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, we assign an extended real number R, called the radius of convergence, satisfying the following properties:

1. $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely on $B(a, R)$.
2. If $0 < S < R$, then $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely and uniformly on $\{z : |z-a| \leq S\}$.
3. If $|z-a| > R$, then $\sum_{n=0}^{\infty} a_n(z-a)^n$ does not converge.

Proof: Define $R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} \in [-\infty, \infty]$ with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. We claim that this R has the desired properties.

1. Let $|z-a| < R$. Then, choose $r \in \mathbb{R}$ such that $|z-a| < r < R$. As $\frac{1}{R} < \frac{1}{r}$, we can write $\frac{1}{r} = \frac{1}{R} + \epsilon$, where $\epsilon = \frac{1}{r} - \frac{1}{R} > 0$. Then, by definition of $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, there exists $m \in \mathbb{N}$ such that

$$|a_n|^{\frac{1}{n}} < \frac{1}{R} + \epsilon = \frac{1}{r}, \ \forall n \geq m.$$

Hence, it follows that $|a_n| < \frac{1}{r^n}, \ \forall n \geq m$. Therefore, we have

$$|a_n(z-a)^n| \leq \left(\frac{|z-a|}{r}\right)^n, \ \forall n \geq m$$

As $\frac{|z-a|}{r} < 1$, we get that $\sum_{n=0}^{\infty} \left(\frac{|z-a|}{r}\right)^n$ converges. (See Theorem 1.3.31.) Thus, by comparison test (Theorem 1.3.30), we get that the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely.

2. If $|z-a| < S < R$, then choose $r \in \mathbb{R}$ such that $S < r < R$. Arguing as before, we get that there exists $m \in \mathbb{N}$ such that $|a_n| < \frac{1}{r^n}, \ \forall n \geq m$. Hence, it follows that

$$|a_n(z-a)^n| \leq \left(\frac{S}{r}\right)^n, \ \forall z \text{ with } |z-a| \leq S, \ \forall n \geq m.$$

Since $\frac{S}{r} < 1$, we get that the series $\sum_{n=0}^{\infty} \left(\frac{S}{r}\right)^n$ converges. Again by comparison test, we get that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely and uniformly on $|z-a| \leq S$ as the convergence of the series $\sum_{n=0}^{\infty} \left(\frac{S}{r}\right)^n$ used in the comparison test is independent of z.

3. If $|z-a| > R$, then we choose $t \in \mathbb{R}$ such that $|z-a| > t > R$. Then, $\frac{1}{t} < \frac{1}{R}$. From the definition of $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, there exists a subsequence $(|a_{n_k}|^{\frac{1}{n_k}})$ of $(|a_n|^{\frac{1}{n}})$, which converges to $\frac{1}{R}$. Then for $\epsilon = \frac{1}{R} - \frac{1}{t} > 0$, there exists $m \in \mathbb{N}$ such that

$$\left| |a_{n_k}|^{\frac{1}{n_k}} - \frac{1}{R} \right| < \frac{1}{R} - \frac{1}{t}, \ \forall k \geq m.$$

Therefore, we have $|a_{n_k}|^{\frac{1}{n_k}} > \dfrac{1}{t},\ \forall k \geq m$, which implies

$$|a_{n_k}| > \frac{1}{t^{n_k}},\ \forall k \geq m.$$

Hence, for $|z-a| > t$ and $k \geq m$, we have

$$|a_{n_k}(z-a)^{n_k}| > \left(\frac{|z-a|}{t}\right)^{n_k} > 1,\ \forall k \geq m.$$

Thus, the subsequence $(a_{n_k}(z-a)^{n_k})$ of $(a_n(z-a)^n)$ does not converge to 0. Hence, $(a_n(z-a)^n)$ itself does not converge to 0. Thus, $\sum_{n=0}^{\infty} a_n(z-a)^n$ diverges by using Result 1.3.27. □

THEOREM 2.3.3 Let the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$ be R. Then

1. the radius of convergence of $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$ is also R,

2. if $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ on $|z-a| < R$, then f is analytic on $|z-a| < R$ and $f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}$.

Proof: Recall that $\dfrac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

1. First, we prove that $n^{\frac{1}{n}} \to \infty$ as $n \to \infty$. If $r_n = n^{\frac{1}{n}} - 1$, $\forall n \in \mathbb{N}$, then we have for each $n \geq 2$,

$$n = (r_n+1)^n = 1 + nC_1 r_n + nC_2 r_n^2 + \cdots + r_n^n > \frac{n(n-1)}{2} r_n^2,$$

which implies that $r_n < \sqrt{\dfrac{2}{n-1}} \to 0$ as $n \to \infty$. Therefore, $n^{\frac{1}{n}} - 1 \to 0$ as $n \to \infty$. Hence, our claim follows. Using Theorem 1.3.19, we get $\limsup_{n\to\infty} n^{\frac{1}{n}} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = R$.

2. Put $g(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}$, $S_m(z) = \sum_{n=0}^{m} a_n(z-a)^n$, and $R_m(z) = \sum_{n=m+1}^{\infty} a_n(z-a)^n$, $\forall z \in B(a,R)$ and $\forall m \in \mathbb{N}$. Obviously, we have the following:

   (a) $f(z) = S_m(z) + R_m(z)$, $\forall m \in \mathbb{N}$ on $B(a,R)$.

   (b) $S'_m(z) \to g(z)$ as $m \to \infty$ on $B(a,R)$.

   (c) $\sum_{n=m+1}^{\infty} n|a_n|r^{n-1} \to 0$ as $m \to \infty$ for $0 \le r < R$, using Result 1.3.26.

   For a fixed $z_0 \in B(a,R)$ and any $z \in B(a,R)$, choose $r \in \mathbb{R}$ such that

$$\max\{|z-a|, |z_0-a|\} < r < R.$$

   Choose $m_1, m_2 \in \mathbb{N}$ such that

$$\left|S'_m(z_0) - g(z_0)\right| < \frac{\epsilon}{3}, \ \forall m \ge m_1 \tag{2.4}$$

   and

$$\sum_{n=m+1}^{\infty} n|a_n|r^{n-1} < \frac{\epsilon}{3}, \ \forall m \ge m_2. \tag{2.5}$$

   For $m \ge m_2$, we have

$$\begin{aligned}
\left|\frac{R_m(z) - R_m(z_0)}{z - z_0}\right| &\le \sum_{n=m+1}^{\infty} |a_n| \frac{|(z-a)^n - (z_0-a)^n|}{|z-z_0|} \\
&\le \sum_{n=m+1}^{\infty} |a_n| \sum_{k=0}^{n-1} |(z-a)^k (z_0-a)^{n-k-1}| \\
&\quad \text{(using } (A-B)^n = (A-B)\sum_{k=0}^{n-1} A^k B^{n-k}\text{)} \\
&< \sum_{n=m+1}^{\infty} n|a_n|r^{n-1} < \frac{\epsilon}{3}.
\end{aligned} \tag{2.6}$$

   From the definition of $S'_m(z_0)$, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{S_m(z) - S_m(z_0)}{z - z_0} - S'_m(z_0)\right| < \frac{\epsilon}{3} \text{ whenever } 0 < |z - z_0| < \delta. \tag{2.7}$$

For $m > \max\{m_1, m_2\}$ and for z with $0 < |z - z_0| < \delta$, we get

$$\begin{aligned}
&\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| \\
&= \left|\frac{S_m(z)-S_m(z_0)}{z-z_0} - S'_m(z_0) + \frac{R_m(z)-R_m(z_0)}{z-z_0} + S'_m(z_0) - g(z_0)\right| \\
&\leq \left|\frac{S_m(z)-S_m(z_0)}{z-z_0} - S'_m(z_0)\right| + \left|\frac{R_m(z)-R_m(z_0)}{z-z_0}\right| + \left|S'_m(z_0) - g(z_0)\right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Hence, f is differentiable and $f' = g$. □

COROLLARY 2.3.4 Let the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converge in $|z-a| < R$ for some $R > 0$. If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, $\forall z \in B(a,R)$, then $a_n = \frac{f^{(n)}(a)}{n!}$, $\forall n = 0, 1, 2, 3, \ldots$

Proof: Applying the previous theorem repeatedly, first we note that $f^{(m)}$ exists for every $m = 0, 1, 2, 3, \ldots$ and

$$\begin{aligned}
f^{(m)}(z) &= \frac{d^m}{dz^m}\left(\sum_{n=0}^{m-1} a_n(z-a)^n + \sum_{n=m}^{\infty} a_n(z-a)^n\right) \\
&= \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_n(z-a)^{n-m} \\
&= a_m m! + \sum_{n=m+1}^{\infty} n(n-1)\cdots(n-m+1)a_n(z-a)^{n-m}
\end{aligned}$$

and hence, $f^{(m)}(a) = a_m m!$. Thus, the corollary follows. □

COROLLARY 2.3.5 $\sum_{n=0}^{\infty} a_n(z-a)^n = \sum_{n=0}^{\infty} b_n(z-a)^n$, $\forall z \in B(a,R)$ iff $a_n = b_n$, $\forall n = 0, 1, 2, 3, \ldots$

Proof: If $a_n = b_n$, $\forall n = 0, 1, 2, 3, \ldots$, then obviously we get

$$\sum_{n=0}^{\infty} a_n(z-a)^n = \sum_{n=0}^{\infty} b_n(z-a)^n.$$

To prove the converse, let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, $\forall z \in B(a,R)$. If $a_n = b_n, \forall n = 0,1,2,3,\ldots$ then by using previous corollary, we get $a_n = \frac{f^{(n)}(a)}{n!} = b_n$, $\forall n = 0,1,2,3,\ldots$ □

Remark 2.3.6: If f is an analytic function on $B(a,R)$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

is called the Taylor's series of f. The corollary 2.3.5 states that if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \ \forall z \in B(a,R),$$

then the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ is same as the Taylor's series of f.

Example 2.3.7 Find the radius of convergence of the following power series:

1. $\sum_{n=0}^{\infty} n^n z^n$;
2. $\sum_{n=0}^{\infty} n^2 z^n$;
3. $\sum_{n=0}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2} z^n$;
4. $\sum_{n=0}^{\infty} (4+i3)^n z^n$.

Solution:

1. Let $a_n = n^n$, $\forall n = 0,1,2,\ldots$. As $|a_n|^{1/n} = n \to \infty$ as $n \to \infty$, the radius of convergence of this power series is 0.
2. Let $a_n = n^2$, $\forall n = 0,1,2,\ldots$. As $|a_n|^{1/n} = n^{2/n} \to \infty$ as $n \to \infty$, the radius of convergence of this power series is 0.
3. As $\lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$, the radius of convergence of this power series is e.

4. As $\lim_{n\to\infty}(|4+i3|^n)^{\frac{1}{n}} = |4+i3| = 5$, the radius of convergence of $\sum_{n=0}^{\infty}(4+i3)^n z^n$ is $\frac{1}{5}$.

Exercise 2.3.8

1. $\sum_{n=0}^{\infty}\frac{2^n}{n!}z^n$;

2. $\sum_{n=0}^{\infty}2^{\sqrt{n}}z^n$;

3. $\sum_{n=0}^{\infty}\frac{n+1}{(n+2)(n+3)}z^n$ (Hint. Find $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ which is same as $\lim_{n\to\infty}\sqrt[n]{a_n}$.);

4. $\sum_{n=0}^{\infty}\frac{1}{n^2}z^n$;

5. $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}z^n$.

Answer: 1. ∞; 2. 1; 3. 1; 4. 1; 5. 4.

THEOREM 2.3.9 (Abel's limit theorem)

Let $\sum_{n=0}^{\infty}a_n$ converge. If $f(z)=\sum_{n=0}^{\infty}a_n z^n$, then $f(z)\to f(1)$ as $z\to 1$ such that $\frac{|1-z|}{1-|z|}$ is bounded.

Proof: First we note that, as $\sum_{n=0}^{\infty}a_n$ converges at 1, the radius R of convergence of $\sum_{n=0}^{\infty}a_n$ is at least 1 by Abel's theorem. If $R>1$, then f is continuous at 1, and hence, the theorem follows. Therefore, assume that $R=1$.

Case 1: $f(1)=\sum_{n=0}^{\infty}a_n = 0$.

If $\alpha_n=\sum_{k=0}^{n}a_k$ and $S_n(z)=\sum_{k=0}^{n}a_k z^k$, $\forall z\in B(0,1)$ and $\forall n=0,1,2,\ldots,$

then for $z \in B(0,1)$ with $\dfrac{|1-z|}{1-|z|} \leq K$, for some $K > 0$, we get

$$\begin{aligned}
S_n(z) &= \sum_{k=0}^{n} a_k z^k \\
&= \alpha_0 + \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) z^k \\
&= \alpha_0 + \sum_{k=1}^{n} \alpha_k z^k - \sum_{k=1}^{n} \alpha_{k-1} z^k \\
&= \alpha_0 + \sum_{k=1}^{n-1} \alpha_k z^k + \alpha_n z^n - \alpha_0 z - \sum_{k=1}^{n-1} \alpha_k z^{k+1} \\
&= \alpha_0 (1-z) + \sum_{k=1}^{n-1} \alpha_k (z^k - z^{k+1}) + \alpha_n z^n \\
&= \alpha_0 (1-z) + \sum_{k=1}^{n-1} \alpha_k z^k (1-z) + \alpha_n z^n \\
&= (1-z) \sum_{k=0}^{n-1} \alpha_k z^k + \alpha_n z^n.
\end{aligned}$$

Allowing $n \to \infty$ on both sides, we get

$$f(z) = (1-z) \sum_{k=0}^{\infty} \alpha_k z^k + \lim_{n \to \infty} \alpha_n z^n = (1-z) \sum_{k=0}^{\infty} \alpha_k z^k,$$

as $\alpha_n \to f(1) = 0$ as $n \to \infty$ and $|z| < 1$. (cf. Proof of Example 1.3.31.)

Again using $\alpha_n \to 0$ as $n \to \infty$, given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|\alpha_n| < \dfrac{\epsilon}{2K}$, $\forall n \geq m$ and choose δ such that

$$0 < \delta < \frac{\epsilon}{2\left(1 + \sum_{k=1}^{m-1} |\alpha_n|\right)}.$$

If $0 < |z-1| < \delta$ and $\dfrac{|1-z|}{1-|z|} \le K$, then

$$\begin{aligned}
|f(z)| &\le |1-z|\left(\sum_{k=0}^{m-1}|\alpha_k z^k| + \sum_{k=m}^{\infty}|\alpha_k z^k|\right)\\
&< |1-z|\left(\sum_{k=0}^{m-1}|\alpha_k| + \frac{\epsilon}{2K}\sum_{k=m}^{\infty}|z^k|\right)\\
&< \delta\sum_{k=0}^{m-1}|\alpha_k| + \frac{\epsilon}{2K}|z^m||1-z|\sum_{k=0}^{\infty}|z^k|\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2K}\frac{|1-z|}{1-|z|}\\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, the theorem follows in this case.

Case 2: $f(1) \neq 0$

If $g(z) = f(z) - f(1) = a_0 - f(1) + \sum_{n=1}^{\infty} a_n z^n$, then $g(1) = 0$. Then, by Case 1, we obtain

$$g(z) \to g(1) \text{ as } z \to 1 \text{ with } \frac{|1-z|}{1-|z|} \text{ is bounded,}$$

and hence,

$$f(z) - f(1) \to 0 \text{ as } z \to 1 \text{ with } \frac{|1-z|}{1-|z|} \text{ is bounded.}$$

Thus, the proof is complete. □

2.4 EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Definition 2.4.1 The unique solution of the initial value problem $f' = f$ with $f(0) = 1$ is called the exponential function and is denoted by exp.

LEMMA 2.4.2 $\exp(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ for $z \in \mathbb{C}$ at which the series converges.

Proof: Let $\exp(z) = \sum_{n=0}^{\infty} a_n z^n$. As it is the solution of the above initial value problem, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_{n-1} z^{n-1} = \sum_{n=1}^{\infty} n a_n z^{n-1}, \text{ and } a_0 = f(0) = 1.$$

Hence, $a_n = \dfrac{a_{n-1}}{n}$, $n \in \{0, 1, 2, \ldots\}$, and $a_0 = f(0) = 1$. We claim that $a_n = \dfrac{1}{n!}$, $\forall n = 0, 1, 2, 3, \ldots$

We prove our claim by induction on n. For $n = 0$, we have $a_0 = 1$. Assume that the claim holds for some $n \in \mathbb{N}$. Now,

$$a_{n+1} = \frac{a_n}{n+1} = \frac{1}{(n+1)n!} = \frac{1}{(n+1)!}.$$

Thus, our claim follows. Therefore, $\exp(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$, $\forall z \in B(0, R)$, where R is the radius of convergence of $\sum_{n=0}^{\infty} \dfrac{z^n}{n!}$. □

LEMMA 2.4.3 The radius of convergence of $\exp(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ is ∞.

Proof: First, we note that

$$\lim_{n\to\infty} \frac{\dfrac{1}{(n+1)!}}{\dfrac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, applying Theorems 1.3.17 and 1.3.20, we get

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \frac{\dfrac{1}{(n+1)!}}{\dfrac{1}{n!}} \\ &\leq \liminf_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} \\ &\leq \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} \end{aligned}$$

$$\begin{aligned} &\leq \limsup_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\ &= 0 \end{aligned}$$

and hence, $\limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0$.

Therefore, the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is ∞. □

LEMMA 2.4.4 For every $z_1, z_2 \in \mathbb{C}$, $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$.

Proof: As

$$\frac{d}{dz}(\exp(z) \cdot \exp(w - z)) = \exp(z) \cdot \exp(w - z) - \exp(z) \cdot \exp(w - z) = 0,$$

we get $\exp(z) \cdot \exp(w - z)$ as a constant, say K. Putting $z = 0$ and using $\exp(0) = 1$, we get $K = \exp(w)$. Thus, we obtain

$$\exp(z) \cdot \exp(w - z) = \exp(w).$$

Replacing z by z_1 and w by $z_1 + z_2$, we get $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$. □

The above property of exponential function is called the addition theorem for exp and is the reason for denoting $\exp(z)$ by e^z.

COROLLARY 2.4.5 For every $z \in \mathbb{C}$, $\exp(z) \neq 0$, and its multiplicative inverse is $\exp(-z)$.

Proof: As for every $z \in \mathbb{C}$, $1 = \exp(0) = \exp(z - z) = \exp(z) \cdot \exp(-z)$, we get $\exp(z) \neq 0$, and its multiplicative inverse is $\exp(-z)$. □

LEMMA 2.4.6 $\exp : \mathbb{R} \to (0, \infty)$ is a bijection.

Proof: First, note that $\exp(0) = 1$, for $x > 0$, we have $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1$ and by previous corollary, for $x < 0$, $\exp(x) = \frac{1}{\exp(-x)} \in (0, 1)$, since $-x > 0 \Rightarrow \exp(-x) > 1$. Therefore, we have $\exp(x) \geq 1$ if $x \geq 0$ and $0 < \exp(x) < 1$ if $x < 0$. So, exp is a mapping from $\mathbb{R}$ into $(0, \infty)$. To prove $\exp : \mathbb{R} \to (0, \infty)$ is onto, let $y > 0$ be arbitrary.

Case (1): $y = 1$. Then, by definition, we have $0 \in \mathbb{R}$ such that $\exp(0) = 1$.

Case (2): $y > 1$. Then, by definition, we have $\exp'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} > x > 0$, and hence, $\exp(x) \to \infty$ as $x \to \infty$. Using this fact, we can choose $s > 0$ such that $\exp(s) > y > 1 = \exp(0)$. As exp is differentiable, it is continuous. Therefore, by intermediate value theorem (Theorem 1.6.30), there exists $x \in (0, s)$ such that $\exp(x) = y$.

Case (3): $y < 1$. Then, $\frac{1}{y} > 1$. Therefore, by Case (2), there exists $t > 0$ such that $\exp(t) = \frac{1}{y}$. Therefore, $\exp(-t) = (\exp(t))^{-1} = y$.

For every $x \in \mathbb{R}$, we have $\exp'(x) = \exp(x) > 0$, and hence, exp is a strictly increasing function (by Lemma 2.2.18), that is, $x < y \Rightarrow \exp(x) < \exp(y)$. Hence, exp is an one-to-one function. □

LEMMA 2.4.7 $|\exp(z)| = 1$ iff $z = iy$ for some $y \in \mathbb{R}$.

Proof: Assume that $z = iy$ for some $y \in \mathbb{R}$. From the power series representation of $\exp(z)$, it is obvious that $\overline{\exp(z)} = \exp(\bar{z})$, $\forall z \in \mathbb{C}$. Therefore,

$$|\exp(iy)|^2 = \exp(iy) \cdot \exp(\overline{iy}) = \exp(iy - iy) = \exp(0) = 1$$

using Lemma 2.4.4.

Conversely, assume that if $z = x + iy$ and $|\exp(z)| = 1$. Then, $1 = |\exp(x) \cdot \exp(iy)| = \exp(x)$ implies that $x = 0$ since exp is a bijection from $\mathbb{R}$ onto $(0, \infty)$. Thus, $z = iy$ for some real $y \in \mathbb{R}$. □

Definition 2.4.8 For every $z \in \mathbb{C}$, define

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{i2}.$$

By definition, both cos and sin are differentiable functions on $\mathbb{C}$, and the power series representations of these two functions are $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ and $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2n+1)}}{(2n+1)!}$.

LEMMA 2.4.9 For every $z \in \mathbb{C}$,

1. $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$.
2. $\exp(iz) = \cos(z) + i\sin(z)$. (Euler's formula)[1].
3. $(\cos(z) + i\sin(z))^n = \cos nz + i\sin nz$, $\forall n \in \mathbb{Z}$ (de Moivre's formula).

Proof: By direct computation, we get (1) and the Euler's formula as given below.

$$\begin{aligned}\sin(-z) &= \frac{\exp(-iz) - \exp(iz)}{i2} = -\frac{\exp(iz) - \exp(-iz)}{i2} = -\sin(z)\\ \cos(-z) &= \frac{\exp(-iz) + \exp(iz)}{2} = \cos(z)\end{aligned}$$

$$\cos(z) + i\sin(z) = \frac{\exp(iz) + \exp(-iz)}{2} + i\frac{\exp(iz) - \exp(-iz)}{i2} = \exp(iz).$$

Next, by using Lemma 2.4.4 and Euler's formula, for $n \in \mathbb{N}$, we get

$$(\cos(z) + i\sin(z))^n = (\exp(iz))^n = \exp(niz) = \cos(nz) + i\sin(nz).$$

Hence, for $n = -1$, we get

$$\begin{aligned}(\cos(z) + i\sin(z))^{-1} &= (\exp(iz))^{-1}\\ &= \exp(-iz)\\ &= \cos(-z) + i\sin(-z)\\ &= \cos(z) - i\sin(z).\end{aligned}$$

If n is a negative integer, then $-n \in \mathbb{N}$ and $n = -(-n)$. Therefore,

$$\begin{aligned}(\cos(z) + i\sin(z))^n &= (\cos(z) + i\sin(z))^{-(-n)}\\ &= (\cos(-nz) + i\sin(-nz))^{-1}\\ &= \cos(nz) + i\sin(nz).\end{aligned}$$

Thus, the de Moivre's formula is obtained. □

LEMMA 2.4.10 For every $z \in \mathbb{C}$, $\sin^2(z) + \cos^2(z) = 1$.

[1] Warning! $\cos(z) \neq \text{Re } e^{iz}$ and $\sin(z) \neq \text{Im } e^{iz}$ for $z \in \mathbb{C}$.

Proof: By direct computation, we get

$$\begin{aligned}
&\sin^2(z)+\cos^2(z)\\
&= \frac{(\exp(iz)-\exp(-iz))^2}{(i2)^2}+\frac{(\exp(iz)+\exp(-iz))^2}{4}\\
&= \frac{-(\exp(iz))^2-(\exp(-iz))^2+2+(\exp(iz))^2+(\exp(-iz))^2+2}{4}\\
&= \frac{4}{4}=1.
\end{aligned}$$

□

Example 2.4.11 Expand $\cos(7\theta)$ as a polynomial in $\cos(\theta)$.
Solution:
By de Moivre's formula, we have

$$\begin{aligned}
\cos(7\theta) &= \text{Re}\,(\cos(7\theta)+i\sin(7\theta))\\
&= \text{Re}\,(\cos(\theta)+i\sin(\theta))^7\\
&= \text{Re}\sum_{k=0}^{7} 7C_k\cos^k(\theta)(i\sin(\theta))^{7-k}\\
&= \cos^7(\theta)-21\cos^5(\theta)\sin^2(\theta)+35\cos^3(\theta)\sin^4(\theta)\\
&\quad -7\cos(\theta)\sin^6(\theta)\\
&= \cos^7(\theta)-21\cos^5(\theta)(1-\cos^2(\theta))+35\cos^3(\theta)(1-\cos^2(\theta))^2\\
&\quad -7\cos(\theta)(1-\cos^2(\theta))^3\\
&= \cos^7(\theta)-21\cos^5(\theta)(1-\cos^2(\theta))\\
&\quad +35\cos^3(\theta)(1-2\cos^2(\theta)+\cos^4(\theta))\\
&\quad -7\cos(\theta)(1-3\cos^2(\theta)+3\cos^4(\theta)-\cos^6(\theta))\\
&= 64\cos^7(\theta)-112\cos^5(\theta)+56\cos^3(\theta)-7\cos(\theta).
\end{aligned}$$

LEMMA 2.4.12 For every $z, w \in \mathbb{C}$, (1) $\sin(z+w) = \sin(z)\cos(w)+\cos(z)\sin(w)$ and (2) $\cos(z+w)=\cos(z)\cos(w)-\sin(z)\sin(w)$.

Proof: As

$$\begin{aligned}
i2\sin(z+w) &= \exp(i(z+w))-\exp(-i(z+w))\\
&= \exp(iz)\exp(iw)-\exp(-iz)\exp(-iw)\\
&= (\cos(z)+i\sin(z))(\cos(w)+i\sin(w))\\
&\quad -(\cos(z)-i\sin(z))(\cos(w)-i\sin(w))\\
&= (\cos(z)\cos(w)-\sin(z)\sin(w))+i(\cos(z)\sin(w)
\end{aligned}$$

$$
\begin{aligned}
& + \sin(z)\cos(w)) - (\cos(z)\cos(w) - \sin(z)\sin(w)) \\
& + i(\cos(z)\sin(w) + \sin(z)\cos(w)) \\
= \; & i2(\cos(z)\sin(w) + \sin(z)\cos(w)),
\end{aligned}
$$

we get $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$.
Similarly,

$$
\begin{aligned}
2\cos(z+w) &= \exp(i(z+w)) + \exp(-i(z+w)) \\
&= \exp(iz)\exp(iw) + \exp(-iz)\exp(-iw) \\
&= (\cos(z) + i\sin(z))(\cos(w) + i\sin(w)) \\
&\quad + (\cos(z) - i\sin(z))(\cos(w) - i\sin(w)) \\
&= (\cos(z)\cos(w) - \sin(z)\sin(w)) + i(\cos(z)\sin(w) \\
&\quad + \sin(z)\cos(w)) + (\cos(z)\cos(w) - \sin(z)\sin(w)) \\
&\quad - i(\cos(z)\sin(w) + \sin(z)\cos(w)) \\
&= 2(\cos(z)\cos(w) - \sin(z)\sin(w)).
\end{aligned}
$$

□

COROLLARY 2.4.13 Prove that for $z, w \in \mathbb{C}$, $\sin(z-w) = \sin(z)\cos(w) - \cos(z)\sin(w)$ and $\cos(z-w) = \cos(z)\cos(w) + \sin(z)\sin(w)$.

Proof: In the previous lemma, replacing w by $-w$ and using

$$\sin(-w) = -\sin(w),\ \cos(-w) = \cos(w),$$

we obviously get this corollary. □

LEMMA 2.4.14 (1) $\dfrac{d}{dz}\sin(z) = \cos(z)$ and (2) $\dfrac{d}{dz}\cos(z) = -\sin(z)$.

Proof: By direct calculation, we obtain

$$
\begin{aligned}
1.\quad \frac{d}{dz}\sin(z) &= \frac{d}{dz}\left(\frac{\exp(iz) - \exp(-iz)}{i2}\right) = \frac{i\exp(iz) + i\exp(-iz)}{i2} \\
&= \frac{\exp(iz) + \exp(-iz)}{2} = \cos z. \\
2.\quad \frac{d}{dz}\cos z &= \frac{d}{dz}\left(\frac{\exp(iz) + \exp(-iz)}{2}\right) = \frac{i\exp(iz) - i\exp(-iz)}{2} \\
&= -\left(\frac{\exp(iz) - \exp(-iz)}{i2}\right) = -\sin z.
\end{aligned}
$$

□

RESULT 2.4.15 There exists a least $r_0 > 0$ such that $\cos(r_0) = 0$.

Proof: For every $r > 0$, by mean-value theorem, we have $\sin(r) - \sin(0) = \cos(t)$, for some $t \in (0, r)$, which implies that $\sin(r) \leq r, \forall r > 0$ as $|\cos(t)| \leq |\exp(it)| = 1$ and $\sin(0) = \dfrac{\exp(0) - \exp(0)}{i2} = 0$. Applying $\int_0^r$ on both sides of $\sin(r) < r$, we get

$$[-\cos(r)]_0^r \leq \left[\frac{r^2}{2}\right]_0^r \Rightarrow -\cos(r) + 1 \leq \frac{r^2}{2} \Rightarrow \cos(r) \geq 1 - \frac{r^2}{2}.$$

Applying $\int_0^r$ on both sides of the resulting inequalities repeatedly, we get

$$\sin(r) \geq r - \frac{r^3}{6} \Rightarrow -\cos(r) + 1 \geq \frac{r^2}{2} - \frac{r^4}{24} \Rightarrow \cos(r) \leq 1 - \frac{r^2}{2} + \frac{r^4}{24}.$$

Therefore, $\cos(\sqrt{3}) \leq 1 - \dfrac{3}{2} + \dfrac{9}{24} < 0$, and from the definition, we obtain $\cos(0) = 1$. Therefore, by intermediate value theorem (Theorem 1.6.30), there exists $r_0 \in (0, \sqrt{3})$ such that $\cos(r_0) = 0$.
Next, we claim that if $0 < r < r_0 < \sqrt{3}$, then $\cos(r) \neq 0$. From one of the above inequalities, we get

$$\sin(r) \geq r - \frac{r^3}{6} = r\left(1 - \frac{r^2}{6}\right) > r\left(1 - \frac{(\sqrt{3})^2}{6}\right) = \frac{r}{2} > 0.$$

Therefore, $\dfrac{d}{dr}\cos(r) = -\sin(r) < 0$, and hence, cos is strictly decreasing on $(0, r_0)$, applying Lemma 2.2.18. Since $\sin(r) > 0, \forall r \in (0, r_0)$ and $\sin^2(r) + \cos^2(r) = 1$, we get sin, and it is strictly increasing on $(0, r_0)$. Therefore,

$$0 < \sin(r) < \sin(r_0) = \sqrt{1 - \cos^2(r_0)} = 1.$$

Therefore, $\cos^2(r) = 1 - \sin^2(r) > 0$, and hence, $\cos(r) \neq 0$. □

Definition 2.4.16 Define $\pi = 2r_0$, where r_0 is the least positive real number such that $\cos(r_0) = 0$.

Example 2.4.17 Prove that $\cos\left(\dfrac{\pi}{2}\right) = 0$, $\sin\left(\dfrac{\pi}{2}\right) = 1$, $\cos(\pi) = -1$, $\sin(\pi) = 0$, $\cos(2\pi) = 1$, and $\sin(2\pi) = 0$.

By definition, $\pi = 2r_0$, where r_0 is the least positive real number such that $\cos(r_0) = 0$. Therefore,

$$\cos\left(\frac{\pi}{2}\right) = \cos(r_0) = 0,\ \sin\left(\frac{\pi}{2}\right) = \pm\sqrt{1 - \cos^2\left(\frac{\pi}{2}\right)} = \pm 1,$$

but $\sin\left(\frac{\pi}{2}\right) = \sin(r_0) > 0$, we have $\sin\frac{\pi}{2} = 1$. Therefore, $\exp\left(i\frac{\pi}{2}\right) = i$, which implies that

$$\exp(i\pi) = i^2 = -1 \text{ and } \exp(i2\pi) = (-1)^2 = 1.$$

Equating real and imaginary parts in the above two equations, we can get the required identities.

Exercise 2.4.18 Prove that for every $z \in \mathbb{C}$ in the following:

1. $\sin\left(z + \frac{\pi}{2}\right) = \cos(z)$.
2. $\cos\left(z + \frac{\pi}{2}\right) = -\sin(z)$.
3. $\sin\left(z - \frac{\pi}{2}\right) = -\cos(z)$.
4. $\cos\left(z - \frac{\pi}{2}\right) = \sin(z)$.
5. $\sin(z + \pi) = -\sin(z)$.
6. $\cos(z + \pi) = -\cos(z)$.
7. $\sin(z - \pi) = -\sin(z)$.
8. $\cos(z - \pi) = -\cos(z)$.

Hint. Apply Result 2.4.12 and Corollary 2.4.13.

LEMMA 2.4.19 If $0 < t < 2\pi$, then $\exp(it) \neq 1$.

Proof: If $0 < t < 2\pi$ and $\exp(it/4) = u + iv$, then using the following:

1. $0 < \frac{t}{4} < \frac{\pi}{2}$,
2. $\exp(it/4) = \cos(t/4) + i\sin(t/4)$,
3. $\frac{\pi}{2}$ is the least positive real number such that $\cos\left(\frac{\pi}{2}\right) = 0$,
4. $\sin > 0$ on $\left(0, \frac{\pi}{2}\right)$,

we get $0 < u < 1$ and $0 < v < 1$. As $u + iv$ is not equal to any of $1, -1, i$, and $-i$, which are the solutions of $z^4 = 1$, we obtain that $\exp(it/4) = u + iv$ is not a solution of $z^4 = 1$. Therefore, $\exp(it) \neq 1$. □

RESULT 2.4.20 $\exp(z) = 1$ iff $z = 2ki\pi$ for some $k \in \mathbb{Z}$.

Proof: Let $z = 2ki\pi$ for some $k \in \mathbb{Z}$. If $k = 0$, then by definition, $\exp(2ki\pi) = \exp(0) = 1$. If k is a positive integer, then

$$\exp(2ki\pi) = [\exp(i2\pi)]^k = 1.$$

If k is a negative integer, then $k = -(-k)$ and $-k$ is a positive integer. Therefore, $\exp(2ki\pi) = [\exp(2(-k)i\pi)]^{-1} = 1$.

To prove the converse, let z be such that $\exp(z) = 1$. Then, $1 = |\exp(z)|$. Therefore, using Lemma 2.4.7, we have $z = iy$ for some $y \in \mathbb{R}$. We claim that $y = 2k\pi$ for some $k \in \mathbb{Z}$. Suppose if it is not so, then there exists a unique integer n such that $2n\pi < y < 2(n+1)\pi$. Therefore,

$$\begin{aligned} 1 &= \exp(iy) \\ &= \exp(i(y - 2n\pi + 2n\pi)) \\ &= \exp(i(y - 2n\pi)) \exp(i2n\pi) \\ &= \exp(i(y - 2n\pi)), \end{aligned}$$

which is a contradiction to Lemma 2.4.19 as $0 < y - 2n\pi < 2\pi$. Thus, $z = 2ki\pi$ for some $k \in \mathbb{Z}$. □

Now, we define periodic function, which will be studied in detail in Chapter 8.

Definition 2.4.21 Let $f : \Phi \to \mathbb{C}$ and let $a \in \Phi$. We say that 'a' is a period of f if $f(z + a) = f(z)$, $\forall z \in \Phi$, where $\Phi = \mathbb{C}$ or $\mathbb{R}$. A function is said to be a periodic function if it has a non-zero period.

Remark 2.4.22: From addition theorem of exp function and Result 2.4.20, we conclude that $i2k\pi$ are periods of exp.

Now, we prove a simple but useful lemma on the integrals of a periodic function.

LEMMA 2.4.23 If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $p > 0$ and $a \in \mathbb{R}$, then $\int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx$.

Proof: By Archimedian property, we can find a unique $m \in \mathbb{Z}$ such that $m \leq \frac{a}{p} < (m+1)$. Then, we have $(m+1)p < a + p$ and $0 \leq a - mp < p$. Hence,

$$\begin{aligned}
\int_a^{a+p} f(x)\,dx &= \int_a^{(m+1)p} f(x)\,dx + \int_{(m+1)p}^{a+p} f(x)\,dx \\
&= \int_{a-mp}^{p} f(y-mp)\,dy + \int_0^{a-mp} f(z-(m+1)p)\,dz \\
&\quad \text{(by putting } y = x - mp \text{ and } z = x - (m+1)p\text{)} \\
&= \int_{a-mp}^{p} f(y)\,dy + \int_0^{a-mp} f(z)\,dz \text{ (since } p \text{ is a period of } f\text{)} \\
&= \int_0^{p} f(y)\,dy.
\end{aligned}$$

Hence, the lemma follows. □

RESULT 2.4.24 $|z| = 1$ iff $z = \exp(iy)$ for some $y \in \mathbb{R}$.

Proof: If $z = \exp(iy)$ for some $y \in \mathbb{R}$, then using Lemma 2.4.7, we get $|z| = 1$. Conversely, assume that $|z| = 1$. As $\cos(0) = 1$ and $\cos(\frac{\pi}{2}) = 0$ (see Example 2.4.17), by intermediate value theorem (Theorem 1.6.30), for every $t \in [0, 1]$, there exists $\theta \in [0, \frac{\pi}{2}]$ such that $\cos\theta = t$. Since $|\text{Re } z| \leq |z|$, we have $|\text{Re } z| \leq 1$; similarly, we have $|\text{Im } z| \leq 1$.

Case 1: If $z = 1$, then we choose $\theta = 0$; obviously, we have $\exp(0) = 1$.
If $z = i$, then we choose $\theta = \frac{\pi}{2}$ so that $\exp\left(i\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$ as $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.
If $z = -1$, then we choose $\theta = \pi$ so that $\exp(i\pi) = \cos(\pi) + i\sin(\pi) = -1$ as $\cos(\pi) = -1$ and $\sin(\pi) = 0$.
If $z = -i$, then we choose $\theta = -\frac{\pi}{2}$ so that $\exp\left(-i\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = -i$ as $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

Case 2: $0 < \text{Re } z < 1$, $0 < \text{Im } z < 1$. Then, there exists $\theta \in \left(0, \frac{\pi}{2}\right)$ such that $\text{Re } z = \cos(\theta)$. Using $\sin(\theta) > 0$ in $\left(0, \frac{\pi}{2}\right)$, we get

$$\text{Im } z = \sqrt{1 - (\text{Re z})^2} = \sqrt{1 - \cos^2(\theta)} = \sin(\theta),$$

and hence,

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta) = \text{Re}\, z + i\text{Im}\, z = z$$

Case 3: $-1 < \text{Re}\, z < 0$, $0 < \text{Im}\, z < 1$. Then, $-\text{Re}\, z = \cos(\theta)$ for some $\theta \in \left(0, \frac{\pi}{2}\right)$. As $\sin(\theta) > 0$ in $\left(0, \frac{\pi}{2}\right)$, we get

$$\text{Im}\, z = \sqrt{1 - (\text{Re z})^2} = \sqrt{1 - \cos^2(\theta)} = \sin(\theta),$$

and hence,

$$\begin{aligned} \exp(i(\pi - \theta)) &= \exp(i\pi)\exp(-i\theta) \\ &= (\cos(\pi) + i\sin(\pi))(\cos(\theta) - i\sin(\theta)) \\ &= -\cos(\theta) + i\sin(\theta) \\ &= \text{Re}\, z + i\text{Im}\, z \\ &= z. \end{aligned}$$

Case 4: $-1 < \text{Re}\, z < 0$, $-1 < \text{Im}\, z < 0$. Then, $-\text{Re}\, z = \cos(\theta)$ for some $\theta \in \left(0, \frac{\pi}{2}\right)$. As $\sin(\theta) > 0$ in $\left(0, \frac{\pi}{2}\right)$, we get

$$-\text{Im}\, z = \sqrt{1 - (\text{Re z})^2} = \sqrt{1 - \cos^2(\theta)} = \sin(\theta),$$

and hence,

$$\begin{aligned} \exp(i(-\pi + \theta)) &= \exp(-i\pi)\exp(i\theta) \\ &= (-1)(\cos(\theta) + i\sin(\theta)) \\ &= -\cos(\theta) - i\sin(\theta) \\ &= \text{Re}\, z + i\text{Im}\, z \\ &= z. \end{aligned}$$

Case 5: $0 < \text{Re}\, z < 1$, $-1 < \text{Im}\, z < 0$. Then, $\text{Re}\, z = \cos(\theta)$ for some $\theta \in (0, \frac{\pi}{2})$. As $\sin(\theta) > 0$ in $(0, \frac{\pi}{2})$, we get

$$-\text{Im}\, z = \sqrt{1 - (\text{Re z})^2} = \sqrt{1 - \cos^2(\theta)} = \sin(\theta),$$

and hence,

$$\exp(-i\theta) = \cos(\theta) - i\sin(\theta) = \text{Re}\, z + i\text{Im}\, z = z.$$

Hence, the result follows. □

COROLLARY 2.4.25 Every non-zero complex number, z can be written as $z = |z| \exp(i\theta)$ for some $\theta \in (-\pi, \pi]$.

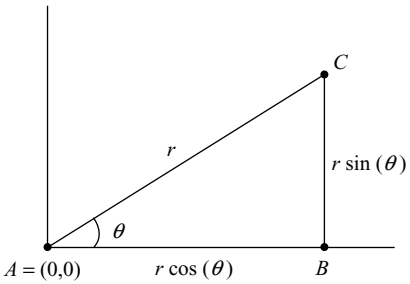
Proof: As $z \neq 0$, we write $z = |z|\dfrac{z}{|z|}$, where $\left|\dfrac{z}{|z|}\right| = 1$. Therefore, from the proof of previous result, we get $\dfrac{z}{|z|} = \exp(iy)$ for some $y \in (-\pi, \pi]$. Hence, the corollary follows. □

RESULT 2.4.26 If $\theta \in (0, \frac{\pi}{2})$, then, in a right-angled triangle with an angle θ,

1. $\sin\theta$ is equal to opposite side divided by hypotenuse.

2. $\cos\theta$ is equal to adjacent side divided by hypotenuse.

Proof: Let the given triangle be ΔABC with $\angle A = \theta$, $\angle B = \frac{\pi}{2}$, and hypotenuse $|\vec{AC}| = r$. Fix the origin as A and the real axis as the line passing through A and B.



Then by Euler's formula, we have $C = r\exp(i\theta) = (\cos(\theta) + i\sin(\theta))$. (*Note that the polar form* $r\exp(i\theta)$ *was used just as a symbol in Chapter 1 and by the previous corollary it is justified that it actually means in terms of the exponential function. Therefore, we can use the Euler's formula.*) Hence, $\overline{AB} = r\cos(\theta)$ and $|\vec{BC}| = r\sin(\theta)$. Therefore, $\sin(\theta) = \dfrac{|\vec{BC}|}{|\vec{AC}|} = \dfrac{\text{opposite side}}{\text{hypotenuse}}$ and $\cos(\theta) = \dfrac{|\vec{AB}|}{|\vec{AC}|} = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$. □

Definition 2.4.27 Define the following:

1. $\tan(z) = \dfrac{\sin(z)}{\cos(z)}, \forall z \in \mathbb{C} \setminus \{z : \cos(z) = 0\}$.
2. $\csc(z) = \dfrac{1}{\sin(z)}, \forall z \in \mathbb{C} \setminus \{z : \sin(z) = 0\}$.
3. $\sec(z) = \dfrac{1}{\cos(z)}, \forall z \in \mathbb{C} \setminus \{z : \cos(z) = 0\}$.
4. $\cot(z) = \dfrac{1}{\tan(z)} = \dfrac{\cos(z)}{\sin(z)}, \forall z \in \mathbb{C} \setminus \{z : \sin(z) = 0\}$.

Example 2.4.28 Find the domains of tan, csc, sec, and cot.

We have already proved that $\dfrac{\pi}{2}$ is the least positive real number at which cos vanishes. See Result 2.4.15. Therefore, for any $k \in \mathbb{Z}$,

$$\begin{aligned}
\cos\left(\frac{\pi}{2} + k\pi\right) &= \frac{\exp\left(i\left(\frac{\pi}{2} + k\pi\right)\right) + \exp\left(-i\left(\frac{\pi}{2} + k\pi\right)\right)}{2} \\
&= \frac{i\exp(ik\pi) - i\exp(-ik\pi)}{2} \\
&= \frac{i(-1)^k - i(-1)^{-k}}{2} \\
&= 0 \qquad (\text{since } \pm 1 = (\pm 1)^{-1}).
\end{aligned}$$

Conversely, if $z \in \mathbb{C}$ such that $\cos(z) = 0$, then $\sin(z) = \pm 1$ from $\sin^2(z) + \cos^2(z) = 1$. Hence, using Result 2.4.20, it follows that

$$\begin{aligned}
\cos(z) + i\sin(z) = \pm i \quad &\Rightarrow \quad \exp(iz) = \exp\left(i\frac{\pi}{2}\right) \text{ or } \exp\left(i\frac{3\pi}{2}\right) \\
&\Rightarrow \quad \exp\left(i\left(z - \frac{\pi}{2}\right)\right) = 1 \text{ or } \exp\left(i\left(z - 3\frac{\pi}{2}\right)\right) = 1 \\
&\Rightarrow \quad z - \frac{\pi}{2} = 2n\pi \text{ or } z - \frac{3\pi}{2} = 2n\pi,\ n \in \mathbb{Z} \\
&\Rightarrow \quad z = (4n+1)\frac{\pi}{2} \text{ or } z = (4n+3)\frac{\pi}{2},\ n \in \mathbb{Z} \\
&\Rightarrow \quad z = (2(2n)+1)\frac{\pi}{2} \text{ or} \\
&\qquad\quad z = (2(2n+1)+1)\frac{\pi}{2},\ n \in \mathbb{Z}
\end{aligned}$$

$$\Rightarrow \quad z = (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}$$
$$\Rightarrow \quad z = k\pi + \frac{\pi}{2},\ k \in \mathbb{Z}.$$

Therefore,

$$\{z \in \mathbb{C} : \cos(z) = 0\} = \left\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\right\}.$$

Similarly, we get

$$\begin{aligned}
\sin(z) = 0 &\Leftrightarrow \cos(z) = \pm 1 \\
&\Leftrightarrow \exp(iz) = \pm 1 = \exp(0) \text{ or } \exp(i\pi) \\
&\Leftrightarrow \exp(iz) = 1 \text{ or } \exp(i(z-\pi)) = 1 \\
&\Leftrightarrow z = 2n\pi \text{ or } z = 2n\pi + \pi,\ n \in \mathbb{Z} \\
&\Leftrightarrow z = k\pi,\ k \in \mathbb{Z}.
\end{aligned}$$

Therefore,

$$\{z \in \mathbb{C} : \sin(z) = 0\} = \{k\pi : k \in \mathbb{Z}\}.$$

Therefore, the domain of tan and sec is same as $\mathbb{C} \setminus \left\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\right\}$ and that of cot and csc is $\mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$.

THEOREM 2.4.29 $\lim\limits_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$, $\lim\limits_{x \to -\frac{\pi}{2}^+} \tan(x) = -\infty$, and $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a bijection.

Proof: We know that

$$\cos(0) = 1,\ \cos\left(\frac{\pi}{2}\right) = 0,\ \text{and } \cos(x) \neq 0,\ \forall x \in \left(0, \frac{\pi}{2}\right).$$

Hence, $\cos(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ because if $\cos(y) < 0$ for some $0 < y < \frac{\pi}{2}$, then by intermediate value theorem, we get cos, which vanishes at a point between 0 and y, which is a contradiction to the fact that cos $\neq 0$ on $\left(0, \frac{\pi}{2}\right)$. As $\sin' = \cos > 0$ on $\left(0, \frac{\pi}{2}\right)$, by Lemma 1.6.8, we have sin, which is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $\sin(0) = 0$, we have sin > 0 on $\left(0, \frac{\pi}{2}\right)$. Now using the continuity of these functions at $\frac{\pi}{2}$, we get $\lim\limits_{x \to \frac{\pi}{2}} \sin(x) = 1$ and $\lim\limits_{x \to \frac{\pi}{2}} \cos(x) = 0$. Therefore, for a given $M > 0$, we choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < \left|x - \frac{\pi}{2}\right| < \delta_1 \Rightarrow |\sin(x) - 1| < \frac{1}{2} \Rightarrow |\sin(x)| > 1 - \frac{1}{2} = \frac{1}{2}$$

and

$$0 < \left|x - \frac{\pi}{2}\right| < \delta_2 \Rightarrow |\cos(x)| < \frac{1}{2M}.$$

If $\delta = \min\{\delta_1, \delta_2, \frac{\pi}{2}\}$, then $\delta > 0$, and

$$\frac{\pi}{2} - \delta < x < \frac{\pi}{2} \Rightarrow \frac{|\sin(x)|}{|\cos(x)|} = \frac{\sin(x)}{\cos(x)} > \frac{1}{2\cos(x)} > M.$$

Therefore, $\lim_{z \to \frac{\pi}{2}^-} \tan(x) = +\infty$.

Similarly, using the continuity of cos and sin at $\frac{\pi}{2}$, we obtain that for a given $K < 0$, there exists $\delta_3 > 0$ such that

$$\begin{aligned} 0 < \left|x - \frac{\pi}{2}\right| < \delta_3 \quad &\Rightarrow \quad |\cos(x)| < \frac{1}{-2K} \Rightarrow 2\cos x < \frac{1}{-k} \\ &\Rightarrow \quad \frac{1}{2\cos x} > -k \Rightarrow \frac{1}{-2\cos x} < k. \end{aligned}$$

If $r = \min\{\delta_1, \delta_3, \frac{\pi}{2}\}$, then $r > 0$, and $-\frac{\pi}{2} < x < -\frac{\pi}{2} + r$ implies

$$\frac{|\sin(x)|}{|\cos(x)|} = \frac{-\sin(x)}{\cos(x)} > \frac{1}{2\cos(x)} \Rightarrow \frac{\sin(x)}{\cos(x)} < \frac{1}{-2\cos(x)} < K$$

since sin < 0 and cos > 0 on $\left(-\frac{\pi}{2}, 0\right)$. Therefore, $\lim_{z \to -\frac{\pi}{2}^+} \tan(x) = -\infty$.

We know that $\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$ and is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Given any $y > 0$, using $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$, we find $t \in \left(0, \frac{\pi}{2}\right)$ such that $\tan(t) > y$. Then by using intermediate value theorem, there exists $x \in (0, t)$ such that $\tan(x) = y$. Similarly, for a given $y < 0$, using $\lim_{x \to -\frac{\pi}{2}^+} \tan(x)$, we can find $x \in \left(-\frac{\pi}{2}, 0\right)$ such that $\tan(x) = y$. Thus, tan : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is onto. Since, $\tan'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$ and $\cos(x) \neq 0, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get that tan is strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, tan : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is one-to-one. Thus, the theorem follows. □

THEOREM 2.4.30 The inverse arctan of tan is continuous from $\mathbb{R}$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proof: Let $x \in \mathbb{R}$ be arbitrarily fixed. Since $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a bijection, there exists unique $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(t) = x$. For a given $\epsilon > 0$, we choose $\epsilon_1 > 0$ such that $0 < \epsilon_1 < \min\left\{\epsilon, \frac{\pi}{2} - t, t + \frac{\pi}{2}\right\}$. As $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is continuous and strictly increasing, we obtain the image of $(t - \epsilon_1, t + \epsilon_1)$ under tan is exactly equal to $(\tan(t - \epsilon_1), \tan(t + \epsilon_1))$ and it contains $x = \tan(t)$. Hence, we find $\delta > 0$ such that

$$(x - \delta, x + \delta) \subseteq (\tan(t - \epsilon_1), \tan(t + \epsilon_1)) = \tan((t - \epsilon_1, t + \epsilon_1))$$

which implies that

$$\arctan((x - \delta, x + \delta)) \subseteq (t - \epsilon_1, t + \epsilon_1) = (\arctan(x) - \epsilon_1, \arctan(x) + \epsilon_1)$$

In other words, we have

$$|y - x| < \delta \Rightarrow |\arctan(y) - \arctan(x)| < \epsilon_1 < \epsilon$$

As $x \in \mathbb{R}$ is arbitrary, the theorem follows. □

Exercise 2.4.31 Prove that $1 + \tan^2(z) = \sec^2(z)$, $1 + \cot^2(z) = \csc^2(z)$, and $\tan(z + w) = \dfrac{\tan(z) + \tan(w)}{1 - \tan(z)\tan(w)}$, $\forall z, w \in \mathbb{C}$.

Exercise 2.4.32 Prove that $\arctan x \to \dfrac{\pi}{2}$ as $x \to +\infty$ and $\arctan x \to -\dfrac{\pi}{2}$ as $x \to -\infty$.

Exercise 2.4.33

1. Prove that $\tan(-z) = -\tan(z)$, $\cot(z) = -\cot(z)$, $\sec(-z) = \sec(z)$, and $\csc(-z) = -\csc(z)$.
2. Prove the following identities:
   (a) $\tan\left(z - \frac{\pi}{2}\right) = -\cot(z)$
   (b) $\cot\left(z - \frac{\pi}{2}\right) = -\tan(z)$
   (c) $\tan\left(z + \frac{\pi}{2}\right) = -\cot(z)$
   (d) $\cot\left(z + \frac{\pi}{2}\right) = -\tan(z)$

(e) $\sec\left(z-\frac{\pi}{2}\right)=\csc(z)$

(f) $\csc\left(z-\frac{\pi}{2}\right)=-\sec(z)$

(g) $\sec\left(z+\frac{\pi}{2}\right)=-\csc(z)$

(h) $\csc\left(z+\frac{\pi}{2}\right)=\sec(z)$

(i) $\tan(z-\pi)=\tan(z)$

(j) $\cot(z-\pi)=\cot(z)$

(k) $\tan(z+\pi)=\tan(z)$

(l) $\cot(z+\pi)=\cot(z)$

(m) $\sec(z-\pi)=-\sec(z)$

(n) $\csc(z-\pi)=-\csc(z)$

(o) $\sec(z+\pi)=-\sec(z)$

(p) $\csc(z+\pi)=-\csc(z)$.

Hint. Use Exercise 2.4.18.

2.5 HYPERBOLIC FUNCTIONS

Definition 2.5.1 Define

$$\cosh(z)=\frac{\exp(z)+\exp(-z)}{2} \text{ and } \sinh(z)=\frac{\exp(z)-\exp(-z)}{2},\ \forall z\in\mathbb{C}.$$

LEMMA 2.5.2 For every $z\in\mathbb{C}$, we have $\cosh(z)+\sinh(z)=\exp(z)$.

Proof: For each $z\in\mathbb{C}$, we get

$$\begin{aligned}\cosh(z)+\sinh(z) &= \frac{\exp(z)+\exp(-z)}{2}+\frac{\exp(z)-\exp(-z)}{2}\\ &= \frac{2\exp(z)}{2}=\exp(z).\end{aligned}$$

Hence, the lemma follows. □

LEMMA 2.5.3 $\cosh^2(z)-\sinh^2(z)=1,\ \forall z\in\mathbb{C}$.

Proof: For $z \in \mathbb{C}$, we have

$$\begin{aligned}
&\cosh^2(z) - \sinh^2(z) \\
&= \frac{(\exp(z) + \exp(-z))^2}{4} - \frac{(\exp(z) - \exp(-z))^2}{4} \\
&= \frac{(\exp(z))^2 + (\exp(-z))^2 + 2 - (\exp(z))^2 - (\exp(-z))^2 + 2}{4} \\
&= \frac{4}{4} = 1.
\end{aligned}$$

□

LEMMA 2.5.4 sinh and cosh are differentiable and $\sinh' = \cosh$ and $\cosh' = \sinh$.

Proof: As exp is a differentiable function on $\mathbb{C}$, cosh and sinh are differentiable functions on $\mathbb{C}$ and

$$\cosh'(z) = \frac{d}{dz}\left(\frac{\exp(z) + \exp(-z)}{2}\right) = \frac{\exp(z) - \exp(-z)}{2} = \sinh(z)$$

$$\sinh'(z) = \frac{d}{dz}\left(\frac{\exp(z) - \exp(-z)}{2}\right) = \frac{\exp(z) + \exp(-z)}{2} = \cosh(z).$$

Hence, the lemma follows. □

Exercise 2.5.5 Prove that the power series representations of $\cosh(z)$ and $\sinh(z)$ are $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

Exercise 2.5.6 For every $z, w \in \mathbb{C}$, prove the following

1. $\sinh(-z) = -\sinh(z)$.
2. $\cosh(-z) = \cosh(z)$.
3. $\sinh(z + w) = \sinh(z)\cosh(w) + \sinh(z)\cosh(w)$.
4. $\cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$.
5. $\sinh(z - w) = \sinh(z)\cosh(w) - \sinh(z)\cosh(w)$.
6. $\cosh(z - w) = \cosh(z)\cosh(w) - \sinh(z)\sinh(w)$.

Definition 2.5.7 Define the following:

1. $\tanh(z) = \dfrac{\sinh(z)}{\cosh(z)}$, $\forall z \in \{z \in \mathbb{C} : \cosh(z) \neq 0\}$.

2. $\operatorname{csch}(z) = \dfrac{1}{\sinh(z)}$, $\forall z \in \{z \in \mathbb{C} : \sinh(z) \neq 0\}$.

3. $\operatorname{sech}(z) = \dfrac{1}{\cosh(z)}$, $\forall z \in \{z \in \mathbb{C} : \cosh(z) \neq 0\}$.

4. $\coth(z) = \dfrac{1}{\tanh(z)}$, $\forall z \in \{z \in \mathbb{C} : \sinh(z) \neq 0\}$.

LEMMA 2.5.8 For $z, w \in \{z \in \mathbb{C} : \cosh(z) \neq 0\}$,

$$\tanh(z+w) = \frac{\tanh(z) + \tanh(w)}{1 + \tanh(z)\tanh(w)} \text{ and } \tanh(z-w) = \frac{\tanh(z) - \tanh(w)}{1 - \tanh(z)\tanh(w)}.$$

Proof: As

$$\tanh(z+w) = \frac{\sinh(z+w)}{\cosh(z+w)} = \frac{\sinh(z)\cosh(w) + \sinh(z)\cosh(w)}{\cosh(z)\cosh(w) + \sinh(z)\sinh(w)},$$

dividing the numerator and denominator by $\cosh(z)\cosh(w)$, we get

$$\tanh(z+w) = \frac{\dfrac{\sinh(z)}{\cosh(z)} + \dfrac{\sinh(w)}{\cosh(w)}}{1 + \dfrac{\sinh(z)}{\cosh(z)}\dfrac{\sinh(w)}{\cosh(w)}} = \frac{\tanh(z) + \tanh(w)}{1 + \tanh(z)\tanh(w)}. \tag{2.8}$$

Since

$$\tanh(-w) = \frac{\sinh(-w)}{\cosh(-w)} = -\frac{\sinh(w)}{\cosh(w)} = -\tanh(w),$$

by replacing w by $-w$ in equation (2.8), we get

$$\tanh(z-w) = \frac{\tanh(z) - \tanh(w)}{1 - \tanh(z)\tanh(w)}$$

□

Exercise 2.5.9 Prove that for every $z \in \mathbb{C}$, $1 - \tanh^2(z) = \operatorname{sech}^2(z)$ and $\coth^2(z) - 1 = \operatorname{csch}^2(z)$.

Finally, we present the relation between the trigonometric functions and hyperbolic functions.

RESULT 2.5.10 For every $z \in \mathbb{C}$, $\sin(iz) = i\sinh(z)$ and $\cos(iz) = \cosh(z)$.

The proof of the above result is easy, and hence, it is left to the reader as a simple exercise.

3

Rational Functions and Multivalued Functions

3.1 POLYNOMIALS AND RATIONAL FUNCTIONS

Definition 3.1.1 A polynomial is a function $P : \mathbb{C} \to \mathbb{C}$, which is of the form $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $\forall z \in \mathbb{C}$, where $a_i \in \mathbb{C}$, $\forall i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$.

1. If n is the largest positive integer such that $a_n \neq 0$, then we call n the degree of the polynomial P. We define the degree of a constant polynomial by 0. We denote the degree of P by deg P.
2. If $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$, then we call z_0 a zero of P.

Now we state (without proof) a particular version of the Euclidean algorithm, which is an useful result from algebra.

RESULT 3.1.2 (Division algorithm for polynomials)
Let P and Q be polynomials with complex coefficients. Then there exist unique polynomials A and B such that $P = AQ + B$ with either $B = 0$ or deg $B <$ deg Q.

LEMMA 3.1.3 If z_0 is a zero of P, then we can write $P(z) = (z - z_0)Q(z)$, $\forall z \in \mathbb{C}$ for some polynomial Q.

Proof: By division algorithm, we can find polynomials $Q(z)$ and $R(z)$ such that

$$P(z) = (z - z_0)Q(z) + R(z),\ \forall z \in \mathbb{C}$$

with $0 < \deg R(z) < \deg (z - z_0)$ or $R(z) = 0,\ \forall z \in \mathbb{C}$. As $\deg (z - z_0)$ is 1, $R(z)$ must be a constant and is equal to

$$R(z_0) = P(z_0) - (z_0 - z_0)Q(z_0) = 0.$$

Therefore, the lemma follows. □

Definition 3.1.4 If $m \in \mathbb{N}$ is such that

$$P(z) = (z - z_0)^m Q(z),\ \forall z \in \mathbb{C}$$

for some polynomial Q such that $Q(z_0) \neq 0$, then m is called the order of the zero z_0 of P.

RESULT 3.1.5 If m is the order of a zero of a non-constant polynomial P, then $P^{(k)}(z_0) = 0, \forall k \in \{1, 2, 3, \ldots, m - 1\}$ and $P^{(m)}(z_0) \neq 0$.

Proof: Let z_0 be a zero of order m, then by definition, we have

$$P(z) = (z - z_0)^m Q(z),\ \forall z \in \mathbb{C}$$

for some polynomial Q. Using the Leibniz rule(Result 2.1.13), we have

$$\frac{d^k}{dz^k}\left((z - z_0)^m Q(z)\right) = \sum_{j=0}^{k} {}_kC_j \frac{d^j}{dz^j}(z - z_0)^m \cdot \frac{d^{k-j}}{dz^{k-j}}Q(z),\ \forall k \in \mathbb{N}.$$

For $1 \leq k \leq m - 1$ and $0 \leq j \leq k$, we have

$$\frac{d^j}{dz^j}(z - z_0)^m = m(m-1)\cdots(m - j + 1)(z - z_0)^{m-j} \text{ and } m - j > 0$$

and hence, we get $P^{(k)}(z_0) = 0,\ 1 \leq k \leq m - 1$. Furthermore, using the same observation, we obtain

$$P^{(m)}(z_0) = \left[\frac{d^m}{dz^m}(z - z_0)^m\right] Q(z) = m!Q(z_0) \neq 0.$$

Thus, the result follows. □

Converse of the above result is also true and is given by the following result.

RESULT 3.1.6 Let P be a non-constant polynomial and $z_0 \in \mathbb{C}$. If $m \in \mathbb{N}$ is such that $P^{(k)}(z_0) = 0$, $\forall k \in \{1, 2, 3 \ldots, m-1\}$ and $P^{(m)}(z_0) \neq 0$, then the order of zero of P at z_0 is m.

Proof: If $m = 1$, then using Lemma 3.1.3, we have

$$P(z) = (z - z_0)P_1(z), \ \forall z \in \mathbb{C}$$

for some polynomial P_1. Using

$$P'(z) = (z - z_0)P_1'(z) + P_1(z), \ \forall z \in \mathbb{C},$$

we get $0 \neq P'(z_0) = P_1(z_0)$.
If $m = 2$, using $P(z_0) = 0$, then we write

$$P(z) = (z - z_0)P_1(z), \ \forall z \in \mathbb{C}$$

and we get $0 = P'(z_0) = P_1(z_0)$. Therefore,

$$P_1(z) = (z - z_0)P_2(z), \ \forall z \in \mathbb{C}$$

for some polynomial P_2, and hence,

$$P(z) = (z - z_0)^2 P_2(z), \ \forall z \in \mathbb{C}.$$

Using

$$P''(z) = (z - z_0)^2 P_2''(z) + 4(z - z_0)P_2'(z) + 2P_2(z), \ \forall z \in \mathbb{C},$$

we get $0 \neq P''(z_0) = P_2(z_0)$.
Proceeding further at the $(m-1)^{\text{st}}$ stage using $P^{(m-1)}(z_0) = 0$, we get

$$P(z) = (z - z_0)^m P_m(z), \ \forall z \in \mathbb{C}$$

for some polynomial P_m. Therefore, for each $z \in \mathbb{C}$, we have

$$\begin{aligned} P^{(m)}(z) &= \sum_{j=0}^{m} {}^mC_j \frac{d^j}{dz^j} \left[(z - z_0)^m\right] P_m^{(m-j)}(z) \\ &= \sum_{j=0}^{m} {}^mC_j \left[\prod_{\nu=0}^{j-1} (m - \nu)\right] (z - z_0)^{m-j} P_m^{(m-j)}(z). \end{aligned}$$

Thus, $0 \neq P^{(m)}(z_0) = m! P_m(z_0) \Rightarrow P_m(z_0) \neq 0$. Therefore, m is the order of zero at z_0 for P. □

Remark 3.1.7: Another simple proof of the previous result is given in Corollary 6.1.4.

Next we discuss a theorem that states a relation between the zeroes of a polynomial P with the zeroes of its derivative. For that purpose, now we discuss how to represent a half plane described by a straight line. We know that the equation of a straight line, which has the slope as the argument of a non-zero complex number b and it passes through the complex number a, is given by $z = a + tb$, $t \in \mathbb{R}$. Hence, we have

$$\begin{aligned} z \text{ lies on the straight line} \quad &\text{iff} \quad \frac{z-a}{b} \text{ is a real number} \\ &\text{iff} \quad \operatorname{Im} \frac{z-a}{b} = 0. \end{aligned}$$

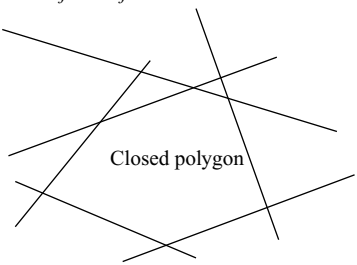
Therefore, z is not on the straight line iff $\operatorname{Im} \dfrac{z-a}{b} \neq 0$. Hence,

$$\left\{ z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} > 0 \right\} \text{ and } \left\{ z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} < 0 \right\}$$

are the two half planes determined by the straight line $z = a + tb$, $t \in \mathbb{R}$.

THEOREM 3.1.8 (Lucas' theorem)
If every zero of a polynomial P is enclosed by a closed polygon K in $\mathbb{C}$, then every zero of P' is also enclosed by K.

Proof: Let the closed polygon K is formed by the straight lines $z = a_j + tb_j,\ t \in \mathbb{R}, j = 1, 2, \ldots, m$. That is, the region enclosed by the given polygon is the intersection of m half planes each of which is determined by one of these m straight lines $z = a_j + tb_j,\ t \in \mathbb{R}$.



Therefore, so to conclude this theorem, we shall show that every zero of P lies in a half plane described by $z = a_j + tb_j,\ t \in \mathbb{R}$ and every zero of P' also lies in the same half plane for all $j = 1, 2, \ldots, m$. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

are the zeroes of P including multiplicities. Then, using Lemma 3.1.3 and fundamental theorem of algebra (Theorem 4.3.12), we get

$$P(z) = c(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n),\ \forall z \in \mathbb{C} \text{ and for some } c \in \mathbb{C}.$$

Therefore, either by direct calculation or by finding $\dfrac{d}{dz}\log P(z)$ (with less rigor), we get

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \cdots + \frac{1}{z-\alpha_n},\ \forall z \in \mathbb{C}. \tag{3.1}$$

Let $1 \le j \le m$ and let $H_j = \left\{ z \in \mathbb{C} : \operatorname{Im} \dfrac{z-a_j}{b_j} > 0 \right\}$ be a half plane described by the straight line $z = a_j + tb_j,\ t \in \mathbb{R}$. Now assume that $\alpha_k \in H_j$, $\forall k \in \{1, 2, \ldots, n\}$. Therefore, we have

$$\operatorname{Im} \frac{\alpha_k - a_j}{b_j} > 0,\ \forall 1 \le k \le n.$$

If $z_0 \notin H_j$, then $\operatorname{Im} \dfrac{z_0 - a_j}{b_j} \le 0$, and hence,

$$\operatorname{Im} \frac{z_0 - \alpha_k}{b_j} = \operatorname{Im} \frac{z_0 - a_j}{b_j} - \operatorname{Im} \frac{\alpha_k - a_j}{b_j} < 0.$$

As $(\operatorname{Im} w) \times (\operatorname{Im} w^{-1}) < 0$, for any complex number w with $\operatorname{Im} w \neq 0$, we have $\operatorname{Im} \dfrac{b_j}{z_0 - \alpha_k} > 0$. Therefore,

$$\operatorname{Im} \frac{b_j P'(z)}{P(z)} = \sum_{k=1}^{n} \operatorname{Im} \frac{b_j}{z_0 - b_k} > 0 \Rightarrow b_j P'(z_0) \neq 0 \Rightarrow P'(z_0) \neq 0.$$

Thus, if $P'(z_0) = 0$, then $z_0 \in H_j$. Therefore, every zero of P' lies in H_j. As j is arbitrary, every zero of P' lies in the closed polygon K. □

Definition 3.1.9 A function $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called a rational function if it is a quotient of two polynomials.

Usually, we write a rational function in the reduced form in the sense that if $R = \dfrac{P}{Q}$, then P and Q are polynomials without common zeroes.

LEMMA 3.1.10 If R is any rational function, then $\lim_{z\to\infty} R(z)$ exists in $\mathbb{C}_\infty$.

Proof: Let $R = \dfrac{P}{Q}$, where

$$P(z) = \sum_{k=0}^{m} a_k z^k \text{ and } Q(z) = \sum_{j=0}^{n} b_j z^j,\ \forall z \in \mathbb{C}$$

with $a_m \neq 0$ and $b_n \neq 0$. We also assume that P and Q have no common zeroes. Then, we have

$$\begin{aligned}
\lim_{z\to\infty} R(z) &= \lim_{z\to\infty} \frac{\sum_{k=0}^{m} a_k z^k}{\sum_{j=0}^{n} b_j z^j} \\
&= \lim_{z\to\infty} z^{m-n} \frac{\sum_{k=0}^{m} a_k z^{k-m}}{\sum_{j=0}^{n} b_j z^{j-n}} \\
&= \lim_{z\to\infty} z^{m-n} \frac{a_m}{b_n} \\
&= \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n > m \\ \infty & \text{if } n < m. \end{cases}
\end{aligned}$$

Definition 3.1.11 The value of a rational function R at ∞ is defined by $\lim_{z\to\infty} R(z)$.

Definition 3.1.12 Let R be a rational function and $z_0 \in \mathbb{C}_\infty$, then

1. z_0 is said to be a zero of R if $R(z_0) = 0$.
2. z_0 is said to be a pole of R if $R(z_0) = \infty$.

Note that if $R = \frac{P}{Q}$ is written in the reduced form, then every zero of P is a finite zero of R and every zero of Q is a finite pole of R. Furthermore, ∞ is a zero of R if $\deg P < \deg Q$ and is a pole if $\deg P > \deg Q$.

Definition 3.1.13 Let $R = \dfrac{P}{Q}$ be a rational function, $z_0 \in \mathbb{C}_\infty$ be a zero of R, and $k \in \mathbb{N}$.

1. If $z_0 \in \mathbb{C}$, then we say that R has a zero of order k at z_0 if z_0 is a zero of order k for the polynomial P; and we say that R has a pole of order k at z_0 if z_0 is a zero of order k for the polynomial Q.
2. We say that R has a zero of order k at ∞ if 0 is a zero of order k for the rational function R_1; and we say that R has a pole of order k at ∞ if 0 is a zero of order k for the rational function R_1, where $R_1(z) = R\left(\frac{1}{z}\right)$.

Note that if $R = \dfrac{P}{Q}$, then R has a zero of order k at ∞ if $\deg P = k + \deg Q$ and has a pole of order k at ∞ if $\deg Q = k + \deg P$.

Remark 3.1.14: Let z_j, $j = 1, 2, \ldots, n$ be the distinct zeroes (poles) of a rational function R. If k_j is the order of the zero (pole) at z_j, $\forall j = 1, 2, \ldots, n$, then we mean the number of zeroes (poles) of R including multiplicities by $\sum_{j=1}^{n} k_j$.

RESULT 3.1.15 Let $R = \frac{P}{Q}$ be in the reduced form, where P and Q are polynomials of degree m and n, respectively. If μ is the number of zeroes of R including multiplicities in $\mathbb{C}_\infty$ and ν is the number of poles of R including multiplicities in $\mathbb{C}_\infty$, then $\mu = \nu = \max\{m, n\}$.

Proof: Obviously, there are m finite zeroes of R including multiplicities and n finite poles of R including multiplicities.

Case 1: $m = n$. In this case, ∞ is neither a pole nor a zero for R. Therefore, $\mu = m = n = \nu$.

Case 2: $m > n$. In this case, it is easy to verify that ∞ is a pole for R of order $m - n$. Therefore, $\mu = m$ and $\nu = n + m - n = m$.

Case 3: $m < n$. In this case, it is easy to verify that ∞ is a zero for R of order $n - m$. Therefore, $\mu = m + n - m = n$ and $\nu = n$.

From the above discussions, we get $\mu = \nu = \max\{m, n\}$. □

LEMMA 3.1.16 Let R be a rational function. R is a polynomial iff the only possible pole of R is ∞.

Proof: Let R be a polynomial. If R is a constant polynomial, then it has no poles. If R is a non-constant polynomial, then obviously $R(\infty) = \infty$, and

hence, ∞ is a pole of R. For every $c \in \mathbb{C}$, as $\lim_{z \to c} R(z) = R(c) \neq \infty$, it has no other poles. Conversely, assume that R has only one pole at ∞. If $R = \dfrac{P}{Q}$, where P and Q are polynomials without common zeroes. If Q has a zero z_0 in $\mathbb{C}$, then z_0 becomes a pole for R, which is a contradiction. Therefore, Q must not have a zero in $\mathbb{C}$, and hence, by the fundamental theorem of algebra (Theorem 4.3.12), we get that Q is a constant. Thus, R is a polynomial. □

Definition 3.1.17 Let $z_1, z_2, \ldots, z_n$ be the distinct finite poles of a rational function R, then the expression

$$R(z) = G(z) + \sum_{j=1}^{n} F_j\left(\frac{1}{z - z_j}\right), \ \forall z \in \mathbb{C}$$

is called the partial fraction expansion of R if G is a polynomial and F_j are polynomials with $F_j(0) = 0$.

THEOREM 3.1.18 Every rational function has partial fraction expansion.

Proof: Let $R = \dfrac{P}{Q}$ be in reduced form. Hence, by division algorithm for polynomials (Result 3.1.2), we can write $P = AQ + B$, where A and B are polynomials such that either $B = 0$ or $\deg B < \deg Q$. Therefore, $R = A + \dfrac{B}{Q}$. Let $C = A - A(0)$ and $D = A(0) + \dfrac{B}{Q}$. Hence, we can write $R = C + D$, where C is a polynomial with $C(0) = 0$ and D is a rational function such that $D(\infty) \neq \infty$, as $D = \dfrac{A(0)Q + B}{Q}$ and degree of $(A(0)Q + B) \leq$ degree of Q.

We note that if $R(\infty)$ is finite, then $C = 0$, and if $R(\infty) = \infty$, then degree of $P >$ degree of $Q \Rightarrow C$ is a non-constant polynomial. ..... (I)

For every $j = 1, 2, \ldots, n$, let $R_j(w) = R\left(z_j + \frac{1}{w}\right), \ \forall w \in \mathbb{C}_\infty$, where $z_1, z_2, \ldots, z_n$ are distinct finite poles of R. Note that $R_j(\infty) = \infty$; then by previous argument, there exist non-constant polynomial C_j with $C_j(0) = 0$ and rational function D_j with $D_j(\infty) \neq \infty$ such that

$$R_j(w) = C_j(w) + D_j(w), \ \forall w \in \mathbb{C}_\infty, \ \forall j = 1, 2, \ldots, n.$$

Applying the change of variable $w = \dfrac{1}{z - z_j}$, we get

$$\begin{aligned} R(z) &= R\left(z_j + \tfrac{1}{w}\right) \\ &= R_j(w) \\ &= C_j(w) + D_j(w) \\ &= C_j\left(\frac{1}{z - z_j}\right) + D_j\left(\frac{1}{z - z_j}\right) \\ &= E_j(z) + F_j(z), \text{(say)}\ \forall z \in \mathbb{C}_\infty \end{aligned}$$

and $F_j(z_j) \neq \infty, j = 1, 2, \ldots, n$. Now the rational function

$$R - C - \sum_{j=1}^{n} E_j \tag{3.2}$$

has possible poles from $\{z_1, z_2, \ldots, z_n, \infty\}$.
Note that

1. $R(z_j) = \infty$,
2. from $R_j(\infty) = \infty$, by the observation (I), we get C_j is a non-constant polynomial, and hence, $C_j(\infty) = \infty \Rightarrow E_j(z_j) = C_j(\infty) = \infty$.

Although $R(z_j) = \infty$ and $E_j(z_j) = \infty$, as $R(z_j) - E_j(z_j) = F_j(z_j) \neq \infty$, it follows that z_j is not a pole for the rational function in equation(3.2). Next from the following statements:

1. If $R(\infty) = \infty$, then using (I), we get $C(\infty) = \infty$, but as
$$(R - C)(\infty) = D(\infty) \neq \infty,$$
we get that ∞ is not a pole for R.
2. If $R(\infty) \neq \infty$, then using (I), we get $C = 0$, and hence, $(R - C)(\infty) \neq \infty$.

we conclude that ∞ is not a pole for equation(3.2).

Hence, $R - C - \sum_{j=1}^{n} E_j$ has no poles in $\mathbb{C}_\infty$. Therefore, by Result 3.1.15, we get $R - C - \sum_{j=1}^{n} E_j$ is a constant say α. Hence,

$$R = (C + \alpha) + \sum_{j=1}^{n} E_j \Rightarrow R(z) = G(z) + \sum_{j=1}^{n} F_j\left(\frac{1}{z - z_j}\right),\ \forall z \in \mathbb{C}$$

where $G = C + \alpha$. □

3.2 LINEAR FRACTIONAL TRANSFORMS

Definition 3.2.1 A linear fractional transform (or Möbius transfrom) T is of the form $T(z) = \dfrac{az+b}{cz+d}$, where $a, b, c,$ and $d \in \mathbb{C}$ such that $ad - bc \neq 0$.

The condition $ad - bc \neq 0$ states that $az + b$ and $cz + d$ do not have a common zero. For if $ad - bc = 0$, then $\dfrac{-b}{a} = \dfrac{-d}{c}$. However, as $\dfrac{-b}{a}$ is the zero of $az + b$ and $\dfrac{-d}{c}$ is the zero of $cz + d$, it would follow that $az + b$ and $cz + d$ have a common zero.

Definition 3.2.2 (Elementary linear fractional transforms)

1. Translation: $T(z) = z + a$, $\forall z \in \mathbb{C}_\infty$ for some $a \in \mathbb{C}$.
2. Rotation: $T(z) = e^{i\theta}z$, $\forall z \in \mathbb{C}_\infty$ for some $\theta \in [0, 2\pi)$.
3. Scaling: $T(z) = rz$, $\forall z \in \mathbb{C}_\infty$ for some $r > 0$.
   (a) If $r > 1$, then this transform is called magnification.
   (b) If $r < 1$, then this transform is called contraction.
4. Inversion: $T(z) = \dfrac{1}{z}$, $\forall z \in \mathbb{C}_\infty$.

LEMMA 3.2.3 Every linear fractional transform can be written as a suitable composition of these elementary linear fractional transforms.

Proof: Let $T(z) = \dfrac{az+b}{cz+d}$ be given.

Case 1: If $c = 0$, then $T(z) = \dfrac{a}{d}z + \dfrac{b}{d}$. Here, we note that $a \neq 0$ and $d \neq 0$ from $ad - bc \neq 0$. Let $\dfrac{a}{d} = re^{i\theta}$. Therefore, $T = T_1 \circ T_2 \circ T_3$, where $T_3(z) = rz$, $T_2(z) = e^{i\theta}z$, and $T_1(z) = z + \dfrac{b}{d}$.

Case 2: If $c \neq 0$, then

$$T(z) = \frac{\dfrac{a}{c}z + \dfrac{b}{c}}{z + \dfrac{c}{d}}$$

$$= \frac{\frac{a}{c}\left(z+\frac{d}{c}\right)+\frac{b}{c}-\frac{ad}{c^2}}{z+\frac{d}{c}}$$

$$= \frac{a}{c}+\left(\frac{bc-ad}{c^2}\right)\frac{1}{z+\frac{d}{c}}.$$

As $\frac{bc-ad}{c^2} \neq 0$, we can write $\frac{bc-ad}{c^2} = \rho e^{i\phi}$. Therefore, $T = T_1 \circ T_2 \circ T_3 \circ T_4 \circ T_5$, where $T_5(z) = z + \frac{d}{c}$, $T_4(z) = \frac{1}{z}$, $T_3(z) = e^{i\phi}z$, $T_2(z) = \rho z$, and $T_1(z) = z + \frac{a}{c}$. □

THEOREM 3.2.4 Every linear fractional transform maps circles and lines to circles and lines.

Proof: To prove this theorem, we shall first show that the elementary linear fractional transforms map circles and lines to circles and lines.

1. The translation operator $w = z + c$ maps the line $z = a + tb$ into the line $w = (a + c) + tb$, and it maps the circle $z = a + re^{it}$ into the circle $w = (a + c) + re^{it}$. Geometrically,

   (a) the line passing through a with slope m is mapped into the line passing through $a + c$ with slope m.

   (b) the circle with centre a and radius r is mapped to the circle with centre $a + c$ and radius r.

2. The rotation operator $w = e^{i\theta}z$ maps the line $z = a + tb$ into the line $w = (e^{i\theta}a) + t(e^{i\theta}b)$, and it maps the circle $z = a + re^{it}$ into the circle $z = (e^{i\theta}a) + re^{it}$. Geometrically,

   (a) the line passing through a with slope m is mapped to the line passing through $e^{i\theta}a$ with slope $m + \theta$.

   (b) the circle with centre a and radius r is mapped to the circle with centre $e^{i\theta}a$ and radius r.

3. The scaling operator ρz maps the line $z = a + tb$ into the line $z = (\rho a) + tb$, and it maps the circle $z = a + re^{it}$ into the circle $z = (\rho a) + r\rho e^{it}$. Geometrically,

(a) the line passing through a with slope m is mapped to the line passing through ρa with slope m.

(b) the circle with centre a and radius r is mapped to the circle with centre ρa and radius $r\rho$.

4. The combined equation of a line and circle (with non-zero centre) is $az\bar{z} + b\bar{z} + \bar{b}z + c = 0$, whose image under the inversion operator $w = \dfrac{1}{z}$ is $cw\bar{w} + bw + \bar{b}\bar{w} + a = 0$, which is again a combined equation of a line and circle (with non-zero centre). As $az\bar{z} + b\bar{z} + \bar{b}z + c = 0$ represents a circle (with non-zero centre) if $a \neq 0$ and represents a line if $a = 0$ and it passes through c, geometrically we have the following statements:

   (a) a line passing through 0 is mapped onto a line passing through 0.

   (b) a line not passing through 0 is mapped onto a circle (with non-zero centre) passing through 0.

   (c) a circle (with non-zero centre) passing through 0 is mapped onto a line not passing through 0.

   (d) a circle (with non-zero centre) not passing through 0 is mapped onto a circle (with non-zero centre) not passing through 0.

   If we consider a circle with centre zero radius $r > 0$, then it is mapped to a circle with centre zero and radius $\dfrac{1}{r}$, as

$$z = r\exp(i\theta),\ \theta \in [-\pi, \pi] \Rightarrow w = \frac{1}{r}\exp(-i\theta),\ \theta \in [-\pi, \pi].$$

As every linear fractional transform is a composition of these elementary linear fractional transforms, the theorem follows. □

RESULT 3.2.5 Let T and S be linear fractional transforms, then

1. $T \circ S$ is a linear fractional transform.

2. T^{-1} exists, and T^{-1} is also a linear fractional transform.

Proof: Let $T(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ and $S(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$, where $a_i, b_i, c_i,$ and $d_i \in \mathbb{C}$ with $a_i d_i - b_i c_i \neq 0$, for $i = 1, 2$. As

$$(T \circ S)(z) = \frac{a_1\left(\dfrac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\dfrac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + c_2 d_1)z + (c_1 b_2 + d_1 d_2)},$$

which is of the form $\dfrac{Az + B}{Cz + D}$, and

$$\begin{aligned} AD - BC &= (a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + b_1 d_2)(c_1 a_2 + c_2 d_1) \\ &= a_1 a_2 c_1 b_2 + a_1 a_2 d_1 d_2 + b_1 c_2 c_1 b_2 + b_1 c_2 d_1 d_2 \\ &\quad - a_1 b_2 c_1 a_2 - a_1 b_2 c_2 d_1 - b_1 d_2 c_1 a_2 - b_1 d_2 c_2 d_1 \\ &= a_1 a_2 d_1 d_2 + b_1 c_2 c_1 b_2 - a_1 b_2 c_2 d_1 - b_1 d_2 c_1 a_2 \\ &= a_1 d_1 (a_2 d_2 - b_2 c_2) + b_1 c_1 (b_2 c_2 - a_2 d_2) \\ &= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0. \end{aligned}$$

Therefore, $T \circ S$ is a linear fractional transform.
The equation $w = \dfrac{az + b}{cz + d}$ implies that

$$w(cz + d) = az + b \Rightarrow z(cw - a) = -dw + b \Rightarrow z = \frac{dw - b}{-cw + a}$$

which is denoted by $T^{-1}(w)$. Clearly, T^{-1} is a linear fractional transform as $da - (-b)(-c) \neq 0$. □

From the previous result, we get the collection of linear fractional transforms, which becomes a group with respect to composition '$\circ$' of functions, as the other axioms of group are obvious. It is not an abelian group because if $T(z) = 2z$ and $S(z) = \dfrac{1}{z}$, $\forall z \in \mathbb{C}_\infty$, then $T \circ S \neq S \circ T$.

THEOREM 3.2.6 If T is a linear fractional transform such that T is not identity, then T has at most two fixed points in $\mathbb{C}$.

Proof: Let $T(z) = \dfrac{az + b}{cz + d}$, $\forall z \in \mathbb{C}_\infty$ with $ad - bc \neq 0$. The fixed points of T are the solutions of $z = \dfrac{az + b}{cz + d} \Leftrightarrow cz^2 + (d - a)z - b = 0$.
If $c \neq 0$, then the equation has at most two solutions.

If $c = 0$ and $d - a \neq 0$, then from $ad - bc \neq 0$, we get $a \neq 0$ and $d \neq 0$. Hence,

$$T(z) - z = \left(\frac{a}{d} - 1\right) z + \frac{b}{d} \text{ with } 0 \neq \frac{a}{d} \neq 1.$$

Therefore, $z = \left(1 - \frac{a}{d}\right)^{-1} \frac{b}{d}$ is the only fixed point of T in $\mathbb{C}$.
If $c = 0$, $d - a = 0$, and $b \neq 0$, then $d \neq 0$, and hence, $T(z) = z + \frac{b}{d}$ with $\frac{b}{d} \neq 0$, which has no fixed point.
The case $c = 0$, $d - a = 0$, and $b = 0$ is not possible as T is not the identity transformation. Thus, the theorem follows. □

Example 3.2.7 If $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a bijection such that T maps lines and circles to lines and circles, then T need not be a linear fractional transform.
If $T(z) = \bar{z}$, $\forall z \in \mathbb{C}_\infty$, then T is a bijection from $\mathbb{C}_\infty$ onto itself and T maps the line (circle) $a|z|^2 + b\bar{z} + \overline{b}z + c = 0$ onto the line (circle) $a|z|^2 + bz + \overline{bz} + c = 0$. By previous theorem, T is not a linear fractional transform, as it has every real number as a fixed point.

THEOREM 3.2.8 Every linear fractional transform $w = T(z)$, which has only two distinct fixed points $\alpha, \beta \in \mathbb{C}$, satisfies the equation $\dfrac{w - \alpha}{w - \beta} = \gamma \dfrac{z - \alpha}{z - \beta}$ for some $\gamma \in \mathbb{C}$.

Proof: Let $w = T(z) = \dfrac{az + b}{cz + d}$, $\forall z \in \mathbb{C}_\infty$. As α and β are the fixed points of T, we have $\alpha = \dfrac{a\alpha + b}{c\alpha + d}$ and $\beta = \dfrac{a\beta + b}{c\beta + d}$. Therefore,

$$\begin{aligned} w - \alpha &= \frac{az + b}{cz + d} - \frac{a\alpha + b}{c\alpha + d} = \frac{(ad - bc)(z - \alpha)}{(cz + d)(c\alpha + d)} \\ w - \beta &= \frac{az + b}{cz + d} - \frac{a\beta + b}{c\beta + d} = \frac{(ad - bc)(z - \beta)}{(cz + d)(c\beta + d)} \end{aligned}$$

Therefore,

$$\frac{w - \alpha}{w - \beta} = \gamma \frac{z - \alpha}{z - \beta}$$

where $\gamma = \dfrac{c\beta + d}{c\alpha + d}$. □

THEOREM 3.2.9 Every linear fractional transform $w = T(z)$, which has only one fixed point $\alpha \in \mathbb{C}$, satisfies the equation $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + \gamma$ for some $\gamma \in \mathbb{C}$.

Proof: As in the proof of previous theorem, if $w = T(z) = \dfrac{az+b}{cz+d}, \forall z \in \mathbb{C}_\infty$, then we have

$$\alpha = \frac{a\alpha + b}{c\alpha + d} \Rightarrow \alpha(c\alpha - a) = b - d\alpha.$$

Since,

$$\begin{aligned} w - \alpha &= \frac{az+b}{cz+d} - \alpha \\ &= \frac{az + b - \alpha cz - \alpha d}{cz+d} \\ &= \frac{(a - c\alpha)z + \alpha(c\alpha - a)}{cz+d} \\ &= \frac{(a-c\alpha)(z-\alpha)}{cz+d} \end{aligned}$$

we have

$$\frac{1}{w-\alpha} - \frac{1}{z-\alpha} = \frac{1}{z-\alpha}\left(\frac{cz+d}{a-c\alpha} - 1\right) = \frac{1}{z-\alpha}\frac{cz+d-a+c\alpha}{a-c\alpha}.$$

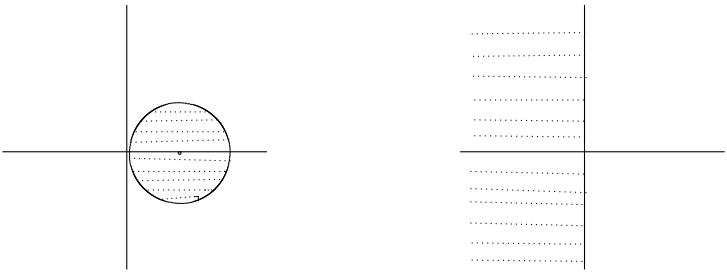
As, α is the double root of the quadratic equation, $cz^2 + (d-a)z - b = 0$, we have $2\alpha = \dfrac{a-d}{c} \Rightarrow a - d = 2\alpha c$. Therefore,

$$\begin{aligned} \frac{1}{w-\alpha} - \frac{1}{z-\alpha} &= \frac{1}{z-\alpha}\frac{cz - 2\alpha c + c\alpha}{a - c\alpha} \\ &= \frac{1}{z-\alpha}\frac{c(z-\alpha)}{a-c\alpha} \\ &= \frac{c}{a-c\alpha} \\ &= \gamma \text{ (say)} \end{aligned}$$

Hence, $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + \gamma$. □

PROBLEM 3.2.10 Find the image of $|z-2| < 2$ under the linear fractional transform $T(z) = \frac{z}{2z-8}$.

Solution: As the point 4 lies on the circle $|z-2|=2$ and $T(4)=\infty$, the image of the circle is a straight line. We see that 0 and $2+2i$ lie on the circle and $f(0)=0, f(2+2i)=\frac{2+2i}{4i-4}=\frac{-i}{2}$. Hence, the image of the circle $|z-2|=2$ is the imaginary axis. As the centre 2 of this circle is mapped to $T(2)=\frac{-1}{2}$, which lies in the left half plane, the image of $|z-2|<2$ under the linear fractional transform $T(z)=\frac{z}{2z-8}$ is the left half plane $\{z\in\mathbb{C}:\operatorname{Re} z<0\}$.



PROBLEM 3.2.11 Determine the image of $\{z:\operatorname{Re} z<0 \text{ and } \operatorname{Im} z>0\}$ under the linear fractional transform $S(z)=\frac{z+i}{z-i}$.

Solution: We first find the images of the real and imaginary axes under $\frac{z+i}{z-i}$. As $|T(x)|=\left|\frac{x+i}{x-i}\right|=1$, for all $x\in\mathbb{R}$, the image of the real axis is the unit circle. As $T(i)=\infty$, the image of the imaginary axis under the linear fractional transform is a straight line. Seeing $T(2i)=3$ and $T(3i)=2$, we conclude that the imaginary axis is mapped onto the real axis. As $-1+i$ lies on the second quadrant $\{z:\operatorname{Re} z<0 \text{ and } \operatorname{Im} z>0\}$ and $T(-1+i)=1-2i$ (which lies outside the circle and below the real axis), we conclude that the image of the second quadrant under $\frac{z+i}{z-i}$ is $\{z\in\mathbb{C}:|z|>1 \text{ and } \operatorname{Im} z<0\}$.

Definition 3.2.12 For given four distinct extended complex numbers z_1, z_2, z_3, and z_4, we define the cross ratio (z_1, z_2, z_3, z_4) by $T(z_1)$, where T is a linear fractional transform that maps z_2, z_3, and z_4 to $1, 0$, and ∞, respectively. More explicitly,

$$(z_1, z_2, z_3, z_4) = \begin{cases} \dfrac{z_1 - z_3}{z_1 - z_4} \times \dfrac{z_2 - z_4}{z_2 - z_3} & \text{if } z_1, z_2, z_3, z_4 \in \mathbb{C} \\ \dfrac{z_2 - z_4}{z_2 - z_3} & \text{if } z_1 = \infty,\ z_2, z_3, z_4 \in \mathbb{C} \\ \dfrac{z_1 - z_3}{z_1 - z_4} & \text{if } z_2 = \infty,\ z_1, z_3, z_4 \in \mathbb{C} \\ \dfrac{z_2 - z_4}{z_1 - z_4} & \text{if } z_3 = \infty,\ z_1, z_2, z_4 \in \mathbb{C} \\ \dfrac{z_1 - z_3}{z_2 - z_3} & \text{if } z_4 = \infty,\ z_1, z_2, z_3 \in \mathbb{C}. \end{cases}$$

Example 3.2.13 Evaluate the following cross ratios:

1. $(1 + i, 2 - i, 3, -i)$.

2. $(2 + i, i, 5 - i2, \infty)$.

3. $(1 + i2, 2 + i3, \infty, i)$.

4. $(2, \infty, 1 - i, 3 + i)$.

5. $(\infty, 1 - i, 1 + i, i)$.

Solution:

1. $$\begin{aligned} (1 + i, 2 - i, 3, -i) &= \frac{1 + i - 3}{1 + i + i} \times \frac{2 - i + i}{2 - i - 3} \\ &= \frac{-4 + i2}{1 - i3} = -1 - i. \end{aligned}$$

2. $(2+i, i, 5-i2, \infty) = \dfrac{2+i-5+i2}{i-5+i2}$

$$= \frac{-3-i3}{-5+i3} = \frac{3}{17}(4-i).$$

3. $(1+i2, 2+i3, \infty, 1) = \dfrac{2+i3-1}{1+i2-1}$

$$= \frac{1+i3}{i2} = \frac{1}{2}(3-i).$$

4. $(2, \infty, 1-i, 3+i) = \dfrac{2-1+i}{2-3-i}$

$$= \frac{1+i}{-1-i} = -1.$$

5. $(\infty, 1-i, 1+i, i) = \dfrac{1-i-i}{1-i-1-i}$

$$= \frac{1-i2}{-i2} = \frac{1}{2}(2+i).$$

Exercise 3.2.14 Find the following cross ratios:

1. $(2+i3, 1+i, 3-i, 1-i4)$.
2. $(1+i3, 5+i, 2-i3, \infty)$.
3. $(3+i2, 1-i2, \infty, i5)$.
4. $(1-i2, \infty, 1+i4, 2+i)$.
5. $(\infty, 2-i, 3+i4, i3)$.

Answers: 1. $\frac{19}{20} - i\frac{1}{5}$, 2. $\frac{1}{25}(21+i22)$, 3. $\frac{4}{3} - i$, 4. $\frac{1}{5}(9+i3)$, 5. $\frac{1}{13}(9+i7)$.

RESULT 3.2.15 Let z_1, z_2, z_3, and z_4 be extended complex numbers, then

1. $\overline{(z_1, z_2, z_3, z_4)} = (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4})$.
2. $(S(z_1), S(z_2), S(z_3), S(z_4)) = (z_1, z_2, z_3, z_4)$ for every linear fractional transform S.

Proof: Let T be the linear fractional transform that maps z_2, z_3, and z_4 to $1, 0$, and ∞, respectively.

1. If $T(z) = \dfrac{az+b}{cz+d}$ and if $T_1(z) = \dfrac{\overline{a}z+\overline{b}}{\overline{c}z+\overline{d}}$, then $T_1(\overline{z}) = \overline{T(z)}$ for every extended complex number z. Hence,

$$\begin{array}{rcccccl} T_1(\overline{z_2}) & = & \overline{T(z_2)} & = & \overline{1} & = & 1, \\ T_1(\overline{z_3}) & = & \overline{T(z_3)} & = & \overline{0} & = & 0, \\ T_1(\overline{z_4}) & = & \overline{T(z_4)} & = & \overline{\infty} & = & \infty. \end{array}$$

   Therefore, T_1 is the linear fractional transform that maps $\overline{z_2}, \overline{z_3}$, and $\overline{z_4}$ to $1, 0$, and ∞, respectively. Therefore, $(\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) = T_1(\overline{z_1}) = \overline{T(z_1)} = \overline{(z_1, z_2, z_3, z_4)}$.

2. Using Result 3.2.5, we get $T \circ S^{-1}$, which is a linear fractional transform. Clearly, we have $T \circ S^{-1}$ maps Sz_2, Sz_3, and Sz_4 to $1, 0$, and ∞, respectively. Therefore, $(S(z_1), S(z_2), S(z_3), S(z_4)) = (T \circ S^{-1})(S(z_1)) = T(z_1) = (z_1, z_2, z_3, z_4)$. □

THEOREM 3.2.16 Let z_1, z_2, z_3, and z_4 be distinct extended complex numbers, then z_1, z_2, z_3, and z_4 lie on a circle or on a straight line iff (z_1, z_2, z_3, z_4) is real.

Proof: Let T be the linear fractional transform that maps z_2, z_3, and z_4 to $1, 0$, and ∞, respectively, then $T(z_1) = (z_1, z_2, z_3,$ and $z_4)$. Assume that (z_1, z_2, z_3, z_4) is real. Then, $T(z_1), T(z_2) = 1, T(z_3) = 0, T(z_4) = \infty$ lie on the real line. As, T^{-1} is also a linear fractional transform, by Theorem 3.2.4, it maps the real line to a circle or to a straight line. In particular,

$$T^{-1}(T(z_1)) = z_1, T^{-1}(T(z_2)) = z_2, T^{-1}(T(z_3)) = z_3, \text{ and } T^{-1}(T(z_4)) = z_4$$

lie on a circle or on a straight line.

Conversely, assume that $z_1, z_2, z_3,$ and z_4 lie on a circle or on a straight line. Since T is a linear fractional transform, $T(z_1), T(z_2) = 1, T(z_3) = 0,$ and $T(z_4) = \infty$ lie on a circle or on a straight line, say C. As ∞ lies on C, it follows that C is a straight line. Using the fact that it passes through 1 and 0, we conclude that C must be the real line. Hence, $T(z_1)$ lies on the real line itself. Thus, $z_1, z_2, z_3,$ and z_4 are real. □

LEMMA 3.2.17 Let C be a circle and z and w be any two points not on C. If $(w, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ for some three distinct points $z_1, z_2,$ and z_3

on C, then $(w, a_1, a_2, a_3) = \overline{(z, a_1, a_2, a_3)}$ for any other three distinct points $a_1, a_2,$ and a_3 on C.

Proof: Let T be the linear fractional transform that maps $z_1, z_2,$ and z_3 to $1, 0,$ and ∞, respectively. Therefore, $(w, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ can be rewritten as $T(w) = \overline{T(z)}$. As, $a_1, z_1, z_2,$ and z_3 lie on C, by Theorem 3.2.16, $T(a_1)$ is real. Similarly, we can show that $T(a_j)$ is real for $j = 2$ and 3. Hence, using Result 3.2.15, we get

$$\begin{aligned}
\overline{(z, a_1, a_2, a_3)} &= \overline{(T(z), T(a_1), T(a_2), T(a_3))} \\
&= (\overline{T(z)}, \overline{T(a_1)}, \overline{T(a_2)}, \overline{T(a_3)}) \\
&= (\overline{T(z)}, T(a_1), T(a_2), T(a_3)) \\
&= (T(w), T(a_1), T(a_2), T(a_3)) \\
&= (w, a_1, a_2, a_3).
\end{aligned}$$

Hence, the lemma follows. □

Definition 3.2.18 Let C be a circle or a straight line. If $z, w \notin C$, then z and w are said to be symmetric with respect to C if $(w, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ for any three distinct points $z_1, z_2,$ and z_3 on C.

By previous lemma, the symmetry of the points does not depend on the choice of the three points on C.

Example 3.2.19 Explain symmetry with respect to a straight line.
If C is a straight line and z and w are symmetric with respect to C, then by definition, we have

$$(w, z_1, z_2, \infty) = \overline{(z, z_1, z_2, \infty)}$$

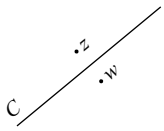
for any pair of distinct points z_1 and z_2 in C. Hence, we have

$$\frac{z - z_2}{z_1 - z_2} = \overline{\left(\frac{w - z_2}{z_1 - z_2}\right)}.$$

This implies that $|z - z_2| = |w - z_2|$ for every z_2 on C and

$$\text{Im}\left(\frac{z - z_2}{z_1 - z_2}\right) = -\text{Im}\left(\frac{w - z_2}{z_1 - z_2}\right)$$

which imply geometrically that z and w lie on different half planes determined by the line C, and they are at equal distance from an arbitrary point on C.

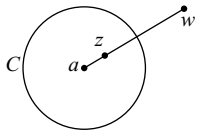


Example 3.2.20 Explain symmetry with respect to a circle.
If C is the circle $|z-a|=r$ and z and w are symmetric with respect to C, then we have $(w,z_1,z_2,z_3)=\overline{(z,z_1,z_2,z_3)}$ for any three distinct points on C. Using Result 3.2.15, we get

$$\begin{aligned}(w,z_1,z_2,z_3) &= \left(\frac{r^2}{w-a},\frac{r^2}{z_1-a},\frac{r^2}{z_2-a},\frac{r^2}{z_3-a}\right)\\ &= \left(\frac{r^2}{w-a},\overline{z_1}-\overline{a},\overline{z_2}-\overline{a},\overline{z_3}-\overline{a}\right)\\ &= \left(\frac{r^2}{w-a}+\overline{a},\overline{z_1},\overline{z_2},\overline{z_3}\right)\\ &= \overline{\left(\left(\frac{r^2}{w-a}+\overline{a}\right),z_1,z_2,z_3\right)}.\end{aligned}$$

Therefore, $z=\overline{\left(\frac{r^2}{w-a}+\overline{a}\right)}$, which implies that $(\overline{z}-\overline{a})(w-a)=r^2$ and $w-a=r^2\dfrac{z-a}{|z-a|^2}$. Hence, $|z-a|\cdot|w-a|=r^2$ and $\arg(w-a)=\arg(z-a)$, provided $z,w\in\mathbb{C}$. If $z=a$, then $w=\infty$.

Geometrically, one of the two points z and w lies inside the circle and the other point lies outside the circle satisfying $|z-a||w-a|=r^2$. Furthermore, $a,w,$ and z are collinear, such that both z and w lie on a same ray starting from a.



RESULT 3.2.21 (Principle of symmetry)
If z and w are symmetric with respect to a circle or to a straight line C, then $T(z)$ and $T(w)$ are symmetric with respect to $T(C)$.

Proof: Let $z_1,z_2,$ and z_3 be three distinct points on C. As z and w are symmetric with respect to C, we have $(w,z_1,z_2,z_3)=\overline{(z,z_1,z_2,z_3)}$. By Result 3.2.15, we get

$$(T(w), T(z_1), T(z_2), T(z_3)) = \overline{(T(z), T(z_1), T(z_2), T(z_3))}.$$

Using Result 3.2.5, we get that T is one-to-one, and hence, $T(z_1), T(z_2)$, and $T(z_3)$ are three distinct points on $T(C)$. Hence, $T(z)$ and $T(w)$ are symmetric with respect to $T(C)$. □

Algorithm 3.2.22 (To find the LFT that maps *a*, *b*, and *c* to *p*, *q*, and *r*, respectively.)

Step 1: Let T be the linear fractional transform that maps $a, b,$ and c into $1, 0,$ and ∞, respectively. That is, $T(z) = \dfrac{(z-b)}{(z-c)}\dfrac{(a-c)}{(a-b)}$.

Step 2: Let S be the linear fractional transform that maps $p, q,$ and r into $1, 0,$ and ∞, respectively. That is, $S(w) = \dfrac{(w-q)}{(w-r)}\dfrac{(p-r)}{(p-q)}$.

Step 3: Then $S^{-1} \circ T$ is the required linear fractional transform, which can be obtained by solving for w from $\dfrac{(w-q)}{(w-r)}\dfrac{(p-r)}{(p-q)} = \dfrac{(z-b)}{(z-c)}\dfrac{(a-c)}{(a-b)}$ in terms of z.

Example 3.2.23 Find the linear fractional transform that maps $1, 2,$ and 3 to $0, i,$ and $-i$, respectively.

Solution:
The linear fractional transform that maps $1, 2,$ and 3 into $1, 0,$ and ∞, respectively, is given by $\dfrac{(z-2)}{(z-3)}\dfrac{(1-3)}{(1-2)}$, and the linear fractional transform that maps $0, i,$ and $-i$ into $1, 0,$ and ∞, respectively, is given by $\dfrac{(w-i)}{(w+i)}\dfrac{(0+i)}{(0-i)}$.

$$\begin{aligned}
\frac{(w-i)}{(w+i)}\frac{(0+i)}{(0-i)} &= \frac{(z-2)}{(z-3)}\frac{(1-3)}{(1-2)} \\
\Rightarrow \quad -\frac{(w-i)}{(w+i)} &= 2\frac{(z-2)}{(z-3)} \\
\Rightarrow \quad -(w-i)(z-3) &= 2(z-2)(w+i) \\
\Rightarrow \quad -w(z-3) - 2w(z-2) &= i2(z-2) - i(z-3) \\
\Rightarrow \quad w(-3z+7) &= i(z-1) \\
\Rightarrow \quad w &= \frac{i(z-1)}{-3z+7}.
\end{aligned}$$

Hence, $R(z) = \dfrac{i(z-1)}{-3z+7}$, $\forall z \in \mathbb{C}_\infty$, is the required linear fractional transform.

Exercise 3.2.24 Find the linear fractional transform that maps $a, b,$ and c to $p, q,$ and r, respectively, in the following:

1. $1, 0,$ and -1 to $1, 1-i2,$ and -1
2. $1, -1,$ and i to $1, i,$ and -1
3. $0, 1,$ and ∞ to $-1, -i,$ and 1
4. $\infty, -1,$ and -2 to $1, \infty,$ and 3
5. $1-i, i,$ and 0 to $-i, 1+i,$ and 1
6. $1, \infty,$ and -2 to $-1, 2,$ and $\dfrac{1}{2}$
7. $-i, 0$ and ∞ to $\infty, -i2,$ and 1
8. $-1, 0,$ and 2 to $-1, 4,$ and 2
9. $-1, \infty,$ and i to $3-i3, 2+i3,$ and $-4+i2$
10. $1+i, 1,$ and i to $\dfrac{3+i4}{5}, \dfrac{-1+i}{2},$ and $\dfrac{2+i3}{2}$.

Answers:

1. $\dfrac{(1+i2)z+1}{z+(1+i2)}$,
2. $\dfrac{(i-1)z+(3+i)}{(1+i3)z+(1-i)}$,
3. $\dfrac{-iz-1}{-iz+1}$,
4. $\dfrac{z-1}{z+1}$,
5. $\dfrac{1}{-z+1}$,
6. $\dfrac{2z+1}{z-4}$,
7. $\dfrac{z+2}{z+i}$,
8. $\dfrac{3z+4}{2z+1}$,
9. $\dfrac{(2+i3)z-1}{z+(1-i)}$,
10. $\dfrac{2z-3}{z+i}$.

Example 3.2.25 Check the symmetry of given z, and w with respect to the given C in the following:

1. $z = -4 + i7$, $w = 8 + i$ and C is the straight line $y = 2x$.
2. $z = 2 + i5$, $w = 3$ and C is the straight line $3x - 4x + 1 = 0$.
3. $z = 8 + i7$, $w = 17 + i19$ and C is the circle $|z - 5 - i3| = 10$.

Solution:

1. We choose the three points $\infty, 0,$ and $1 + 2i$ on $y = 2x$. As

   $$(-4 + i7, \infty, 0, 1 + 2i) = \frac{11 - i3}{10} \text{ and } (8 + i, \infty, 0, 1 + 2i) = \frac{11 + i3}{10},$$

   $-4 + i7$ and $8 + i$ are symmetric with respect to $y = 2x$.

2. We choose the three points $\infty, i,$ and $5 + 4i$ on $3x - 4y + 1 = 0$. As

   $$(2 + i5, \infty, i, 5 + i4) = \frac{1 + 7i}{5} \text{ and } (3, \infty, i, 5 + 4i) = \frac{1 + 7i}{10},$$

   $2 + i5$ and 3 are not symmetric with respect to $3x - 4y + 1 = 0$.

3. We choose the three points $15 + 3i, -5 + 3i,$ and $5 + 13i$ on $|z - 5 - i3| = 10$. As $(8 + 7i, 15 + i3, -5 + i3, 5 + i13) = \frac{7 + 5i}{6}$ and $(17 + i19, 15 + i3, -5 + i3, 5 + i13) = \frac{7 - 5i}{6}$, $8 + 7i$ and $17 + 19i$ are symmetric with respect to $|z - 5 - i3| = 10$.

Example 3.2.26 Prove that if $w = T(z) = \frac{r(z - a)}{r^2 - \overline{a}z}$, then T maps $|z - a| = r$ onto $|w| = 1$ and $T(B(0, r)) = B(0, 1)$.

Proof: Let $|z| = r$. If $w = T(z)$, then we can write $z = re^{i\theta}$. Therefore,

$$|w| = \left|\frac{r(z - a)}{r^2 - \overline{a}z}\right| = \left|\frac{r(re^{i\theta} - a)}{r^2 - \overline{a}re^{i\theta}}\right| = \left|\frac{r - ae^{-i\theta}}{r - \overline{a}e^{i\theta}}\right| = 1.$$

Therefore, T maps $|z| = r$ into $|w| = 1$. As every linear fractional transform maps circles to circles, we conclude that T maps $|z - a| = r$ onto $|w| = 1$. Since $T(B(0, r))$ is a connected subset of $\{w \in \mathbb{C} : |w| \neq 1\}$ and the latter is having exactly two components namely, $B(0, 1)$ and $\{w \in \mathbb{C} : |w| > 1\}$.

Using the observation $T(a) = 0$, it follows that $T(B(0,r)) \subseteq B(0,1)$. Since, T^{-1} is also a linear fractional transform, such that T^{-1} maps $|w| = 1$ onto $|z-a| = r$ and $T^{-1}(0) = a$, by a similar argument, we obtain $T^{-1}(B(0,1)) \subseteq B(0,r)$. Thus, we get $T(B(0,r)) = B(0,1)$. □

PROBLEM 3.2.27 Find all linear fractional transform that maps $|z| = r$ onto $|w| = \rho$.

Solution: Let $w = T(z) = \frac{az+b}{cz+d}$ be a required transformation, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Clearly, 0 and ∞ are symmetric points with respect to the circle $|w| = \rho$; therefore, by symmetric principle, if $T^{-1}(0) = \alpha$ and $T^{-1}(\infty) = \beta$, then α and β are symmetric points with respect to $|z| = r$.

Case 1: $0 \neq \alpha \neq \infty$.

By one of the properties of the symmetric points of a circle, we have $\beta = \frac{r^2}{\overline{\alpha}}$. Putting $w = 0$ and $w = \infty$ in $z = \frac{dw-b}{-cw+a}$, we get $\alpha = -\frac{b}{a}$ and $\frac{r^2}{\overline{\alpha}} = -\frac{d}{c}$. From these observations, we note that $c \neq 0$, otherwise $\overline{\alpha} = 0$, which contradicts the fact that $\alpha \neq 0$. Therefore, $w = \frac{a}{c}\frac{z+\frac{b}{a}}{z+\frac{d}{c}} = \frac{a}{c}\frac{z-\alpha}{z-\frac{r^2}{\overline{\alpha}}} = \frac{a\overline{\alpha}}{c}\frac{z-\alpha}{\overline{\alpha}z-r^2}$. As the image $T(r)$ of r must lie on the circle $|w| = \rho$, we have $\rho = |T(r)| = \left|\frac{a\overline{\alpha}}{c}\frac{r-\alpha}{\overline{\alpha}r-r^2}\right| = \left|\frac{a\overline{\alpha}}{cr}\right|$, which implies $\left|\frac{a\overline{\alpha}}{cr\rho}\right| = 1$. Hence, by Result 2.4.24, we can find $\theta \in \mathbb{R}$ such that $\frac{a\overline{\alpha}}{cr\rho} = \exp(i\theta)$. Thus, $T(z) = \frac{a\overline{\alpha}}{c}\frac{z-\alpha}{\overline{\alpha}z-r^2} = r\rho\exp(i\theta)\frac{z-\alpha}{\overline{\alpha}z-r^2}$.

Case 2: $\alpha = 0$

In this case, immediately, we have $\beta = \infty$ using the symmetry of α and β with respect to $|z| = r$. Now, $T(0) = 0$ and $T(\infty) = \infty$ imply $b = 0$ and $c = 0$, and hence, $a \neq 0 \neq d$. As T maps $|z| = r$ onto $|w| = \rho$, we have $\rho = |w| = |T(z)| = \left|\frac{a}{d}\right| r$. Therefore, there exists $\varphi \in \mathbb{R}$ such that $\frac{a}{d} = \frac{\rho}{r}\exp(i\varphi)$. Thus, $T(z) = \frac{\rho}{r}\exp(i\varphi)z$.

Case 3: $\alpha = \infty$.

Then, we have $\beta = 0$. Now, $T(0) = \infty$ and $T(\infty) = 0$ imply that $d = 0$ and $a = 0$. As T maps $|z| = r$ onto $|w| = \rho$, we have $\rho = |w| = |T(z)| = \left|\frac{b}{c}\right|\frac{1}{r}$. Therefore, there exists $\psi \in \mathbb{R}$ such that $\frac{b}{c} = r\rho\exp(i\psi)$. Thus, $T(z) = r\rho\exp(i\psi)\frac{1}{z}$.

Remark 3.2.28: The linear fractional transforms obtained in Case 1 and Case 2 of the previous problem map $|z| < r$ onto $|w| < \rho$. The

transforms obtained in Case 3 of the previous problem map $|z| < r$ onto $|w| > \rho$.

PROBLEM 3.2.29 Find all linear fractional transform that maps the upper half-plane onto the unit circle.

Solution: Let $w = T(z) = \frac{az+b}{cz+d}$ be a linear fractional transform, which maps upper half-plane onto the unit circle. Clearly, 0 and ∞ are symmetric points with respect to the circle $|w| = 1$. If $T^{-1}(0) = \alpha$ and $T^{-1}(\infty) = \beta$, then α and β are symmetric points with respect to the real axis. Therefore, obviously we have $\alpha, \beta \notin \{0, \infty\}$ and $\beta = \overline{\alpha}$. Using $w = 0$ and $w = \infty$ in $z = \frac{dw-b}{-cw+a}$, we get $\alpha = -\frac{b}{a}$ and $\overline{\alpha} = -\frac{d}{c}$. Therefore, $w = \frac{a}{c}\frac{z+\frac{b}{a}}{z+\frac{d}{c}} = \frac{a}{c}\frac{z-\alpha}{z-\overline{\alpha}}$. As $T(0)$ lies on $|w| = 1$, we have $1 = |T(0)| = \left|\frac{a}{c}\right|$, and hence, $\frac{a}{c} = \exp(i\theta)$ for some $\theta \in \mathbb{R}$. Thus, $T(z) = \exp(i\theta)\frac{z-\alpha}{z-\overline{\alpha}}$.

3.3 BRANCH OF A MULTIVALUED FUNCTION

We shall start this section with the definition logarithm and then we introduce the concept of branch of a multivalued function.

Definition 3.3.1 The inverse function of $\exp : \mathbb{R} \to (0, \infty)$ is called the natural logarithm and is denoted by ln.

Note that the above definition is well defined as $\exp : \mathbb{R} \to (0, \infty)$ is a bijection. See Lemma 2.4.6.

LEMMA 3.3.2 If $a, b \in (0, \infty)$, then $\ln(ab) = \ln a + \ln b$ and $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Proof: Let $x = \ln a$ and $y = \ln b$, then $\exp(x) = a$ and $\exp(y) = b$. By Theorem 2.4.4, we have

$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y) \quad \text{and} \\ \exp(x-y) &= \exp(x)\exp(-y) = \frac{\exp(x)}{\exp(y)}. \end{aligned}$$

Therefore,

$$\ln a + \ln b = x + y = \ln(\exp(x+y)) = \ln(\exp(x)\exp(y)) = \ln(ab)$$

and

$$\ln a - \ln b = x - y = \ln(\exp(x - y)) = \ln\left(\frac{\exp(x)}{\exp(y)}\right) = \ln\left(\frac{a}{b}\right).$$

Hence, the lemma follows. □

RESULT 3.3.3 $\ln : (0, \infty) \to \mathbb{R}$ is differentiable and $\ln' a = \dfrac{1}{a}$.

Proof: First, we prove that $\ln : (0, \infty) \to \mathbb{R}$ is continuous on $(0, \infty)$.
For arbitrary $a, b \in [1, \infty)$, we choose $x, y \in [0, \infty)$ such that $\exp(x) = a$ and $\exp(y) = b$ and then we get $x = \ln a$, $y = \ln b$. Now,

$$\begin{aligned}
|a - b| &= |\exp(x) - \exp(y)| \\
&= \left|\sum_{n=1}^{\infty} \frac{x^n - y_n}{n!}\right| \\
&= |x - y| \left|\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n-1} x^k y^{n-1-k}}{n!}\right| \\
&= |x - y| \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n-1} x^k y^{n-1-k}}{n!} \\
&\geq |x - y| \\
&= |\ln a - \ln b|.
\end{aligned}$$

Therefore, $\ln b \to \ln a$ as $b \to a$, whenever $a, b \in [1, \infty)$.
Suppose, $a, b \in (0, 1]$, then $\dfrac{1}{a}, \dfrac{1}{b} \in [1, \infty)$, and hence,

$$b \to a \Rightarrow \frac{1}{b} \to \frac{1}{a} \Rightarrow \ln\frac{1}{b} \to \ln\frac{1}{a} \Rightarrow -\ln b \to -\ln a \Rightarrow \ln b \to \ln a.$$

Hence, ln is continuous on $(0, \infty)$.
To prove ln is differentiable on $(0, \infty)$, fix $a, b \in (0, \infty)$ arbitrarily. Then let $x = \ln a$ and $\ln b = y$. Now, using $b \to a \Rightarrow y \to x$, we get

$$\lim_{b \to a} \frac{\ln b - \ln a}{b - a} = \lim_{y \to x} \frac{y - x}{\exp(y) - \exp(x)}$$

$$
\begin{aligned}
&= \lim_{y\to x} \frac{\dfrac{1}{\exp(y)-\exp(x)}}{y-x} \\
&= \frac{1}{\lim\limits_{y\to x} \dfrac{\exp(y)-\exp(x)}{y-x}} \\
&= \frac{1}{\exp'(x)} = \frac{1}{\exp(x)} = \frac{1}{a}.
\end{aligned}
$$

Thus, ln is differentiable on $(0,\infty)$ and $\ln' a = \dfrac{1}{a}$. □

We would like to define complex logarithm as the inverse function of exp on $\mathbb{C}$. However, unfortunately, $\exp : \mathbb{C} \to \mathbb{C}$ is neither one-to-one nor onto. However, we can define the complex logarithm of a non-zero complex number as a *multivalued function* as follows.

Definition 3.3.4 For every $w \in \mathbb{C} \setminus \{0\}$, we define $\log w = \ln|w| + i\arg w$.

Note that in the above definition, the imaginary part of log is not single valued.

Definition 3.3.5 Let f be a multivalued function on a subset of Ω. We mean a branch (an analytic branch) of f by a single-valued continuous (or analytic) function g on a sub-region of Ω, if value of g at a point is one of the multivalues of f at the same point.

To define an analytic branch of log, we have to choose a unique value of its imaginary part so that log becomes a single-valued analytic function.

LEMMA 3.3.6 The principal argument given in Definition 1.2.16 is continuous on $\mathbb{C} \setminus \{(a,0) : a \le 0\}$.

Proof: Using the continuity of arctan, we get that arg is continuous on each of the following regions.

1. $\{(a,b) : a > 0,\ b \in \mathbb{R}\}$
2. $\{(a,b) : a < 0,\ b > 0\}$
3. $\{(a,b) : a < 0,\ b < 0\}$

Hence to conclude the proof of this lemma, we shall show that arg is continuous on the positive imaginary axis and on the negative imaginary axis.

Let $(0, b)$ be fixed, where $b > 0$ is arbitrary. If $(a_n, b_n) \to (0, b)$ as $n \to \infty$, with $(a_n, b_n) \neq (0, b)$, $\forall n \in \mathbb{N}$, then, $a_n \to 0$ and $b_n \to b$ as $n \to \infty$, and hence, there exists $M \in \mathbb{N}$ such that $b_n > 0$ for all $n \geq M$.

Case 1: $a_n > 0, \forall n \in \mathbb{N}$

As $\dfrac{b_n}{a_n} \to +\infty$ as $n \to \infty$, we have $\arctan \dfrac{b_n}{a_n} \to \dfrac{\pi}{2}$ as $n \to \infty$. Therefore,

$$\arg((0, b)) = \frac{\pi}{2} = \lim_{n\to\infty} \arctan\left(\frac{b_n}{a_n}\right) = \lim_{n\to\infty} \arg((a_n, b_n)).$$

Case 2: $a_n < 0, \forall n \in \mathbb{N}$

Therefore, using the fact that $\arctan\left(\dfrac{b_n}{|a_n|}\right) \to \dfrac{\pi}{2}$ as $n \to \infty$, we obtain

$$\begin{aligned} \arg((0, b)) &= \frac{\pi}{2} = \pi - \frac{\pi}{2} = \pi - \lim_{n\to\infty} \arctan\left(\frac{b_n}{|a_n|}\right) \\ &= \lim_{n\to\infty} \arg((a_n, b_n)). \end{aligned}$$

Case 3: $a_n \in \mathbb{R}, \forall n \in \mathbb{N}$.

Given $\epsilon > 0$ using Case 1 and Case 2, we find $N_1, N_2 \in \mathbb{N}$ such that $N_1 > M$, $N_2 > M$, and

$$\begin{aligned} |\arg((a_n, b_n)) - \arg((0, b))| &= \left|\arg((a_n, b_n)) - \frac{\pi}{2}\right| < \epsilon, \\ &\quad \forall n \geq N_1 \text{ if } a_n > 0 \\ |\arg((a_n, b_n)) - \arg((0, b))| &= \left|\arg((a_n, b_n)) - \frac{\pi}{2}\right| < \epsilon \\ &\quad \forall n \geq N_2 \text{ if } a_n < 0. \end{aligned}$$

As

$$\begin{aligned} |\arg((a_n, b_n)) - \arg((0, b))| &= \left|\frac{\pi}{2} - \frac{\pi}{2}\right| = 0 < \epsilon, \\ \forall n \in \mathbb{N} \text{ if } a_n &= 0, \end{aligned}$$

it follows that $\left|\arg((a_n, b_n)) - \dfrac{\pi}{2}\right| < \epsilon$, $\forall n \geq \max\{N_1, N_2\}$ if $a_n \in \mathbb{R}$. □

Remark 3.3.7: The principal branch of arg is not continuous at any point of negative real axis.

Indeed, if $(x, 0) \in \mathbb{C}$ with $x < 0$, we have

$$\left(x + \frac{1}{n}, 0\right) \to (x, 0) \text{ and } \left(x - \frac{1}{n}, 0\right) \to (x, 0) \text{ as } n \to \infty$$

but

$$\arg\left(x + \frac{1}{n}, 0\right) \to \frac{\pi}{2} \text{ and } \arg\left(x - \frac{1}{n}, 0\right) \to -\frac{\pi}{2} \text{ as } n \to \infty.$$

LEMMA 3.3.8 If $0 \neq w \in \mathbb{C}$, then there exists a region containing w in which some suitable branch of arg is continuous.

Proof: If $0 \neq w \in \mathbb{C}$, then we can define $\arg_w$ on $\Omega_w = \mathbb{C}\backslash\{r\exp(i\theta) : r > 0\}$, by $\arg_w(z) = \arg(z) + \pi + \theta_w, \forall z \in \Omega_w$, where $\arg(z)$ is the principal argument of z as in Definition 1.2.16, and $\theta_w \in \mathbb{R} \setminus \{\arg(w) + 2k\pi : k \in \mathbb{Z}\}$. Being $\arg_w$ is a translate of principal argument, it is also continouous on Ω_w. □

RESULT 3.3.9 (Principal branch for log)
If $\Omega = \mathbb{C} \setminus \{(x, 0) : x \leq 0\}$ and if $\log(w) = \ln|w| + i\arg(w)$, with $\text{Im}\log(w) \in (-\pi, \pi)$, then $\log : \Omega \to \mathbb{C}$ is analytic.

Proof: First, we note that for every $w \in \Omega$, principal argument of w belongs to $(-\pi, \pi)$, and hence, log is well defined on Ω. As ln is continuous on $(0, \infty)$ and $|\cdot|$ is continuous on $\mathbb{C}$ (Example 1.6.18), we have Re log, which is continuous on Ω. By Lemma 3.3.6, we have Im log, which is continuous on Ω. Therefore, log is continuous on Ω by Lemma 1.6.19. Next, we prove that log is differentiable on Ω. For $w_0, w \in \Omega$, using the continuity of log on Ω, we get

$$\lim_{w \to w_0} \frac{\log(w) - \log(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{\exp(z) - \exp(z_0)},$$

where $z = \log(w)$ and $z_0 = \log(w_0)$

$$= \lim_{z \to z_0} \frac{1}{\dfrac{\exp(z) - \exp(z_0)}{z - z_0}}$$

$$= \frac{1}{\lim\limits_{z \to z_0} \dfrac{\exp(z) - \exp(z_0)}{z - z_0}}$$

$$= \frac{1}{\exp'(z_0)} = \frac{1}{\exp(z_0)} = \frac{1}{z_0}.$$

Thus, log is differentiable on Ω and $\log'(a) = \dfrac{1}{a}$, $\forall a \in \Omega$. □

Definition 3.3.10 For a non-zero complex number z, we define $w = \sqrt{z}$ by the solution of the equation $w^2 = z$. More explicitly, $\sqrt{z} = \pm\sqrt{r}\exp\left(i\dfrac{\theta}{2}\right)$, where $z = r\exp(i\theta)$.

RESULT 3.3.11 (Analytic branch for $\sqrt{\cdot}$)
If $\Omega = \mathbb{C} \setminus \{(x, 0) : x \leq 0\}$ and if $\sqrt{z}$ is defined by the unique value of $\sqrt{z}$ whose real part is positive, then $\sqrt{\cdot} : \Omega \to \mathbb{C}$ is analytic.

Proof: First, we note that every $z \in \Omega$ is never real and negative, and hence, $\sqrt{z}$ is never purely imaginary. Therefore, Re $\sqrt{z} \neq 0$. If $\sqrt{z}$ is defined by the unique value of $\sqrt{z}$ whose real part is positive, and then $\sqrt{z}$ is single valued. First, we claim that $\sqrt{\cdot}$ is continuous on Ω. Let $z_0, z \in \Omega$ and $w = \sqrt{z}, w_0 = \sqrt{z_0}$. Hence, $w^2 = z$ and $w_0^2 = z_0$. Using the definition of $\sqrt{z}$, we get

$$\begin{aligned} |z - z_0| &= |w^2 - w_0^2| = |w - w_0| \cdot |w + w_0| \geq |w - w_0| \cdot \text{Re}\,(w + w_0) \\ &> |w - w_0| \cdot \text{Re}\, w_0 \end{aligned}$$

which implies that

$$|\sqrt{z} - \sqrt{z_0}| = |w - w_0| < \frac{1}{\text{Re}\, w_0}|z - z_0|.$$

Hence, $\sqrt{z} \to \sqrt{z_0}$ as $z \to z_0$. Thus, $\sqrt{\cdot}$ is continuous on Ω.
As

$$\lim_{z \to z_0} \frac{\sqrt{z} - \sqrt{z_0}}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{w^2 - w_0^2} = \lim_{w \to w_0} \frac{1}{w + w_0} = \frac{1}{2w_0} = \frac{1}{2\sqrt{z_0}},$$

we get $\sqrt{\cdot}$ is differentiable and its derivative is $\dfrac{1}{2\sqrt{\cdot}}$. □

Example 3.3.12 Define an analytic branch for $\sqrt{1 + \sqrt{z}}$ and justify that it is analytic.
We know that $\sqrt{z}$ is defined on $\Omega_1 = \mathbb{C} \setminus \{(x, 0) : x \leq 0\}$ as an analytic function by the unique value of $\sqrt{z}$, whose real part is positive. As the range

G of $\sqrt{\cdot}$ is $\{z \in \mathbb{C} : \text{Re}\, z > 0\}$, we get $1 + \sqrt{z} \in G, \forall z \in \Omega_1$. Furthermore, $G \subseteq \Omega_1$, and we can define $\sqrt{1 + \sqrt{z}}$ using the above definition of $\sqrt{\cdot}$ twice as an analytic function.

RESULT 3.3.13 $\arccos(w) = \pm i \log\left(w + \sqrt{w^2 - 1}\right)$

Proof: Let $z = \arccos w$. Then $w = \cos z = \dfrac{\exp(iz) + \exp(-iz)}{2}$. Hence, it follows that

$$2w = \exp(iz) + \exp(-iz) \Rightarrow \exp(i2z) - 2w \exp(iz) + 1 = 0$$

which is a quadratic equation in $\exp(iz)$. Therefore,

$$\exp(iz) = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}.$$

Therefore, $iz = \log(w \pm \sqrt{w^2 - 1}) \Rightarrow z = -i \log(w \pm \sqrt{w^2 - 1})$. As

$$(w + \sqrt{w^2 - 1}) \cdot (w - \sqrt{w^2 - 1}) = w^2 - (w^2 - 1) = 1$$

$\arccos(w) = z = \pm i \log(w + \sqrt{w^2 - 1})$ □

RESULT 3.3.14 $\arcsin(w) = -i \log(iw \pm \sqrt{1 - w^2})$.

Proof: Let $z = \arcsin(w)$. Then $w = \sin(z) = \dfrac{\exp(iz) - \exp(-iz)}{i2}$. Hence, it follows that

$$i2w = \exp(iz) - \exp(-iz) \Rightarrow \exp(i2z) - i2w \exp(iz) - 1 = 0$$

which is a quadratic equation in $\exp(iz)$. Therefore,

$$\exp(iz) = \frac{i2w \pm \sqrt{-4w^2 + 4}}{2} = iw \pm \sqrt{1 - w^2}.$$

Therefore, $iz = \log(iw \pm \sqrt{1 - w^2}) \Rightarrow z = -i \log(iw \pm \sqrt{1 - w^2})$. Thus, $\arcsin w = -i \log(iw \pm \sqrt{1 - w^2})$ □

RESULT 3.3.15 $\sinh^{-1} w = \log\left(w \pm \sqrt{w^2 + 1}\right)$

Proof: If $z = \sinh^{-1} w$, then

$$w = \sinh z = \frac{\exp(z) - \exp(-z)}{2}$$

which implies that $\exp(2z) - 2w\exp(z) - 1 = 0$. Therefore,

$$\exp(z) = \frac{2w \pm \sqrt{4w^2 + 4}}{2} = w \pm \sqrt{w^2 + 1}$$

and hence, $z = \log\left(w \pm \sqrt{w^2 + 1}\right)$. □

RESULT 3.3.16 $\tanh^{-1} w = \frac{1}{2}\log\left(\frac{1+w}{1-w}\right)$.

If $z = \tanh^{-1} w$, then $w = \tanh z = \dfrac{\exp(z) - \exp(-z)}{\exp(z) - \exp(-z)}$,

$$\begin{aligned}
&\Rightarrow \quad \exp(z) - \exp(-z) = w(\exp(z) + \exp(-z)) \\
&\Rightarrow \quad \exp(2z) - 1 = w(\exp(2z) + 1) \\
&\Rightarrow \quad \exp(2z)(1 - w) = 1 + w \\
&\Rightarrow \quad \exp(2z) = \frac{1+w}{1-w}.
\end{aligned}$$

Therefore, $z = \frac{1}{2}\log\left(\frac{1+w}{1-w}\right)$. □

Exercise 3.3.17 Prove that $\cosh^{-1} w = \log\left(w \pm \sqrt{w^2 - 1}\right)$ and $\tan^{-1} w = \frac{1}{i2}\log\left(\frac{1+iw}{1-iw}\right)$.

Example 3.3.18 Find a branch for arccos.
Let $\Omega_1 = \mathbb{C} \setminus \{(x, 0) : |x| \geq 1\}$. Define $\arccos(w) = i\log(w + \sqrt{w^2 - 1})$ with its real part lies between $(-\pi, 0)$ and $\sqrt{w^2 - 1} = i\sqrt{1 - w^2}$, $\forall w \in \Omega_1$. To prove that this branch of arccos is analytic, we first show that $1 - w^2$ and $w + \sqrt{w^2 - 1}$ are never real and negative whenever $w \in \Omega_1$, so that from the principal branch of log, arccos becomes analytic.

$1 - w^2$ is real and less than or equal to 0	iff	w^2 is real and $w^2 \geq 1$
	iff	w is real and $\lvert w\rvert \geq 1$
	iff	$w \notin \Omega_1$.

As $(w+\sqrt{w^2-1})(w-\sqrt{w^2-1}) = w^2-(w^2-1) = 1$, we have

$$\begin{array}{lll}(w+\sqrt{w^2-1}) \text{ is real} & \text{iff} & (w-\sqrt{w^2-1}) \text{ is real}\\ & \text{iff} & w \text{ and } \sqrt{w^2-1} \text{ are real}\\ & \text{iff} & w \text{ is real and } |w| \geq 1\\ & \text{iff} & w \notin \Omega_1.\end{array}$$

If $g(w) = w+\sqrt{w^2-1}$, $\forall w \in \Omega_1$, then g is a continuous function on Ω_1, and hence, $\{w+\sqrt{w^2-1} : w \in \Omega_1\}$ is a connected subset of $\mathbb{C}$. As $w+\sqrt{w^2-1}$ is never real on Ω_1, $\{w+\sqrt{w^2-1} : w \in \Omega_1\}$ lies either in the upper half-plane or in the lower half-plane. As $i \in \Omega_1$ and $g(i) = i+i\sqrt{2}$, which belongs to the upper half-plane, $\{w+\sqrt{w^2-1} : w \in \Omega_1\}$ should be in the upper half-plane, and hence, we define $\log(w+\sqrt{w^2-1})$ as an analytic function on Ω_1 whose imaginary part lies in $(0,\pi)$. Hence, $\arccos(w) = i\log(w+\sqrt{w^2-1})$, $\forall w \in \Omega_1$ is analytic if its real part lies in $(-\pi, 0)$.

Further, the derivative of arccos is obtained by

$$\begin{aligned}\frac{d}{dw}(\arccos(w)) &= \frac{i}{w+\sqrt{w^2-1}} \times \left(1+\frac{2w}{2\sqrt{w^2-1}}\right)\\ &= \frac{i}{\sqrt{w^2-1}} = \frac{i}{i\sqrt{1-w^2}} = \frac{1}{\sqrt{1-w^2}}.\end{aligned}$$

Exercise 3.3.19 Define an analytic branch for arcsin.

Exercise 3.3.20 Define an analytic branch for $\sqrt[n]{\cdot}$. (Hint. Use the branch of log and $\sqrt[n]{z} = \exp\left(\frac{1}{n}\log(z)\right)$.)

3.4 CONFORMAL MAPPING

Definition 3.4.1 (Curve)
If $\varphi : [a,b] \to \mathbb{C}$ is a continuous function, then the image $\gamma = \varphi([a,b])$ of φ is called a curve with initial point $\varphi(a)$ and end point $\varphi(b)$ in $\mathbb{C}$.
We call $\varphi(t)$, $t \in [a,b]$ a parametric equation of the curve γ.

Definition 3.4.2 Let Ω be a region and let $z_0 \in \Omega$. A continuous function $f : \Omega \to \mathbb{C}$ is said to be a

1. conformal mapping of first type at z_0 if for every curve γ with a parametric equation $\varphi(t)$, $t \in [a,b]$ such that $\varphi(t_0) = z_0$ and $\varphi'(t_0) \neq 0$,

we get $(f \circ \varphi)'(t_0) \neq 0$ and

$$\lim_{t \to t_0} \arg \frac{(f \circ \varphi)(t) - (f \circ \varphi)(t_0)}{\varphi(t) - \varphi(t_0)}$$

exists for a suitable branch of arg and is independent of the curve γ.

2. conformal mapping of second type at z_0 if $\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$ exists and is non-zero.

3. conformal mapping if f is conformal of first type at z_0 as well as conformal of second type at z_0.

THEOREM 3.4.3 If f is differentiable at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.

Proof: By assumption, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \neq 0.$$

Let γ be any curve with a parametric equation $\varphi(t)$, $t \in [a, b]$ such that $\varphi(t_0) = z_0$ and $\varphi'(t_0) \neq 0$. By chain rule, we have

$$(f \circ \varphi)'(t_0) = f'(z_0)\varphi'(t_0) \neq 0$$

and as $f'(z_0) \neq 0$, then arg is continuous on a region that contains $f'(z_0)$ with respect to a suitable branch (Lemma 3.3.8), and hence, we get

$$\begin{aligned}
&\lim_{t \to t_0} \arg \frac{(f \circ \varphi)(t) - (f \circ \varphi)(t_0)}{\varphi(t) - \varphi(t_0)} \\
&\quad = \arg \lim_{t \to t_0} \frac{(f \circ \varphi)(t) - (f \circ \varphi)(t_0)}{\varphi(t) - \varphi(t_0)} \\
&\quad = \arg \lim_{t \to t_0} \frac{(f \circ \varphi)(t) - (f \circ \varphi)(t_0)}{t - t_0} \cdot \frac{t - t_0}{\varphi(t) - \varphi(t_0)} \\
&\quad = \arg \lim_{t \to t_0} \frac{(f \circ \varphi)(t) - (f \circ \varphi)(t_0)}{t - t_0} \cdot \frac{t - t_0}{\varphi(t) - \varphi(t_0)} \\
&\quad = \arg \frac{(f \circ \varphi)'(t_0)}{\varphi'(t_0)} \\
&\quad = \arg f'(z_0)
\end{aligned}$$

which is independent of the curve γ. Again using the hypothesis and the continuity of the modulus function (Example 1.6.18), we get

$$\lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|} \neq 0.$$

Thus, f is conformal at z_0. □

THEOREM 3.4.4 If f is conformal at z_0, then f is differentiable at z_0 and $f'(z_0) \neq 0$.

Proof: By assumption, we have

1. for every curve γ with a parametric equation $\varphi(t),\ t \in [a,b]$ such that $\varphi(t_0) = z_0$ and $\varphi'(t_0) \neq 0$, we get $(f \circ \varphi)'(t_0) \neq 0$ and

$$\lim_{t\to t_0} \arg \frac{(f\circ\varphi)(t)-(f\circ\varphi)(t_0)}{\varphi(t)-\varphi(t_0)} = \theta$$

   for some $\theta \in \mathbb{R}$ with respect to a suitable branch of arg and is independent of the curve γ.

2. conformal of second type at z_0. That is, $\lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|} = A$ for some $A \neq 0$.

To prove this theorem, we show that if

$$z_n \to z_0 \text{ as } n \to \infty \text{ with } z_n \neq z_0,\ \forall n \in \mathbb{N}$$

then

$$\frac{f(z_n)-f(z_0)}{z_n-z_0} \to A\exp(i\theta) \text{ as } n \to \infty.$$

First, we choose a curve γ with a parametric equation $\varphi(t),\ t \in [a,b]$ with the following properties:

1. $\varphi(t_0) = z_0$, for some $t_0 \in [a,b]$ and $\varphi(t_n) = z_n$, $\forall n \in \mathbb{N}$ for some sequence (t_n) of distinct points of $[a,b]$ such that $t_n \to t_0$ as $n \to \infty$.

2. $\varphi'(t_0) \neq 0$.

Now, using the assumptions and Lemma 3.3.8,

$$\lim_{n\to\infty} \frac{f(z_n)-f(z_0)}{z_n-z_0}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left|\frac{f(z_n)-f(z_0)}{z_n-z_0}\right| \exp\left(i\arg\left(\frac{f(z_n)-f(z_0)}{z_n-z_0}\right)\right)\\
&= \lim_{n\to\infty} \left|\frac{f(z_n)-f(z_0)}{z_n-z_0}\right| \exp\left(i\lim_{n\to\infty}\arg\left(\frac{f(\varphi(t_n))-f(\varphi((t_0))}{\varphi(t_n)-\varphi(t_0)}\right)\right)\\
&= A\exp(i\theta)\,.
\end{aligned}
$$

Thus, f is differentiable at z_0 and $f'(z_0) = A\exp(i\theta) \neq 0$. □

THEOREM 3.4.5 Let Ω be a region, $z_0 \in \Omega$ and let $f : \Omega \to \mathbb{C}$ be continuous at z_0.

1. If f is conformal of first type at z_0 and f_x, f_y exist at z_0 and they are continuous, then f is differentiable at z_0.
2. If f is conformal of second type at z_0 and f_x, f_y exist at z_0 and they are continuous, then f or $\overline{f}$ is differentiable at z_0.

Proof: Let γ be a curve with a parametric equation $\varphi(t)$, $t \in [a,b]$ such that $\varphi(t_0) = z_0$ and $\varphi'(t_0) \neq 0$. If $\psi(t) = f(\varphi(t))$, $\forall t \in [a,b]$ and if $x'(t_0)$ and $y'(t_0)$ are the real and imaginary parts of $\varphi'(t_0)$, respectively, then ψ is differentiable at t_0 and

$$
\begin{aligned}
\psi'(t_0) &= f_x(\varphi(t_0))x'(t_0) + f_y(\varphi(t_0))y'(t_0)\\
&= f_x(z_0)\left(\frac{\varphi'(t_0)+\overline{\varphi'(t_0)}}{2}\right) + f_y(z_0)\left(\frac{\varphi'(t_0)-\overline{\varphi'(t_0)}}{i2}\right)\\
&= \varphi'(t_0)\left(\frac{f_x(z_0)-if_y(z_0)}{2}\right) + \overline{\varphi'(t_0)}\left(\frac{f_x(z_0)+if_y(z_0)}{2}\right).
\end{aligned}
$$

Thus, we have

$$
\frac{\psi'(t_0)}{\varphi'(t_0)} = \left(\frac{f_x(z_0)-if_y(z_0)}{2}\right) + \left(\frac{f_x(z_0)+if_y(z_0)}{2}\right)\frac{\overline{\varphi'(t_0)}}{\varphi'(t_0)}. \tag{3.3}
$$

which represents an equation of a circle with centre $\dfrac{f_x(z_0)-if_y(z_0)}{2}$ and radius $\dfrac{f_x(z_0)+if_y(z_0)}{2}$, as φ varies through the continuous function on $[a,b]$ with $\varphi(t_0) = z_0$ for some t_0 and $\varphi'(t_0) \neq 0$.

1. Assume that f is conformal of first type at z_0.
   Then, we have

$$
\arg\frac{\psi'(t_0)}{\varphi'(t_0)} = \arg\frac{(f\circ\varphi)'(t_0)}{\varphi(t_0)} = \lim_{t\to t_0}\arg\frac{(f\circ\varphi)(t)-(f\circ\varphi)(t_0)}{\varphi(t)-\varphi(t_0)}
$$

which exists and independent of φ. Therefore, every point on the circle represented by the equation (3.3) has same argument. This is possible iff the radius of the circle is 0. That is, we have

$$\frac{f_x(z_0) + if_y(z_0)}{2} = 0 \Rightarrow f_x(z_0) = -if_y(z_0).$$

Therefore, f satisfies C–R equations at z_0 whence by using Theorem 3.1.48, we get f is differentiable at z_0.

2. Assume that f is conformal of second type at z_0.
   By a similar argument, we get the absolute value of every point on the circle given in the equation (3.3) is constant. This is possible iff the radius of the circle is 0 or the centre of the circle is 0. That is, we have

$$\frac{f_x(z_0) + if_y(z_0)}{2} = 0 \quad \text{or} \quad \frac{f_x(z_0) - if_y(z_0)}{2} = 0.$$

   Just now we have seen that $\frac{f_x(z_0) + if_y(z_0)}{2} = 0$ implies that f is differentiable at z_0. If $\frac{f_x(z_0) - if_y(z_0)}{2} = 0$, then we have

$$f_x(z_0) = if_y(z_0) \Rightarrow \left(\overline{f}\right)_x(z_0) = -i\left(\overline{f}\right)_y(z_0)$$

   and hence, $\overline{f}$ satisfies the C–R equation at z_0. Furthermore, as f has continuous partial derivatives at z_0, $\overline{f}$ also has continuous partial derivatives at z_0. Thus, $\overline{f}$ is differentiable at z_0, by Theorem 2.2.28. □

Example 3.4.6 The function exp maps the rectangle $\{z \in \mathbb{C} : 1 < \operatorname{Re} z < 2, \frac{\pi}{6} < \operatorname{Im} z < \frac{\pi}{3}\}$ onto the part of angular sector $\{w \in \mathbb{C} : a < |w| < b, \frac{\pi}{6} < \arg w < \frac{\pi}{3}\}$.
Let $z = x + iy$. Under the exponential function,

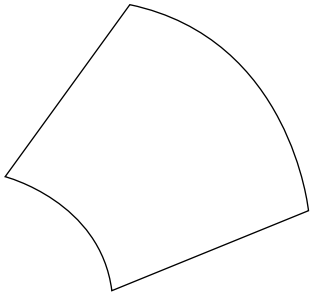
1. the horizontal line segment $\{(x, \frac{\pi}{6}) : 1 < x < 2\}$ is mapped onto $\{\exp(x)\exp(i\frac{\pi}{6}) : 1 < x < 2\}$, which is the line segment with slope $\frac{\pi}{6}$ with the positive real axis from $|w| = \exp(1)$ to $|w| = \exp(2)$.

2. similarly, the horizontal line segment $\{(x, \frac{\pi}{3}) : 1 < x < 2\}$ is mapped onto $\{\exp(x)\exp(i\frac{\pi}{3}) : 1 < x < 2\}$, which is the line segment with slope $\frac{\pi}{3}$ with the positive real axis from $|w| = \exp(1)$ to $|w| = \exp(2)$.

3. the vertical line segment $\{(1,y) : \frac{\pi}{6} < y < \frac{\pi}{3}\}$ is mapped onto $\{\exp(1)\exp(iy) : \frac{\pi}{6} < y < \frac{\pi}{3}\}$, which is the part of the circular arc with radius $\exp(1)$ with argument varies from $\frac{\pi}{6}$ to $\frac{\pi}{3}$.

4. similarly, the vertical line segment $\{(2,y) : \frac{\pi}{6} < y < \frac{\pi}{3}\}$ is mapped onto $\{\exp(2)\exp(iy) : \frac{\pi}{6} < y < \frac{\pi}{3}\}$, which is the part of the circular arc with radius $\exp(2)$ with argument varies from $\frac{\pi}{6}$ to $\frac{\pi}{3}$.

Thus, the image of the rectangle under exp is shown as in the following diagram.



Exercise 3.4.7

1. Prove that the map $f(z) = z^2$, $\forall z \in \mathbb{C}$ maps the first quadrant $\{(x,y) \in \mathbb{C} : x > 0, y > 0\}$ onto the upper half-plane $\{(x,y) \in \mathbb{C} : y > 0\}$.

2. Prove that the map $f(z) = \dfrac{z-1}{z+1}$, $\forall z \in \mathbb{C} \setminus \{-1\}$ maps the upper half-plane $\{(x,y) \in \mathbb{C} : y > 0\}$ onto the unit circle $\{z \in \mathbb{C} : |z| < 1\}$.

3.5 ELEMENTARY RIEMANN SURFACE

The functions $z \mapsto z^{1/n}$, log and arccos, and so on are not single-valued functions because $z \mapsto z^n$, exp and cos are not injective. Therefore, in the previous section, we have defined branches of each of these multivalued functions by omitting all but one of its images at every point. However, Riemann introduced an alternate concept, called elementary surfaces corresponding to every multivalued function, using which the multivalued function becomes a bijective single-valued function between a region of the complex plane onto a suitable surface.

A Riemann surface corresponding to a non-injective function f, intuitively speaking, is a surface obtained by attaching domains (commonly

called sheets) of all the different branches of the multivalued function f^{-1} so that f becomes a bijection from $\mathbb{C}$ onto the surface (or equivalently, f^{-1} can be defined as a single-valued function from the surface into $\mathbb{C}$) without losing any of the multivalued information.

The rigorous definition of elementary Riemann surface uses terminologies from *topology*, which are not within the scope of this book. For the sake of completeness, here we provide the rigorous definition of a Riemann surface.

Definition 3.5.1 A Riemann surface is a two-dimensional real analytic manifold.[1]

Example 3.5.2 Construct the Riemann surface corresponding to $z \mapsto z^n$ for some $n \in \mathbb{N}$ with $n > 1$.

We first note that $z \mapsto z^n$ is not injective; indeed, for each $\theta \in \mathbb{R}$, all of the following pairwise distinct n points $r \exp\left(i\left(\frac{\theta+2\pi}{k}\right)\right)$, $k = 0, 1, 2, \ldots, n-1$, are mapped to the same value $r \exp(i\theta)$. We also observe that the function $z \mapsto z^n$ maps positive real axis onto itself. Although any sector

$$\left\{z \in \mathbb{Z} \setminus \{0\} : a < \arg z < a + \frac{2\pi}{n}\right\}$$

is mapped onto the same set $\mathbb{C} \setminus \{(x, 0) : x \geq 0\}$, for an arbitrary $a \in \mathbb{R}$, we distinguish the images of the sectors

$$\left\{z \in \mathbb{Z} \setminus \{0\} : \frac{2k\pi}{n} < \arg z < \frac{2(k+1)\pi}{n}\right\}$$

by denoting as

$$\{w \in \mathbb{C} \setminus \{0\} : 2k\pi < \arg w < 2(k+1)\pi\}$$

in n different complex planes, and we call them by sheet-k for every $k \in 0, 1, 2, \ldots, n-1$. As the boundary $\arg z = \frac{2(k+1)\pi}{n}$ in sheet-k and the boundary $\arg z = \frac{2(k+1)\pi}{n}$ in sheet-$(k+1)$ are common, by using these n sheets, we construct the Riemann surface as follows:

1. The edge $\arg z = \frac{2(k+1)\pi}{n}$ at the bottom of the slit in sheet-k and $\arg z = \frac{2(k+1)\pi}{n}$ at the top of the slit in sheet-$(k+1)$ should be pasted, for all $k = 0, 1, 2, \ldots, n-2$.

[1] A studious reader can refer to any book on topology to know the definitions of the terminologies used in the above definition.

2. The boundary $\arg z = 2\pi$ at the bottom of the slit in sheet-$(n-1)$ should be pasted with the boundary $\arg z = 0$ at the top of the slit in sheet-0.

3. The origin at every sheet should be glued as a single point in the surface.

Now, if $\arg z$ traverses in the z-plane from 0 to 2π through $\frac{2\pi}{k}$, $k = 1, 2, \ldots, n-1$, correspondingly, $\arg z^n$ navigates from 0 to 2π in sheet-0, then it moves to the successive sheets through the pasted edges $\arg z = \frac{2(k+1)\pi}{n}$, and finally, it comes back to the top edge of the slit in sheet-0. In other words, we have shown geometrically that the function $z \mapsto z^n$ is a continuous bijection from $\mathbb{C}$ onto the surface.

Example 3.5.3 Construct the Riemann surface associated with cos.

We observe that the non-negative imaginary axis $\{(0,y) : y \geq 0\}$ is mapped into $\{(u,0) : u \geq 1\}$ because

$$\begin{aligned} \cos(0+iy) &= \cos(0)\cosh(y) - i\sin(0)\sinh(y) = \cosh(y) \\ &= \frac{\exp(y)+\exp(-y)}{2} \geq 1. \end{aligned}$$

In fact, we see that $\cos : \{(0,y) : y \geq 0\} \to \{(u,0) : u \geq 1\}$ is onto. Indeed, for a given $u > 1$, using $\cosh(y) \to +\infty$ as $y \to \infty$, we find $y > 0$ such that $\cosh(y) > u$. Therefore, applying the intermediate value theorem, we can find $y_0 \in (0,y)$ such that $\cosh(y_0) = u$. Being cosh an even function, we also observe that $\{(0,y) : y \leq 0\}$ is mapped onto $\{(u,0) : u \geq 1\}$. By a similar argument, one can verify that cos maps

- both of $\{(\pi,y) : y \geq 0\}$ and $\{(\pi,y) : y \leq 0\}$ onto $\{(u,0) : u \leq 1\}$,
- $\{(x,y) : 0 < x < \pi,\ y > 0\}$ onto the lower half-plane $\{(u,v) : v < 0\}$,
- $\{(x,y) : 0 < x < \pi,\ y < 0\}$ onto the upper half-plane $\{(u,v) : v > 0\}$,
- $\{(x,y) : 0 < x < \pi,\ y = 0\}$ onto the line segment $\{(u,0) : |u| < 1\}$.

Now we claim that $\cos : \{(x,y) : 0 < x < \pi\} \to \mathbb{C} \setminus \{(u,v) \in \mathbb{C} : |u| \geq 0,\ v = 0\}$ is a bijection. As, the injectivity of cos on $\{(x,y) : 0 < x < \pi\}$ is obvious, we only verify that surjectivity of cos. Let $w \in \mathbb{C} \setminus \{(u,v) \in \mathbb{C} : |u| \geq 0,\ v = 0\}$ be arbitrary and let $\theta = \left(w \pm \sqrt{w^2-1}\right)$. If $\theta \in (0,\pi)$, then we take $z = \theta - i\log\left|w \pm \sqrt{w^2-1}\right|$ so that $z \in \{(x,y) : 0 < x < \pi\}$

and $\cos z = w$. Otherwise, $\theta \in (\pi, 2\pi)$. In this case, we choose the required pre-image by $z = 2\pi - \theta - i \log \left| w \pm \sqrt{w^2 - 1} \right|$.
By a similar argument, we obtain $\cos : \{(x, y) : (k-1)\pi < x < k\pi\} \to \mathbb{C} \setminus \{(u, v) \in \mathbb{C} : |u| \geq 0,\ v = 0\}$ is a bijection, $\forall k \in \mathbb{N}$, with the additional information that

- the upper and lower half-strips are mapped onto the lower and upper half-planes, respectively, when k is odd,
- the upper and lower half-strips are mapped onto themselves when k is even.

Now we are ready to construct the Riemann surface associated with cos.

- We take countably infinite number of complex planes with two slits from 1 to ∞ and -1 to ∞ and we denote them by

  $$\{w \in \mathbb{C} \setminus \{0\} : (2k+1)\pi \neq \arg w \in (2k\pi, (2k+2)\pi)\} \cup \{(u, 0) \in \mathbb{C} : |u| \leq 1\},$$

  called sheet-k, $\forall k \in \mathbb{Z}$.
- Paste upper edge on the slit on the left side of sheet-k with lower edge on the slit on the left side of sheet-$(k+1)$ and lower edge on the slit on the left side of sheet-k with upper edge on the slit on the left side of sheet-$(k+1)$.
- Paste upper edge on the slit on the right side of sheet-k with lower edge on the slit on the right side of sheet-$(k+1)$ and lower edge on the slit on the right side of sheet-k with upper edge on the slit on the right side of sheet-$(k+1)$.
- By gluing the line segment $\{(u, 0) : |u| \leq 1\}$ in every sheet altogether, identify it as a single line segment on the surface.

Then, cos becomes a continuous bijection from $\mathbb{C}$ onto the surface.

Exercise 3.5.4 Construct the Riemann surface corresponding to $\exp(z)$.

4

Complex Integration

4.1 LINE INTEGRAL

In this section, first we recall some definitions and results (without proofs) from the theory of Riemann integration of real-valued function on $[a, b]$.

Definition 4.1.1 Let $f : [a, b] \to \mathbb{R}$ be a bounded function, where $-\infty < a < b < +\infty$.

1. By a partition of $[a, b]$, we mean a finite subset $\{t_0, t_1, t_2, \ldots, t_n\}$ with $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.
2. Define $U(P, f) = \sum_{k=1}^{n} \left[\sup_{t \in [t_{k-1}, t_k]} f(t) \right] (t_k - t_{k-1})$ and $L(P, f) = \sum_{k=1}^{n} \left[\inf_{t \in [t_{k-1}, t_k]} f(t) \right] (t_k - t_{k-1})$.
3. f is Riemann integrable if $\inf_P U(P, f) = \sup_P L(P, f)$ and this value is denoted by $\int_a^b f(t)\, dt$.

THEOREM 4.1.2 Let $f : [a, b] \to \mathbb{R}$ be a bounded real-valued function. Then f is Riemann integrable on $[a, b]$ iff given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

THEOREM 4.1.3 If f, f_1, and f_2 are Riemann integrable functions on $[a, b]$, then

1. $\int_a^b (f_1 + f_2)(t)\, dt = \int_a^b f_1(t)\, dt + \int_a^b f_2(t)\, dt$,

2. $\int_a^b (\alpha f)(t)\,dt = \alpha \int_a^b f(t)\,dt, \forall \alpha \in \mathbb{R}$,

3. $\int_a^b f_1(t)\,dt \leq \int_a^b f_2(t)\,dt$, whenever $f_1 \leq f_2$ on $[a, b]$,

4. $\int_a^b f(t)\,dt = \int_c^b f(t)\,dt + \int_a^c f(t)\,dt$, provided $a < c < b$.

RESULT 4.1.4 If $f : [a, b] \to \mathbb{R}$ is bounded and continuous except at a finite number of points of $[a, b]$, then f is Riemann integrable.

RESULT 4.1.5 If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then $|f|$ is also Riemann integrable over $[a, b]$ and $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$.

Definition 4.1.6 Let $f = u + iv$ be a complex-valued continuous function on $[a, b]$, where u and v are real-valued continuous functions on $[a, b]$. We define

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

where $\int_a^b u(t)\,dt$ and $\int_a^b v(t)\,dt$ are Riemann integrals of u and v, respectively, on $[a, b]$.

RESULT 4.1.7 If $f : [a, b] \to \mathbb{C}$, $g : [a, b] \to \mathbb{C}$ are continuous and $c \in \mathbb{C}$, then $\int_a^b (f + g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$ and $\int_a^b (cf)(t)\,dt = c\int_a^b f(t)\,dt$.

Proof: Let $f = u_1 + iv_1$, $g = u_2 + iv_2$ and $c = \alpha + i\beta$. Then

$$\int_a^b (f + g)(t)\,dt$$

$$= \int_a^b (u_1 + u_2)(t)\,dt + i\int_a^b (v_1 + v_2)(t)\,dt$$

$$= \left(\int_a^b u_1(t)\,dt + \int_a^b u_2(t)\,dt \right) + i \left(\int_a^b v_1(t)\,dt + \int_a^b v_2(t)\,dt \right)$$

$$= \left(\int_a^b u_1(t)\,dt + i \int_a^b v_1(t)\,dt \right) + \left(\int_a^b u_2(t)\,dt + i \int_a^b v_2(t)\,dt \right)$$

$$= \int_a^b f(t)\,dt + \int_a^b g(t)\,dt,$$

$$\int_a^b (cf)(t)\,dt,$$

$$= \int_a^b ((\alpha + i\beta)(u + iv))(t)$$

$$= \int_a^b (\alpha u(t) - \beta v(t))\,dt + i \int_a^b (\alpha v(t) + \beta u(t))\,dt$$

$$= \alpha \int_a^b u(t)\,dt - \beta \int_a^b v(t)\,dt + i \left(\alpha \int_a^b v(t)\,dt + \beta \int_a^b u(t)\,dt \right)$$

$$= (\alpha + i\beta) \left(\int_a^b u(t)\,dt + i \int_a^b v(t)\,dt \right)$$

$$= c \int_a^b f(t)\,dt.$$

Hence, the result follows. □

LEMMA 4.1.8 If $f : [a, b] \to \mathbb{C}$ is a continuous function, then $\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$.

Proof: If $\int_a^b f(t)\,dt = 0$, then obviously the inequality follows. If $\int_a^b f(t)\,dt \neq 0$, then we can write it as $\int_a^b f(t)\,dt = re^{i\theta}$. Then, using the previous theorem, we get

$$\begin{aligned}
\left|\int_a^b f(t)\,dt\right| &= r \\
&= e^{-i\theta}\int_a^b f(t)\,dt \\
&= \int_a^b e^{-i\theta} f(t)\,dt \\
&= \text{Re}\left(\int_a^b e^{-i\theta} f(t)\,dt\right) \\
&= \int_a^b \text{Re}\,(e^{-i\theta} f(t))\,dt \\
&\leq \int_a^b \left|e^{-i\theta} f(t)\right|\,dt \\
&= \int_a^b |f(t)|\,dt.
\end{aligned}$$

□

RESULT 4.1.9 If $f : [a,b] \to \mathbb{C}$ is Riemann integrable and $|f(t)| \leq M$, $\forall t \in [a,b]$, then $\int_a^b |f(t)|\,dt \leq M(b-a)$.

Proof of this result follows immediately from the previous Lemma.

RESULT 4.1.10 Let f_n and f be complex-valued Riemann integrable functions on $[a,b]$, $\forall n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$, then $\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/(b-a)\ \forall\, n \geq N$ and $\forall\, x \in [a,b]$.

Now for $n \geq N$,

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \int_a^b |f_n(x) - f(x)|dx < \frac{\epsilon}{b-a} \int_a^b dx = \epsilon.$$

Therefore, $\int_a^b f_n(x)\,dx \to \int_a^b f dx$ as $n \to \infty$. □

THEOREM 4.1.11 If $f : [a,b] \to \mathbb{C}$ is continuous and $g(t) = \int_a^t f(\tau)\,d\tau,\ \forall t \in [a,b]$, then g is differentiable on $[a,b]$ and $g' = f$.

Proof: Let $t \in [a,b)$ be arbitrary. Since f is continuous at t, given $\epsilon > 0$, there exists $\delta > 0$ such that $[t, t+\delta] \subset [a,b)$ and $t \leq s < t+\delta \Rightarrow |f(s) - f(t)| < \epsilon$. For $0 < h < \delta$, we have

$$\begin{aligned}
\left| \frac{g(t+h) - g(t)}{h} - f(t) \right| &= \left| \frac{\int_a^t f(\tau)\,d\tau - \int_a^{t+h} f(\tau)\,d\tau - \int_t^{t+h} f(t)\,d\tau}{h} \right| \\
&= \frac{1}{h} \left| \int_t^{t+h} (f(\tau) - f(t))\,d\tau \right| \\
&\leq \frac{1}{h} \int_t^{t+h} |f(\tau) - f(t)|\,d\tau \\
&= \frac{1}{h} h\epsilon = \epsilon, \quad \text{since } |\tau - t| < h < \delta.
\end{aligned}$$

Similarly, we can prove that for every $t \in (a,b]$, there exits $\delta > 0$ such that $\left| \frac{g(t-h) - g(t)}{-h} - f(t) \right| < \epsilon$, whenever $0 < h < \delta$. Therefore, we have proved that

$$\lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = f(t),\ \forall t \in [a,b].$$

Thus, $g' = f$ on $[a,b]$. □

THEOREM 4.1.12 (Fundamental theorem of calculus)
If $f : [a,b] \to \mathbb{C}$ is differentiable on $[a,b]$ and f' is Riemann itegrable on $[a,b]$, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.

Proof: First, we prove this theorem for the case that f is a real-valued Riemann integrable function on $[a,b]$. By Theorem 4.1.2, given $\epsilon > 0$, there exists a partition $P = \{t_0, t_1, t_2, \ldots, t_n\}$ of $[a,b]$ such that $U(P,f') - L(P,f') < \epsilon$. By mean value theorem, there exists $s_k \in (t_{k-1}, t_k)$ such that

$$f(t_k) - f(t_{k-1}) = f'(s_k)(t_k - t_{k-1}),\ \forall k = 1, 2, \ldots, n. \tag{4.1}$$

If $M_k = \sup_{t_{k-1}<t<t_k} f'(t)$ and $m_k = \inf_{t_{k-1}<t<t_k} f'(t)$, $\forall k = 1, 2, \ldots, n$, then we have $m_k \le f'(s_k) \le M_k, \forall k = 1, 2, \ldots, n$, which implies that

$$\begin{aligned} L(P, f') &= \sum_{k=1}^{n} m_k(t_k - t_{k-1}) \\ &\le \sum_{k=1}^{n} f'(s_k)(t_k - t_{k-1}) \\ &\le \sum_{k=1}^{n} M_k(t_k - t_{k-1}) \\ &= U(P, f'). \end{aligned}$$

On the other hand, by the definition of Riemann integral, we also have $L(P, f') \le \int_a^b f'(t)\,dt \le U(P, f')$, and hence,

$$\left| \int_a^b f'(t)\,dt - \sum_{k=1}^{n} f'(s_k)(t_k - t_{k-1}) \right| \le U(P, f) - L(P, f).$$

Therefore, using equation (4.1), we get

$$\left| \int_a^b f'(t)\,dt - f(a) + f(b) \right| = \left| \int_a^b f'(t)\,dt - \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \right|$$

$$\begin{aligned} &= \left| \int_a^b f'(t)\,dt - \sum_{k=1}^{n} f'(s_k)(t_k - t_{k-1}) \right| \\ &\leq U(P,f) - L(P,f) < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get $\int_a^b f'(t)\,dt = f(b) - f(a)$.

If f is a complex-valued function, let $f = u + iv$, where u and v are real-valued Riemann integrable functions. Hence, using this theorem for u and v, we obtain

$$\begin{aligned} \int_a^b f'(t)\,dt &= \int_a^b u'(t)\,dt + i\int_a^b v'(t))\,dt \\ &= (u(b) - u(a)) + i(v(b) - v(a)) \\ &= f(b) - f(a). \end{aligned}$$

Hence, the theorem follows. □

Example 4.1.13 Find $\int_0^2 [(1 + 9t^2) + i(4t + 2)]\,dt$.

$$\begin{aligned} \int_0^2 [(1 + 9t^2) + i(4t + 2)]\,dt &= \int_0^2 (1 + 9t^2)\,dt + i\int_0^2 (4t + 2)\,dt \\ &= \left[t + 3t^3\right]_0^2 + i\left[2t^2 + 2t\right]_0^2 \\ &= 26 + 12i. \end{aligned}$$

Definition 4.1.14 (Piecewise smooth curve)
A curve γ with a parametric equation $\varphi(t),\ t \in [a, b]$ is called a piecewise smooth curve if φ is differentiable and φ' is continuous for all but finite number of points $\{t_j : 1 \leq j \leq n\}$ of $[a, b]$. Furthermore, φ has both left limit and right limit at each t_j, $1 \leq j \leq n$. Hereafter by a curve, we mean a piecewise smooth curve.

Definition 4.1.15 A curve with a parametric equation $\varphi(t),\ t \in [a, b]$ is said to be

1. a closed curve if $\varphi(a) = \varphi(b)$.
2. a simple curve if φ is one-to-one on $[a, b]$.
3. a simple closed curve if it is closed and $\phi(t) = \phi(s) \Rightarrow$ either $t = s$ or $\{t, s\} = \{a, b\}$.

Definition 4.1.16 Let Ω be a region and let γ be a curve in Ω. If f is a continuous function on Ω, then define the line integral

$$\int_{\gamma} f(z)\,dz = \int_{a}^{b} f(\varphi(t))\varphi'(t)\,dt,$$

where $\varphi(\tau),\ \tau \in [a, b]$ is a parametric equation of γ.

Definition 4.1.17 Let γ be a curve in a region Ω. If f is a continuous function on Ω and g has a continuous partial derivatives on Ω, then define $\int_{\gamma} f\,dg = \int_{a}^{b} f\left(\frac{\partial g}{\partial x}\,dx - \frac{\partial g}{\partial y}\,dy\right)$. More explicitly, if $\varphi(t),\ t \in [\alpha, \beta]$ is the parametric equation of γ, then

$$\int_{\gamma} f\,dg = \int_{\alpha}^{\beta} f(\phi(t))\left(\frac{\partial g}{\partial x}(\phi(t))\,(\mathrm{Re}\ \varphi)'(t) - \frac{\partial g}{\partial y}(\phi(t))\,(\mathrm{Im}\ \varphi)'(t)\right)\,dt.$$

Example 4.1.18 Find $\int_{\gamma} f(z)\,dz$, where $f(z) = z + 1,\ \forall z \in \mathbb{C}$ and $\varphi(t) = t + it,\ t \in [0, 1]$.

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \int_{0}^{1} f((\varphi(t))\varphi'(t)\,dt \\
&= \int_{0}^{1} (\varphi(t) + 1)(1 + i)\,dt \\
&= \int_{0}^{1} ((1 + t) + it)(1 + i)\,dt \\
&= \int_{0}^{1} [((1 + t) - t) + i(1 + t + t)]\,dt
\end{aligned}$$

$$\begin{aligned} &= \int_0^1 (1 + i(2t+1))\, dt \\ &= \int_0^1 dt + i\int_0^1 (2t+1)\, dt \\ &= 1 + i2. \end{aligned}$$

Example 4.1.19 If $f(z) = z^2$ and γ is the parabola given by $2t + i(t^2 + 1)$ $t \in \mathbb{R}$, then find $\int_\sigma f(z)\, dz$, where σ is the part of γ from i to $2 + i2$.

First, we note that the points i and $2 + i2$ correspond to $t = 0$ and $t = 1$, respectively, and hence, the parametric equation of σ is $\varphi(t) = 2t + i(t^2 + 1)$, $t \in [0, 1]$. Now, for every $t \in [0, 1]$, we have

$$\begin{aligned} \varphi'(t) &= 2 + i2t \\ f(\varphi(t)) &= (2t + i(t^2+1))^2 \\ &= (4t^2 - (t^2+1)^2) + i2t(t^2+1) \\ &= (-t^4 + 2t^2 - 1) + i(2t^3 + 2t) \\ f(\varphi(t))\varphi'(t) &= 2(-t^4 + 2t^2 - 1) - 2t(2t^3 + 2t) \\ &\quad + i(4t^3 + 4t - 2t^5 + 4t^3 - 2t) \\ &= (-6t^4 - 2) + i(-2t^5 + 8t^3 + 2t). \\ \int_\sigma f(z)\, dz &= \int_0^1 f((\varphi(t))\varphi'(t)\, dt \\ &= \int_0^1 (-6t^4 - 2)\, dt + i\int_0^1 (-2t^5 + 8t^3 + 2t)\, dt \\ &= \left(-\frac{6}{5} - 2\right) + i\left(-\frac{1}{3} + 2 + 1\right) \\ &= -\frac{16}{5} + i\frac{8}{3}. \end{aligned}$$

Example 4.1.20 Find $\int_\gamma (z+3)\, dz$, where γ is the circular arc given by $\varphi(t) = i + 2e^{it},\ t \in [0, \pi]$.

Solution:

$$\begin{aligned}\int_{\gamma}(z+3)\,dz &= \int_0^{\pi}[(i+2e^{it})+3]i2e^{it}\,dt\\ &= i2\int_0^{\pi}[(3+i)e^{it}+2e^{i2t})]\,dt\\ &= i2\left[-(3+i)ie^{it}+-ie^{i2t}\right]_0^{\pi}\\ &= -12-i4.\end{aligned}$$

Exercise 4.1.21

1. Evaluate $\int_{\gamma} f(z)\,dz$, where f and the parametric equation '$\varphi(t)$, $t\in[a,b]$' of γ are given as follows:

   (1) $f(z)=z^3$, $\varphi(t)=1+it$, $t\in[0,1]$,

   (2) $f(z)=\bar{z}$, $\varphi(t)=\exp(i\pi t)$, $t\in[0,1]$,

   (3) $f(z)=\dfrac{1}{z}$, $\varphi(t)=1+2t+it^2$, $t\in[0,1]$,

   (4) $f(z)=z\exp(z^2)$, $\varphi(t)=\sqrt{t}$, $t\in[1,4]$,

   (5) $f(z)=\cos(z)$, $\varphi(t)=t+it^2$, $t\in[0,1]$.

   Answers: (1) $\dfrac{-5}{4}$; (2) $i\pi$; (3) $\log(3+i)$; (4) $\dfrac{\exp(1)}{2}(\exp(3)-1)$; (5) $\sin(1+i)$.

2. Evaluate $\int_{\gamma} f(z)\,dz$, where $f(z)=z^2+z$, where γ is the part of the parabola $x=y^2$ from $(0,0)$ to $(4,2)$.

   Answer: $\dfrac{2}{3}(17+i56)$.

3. Evaluate $\int_{\gamma} f(z)\,dz$, where $f(z)=\dfrac{1}{z-a}$, where γ is the part of the circle with center $a\in\mathbb{C}$ and radius $r>0$ from $a+r$ to $a+ir$.

   Answer: $i\dfrac{\pi}{2}$.

Definition 4.1.22 Let Ω be a region and let γ be a curve in Ω. If f is a continuous function on Ω, then define $\int_{\gamma} f(z)\,|dz| = \int_a^b f(\varphi(t))|\varphi'(t)|\,dt$, where $\varphi(t),\ t \in [a,b]$ is a parametric equation of γ.

THEOREM 4.1.23 If f is continuous on Ω and γ is a curve in Ω, then $\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\ |dz|$.

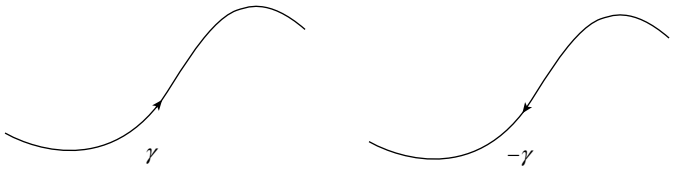
Proof: Using Lemma 4.1.8, if $\varphi(t),\ t \in [a,b]$ is a parametric equation of γ, then

$$\left|\int_{\gamma} f(z)\,dz\right| = \left|\int_a^b f(\varphi(t))\varphi'(t)\,dt\right| \leq \int_a^b |f(\varphi(t))\varphi'(t)|\,dt = \int_{\gamma} |f(z)|\ |dz|.$$

Hence, the theorem follows. □

Definition 4.1.24 If a curve γ is given by $\varphi(t),\ t \in [a,b]$, then the opposite curve $-\gamma$ is defined by $\psi(t) = \varphi(a+b-t),\ t \in [a,b]$.

Geometrically, $-\gamma$ is obtained from γ just by changing its direction in the opposite sense.



LEMMA 4.1.25 If f is continuous on Ω and γ is a curve in Ω, then $\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$.

Proof: If $\varphi(t),\ t \in [a,b]$ is a parametric equation of γ and $\psi(t) = \varphi(a+b-t),\ t \in [a,b]$ is the parametric equation of $-\gamma$, then using the change of variable $s = a+b-t$ in the following, we get

$$\begin{aligned}\int_{-\gamma} f(z)\,dz &= \int_a^b f(\psi(t))\psi'(t)\,dt \\ &= \int_a^b f(\varphi(a+b-t))\varphi'(a+b-t)\,(-dt)\end{aligned}$$

$$\begin{aligned} &= \int_b^a f(\varphi(s))\varphi'(s)\,ds \\ &= -\int_a^b f(\varphi(s))\varphi'(s)\,ds \\ &= -\int_\gamma f(z)\,dz. \end{aligned}$$

Hence, the lemma follows. □

Definition 4.1.26 (Arc length)
If γ is curve with parametric equation $z(t),\ t \in [a,b]$, then the length of the curve is defined by $L(\gamma) = \sup_P |z(t_k) - z(t_{k-1})|$, where P varies over all partitions of $[a,b]$.

A curve with finite length is called a rectifiable curve.

THEOREM 4.1.27 If γ is a smooth curve with the parametric equation $z(\tau),\ \tau \in [a,b]$, then it is rectifiable and $L(\gamma) = \int_a^b |z'(t)|\,dt$.

Proof: For each fixed $t \in [a,b]$, let $\gamma_{a,t}$ be the part of the curve γ with parametric equation $z(\tau) = u(\tau) + iv(\tau),\ \tau \in [a,t]$, where u and v are real-valued differentiable function on $[a,b]$. Define $S : [a,b] \to \mathbb{R}$ by $S(t) = L(\gamma_{a,t}),\ \forall t \in [a,b]$. We shall show that S is a differentiable function and its $S'(t) = |z'(t)|,\ \forall t \in [a,b]$. Therefore, using the uniform continuity of z', given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} |\tau - t| < \delta \quad &\Rightarrow \quad |z'(\tau) - z'(t)| < \frac{\epsilon}{\sqrt{2}} \\ &\Rightarrow \quad |u'(\tau) - u'(t)| < \frac{\epsilon}{\sqrt{2}} \text{ and } |v'(\tau) - v'(t)| < \frac{\epsilon}{\sqrt{2}}. \end{aligned} \tag{4.2}$$

Therefore, for $|\tau - t| < \delta$ and $|\sigma - t| < \delta$, we have

$$\begin{aligned} \big|\, |(u'(\tau), v'(\sigma))| - |z'(t)| \,\big| &= \big|\, |(u'(\tau), v'(\sigma))| - |(u'(t), v'(t))| \,\big| \\ &\le |(u'(\tau), v'(\sigma)) - (u'(t), v'(t))| \\ &= \sqrt{(u'(\tau) - u'(t))^2 + (v'(\sigma) - v'(t))^2} \\ &< \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} \\ &= \epsilon. \end{aligned}$$

Hence, it follows that

$$|z'(t)| - \epsilon < |(u'(\tau), v'(\sigma))| < |z'(t)| + \epsilon, \forall \sigma, \tau \in (t-\delta, t+\delta). \tag{4.3}$$

Let $0 < s < \delta$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be an arbitrary partition of $[t, t+s]$. Then, applying mean-value theorem (Theorem 2.1.18) for u and v, we get

$$\begin{aligned} \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})| &= \sum_{k=1}^{n} \sqrt{(u(t_k) - u(t_{k-1}))^2 + (v(t_k) - v(t_{k-1}))^2} \\ &= \sum_{k=1}^{n} (t_k - t_{k-1}) |(u'(x_k), v'(y_k))|, \end{aligned} \tag{4.4}$$

for some $x_k, y_k \in (t_{k-1}, t_k)$.
Therefore,

$$\begin{aligned} s(|z'(t)| - \epsilon) &= (t_n - t_0)(|z'(t)| - \epsilon) \\ &= \sum_{k=1}^{n} (t_k - t_{k-1})(|z'(t)| - \epsilon) \\ &< \sum_{k=1}^{n} (t_k - t_{k-1}) |(u'(x_k), v'(y_k))| \text{ (by using (4.3))} \\ &= \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})| \text{ (by using (4.4))} \\ &\leq \sum_{k=1}^{n} (t_k - t_{k-1}) |z'(s_k)|, \text{ for some } s_k \in (t_{k-1}, t_k) \\ &\quad \text{(by using Theorem 2.1.19 for the function } z.) \\ &< (t_n - t_0)(|z'(t)| + \epsilon) \\ &\quad \text{(since } |s_k - t| < \delta, \text{ by using (4.3))} \\ &= s(|z'(t)| + \epsilon). \end{aligned}$$

Hence, we proved that

$$s(|z'(t)| - \epsilon) \leq \sup_P \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})| \leq s(|z'(t)| + \epsilon),$$

where P varies over all partitions of $[t, t+s]$. Therefore, if $\gamma_{t,\, t+s}$ is the part of gamma, whose parametric equation is $z(\tau),\ \tau \in [t, t+s]$, then it follows that

$$s(|z'(t)| - \epsilon) \le L(\gamma_{t,t+s}) = (S(t+s) - S(t)) \le s(|z'(t)| + \epsilon),$$

which implies that $\left|\dfrac{S(t+s)-S(t)}{s} - |z'(t)|\right| < \epsilon$. Similarly, we can show that if $0 < s < \delta$, $\left|\dfrac{S(t-s)-S(t)}{(-s)} - |z'(t)|\right| < \epsilon$. That is, we have proved that $S'(t) = \lim\limits_{s\to 0} \dfrac{S(t+s)-S(t)}{s} = |z'(t)|$. Therefore, by Theorem 4.1.11, we get

$$S(t) = \int_a^t S'(\tau)\, d\tau = \int_a^t S'(\tau)\, d\tau, \ \forall t \in [a,b].$$

In particular, $L(\gamma) = S(b) = \int_a^b S'(\tau)\, d\tau$. □

LEMMA 4.1.28 Let $f(z)$ be a continuous function on a region Ω. If there exists an analytic function F on Ω such that $F' = f$, then

$$\int_\gamma f(z)\, dz = F(\varphi(b)) - F(\varphi(a)),$$

for every curve γ in Ω, where $\varphi(t),\ t \in [a,b]$ is a parametric equation of γ.

Proof: Define $h(t) = F(\varphi(t)),\ \forall t \in [a,b]$, then h is piecewise differentiable and $h'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$ for every t at which φ is differentiable. By fundamental theorem of calculus (Theorem 4.1.12),

$$\begin{aligned}\int_\gamma f(z)\, dz &= \int_a^b f(\varphi(t))\varphi'(t)\, dt \\ &= \int_a^b h'(t)\, dt \\ &= h(b) - h(a) \\ &= F(\varphi(b)) - F(\varphi(a)).\end{aligned}$$

Hence, the lemma follows. □

COROLLARY 4.1.29 Let $f(z)$ be a continuous function on a region Ω. If there exists an analytic function F on Ω such that $F' = f$, then $\int_\gamma f(z)\, dz = 0$, for every closed curve γ in Ω.

Proof: Since γ is a closed curve, if $\varphi(t),\ t \in [a,b]$ is a parametric equation of γ, then $\varphi(a) = \varphi(b)$. Hence, using the previous lemma, we get

$$\int_{\gamma} f(z)\,dz = F(\varphi(b)) - F(\varphi(a)) = 0.$$

Converse of the above corollary is also true, which follows. □

RESULT 4.1.30 If f is a continuous function on a region Ω such that $\int_{\gamma} f(z)\,dz = 0$, for every closed curve γ in Ω, then f is the derivative of an analytic function F on Ω.

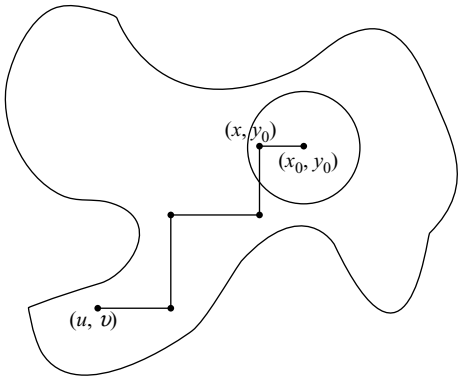
Proof: Let $w = (u,v) \in \Omega$ be arbitrary. Then, for a given $z_0 = (x_0, y_0) \in \Omega$ and an $\epsilon > 0$, choose $\delta > 0$ such that $B(z_0, \delta) \subseteq \Omega$ and

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta. \tag{4.5}$$

Let P_{z_0} be the polygon joining w and z_0 such that

1. the line segments of P_{z_0} are parallel to the coordinate axes.
2. the line segment incident with z_0 is horizontal and completely contained in $B(z_0, \delta)$.
3. $P_{z_0} \subset \Omega$.

Such a polygon exists by Theorem 1.4.31.



Define $F(z_0) = \int\limits_{P_{z_0}} f(z)\,dz$. We claim that $\frac{\partial F}{\partial x}(x_0, y_0) = f(x_0, y_0)$. Let (x, y_0) be the point adjacent with (x_0, y_0) in the polygon P_{z_0}. As the parametric equation of $L_{(x,y_0),(x_0,y_0)}$ and $L_{(x,y_0),(x_0+h,y_0)}$ are

$$\varphi(t) = t + iy_0 = (t, iy_0),\ t \in [x, x_0]$$

and

$$\psi(t) = t + iy_0 = (t, iy_0),\ t \in [x, x_0 + h],$$

respectively, for each $h \in \mathbb{R}$ such that $|h| < \delta$,

$$\begin{aligned}
\frac{F(x_0+h, y_0) - F(x_0, y_0)}{h} &= \frac{\int\limits_{P_{z_0+h}} f(z)\,dz - \int\limits_{P_{z_0}} f(z)\,dz}{h} \\
&= \frac{\int\limits_{L_{(x,y_0),(x_0+h,y_0)}} f(z)\,dz - \int\limits_{L_{(x,y_0),(x_0,y_0)}} f(z)\,dz}{h} \\
&= \frac{\int\limits_{x}^{x_0+h} f(t, y_0)\,dt - \int\limits_{x}^{x_0} f(t, y_0)\,dt}{h} \\
&= \frac{1}{h} \int\limits_{x_0}^{x_0+h} f(t, y_0)\,dt \\
&= \frac{1}{h} \int\limits_{0}^{h} f(x_0 + s, y_0)\,ds. \qquad (4.6)
\end{aligned}$$

Now, using the equation (4.6), we get

$$\begin{aligned}
&\left| \frac{F(x_0+h, y_0) - F(x_0, y_0)}{h} - f(x_0, y_0) \right| \\
&= \left| \frac{1}{h} \int\limits_{0}^{h} f(x_0 + s, y_0)\,ds - f(x_0, y_0) \right| \\
&= \left| \frac{1}{h} \int\limits_{0}^{h} (f(x_0 + s, y_0) - f(x_0, y_0))\,ds \right| \\
&\leq \frac{1}{|h|} \int\limits_{0}^{h} |f(x_0 + s, y_0) - f(x_0, y_0)|\,|ds|
\end{aligned}$$

$$< \frac{1}{|h|}\int_0^h \epsilon\,|ds|$$
$$= \frac{|h|\epsilon}{|h|} = \epsilon,$$

as $|(x_0+s,y_0)-(x_0,y_0)| = |t| < |h| < \delta$ and by using equation (4.5). Thus, our claim is proved. Next for every $z_0 \in \Omega$, we define $G(z_0) = \int\limits_{Q_{z_0}} f(z)\,dz$, where Q_{z_0} is a polygon joining w and z_0 such that

1. the line segments of Q_{z_0} are parallel to the coordinate axes.
2. the line incident with z_0 is vertical and completely contained in $B(z_0,\delta)$.
3. $Q_{z_0} \subseteq \Omega$.

Next we claim that $\frac{\partial G}{\partial y}(x_0,y_0) = if(x_0,y_0)$. We fix $\epsilon > 0$, $\delta > 0$ and $h \in \mathbb{R}$ as before. As the parametric equation of $L_{(x_0,v),(x_0,y_0)}$ is

$$\varphi(t) = x_0 + it = (x_0,t),\ t \in [v,y_0],$$

by a similar argument, we get

$$\left|\frac{G(x_0,y_0+h)-G(x_0,y_0)}{h} - if(x_0,y_0)\right|$$
$$= \left|\frac{1}{h}\int_0^h f(x_0,y_0+t)\,idt - if(x_0,y_0)\right|$$
$$\leq \frac{1}{|h|}\int_0^h |f(x_0,y_0+t)-f(x_0,y_0)|\,|dt| < \epsilon.$$

Since

$$F(z_0)-G(z_0) = \int_{P_{z_0}} f(z)\,dz - \int_{Q_{z_0}} f(z)\,dz = \int_{P_{z_0}\cup(-Q_{z_0})} f(z)\,dz$$

and $P_{z_0}\cup(-Q_{z_0})$ is a closed curve, by hypothesis, we get $\int\limits_{P_{z_0}\cup(-Q_{z_0})} f(z)\,dz = 0$, and hence, $F(z_0) - G(z_0) = 0$. In other words, $F = G$ on Ω. Therefore, F has continuous partial derivatives, and they satisfy the C–R equation $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}(= f)$. Thus, F is analytic on Ω (by Theorem 2.2.28) and $F' = \frac{\partial F}{\partial x} = f$. □

4.2 WINDING NUMBER AND CAUCHY'S THEOREMS

LEMMA 4.2.1 For a curve γ and a point z_0 not on γ, $\dfrac{1}{i2\pi}\displaystyle\int_{\gamma}\frac{dz}{z-z_0}$ is an integer.

Proof: Let $\varphi(t)$, $t \in [a,b]$ be a parametric equation of the closed curve γ. Then we have $\varphi(a) = \varphi(b)$ and

$$\frac{1}{i2\pi}\int_{\gamma}\frac{dz}{z-z_0} = \frac{1}{i2\pi}\int_{a}^{b}\frac{\varphi'(t)\,dt}{\varphi(t)-z_0}.$$

If $F(s) = \displaystyle\int_{a}^{s}\frac{\varphi'(t)\,dt}{\varphi(t)-z_0}$, $\forall s \in [a,b]$, then F is a differentiable function on $[a,b]$ and $F'(s) = \frac{\varphi'(s)}{\varphi(s)-z_0}$, $\forall s \in [a,b]$, by Theorem 4.1.11. To conclude the theorem, we shall show that $F(b)$ is an integral multiple of $i2\pi$. If $\psi(s) = \exp(-F(s))(\varphi(s)-z_0)$, $\forall s \in [a,b]$, then

$$\begin{aligned}\psi'(s) &= -\exp(-F(s))F'(s)(\varphi(s)-z_0) + \exp(-F(s))\varphi'(s)\\ &= -\exp(-F(s))\varphi'(s) + \exp(-F(s))\varphi'(s) = 0,\end{aligned}$$

we get that ψ is a constant function on $[a,b]$. Therefore,

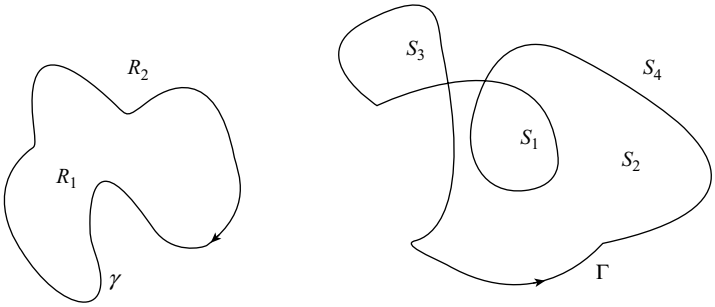
$$\exp(-F(a))(\varphi(a)-z_0) = \exp(-F(b))(\varphi(b)-z_0),$$

and hence, $\exp(-F(a)) = \exp(-F(b)) \Rightarrow \exp(F(b)) = 1$, as $F(a) = 0$. Using Lemma 2.4.20, it follows that $F(b) = i2k\pi$ for some $k \in \mathbb{Z}$. □

Definition 4.2.2 Let γ be a closed curve and z_0 be a point not on γ, then we define the winding number of γ with respect to z_0 or the index of z_0 with respect to γ, by $\text{WN}(\gamma, z_0) = \dfrac{1}{i2\pi}\displaystyle\int_{\gamma}\frac{dz}{z-z_0}$.

Definition 4.2.3 A region determined by a closed curve γ is defined by a component of the complement of γ in the extended complex plane.

The above definition can be understood easily by the following diagram.



Remark 4.2.4: For a given closed curve, among the regions determined by γ, there must be only one unbounded region determined by γ. That region is treated as the region containing ∞ in $\mathbb{C}_\infty$.
In the above diagrams, R_2 and S_4 are the unbounded regions determined by γ and Γ, respectively.

THEOREM 4.2.5 Let γ be a closed curve in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ not on γ. Then,

1. $\text{WN}(-\gamma, z_0) = -\text{WN}(\gamma, z_0)$,
2. $\text{WN}(\gamma, z_0) = \text{WN}(\gamma, w_0)$ if z_0 and w_0 belong to a same region determined by γ,
3. $\text{WN}(\gamma, z_0) = 0$ if z_0 belongs to the unbounded region determined by γ.

Proof: Let $z_0 \notin \gamma$.

1. Using Lemma 4.1.25, we get

$$\text{WN}(-\gamma, z_0) = \frac{1}{i2\pi} \int_{-\gamma} \frac{dz}{z - z_0} = -\frac{1}{i2\pi} \int_{\gamma} \frac{dz}{z - z_0} = -\text{WN}(\gamma, z_0).$$

2. **Case 1.** First, we show that if the line segment L_{z_0,w_0} joining z_0 and w_0 is completely contained in a same region determined by γ. We know that $\log'\left(\dfrac{z - z_0}{z - w_0}\right) = \dfrac{1}{z - z_0} - \dfrac{1}{z - w_0}$, provided the expression $\dfrac{z - z_0}{z - w_0}$ belongs to the domain of log, where it is analytic. From Result

3.3.9, we recall that log is analytic on $\Omega = \mathbb{C} \setminus \{(x,0) : x \leq 0\}$. As

$$\begin{aligned} \frac{z-z_0}{z-w_0} \text{ is real and } \leq 0 \quad &\Rightarrow \quad \frac{z-z_0}{z-w_0} = -k \text{ for some } k > 0 \\ &\Rightarrow \quad z = \frac{z_0 + kw_0}{1+k} \\ &\Rightarrow \quad z \text{ lies on } L_{z_0,w_0}. \end{aligned}$$

Hence, $\log\left(\dfrac{z-z_0}{z-w_0}\right)$ is differentiable on the complement of L_{z_0,w_0}. Since L_{z_0,w_0} does not intersect γ, we obtain that γ belongs to the complement of L_{z_0,w_0}. Thus, using Corollary 4.1.29, we get

$$\int_\gamma \left(\frac{1}{z-z_0} - \frac{1}{z-w_0}\right) dz = \int_\gamma \log'\left(\frac{z-z_0}{z-w_0}\right) dz = 0.$$

Case 2. Next, let z_0 and w_0 be two points in the same region determined by γ. Then, we can join z_0 and w_0 by a polygon $\bigcup_{j=1}^{n} L_{a_{j-1},a_j}$ contained in the same region determined by γ, where $a_0 = z_0$, $a_n = w_0$, and L_{a_{j-1},a_j} is the line segment joining a_{i-1} and a_i. Applying case 1, repeatedly, we get

$$\begin{aligned} \text{WN}(\gamma, z_0) &= \text{WN}(\gamma, a_0) = \text{WN}(\gamma, a_1) = \cdots = \text{WN}(\gamma, a_n) \\ &= \text{WN}(\gamma, w_0). \end{aligned}$$

3. As γ is the continuous image of $[a,b]$, it is a compact subset of $\mathbb{C}$, and hence, it is bounded. Therefore, we choose $M > 0$ such that $\gamma \subseteq B(0,M)$. Let z_0 be the given point belonging to the unbounded region determined by γ. Then for any $w_0 \notin B(0,M)$, z_0 and w_0 belong to the same region determined by γ. Since $\dfrac{1}{z-w_0}$ is an analytic function in $B(0,M)$, using (2) and Cauchy's theorem for simply connected region (Theorem 4.2.14),[1] we have

$$\text{WN}(\gamma, z_0) = \text{WN}(\gamma, w_0) = \frac{1}{i2\pi} \int_\gamma \frac{dz}{z-w_0} = 0.$$

[1] Though Theorem 4.2.14 is proved later, and its proof does not depend on the present theorem.

From the above properties. $\text{WN}(\gamma, z_0)$ is called the winding number of γ with respect to z_0. □

Example 4.2.6 If C is $|\zeta - a| = r$, then $\text{WN}(C, z) = \begin{cases} 1 & \text{if } |z-a| < r \\ 0 & \text{if } |z-a| > r \end{cases}$.
Since the parametric equation of the circle $|\zeta - a| = r$ is given by $\zeta = a + r\exp(i\theta)$, $\theta \in [0, 2\pi]$, we get

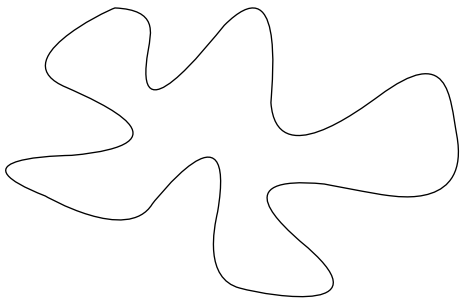
$$\frac{1}{i2\pi} \int\limits_{|\zeta - a| = r} \frac{d\zeta}{\zeta - a} = \frac{1}{i2\pi} \int\limits_{0}^{2\pi} \frac{ir\exp(i\theta)d\theta}{r\exp(i\theta)} = 1.$$

If $|z - a| < r$, then a and z are lying in the same region determined by C, and using Theorem 4.2.5(2), we get $\text{WN}(C, z) = \text{WN}(C, a) = 1$. If $|z - a| > r$, then by using Theorem 4.2.5(3), we get $\text{WN}(C, z) = 0$.

There is an interesting theorem, namely, Jordan curve theorem, which states that the complement of a simple closed curve has exactly to regions. Although it can be realized geometrically, its proof is too lengthy. Hence, we prefer to omit this theorem.

Definition 4.2.7 A region Ω in $\mathbb{C}$ is called a simply connected region if the complement of Ω in the extended complex plane is also connected.

Geometrically, every closed curve in a simply connected region Ω encloses only the points of Ω. We prove this statement rigorously through the following theorem.

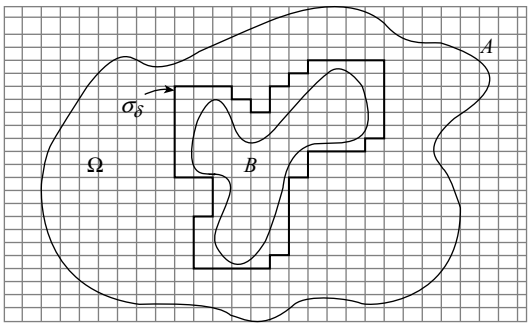


A simply connected region

THEOREM 4.2.8 Let Ω be a region in $\mathbb{C}$. Ω is simply connected iff $\text{WN}(\gamma, z_0) = 0$, for every $z_0 \notin \Omega$ and for every curve γ in Ω.

Proof: Assume that Ω is simply connected, that is, complement of Ω in $\mathbb{C}_\infty$ is connected. Since $\gamma \subseteq \Omega$, we have $\mathbb{C}_\infty\backslash\Omega \subseteq \mathbb{C}_\infty\backslash\gamma$. However, $\mathbb{C}_\infty\backslash\gamma$ is a disjoint union of the regions determined by γ and $\infty \in \mathbb{C}_\infty\backslash\Omega$. Hence, $\mathbb{C}_\infty\backslash\Omega$ is contained in the unbounded region determined by γ. Hence, by Theorem 4.2.5(3), we get $\text{WN}(\gamma, z_0) = 0$, $\forall z_0 \notin \Omega$.

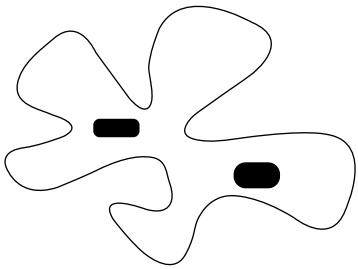
Conversely, assume that $\text{WN}(\gamma, z_0) = 0$, $\forall z_0 \notin \Omega$, for every closed curve γ in Ω. Suppose, $\mathbb{C}_\infty\backslash\Omega$ is not connected. Then, $\mathbb{C}_\infty\backslash\Omega = A \cup B$, where $A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$ and A and B are closed subsets of $\mathbb{C}_\infty\backslash\Omega$. Using $\infty \in \mathbb{C}_\infty\backslash\Omega$, without loss of generality, we assume that $\infty \in A$ and let $b \in B$. Being $\mathbb{C}_\infty\backslash B$ an open set containing ∞, it will be of the form $\mathbb{C}_\infty\backslash K$ for some compact subset K of $\mathbb{C}$. (Cf. Definition 1.5.1.) Thus, B is a compact subset of $\mathbb{C}$, and hence, it is bounded. As $A \setminus \{\infty\}$ and B are disjoint closed subsets of $\mathbb{C}$ with one of them is compact, if we let $\delta = \inf\{|x - y| : x \in A \setminus \{\infty\} \text{ and } y \in B\}$, then $\delta > 0$ (since $A \setminus \{\infty\}$ is closed in $\mathbb{C}$ and by Theorem 1.4.41). Now, we cover the entire complex plane by a net consisting of squares of diameter less than δ such that the point b lies at the centre of a square. Let $\{S_\alpha : \alpha \in \Lambda\}$ be the collection of all squares, which intersect B. Being B is bounded, Λ is a finite set. Then, let $\sigma_\delta = \sum_{\alpha\in\Lambda} \partial S_\alpha$, where ∂S_α is the boundary of the square S_α. After removing the common edges with opposite directions present in σ_δ, it is a closed curve, and it does not intersect B.



We claim that $\sigma_\delta \cap A = \varnothing$. Otherwise, there exists $z_0 \in \sigma_\delta \cap A$. Then, there exists a $\beta \in \Lambda$ such that $S_\beta \cap B \neq \varnothing$ and $z_0 \in S_\beta$. Then, we can choose $w_0 \in S_\beta \cap B$, and hence, $|z_0 - w_0| \geq \delta$. However, $z_0, w_0 \in S_\beta \Rightarrow |z_0 - w_0| < \delta$, which is a contradiction. Hence, σ_δ does not intersect A. Therefore, $\sigma_\delta \subseteq \mathbb{C}\backslash(A \cup B) = \Omega$. From $b \in B \subseteq (\mathbb{C}_\infty\backslash\Omega)$, we have $b \notin \Omega$. Therefore, by assumption, $\text{WN}(\sigma_\delta, b) = 0$. On the other hand, $\text{WN}(\sigma_\delta, b) =$

$\sum_{\alpha\in\Lambda} \text{WN}(\partial S_\alpha, b) = 1$, since $\text{WN}(\partial S_\alpha, b) = \begin{cases} 1 & \text{if } \alpha = \alpha_0 \\ 0 & \text{if } \alpha \neq \alpha_0, \end{cases}$ where $b \in \text{Int } S_{\alpha_0}$.
This is a contradiction. Therefore, $\mathbb{C}_\infty \backslash \Omega$ is connected. Thus, Ω is simply connected. □

Definition 4.2.9 A region Ω is called a multiply connected region, if it is not simply connected.
That is, the complement of Ω in $\mathbb{C}_\infty$ has more than two components or equivalently; there exists a closed curve γ in Ω, which encloses a point of complement of Ω.



A multiply connected region

Definition 4.2.10 Let γ be a closed curve in a region Ω. We say that γ is homologous to 0 in Ω if $n(\gamma, a) = 0$, $\forall a \neq \Omega$. In this case, we write $\gamma \sim 0$ in Ω.

The following result is an immediate consequence of Theorem 4.2.8.

RESULT 4.2.11 Let Ω be a region. Then, Ω is simply connected iff $\gamma \sim 0$ in Ω, $\forall$ closed curve γ in Ω.

In the following theorem, we mean a rectangle R by $[a, b] \times [c, d]$, for some $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. We also use the notation ∂R to denote the boundary of the rectangle R, which is a closed curve.

THEOREM 4.2.12 (Cauchy's theorem for rectangle)
If f is analytic on a region Ω and R is a rectangle contained in Ω, then $\int_{\partial R} f(z)\, dz = 0$.

Proof: Let $I(R)$, $D(R)$, and $L(R)$ denote $\int\limits_{\partial R} f(z)\,dz$, the diameter of R, and the perimeter of R, respectively. If R is subdivided into four equal rectangles $S_1, S_2, S_3,$ and S_4 of same size, then we have

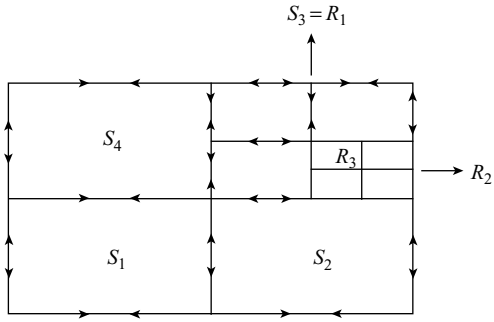
$$I(R) = \sum_{k=1}^{4} I(S_k) \Rightarrow |I(R)| \leq \sum_{k=1}^{4} |I(S_k)|.$$

This implies that there exists at least one $k \in \{1, 2, 3, 4\}$ such that $|I(S_k)| \geq \frac{1}{4}|I(R)|$. Denote that rectangle S_k by R_1. Next, we subdivide R_1 into four equal subrectangles of same size as before and choose the one say R_2 such that $|I(R_2)| \geq \frac{1}{4}|I(R_1)|$. Proceeding further, we get a sequence of rectangles $R \supseteq R_1 \supseteq R_2 \supseteq \cdots$ such that

$$|I(R_{n+1})| \geq \frac{1}{4}|I(R_n)| \quad \Rightarrow \quad |I(R_n)| \geq \frac{1}{4^n}|I(R)|. \tag{4.7}$$

$$L(R_{n+1}) = \frac{1}{2}L(R_n) \quad \Rightarrow \quad L(R_n) = \frac{1}{2^n}L(R). \tag{4.8}$$

$$D(R_{n+1}) = \frac{1}{2}D(R_n) \quad \Rightarrow \quad D(R_n) = \frac{1}{2^n}D(R). \tag{4.9}$$



Since $\{R_n\}$ is a decreasing sequence of compact sets with $D(R_n) \to 0$ as $n \to \infty$, by Cantor's intersection theorem (Theorem 2.4.33),

$$\bigcap_{n=1}^{\infty} R_n = \{z_0\}, \text{ for some } z_0 \in R. \tag{4.10}$$

As f is analytic at z_0, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \frac{\epsilon}{D(R)L(R)}.$$

Therefore, we have

$$0 < |z - z_0| < \delta \Rightarrow \left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| < \frac{\epsilon}{D(R)L(R)}|z - z_0|.$$

Using equations (4.9) and (4.10), we choose $m \in \mathbb{N}$ such that $D(R_m) < \delta$, and hence, $R_m \subseteq B(z_0, \delta)$. If $f_1(z) = 1$ and $f_2(z) = z$, $\forall z \in \mathbb{C}$, then f_1 and f_2 are the derivatives of the analytic functions F_1 and F_2 defined by $F_1(z) = z$ and $F_2(z) = \dfrac{z^2}{2}$, $\forall z \in \mathbb{C}$, respectively. Therefore, by Corollary 4.1.29, we have $\int\limits_{\partial R_m} dz = 0$ and $\int\limits_{\partial R_m} z\,dz = 0$. Now,

$$\begin{aligned}
|I(R)| &\leq 4^m |I(R_m)| \qquad \text{(by equation (4.7))} \\
&= 4^m \left| \int\limits_{\partial R_m} f(z)\,dz \right| \\
&= 4^m \left| \int\limits_{\partial R_m} [f(z) - f(z_0) - (z - z_0)f'(z_0)]\,dz \right| \\
&\leq 4^m \int\limits_{\partial R_m} |f(z) - f(z_0) - (z - z_0)f'(z_0)|\,|dz| \\
&< 4^m \frac{\epsilon}{D(R)L(R)} \int\limits_{\partial R_m} |z - z_0|\,|dz| \\
&< 4^m \frac{\epsilon}{D(R)L(R)} D(R_m) \int\limits_{\partial R_m} |dz| \\
&= 4^m \frac{\epsilon}{D(R)L(R)} D(R_m)L(R_m) \qquad \text{(by Theorem 4.1.27)} \\
&= 4^m \frac{\epsilon}{D(R)L(R)} 4^{-m} D(R)L(R) \qquad \text{(by equations (4.8) and (4.9))} \\
&= \epsilon.
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we get $|I(R)| = 0$. □

THEOREM 4.2.13 Let Ω be a region, $F \subseteq \Omega$ be a finite set and let f be an analytic function on $\Omega \backslash F$. If f is bounded in $B(a, r)\backslash\{a\}$, for some $r_a > 0$, for every $a \in F$, then for every rectangle R such that $\partial R \subset \Omega \backslash F$, $\int\limits_{\partial R} f(z)\,dz = 0$.

Proof: Let $F \cap R = \{a_k : k = 1, 2 \ldots, m\}$. Then, $a_k \in \text{Int } R, \forall k = 1, 2, \ldots, m$. By assumption, there exists $r_k > 0$ such that $B(a_k, r_k) \subseteq \text{Int } R$, and

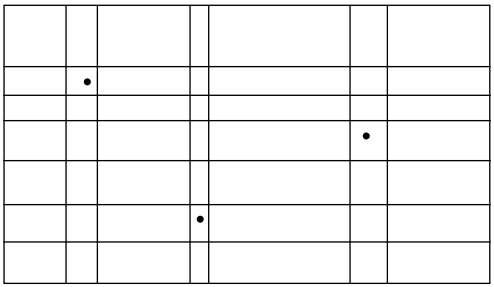
$$0 < |z - a_k| < r_k \Rightarrow |f(z)| \leq M_k, \text{ for some } M_k > 0, \ \forall 1 \leq k \leq m.$$

For each $k = 1, 2, \ldots, m$, we choose $\delta_k \in \mathbb{R}$ such that $0 < \delta_k < \min\left\{\frac{\epsilon}{mM_k}, r_k\right\}$.

Find a subrectangle S_k of R such that $a_k \in \text{Int } S_k$, and its circumference $L(S_k)$ of S_k is less than $\delta_k, \forall 1 \leq k \leq m$. If $D(S_k)$ is the diameter of S_k, then we get

$$z \in \partial S_k \Rightarrow z \in S_k \Rightarrow |z - a_k| \leq D(S_k) \leq L(S_k) < \delta_k. \tag{4.11}$$

Now, extend all the line segments that are used to construct the rectangles S_k up to the boundary of R so that the original rectangle R is subdivided into finite number of rectangles.



Hence, by a similar argument used in the proof of the previous theorem, we get

$$\int_{\partial R} f(z)\, dz = \sum_{k=1}^{m} \int_{\partial S_k} f(z)\, dz + \sum_{j=1}^{p} \int_{\partial T_j} f(z)\, dz,$$

where T_j's are subrectangles of R not containing any a_k. Applying the previous theorem, we get $\int_{\partial T_j} f(z)\, dz = 0, \forall 1 \leq j \leq p$. Now, for every $1 \leq k \leq m$, invoking equation(4.11), we get

$$\begin{aligned}
\left|\int_{\partial S_k} f(z)\, dz\right| &\leq \int_{\partial S_k} |f(z)|\, |dz| \leq \int_{\partial S_k} |f(z)|\, |dz| \\
&\leq \int_{\partial S_k} M_k\, |dz| = M_k L(S_k) \\
&< M_k \delta_k < M_k \tfrac{\epsilon}{mM_k} \\
&= \tfrac{\epsilon}{m}.
\end{aligned}$$

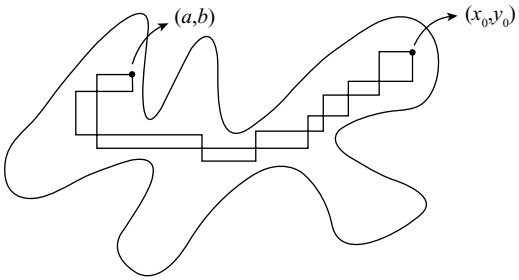
Therefore,

$$\left|\int_{\partial R} f(z)\,dz\right| \leq \sum_{k=1}^{m}\left|\int_{\partial S_k} f(z)\,dz\right| < \sum_{k=1}^{m}\frac{\epsilon}{m} = \epsilon.$$

Thus, $\int_{\partial R} f(z)\,dz = 0$. □

THEOREM 4.2.14 (Cauchy's theorem for a simply connected region)
Let F be a finite[2] subset of a simply connected region Ω and f be an analytic function on $\Omega\backslash F$. If f is bounded in $B(a,r)\backslash\{a\}$, for some $r_a > 0$, for every $a \in F$, then for every closed curve $\gamma \subset \Omega\backslash F$, we get $\int_{\gamma} f(z)\,dz = 0$.

Proof: First we show that there exists an analytic function Φ on Ω such that $\Phi' = f$. This part of this theorem is almost similar to that of the proof of Result 4.1.30. Let $(a,b) \in \Omega\backslash F$ be arbitrarily fixed. For every $(x_0,y_0) \in \Omega\backslash F$, let $\gamma_{(x_0,y_0)} \subseteq \Omega\backslash F$ be a polygon joining (a,b) and (x_0,y_0) consisting of the line segments parallel to the coordinate axes and the line segment incident with (x_0,y_0), which is horizontal (such a polygon exists by Theorem 1.4.31). Furthermore, as F is a finite set, we can choose the polygon such that it does not pass through any $a \in F$. Define $\Phi(x_0,y_0) = \int_{\gamma_{(x_0,y_0)}} f(z)\,dz$. First, we show that the definition of $\Phi(x_0,y_0)$ is independent of the choice of the polygon $\gamma_{(x_0,y_0)}$.



If $\sigma_{(x_0,y_0)}$ is another polygon joining (a,b) and (x_0,y_0), then $\gamma_{(x_0,y_0)} \cup (-\sigma_{(x_0,y_0)})$ is a finite union of boundary of rectangles. As Ω is simply connected, all rectangles enclosed by $\gamma_{(x_0,y_0)} \cup (-\sigma_{(x_0,y_0)})$ are contained in Ω.

[2]Note that empty set is a finite set, and hence, this theorem is also true for the case $F = \varnothing$.

Hence, by Theorem 4.2.13, we get

$$\int\limits_{\gamma_{(x_0,y_0)}\cup(-\sigma_{(x_0,y_0)})} f(z)\,dz = 0 \Rightarrow \int\limits_{\gamma_{(x_0,y_0)}} f(z)\,dz = \int\limits_{\sigma_{(x_0,y_0)}} f(z)\,dz.$$

Thus, Φ is well defined. Let $\epsilon > 0$ be given. Using the continuity of f at (x_0, y_0), choose $\delta > 0$ such that

$$(x,y) \in B((x_0,y_0),\delta) \subset \Omega\backslash F \Rightarrow |f(x,y) - f(x_0,y_0)| < \epsilon.$$

Let $h \in \mathbb{R}$ with $0 < |h| < \delta$. Then, the line segment $L_{(x_0,y_0),(x_0+h,y_0)} \subseteq \Omega$ and is horizontal. Therefore, for $|h| < \delta$, we have

$$\Phi(x_0+h,y_0) = \int\limits_{\gamma_{(x_0,y_0)}\cup L_{(x_0,y_0),(x_0+h,y_0)}} f(z)\,dz,$$

and hence,

$$\begin{aligned}
&\left|\frac{\Phi(x_0+h,y_0)-\Phi(x_0,y_0)}{h} - f(x_0,y_0)\right| \\
&\quad = \left|\frac{1}{h}\int\limits_{L_{(x_0,y_0),(x_0+h,y_0)}} f(z)\,dz - f(x_0,y_0)\right| \\
&\quad = \left|\frac{1}{h}\int\limits_0^h f(x_0+t,y_0)\,dt - f(x_0,y_0)\right| \\
&\quad \le \frac{1}{|h|}\int\limits_0^h |f(x_0+t,y_0)-f(x_0,y_0)|\;|dt| \\
&\quad < \frac{1}{|h|}|h|\epsilon = \epsilon.
\end{aligned}$$

Thus, $\dfrac{\partial\Phi}{\partial x}(x_0,y_0) = f(x_0,y_0)$.

Similarly, we can prove that $\dfrac{\partial\Phi}{\partial y}(x_0,y_0) = if(x_0,y_0)$. Furthermore, we get $\dfrac{\partial\Phi}{\partial x}(x_0,y_0) = -i\dfrac{\partial\Phi}{\partial y}(x_0,y_0) = f(x_0,y_0)$ (the C–R equation), and $\dfrac{\partial\Phi}{\partial x}$, $\dfrac{\partial\Phi}{\partial y}$ are continuous at (x_0,y_0). Therefore, by Theorem 2.2.28, we get that Φ is

differentiable at (x_0, y_0). As $(x_0, y_0) \in \Omega \backslash F$ is arbitrary, we get that Φ is analytic on Ω and $\Phi' = \dfrac{\partial \Phi}{\partial x} = f$. Therefore, using Corollary 4.1.29, we get $\int\limits_{\gamma} f(z)\, dz = 0$, for all closed curve γ in Ω. □

THEOREM 4.2.15 (Cauchy–Goursat theorem)
Let γ be a simple closed curve and f be analytic inside and on γ, then $\int\limits_{\gamma} f(z)\, dz = 0$.

Proof: As f is analytic inside and on γ, for every point z on or inside γ, there exists $r_z > 0$ such that f is analytic on $B(z, r_z)$. If we put Ω as the union of all these open balls $B(z, r_z)$ along with the region enclosed by γ, then Ω is a simply connected region, $\gamma \subset \Omega$, and f is analytic on Ω. Hence, by Cauchy's theorem for simply connected region, we get $\int\limits_{\gamma} f(z)\, dz = 0$. □

4.3 CAUCHY'S INTEGRAL FORMULA

THEOREM 4.3.1 (Cauchy's integral formula)
Let f be an analytic function on a simply connected region Ω. Then, $\text{WN}(\gamma, z_0) f(z_0) = \frac{1}{i2\pi} \int\limits_{\gamma} \dfrac{f(z)}{z - z_0}\, dz$, for every closed curve γ in $\Omega \backslash \{z_0\}$.

Proof: Let $F(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$, $\forall z \in \Omega \backslash \{z_0\}$. Then, F is analytic on $\Omega \backslash \{z_0\}$. Since

$$\lim_{z \to z_0} F(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0),$$

there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < 1.$$

Hence, for every $z \in B(z_0, r) \backslash \{z_0\}$, we have

$$|f(z)| = \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq 1 + |f'(z_0)| < +\infty.$$

Therefore, by Cauchy's theorem for simply connected region, we get $\int_\gamma F(z)\,dz = 0$. Hence, from the definition of $\text{WN}(\gamma, z_0)$, it follows that

$$\begin{aligned}\frac{1}{i2\pi}\int_\gamma \frac{f(z)}{z-z_0}\,dz - \text{WN}(\gamma,z_0)f(z_0) &= \frac{1}{i2\pi}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz \\ &= \frac{1}{i2\pi}\int_\gamma F(z)\,dz = 0.\end{aligned}$$

Hence, the theorem follows. □

THEOREM 4.3.2 Let γ be a curve in $\mathbb{C}$ and φ be a continuous function on γ. If

$$F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}\,d\zeta,$$

for every $z \notin \gamma$ and $n \in \mathbb{N}$, then F_n is differentiable on the complement of γ and $F_n' = nF_{n+1}$, $\forall n \in \mathbb{N}$.

Proof: Let $z_0 \notin \gamma$ be arbitrary. First, we show that if

$$G_{m,n}(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^m(\zeta - z_0)^n}, \forall z \notin \gamma \text{ and } \forall m, n \in \mathbb{N},$$

then $\lim_{z\to z_0} G_{m,n}(z) = G_{m,n}(z_0)$. We choose $\delta > 0$ such that $B(z_0, \delta) \cap \gamma = \varnothing$. Then, we have $|\zeta - z| \geq \frac{\delta}{2}$ and $|\zeta - z_0| \geq \delta$, $\forall z \in B\left(z_0, \frac{\delta}{2}\right)$ and $\forall \zeta \in \gamma$. If $M = \int_\gamma |\phi(\zeta)|\,|d\zeta|$, then $M < \infty$ and

$$\begin{aligned}&|G_{m,n}(z) - G_{m,n}(z_0)| \\ &\quad\leq \int_\gamma \left|\frac{1}{(\zeta - z)^m(\zeta - z_0)^n} - \frac{1}{(\zeta - z_0)^{m+n}}\right| |\phi(\zeta)|\,|d\zeta| \\ &\quad= \int_\gamma \frac{1}{|\zeta - z_0|^n}\left|\frac{1}{(\zeta - z)} - \frac{1}{(\zeta - z_0)}\right| \\ &\qquad\times \left|\sum_{k=0}^{m-1} \frac{1}{(\zeta - z)^{m-1-k}}\frac{1}{(\zeta - z_0)^k}\right| |\phi(\zeta)|\,|d\zeta|\end{aligned}$$

(by using $a^m - b^m = (a-b)\sum_{k=0}^{m-1} a^{m-1-k}b^k$)

$$\leq \frac{|z-z_0|}{\delta^n}\sum_{k=0}^{m-1}\int_\gamma \frac{|\phi(\zeta)|\,|d\zeta|}{|\zeta - z|^{m-k}|\zeta - z_0|^{k+1}}$$

$$\leq \frac{|z-z_0|}{\delta^n}\sum_{k=0}^{m-1}\int_\gamma \frac{2^{m-k}}{\delta^{m-k}\delta^{k+1}}|\phi(\zeta)|\,|d\zeta|$$

$$\leq \frac{M\sum_{k=0}^{m-1}2^{m-k}}{\delta^{m+n+1}}|z-z_0| \to 0 \text{ as } z \to z_0.$$

For $z \in B\left(z_0, \frac{\delta}{2}\right)$, we have

$$\frac{F_n(z) - F_n(z_0)}{z - z_0}$$

$$= \frac{1}{z-z_0}\int_\gamma \left(\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z_0)^n}\right)\phi(\zeta)\,d\zeta$$

$$= \frac{1}{z-z_0}\int_\gamma \left(\frac{1}{(\zeta - z)} - \frac{1}{(\zeta - z_0)}\right)\sum_{k=0}^{n-1}\frac{1}{(\zeta - z)^{n-1-k}}\frac{1}{(\zeta - z_0)^k}\phi(\zeta)\,d\zeta$$

$$= \frac{1}{z-z_0}\int_\gamma \left(\frac{z - z_0}{(\zeta - z)(\zeta - z_0)}\right)\sum_{k=0}^{n-1}\frac{1}{(\zeta - z)^{n-1-k}}\frac{1}{(\zeta - z_0)^k}\phi(\zeta)\,d\zeta$$

$$= \sum_{k=0}^{n-1}\int_\gamma \frac{1}{(\zeta - z)^{n-k}}\frac{1}{(\zeta - z_0)^{k+1}}\phi(\zeta)\,d\zeta$$

$$= \sum_{k=0}^{n-1} G_{n-k,k+1}(z) \to \sum_{k=0}^{n-1} G_{n-k,k+1}(z_0) = \sum_{k=0}^{n-1}\int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta$$

$$= nF_{n+1}(z_0), \text{ as } z \to z_0.$$

This completes the proof of the theorem. □

COROLLARY 4.3.3 (Cauchy's integral formula for derivatives)
Let f be an analytic function on a simply connected region Ω and $z_0 \in \Omega$. Then for every $n \in \mathbb{N}, f^{(n)}$ exists and

$$\text{WN}(\gamma, z_0) f^{(n)}(z_0) = \frac{n!}{i2\pi} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

for every closed curve not passing through z_0 in Ω, and for every $n \in \mathbb{N}$.

Proof: Cauchy's integral formula states that

$$\text{WN}(\gamma, z_0) f(z_0) = \frac{1}{i2\pi} \int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta,$$

where γ is the circle $|z - z_0| = r$, where $r > 0$ is such that $\text{Cl}\, B(z_0, r) \subseteq \Omega$. As f is continuous on γ, by previous theorem, we get that $\frac{1}{i2\pi} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$ is analytic in $B(z_0, r)$, and for every $z \in B(z_0, r)$,

$$\begin{aligned}
\frac{d}{dz}\left(\frac{1}{i2\pi}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta\right) &= \frac{1}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \\
\frac{d^2}{dz^2}\left(\frac{1}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta\right) &= \frac{1 \times 2}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta \\
\frac{d^3}{dz^3}\left(\frac{1}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta\right) &= \frac{1 \times 2 \times 3}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^4}\, d\zeta \\
&\vdots \\
\frac{d^n}{dz^n}\left(\frac{1}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta\right) &= \frac{n!}{i2\pi}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.
\end{aligned}$$

Therefore, $f^{(n)}(z_0) = \frac{n!}{i2\pi} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \forall n \in \mathbb{N}$. □

THEOREM 4.3.4 (Weierstrass theorem on convergence of sequence of analytic functions)
If (f_n) is a sequence of analytic functions on a region Ω such that (f_n) converges to f uniformly on every compact subset of Ω, then f is analytic and (f_n') converges to f' uniformly on every compact subset of Ω.

Proof: Let $z \in \Omega$ be arbitrary. Then, choose $r > 0$ such that Cl $B(z, r) \subseteq \Omega$. Then, by Cauchy's integral formula, we have

$$f_n(w) = \frac{1}{i2\pi} \int\limits_{|\zeta|=r} \frac{f_n(\zeta)}{\zeta - w}\, d\zeta,\ \forall w \in B(z, r)\, \forall n \in \mathbb{N}.$$

By assumption, we have $f_n \to f$ uniformly on $|\zeta| = r$ as $n \to \infty$. For all $w \in B(z, r)$, we have

$$f(w) = \frac{1}{i2\pi} \int\limits_{|\zeta|=r} \frac{f(\zeta)}{\zeta - w}\, d\zeta.$$

Since $f_n \to f$ as $n \to \infty$ uniformly on $|\zeta| = r$ and each f_n is continuous, we have f is continuous on $|\zeta| = r$, by Theorem 1.6.31. Hence, by Theorem 4.3.2, we obtain that f is analytic on $B(z, r)$. As $z \in \Omega$ is arbitrary, f is analytic on Ω. Again by using (f_n) converges uniformly to f on $|\zeta| = r$, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(\zeta) - f(\zeta)| < \epsilon$, $\forall \zeta$ with $|\zeta| = r$, whenever $n \geq N$. By Cauchy's integral formula for derivatives, we have

$$f_n'(w) = \frac{1}{i2\pi} \int\limits_{|\zeta|=r} \frac{f_n(\zeta)}{(\zeta - w)^2}\, d\zeta,\ \forall w \in B(z, r)$$

and

$$f'(w) = \frac{1}{i2\pi} \int\limits_{|\zeta|=r} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta,\ \forall w \in B(z, r),$$

For all $w \in$ Cl $B\left(z, \frac{r}{2}\right)$, and for all $n \geq N$,

$$\left|f_n'(w) - f'(w)\right| \leq \frac{1}{2\pi} \int\limits_{|\zeta|=r} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - w|^2}\, |d\zeta| \leq \frac{1}{2\pi} \int\limits_{|\zeta|=r} \frac{\epsilon}{\frac{r^2}{4}}\, |d\zeta| = \frac{4\epsilon}{r}.$$

Therefore, for every $z \in \Omega$, there exists $\rho_z > 0$ such that $f_n' \to f'$ uniformly on Cl $B(z, \rho_z)$. Let K be a compact subset of Ω. Then, $\{B(z, \rho_z) : z \in K\}$ is

a collection of open sets with $K \subseteq \bigcup_{z\in K} B(z, \rho_z)$. Since K is compact, there exist $z_1, z_2, \ldots, z_m \in K$ such that $K \subseteq \bigcup_{j=1}^{m} B(z_j, \rho_{z_j}) \subseteq \bigcup_{j=1}^{m} \text{Cl}\, B(z_j, \rho_{z_j})$. Since $f_n' \to f'$ uniformly on $\text{Cl}\, B(z_j, \rho_{z_j})$ as $n \to \infty$, for a given $\epsilon > 0$, there exists $N_j \in \mathbb{N}$ such that

$$|f_n'(w) - f'(w)| < \epsilon,\ \forall w \in \text{Cl}\, B(z_j, \rho_{z_j}),\ \forall n \geq N_j,\ \forall 1 \leq j \leq m.$$

If $N_0 = \max\{N_j : 1 \leq j \leq m\}$, then $|f_n'(w) - f'(w)| < \epsilon, \forall w \in K, \forall n \geq N_0$. Thus, (f_n') converges to f' uniformly on every compact subset of Ω. □

The following algorithm is useful to evaluate integrals over a given simple closed curve γ.

Algorithm 4.3.5 (To find $\int_\gamma \phi(z)\, dz$ by Cauchy's integral formula)

Step 1: Let the given function be ϕ and find its poles. (If ϕ is a rational function, the zeroes of the denominator of ϕ are the poles of ϕ.)

Step 2: Check which of the poles are lying outside γ.

Step 3: (a) If there is no pole lying inside γ, then $\int_\gamma \phi(z)\, dz = 0$.

(b) If there is only one pole z_0, which is enclosed by γ, then put

$$f(z) = (z - z_0)^m \phi(z),$$

where m is the order of the pole z_0 of ϕ. Then, we find $\int_\gamma \phi(z)\, dz = \int_\gamma \frac{f(z)}{(z - z_0)^m}\, dz$ using Cauchy's integral formula as follows.

$$\int_\gamma \phi(z)\, dz = \int_\gamma \frac{f(z)}{(z - z_0)^m}\, dz = \frac{i2\pi}{(m-1)!} f^{(m-1)}(z_0).$$

(c) If there are more than one pole, say $z_1, z_2, \ldots, z_n$, enclosed by γ, then put

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_n)^{m_n} \phi(z),$$

where $m_1, m_2, \ldots, m_n$ are the orders of the poles $z_1, z_2, \ldots, z_n$, respectively.

(d) Split $\frac{1}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}}$ into partial fractions as follows.

$$\sum_{k=1}^{m_1}\frac{A_{1,k}}{(z-z_1)^k}+\sum_{k=1}^{m_2}\frac{A_{2,k}}{(z-z_2)^k}+\cdots+\sum_{k=1}^{m_n}\frac{A_{n,k}}{(z-z_n)^k}.$$

(Note that in the field of complex numbers, there is no non-splittable factor. For example, $\frac{1}{z^2+a^2}$ should be written as $\frac{A}{z+ia}+\frac{B}{z-ia}$. Hence, in a partial fraction expansion, a general term should be of the form $\frac{C}{(z-a)^p}$.)

(e) Then write $\int_\gamma \phi(z)\,dz$ as $\sum_{j=1}^{n}\sum_{k=1}^{m_j} A_{j,k}\int_\gamma \frac{f(z)}{(z-z_j)^k}\,dz$.

(f) Doing Step 3(b) repeatedly, we can find the given integral.

Example 4.3.6 Evaluate the following integrals using Cauchy's integral formula:

(1) $\int_{|z|=3}\frac{z^3+1}{z^2+3z-10}\,dz$.

(2) $\int_{|z-3|=1}\frac{z^2+1}{z-1}\,dz$.

(3) $\int_\gamma \frac{z^4-4z-6}{z^2-6z+5}\,dz$, where γ is the square with vertices $(3,3)$, $(-3,3)$, $(-3,-3)$, and $(3,-3)$.

(4) $\int_{|z|=5}\frac{2z+3}{z^2-2z-3}\,dz$.

(5) $\int_{|z-1|=4}\frac{z^2-3}{(z^2+3z+2)(z+6)}\,dz$.

(6) $\int_{|z|=5}\frac{3z+1}{(z-1)(z-2)^2}\,dz$.

(7) $\int_{|z-(1-i)|=3}\frac{\exp(z)(3z^2+4)}{(z-1)^3}\,dz$.

(8) $\int_{|z+1|=2}\frac{z^2+5z+6}{(z^2+1)(z-1+i)}\,dz$.

(9) $\int_{|z|=4}\frac{\exp(z)}{z\left(z-\frac{\pi}{2}\right)(z-\pi)}\,dz$.

(10) $\int_\gamma \frac{z+i}{(z+4)(z-1-i)}\,dz$, where γ is the boundary of $[0,2]\times[0,3]$.

Solution:

(1) Let $\phi(z)=\frac{z^3+1}{z^2+3z-10}$. As $z^2+3z-10=0\Rightarrow z=-5,\ z=2$, the poles of ϕ are -5 and 2. We know that 2 is enclosed by $|z|=3$, and -5 is not enclosed by $|z|=3$ because $|2|=2<3$ and $|-5|=5>3$.

We put $f(z) = \dfrac{z^3+1}{z+5}$. Therefore,

$$\int\limits_{|z|=3} \frac{z^3+1}{z^2+3z-10}\,dz = \int\limits_{|z|=3} \frac{f(z)}{z-2}\,dz = i2\pi f(2) = i2\pi\,\frac{2^3+1}{2+5} = \frac{18i\pi}{7}.$$

(2) Let $\phi(z) = \dfrac{z^2+1}{z-1}$. Then, clearly 1 is the only pole of ϕ and is not enclosed by $|z-3| = 1$ (because $|1-3| = 2 > 1$.) Therefore, $\int\limits_{|z-3|=1} \dfrac{z^2+1}{z-1}\,dz = 0$.

(3) Let $\phi(z) = \dfrac{z^4-4z-6}{z^2-6z+5}$. Since $z^2-6z+5 = 0 \Rightarrow z = 5,\ z = 1$, the poles of ϕ are 5 and 1. Obviously, 1 is enclosed by γ and 5 is not enclosed by γ (check geometrically). We put $f(z) = \dfrac{z^4-4z-6}{z-5}$. Therefore,

$$\int\limits_{\gamma} \frac{z^4-4z-6}{z^2-6z+5}\,dz = \int\limits_{\gamma} \frac{f(z)}{z-5}\,dz = i2\pi f(1) = i2\pi\,\frac{1-4-6}{1-5} = \frac{9i\pi}{2}.$$

(4) Let $\phi(z) = \dfrac{2z+3}{z^2-2z-3}$. Now, $z^2-2z-3 = 0$ implies that $z = -1$, and $z = 3$ are the poles of ϕ, and both are enclosed by $|z| = 5$, because $|-1| = 1 < 5$ and $|3| = 3 < 5$. Therefore, we put $f(z) = 2z+3$. Now, we split $\dfrac{1}{(z+1)(z-3)}$ into partial fractions as follows.

Let $\dfrac{1}{(z+1)(z-3)} = \dfrac{A}{z+1} + \dfrac{B}{z-3} \Rightarrow 1 = A(z-3) + B(z+1)$.

Putting $z = 3$, we get $4B = 1 \Rightarrow B = \dfrac{1}{4}$.

Putting $z = -1$, we get $-4A = 1 \Rightarrow A = \dfrac{-1}{4}$.

Hence, $\dfrac{1}{(z+1)(z-3)} = \dfrac{1}{4}\left(\dfrac{1}{z-3} - \dfrac{1}{z+1}\right)$. Therefore,

$$\int\limits_{|z|=5} \frac{2z+3}{z^2-2z-3}\,dz \quad = \quad \int\limits_{|z|=5} \frac{f(z)}{(z+1)(z-3)}\,dz$$

$$= \frac{1}{4}\left(\int_{|z|=5}\frac{f(z)}{z-3}\,dz - \int_{|z|=5}\frac{f(z)}{z+1}\,dz\right)$$

$$= \frac{i2\pi}{4}\,(f(3)-f(-1)) = \frac{i\pi}{2}(9-1) = 4i\pi.$$

(5) Let $\phi(z) = \dfrac{z^2-3}{(z^2+3z+2)(z+6)}$. As $(z^2+3z+2)(z+6)=0$, it follows that $z=-1, z=-2$, and $z=-6$ are the poles of ϕ. Furthermore, -1 and -2 are enclosed by $|z-1|=4$, and -6 is not enclosed by $|z-1|=4$ because $|-1-1|=2<4$, $|-2-1|=3<4$, and $|-6-1|=7>4$. Hence, we take $f(z)=\dfrac{z^2-3}{z+6}$. Now, we split $\dfrac{1}{(z+1)(z+2)}$ into partial fractions as follows.

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1}+\frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z+1).$$

Putting $z=-2$, we get $-B=1 \Rightarrow B=-1$.

Putting $z=-1$, we get $A=1$.

Hence, $\dfrac{1}{(z+1)(z+2)} = \dfrac{1}{z+1}-\dfrac{1}{z+2}$. Therefore,

$$\int_{|z-1|=4}\frac{z^2-3}{(z^2+3z+2)(z+6)}\,dz = \int_{|z-1|=4}\frac{f(z)}{(z+1)(z+3)}\,dz$$

$$= -\int_{|z-1|=4}\frac{f(z)}{z+2}\,dz + \int_{|z-1|=4}\frac{f(z)}{z+1}\,dz$$

$$= -i2\pi\,(f(-2)-f(-1))$$

$$= -i2\pi\left(\frac{(-2)^2-3}{-2+6}-\frac{(-1)^2-3}{-1+6}\right)$$

$$= -\frac{i\pi}{2}\left(\frac{1}{4}-\frac{2}{5}\right) = -\frac{3i\pi}{40}.$$

(6) If $\phi(z) = \dfrac{3z+1}{(z-1)(z-2)^2}$, then $z=1$ and 2 are the poles of ϕ, and both are enclosed by $|z|=5$. Let $f(z)=3z+1$. Now, we split

$\dfrac{1}{(z-1)(z-2)^2}$ into partial fractions.

$$\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$\Rightarrow 1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$. Putting $z = 1, z = 2$, and $z = 0$, we get, respectively, $A = 1, C = 1, 1 = 4A + 2B - C \Rightarrow B = -1$. Therefore, $\dfrac{1}{(z-1)(z-2)^2} = \dfrac{1}{z-1} - \dfrac{1}{z-2} + \dfrac{1}{(z-2)^2}$. Hence,

$$\begin{aligned}
\int\limits_{|z|=5} \frac{3z+1}{(z-1)(z-2)^2}\,dz &= \int\limits_{|z|=5} \frac{f(z)}{(z-1)(z-2)^2}\,dz \\
&= \int\limits_{|z|=5} \frac{f(z)}{z-1}dz - \int\limits_{|z|=5} \frac{f(z)}{z-2} + \int\limits_{|z|=5} \frac{f(z)}{(z-2)^2}\,dz \\
&= i2\pi \left(f(1) - f(2) + f'(2)\right) \\
&= i2\pi\,(4 - 7 + 3) = 0.
\end{aligned}$$

(7) Let $\phi(z) = \dfrac{\exp(z)(3z^2+4)}{(z-1)^3}$. Then, 1 is the only pole of order 3 for ϕ, which is enclosed by $|z - (1+i)| = 3$, as $|1 - (1-i)| = 1 < 3$. If $f(z) = \exp(z)(3z^2+4)$, then

$$\begin{aligned}
f'(z) &= \exp(z)(3z^2 + 6z + 4), \\
f''(z) &= \exp(z)(3z^2 + 12z + 10) \Rightarrow f''(1) = 25e.
\end{aligned}$$

Therefore, $\displaystyle\int\limits_{|z-(1+i)=3|} \frac{\exp(z)(3z^2+4)}{(z-1)^3}\,dz = \frac{i2\pi}{2!} f''(1) = 25\pi \exp(1)i.$

(8) Let $\phi(z) = \dfrac{z^2+5z+6}{(z^2+1)(z-1+i)}$. Then, $i, -i,$ and $1 - i$ are the poles for ϕ, among which i and $-i$ are enclosed by $|z-1| = 2$, as $|i+1| = \sqrt{2} < 2, |-i+1| = \sqrt{2} < 2$, and $|1-i+1| = \sqrt{5} > 2$. Let $f(z) = \dfrac{z^2+5z+6}{z-1+i}$. Now, we split $\dfrac{1}{(z-i)(z+i)}$ into partial fractions as follows.

$$\frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i} \Rightarrow 1 = A(z+i) + B(z-i).$$

Putting $z = i$ and $z = -i$ we get, respectively, $A = \dfrac{1}{i2}$ and $B = -\dfrac{1}{i2}$. Therefore,

$$\begin{aligned}
&\int\limits_{|z-1|=2} \frac{z^2+5z+6}{(z^2+1)(z-1+i)}dz \\
&\quad = \frac{1}{i2}\left(\int\limits_{|z-1|=2} \frac{f(z)}{z-i}\,dz - \int\limits_{|z-1|=2} \frac{f(z)}{z+i}\,dz\right) \\
&\quad = \pi(f(i)-f(-i)) \\
&\quad = \pi\left(\frac{i^2+i5+6}{i-1+i} - \frac{i^2-i5+6}{-i-1+i}\right) \\
&\quad = \pi\left(\frac{i5+5}{i2-1} - \frac{5-i5}{-1}\right) \\
&\quad = \pi(1-i3+5-i5) = \pi(6-i8).
\end{aligned}$$

(9) Let $\phi(z) = \dfrac{\exp(z)}{z(z-\pi/2)(z-\pi)}$. Then, $0, \pi,$ and $\pi/2$ are the poles for ϕ, and all are enclosed by $|z-1| = 2$, as $|0| < 5$, $|\pi/2| < 5$, and $|\pi| < 5$. Let $f(z) = \exp(z)$. Now, we split $\dfrac{1}{z(z-\pi/2)(z-\pi)}$ into partial fractions as follows.

$$\frac{1}{z(z-\pi/2)(z-\pi)} = \frac{A}{z} + \frac{B}{z-\pi/2} + \frac{C}{z-\pi} \Rightarrow 1 = A(z-\pi/2)(z-\pi) + Bz(z-\pi) + Cz(z-\pi/2).$$

Substituting $z = 0$, $z = \pi/2$, and $z = \pi$, we get, respectively, $A = \dfrac{2}{\pi^2}$, $B = -\dfrac{4}{\pi^2}$, and $C = \dfrac{2}{\pi^2}$. Therefore, we have $\dfrac{1}{z(z-\pi/2)(z-\pi)} = \dfrac{2}{\pi^2}\left(\dfrac{1}{z} - \dfrac{2}{z-\pi/2} + \dfrac{1}{z-\pi}\right)$. Hence,

$$\int\limits_{|z|=5} \frac{\exp(z)}{z(z-\pi/2)(z-\pi)}\,dz$$

$$= \frac{2}{\pi^2}\left(\int_{|z|=5}\frac{f(z)}{z}\,dz - 2\int_{|z|=5}\frac{f(z)}{z-\pi/2}\,dz + \int_{|z|=5}\frac{f(z)}{z-\pi}\,dz\right)$$

$$= i2\pi\,\frac{2}{\pi^2}\,(f(0) - 2f(\pi/2) + f(\pi))$$

$$= \frac{i4}{\pi}(1 - 2\exp(\pi/2) + \exp(\pi)).$$

(10) Let $\phi(z) = \dfrac{z+i}{(z+4)(z-1-i)}$. Then, -4 and $1+i$ are the poles for ϕ. Since the given curve γ is the boundary of the rectangle with vertices $(0,3),(2,0),(2,3)$, and $(0,3)$, $1+i$ is enclosed by γ, and -4 is not enclosed by γ. Therefore, let $f(z) = \dfrac{z+i}{z+4}$. Hence,

$$\int_\gamma \frac{z+i}{(z+4)(z-1-i)}\,dz = \int_\gamma \frac{f(z)}{z-1-i}\,dz$$

$$= i2\pi f(1+i) = i2\pi\,\frac{1+i2}{5+i} = \frac{\pi}{13}(-9+i7).$$

Exercise 4.3.7 Evaluate the following integrals using Cauchy's integral formula:

(1) $\displaystyle\int_{|z|=2}\frac{\exp(z)}{(z+1)(z-3)^2}\,dz.$

(2) $\displaystyle\int_{|z+i2|=2}\frac{dz}{z^2+1}\,dz.$

(3) $\displaystyle\int_{|z|=2}\frac{\sin(z)}{z+i}\,dz.$

(4) $\displaystyle\int_{|z|=1}\frac{\exp(z)}{(z-2)^3}\,dz.$

(5) $\int\limits_{|z+\frac{1}{2}|=1} \dfrac{z^2 \exp(z)}{(z+1)(z-2)}\, dz.$

(6) $\int\limits_{|z-1|=2} \dfrac{z+1}{z^2-9z+20}\, dz.$

(7) $\int\limits_{|z|=3} \dfrac{z \exp(z)}{z-1}\, dz.$

(8) $\int\limits_{|z-(2+i)|=2} \dfrac{z}{(z-(1+i))}\, dz.$

(9) $\int\limits_{|z|=4} \dfrac{\exp(z)}{(z+1)(z-3)^2}\, dz.$

(10) $\int\limits_{|z|=3} \dfrac{z^2+1}{(z^2-1)(z-2)}\, dz.$

(11) $\int\limits_{|z|=2} \dfrac{z}{(z-1)^2}\, dz.$

(12) $\int\limits_{|z-i2|=4} \dfrac{2z+3}{(z-(1-i))^2}\, dz.$

(13) $\int\limits_{|z|=4} \dfrac{e^z}{(z^2+1)^3}\, dz.$

(14) $\int\limits_{|z-1|=4} \dfrac{z+i}{(z-1)^2(z-2)(z-6)}\, dz.$

(15) $\int\limits_{|z-2|=3} \dfrac{\exp(z+1)}{(z+6)^4}\, dz.$

Answers: (1) $\dfrac{i\pi \exp(-1)}{8}$; (2) $-\pi$; (3) $2\pi \sinh(1)$; (4) 0; (5) $\dfrac{i2\pi \exp(-1)}{-3}$; (6) 0; (7) $i2\pi \exp(1)$; (8) $2\pi(-1+i)$; (9) $\dfrac{i\pi}{8}(\exp(-1)+3\exp(3))$; (10) $i2\pi$; (11) $i2\pi$; (12) $4i\pi$; (13) $\dfrac{\pi}{4}(2\sin(1)-3\cos(1)$; (14) $\dfrac{\pi}{50}(1-6i)$; (15) 0

RESULT 4.3.8 (Cauchy's estimate)
If f is analytic on a closed disk Cl $B(a,r)$, then for each $n \in \mathbb{N}$, and $0 < r < \epsilon$, we have $|f^{(n)}(a)| \leq \dfrac{M_r n!}{r^n}$, where $M_r = \sup\limits_{|z|=r} |f(z)|$.

Proof: Let γ be the circle $|z-a| = r$. By Cauchy's integral formula for derivatives, we get

$$\begin{aligned}
\left|f^{(n)}(a)\right| &= \left|\frac{n!}{i2\pi}\int\limits_{\gamma}\frac{f(z)}{(z-a)^{n+1}}\,dz\right| \\
&\leq \frac{n!}{2\pi}\int\limits_{\gamma}\left|\frac{f(z)}{(z-a)^{n+1}}\right|\,dz \\
&\leq \frac{n!}{2\pi}\int\limits_{\gamma}\left|\frac{M_r}{r^{n+1}}\right|\,dz \\
&\leq \frac{M_r n!}{2\pi r^{n+1}}2\pi r = \frac{M_r n!}{r^n}.
\end{aligned}$$

Hence, the result follows. □

THEOREM 4.3.9 (Morera's theorem)
Let f be a continuous function on Ω such that $\int\limits_{\gamma} f(z)\,dz = 0$ for every closed curve γ in Ω, then f is analytic on Ω.

Proof: As $\int\limits_{\gamma} f(z)\,dz = 0$ for every closed curve γ in Ω, using Result 4.1.30, there exists an analytic function F on Ω such that $F' = f$. By Corollary 4.3.3, F' is analytic on Ω, and hence, f is analytic on Ω. □

Definition 4.3.10 An analytic function on $\mathbb{C}$ is called an entire function.

Clearly, every polynomial, exp, sin, and cos are entire functions. Obviously, every constant function is a bounded entire function but the interesting fact is that the converse of this statement is also true, which follows.

THEOREM 4.3.11 (Liouville's theorem)
Every bounded entire function is constant.

Proof: Let f be a bounded entire function such that $\sup_{z\in\mathbb{C}} |f(z)| \leq M$ for some $M > 0$. For an arbitrary $z \in \mathbb{C}$ and for an arbitrary $r > 0$, from Cauchy's estimate, we get

$$|f'(a)| \leq \frac{M_r}{r}, \text{ where } M_r = \sup_{|z|=r} |f(z)|.$$

As $M_r \leq M, \ \forall r > 0$, we have

$$|f'(a)| \leq \frac{M}{r} \to 0 \text{ as } r \to \infty.$$

Thus, $f'(z) = 0, \forall z \in \mathbb{C}$. Hence, by Theorem 2.2.19, we get that f is a constant function. □

THEOREM 4.3.12 (Fundamental theorem of algebra)
Every polynomial of degree n with complex coefficients has exactly n zeroes in $\mathbb{C}$ including multiplicities.

Proof: First, we prove that every non-constant polynomial has a zero in $\mathbb{C}$. Let P be a non-constant polynomial. If P has no zero in $\mathbb{C}$, then $\frac{1}{P}$ is an entire function, as P is an entire function (See Example 2.1.9). Now, we claim that $\frac{1}{P}$ is bounded on $\mathbb{C}$. If $P(z) = \sum_{k=0}^{n} a_k z^k$, with $a_n \neq 0$, then for $|z| \geq 1$, we have

$$\begin{aligned} |P(z)| &\geq \left|a_n z^n\right| - \left|\sum_{k=0}^{n-1} a_k z^k\right| \\ &\geq \left|a_n z^n\right| - \sum_{k=0}^{n-1} \left|a_k z^k\right| \\ &\geq \left|a_n z^n\right| - |z|^{n-1} \sum_{k=0}^{n-1} |a_k| \end{aligned}$$

$$\begin{aligned}
& \text{(since } |z|^k \leq |z|^{n-1} \text{ if } k \leq n-1) \\
\geq\ & |z|^{n-1}\left[|a_n z| - \sum_{k=0}^{n-1} |a_k|\right] \\
\geq\ & |a_n z| - \sum_{k=0}^{n-1} |a_k| \\
\geq\ & 1 \text{ if } |z| \geq \frac{1}{|a_n|}\left[1 + \sum_{k=0}^{n-1} |a_k|\right].
\end{aligned}$$

As $\left|\frac{1}{P}\right|$ is a continuous real-valued function on the compact set

$$\left\{z \in \mathbb{C} : |z| \leq \max\left\{1, \frac{1}{|a_n|}\left[1 + \sum_{k=0}^{n-1} |a_k|\right]\right\}\right\},$$

there exists $M > 0$ such that $\left|\frac{1}{P(z)}\right| \leq M$ on the compact set. Therefore,

$$\left|\frac{1}{P}\right| \leq M + 1 < \infty \text{ on } \mathbb{C},$$

and hence, $\frac{1}{P}$ is a bounded entire function. Therefore, by Liouville's theorem (Theorem 4.3.11), we get that $\frac{1}{P}$ is a constant function. This implies that P is a constant function. This is a contradiction. Therefore, P has a zero in $\mathbb{C}$. If α_1 is a zero of P, then using Lemma 3.1.3, we can write

$$P(z) = (z - \alpha_1)Q_1(z),\ \forall z \in \mathbb{C}.$$

Certainly, degree of Q_1 is $n-1$. If $n-1 > 0$, then Q_1 has a zero say α_2, and hence, we can write

$$P(z) = (z - \alpha_1)(z - \alpha_2)Q_2(z),\ \forall z \in \mathbb{C}.$$

Proceeding further, at the nth stage, we get

$$P(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)Q_n(z),\ \forall z \in \mathbb{C},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the zeroes of P, and Q_n is a polynomial of degree 0, which means that Q_n is a constant. Therefore, P has exactly n zeroes in $\mathbb{C}$. □

4.4 GENERAL VERSION OF CAUCHY'S THEOREM

THEOREM 4.4.1 (General version of Cauchy's theorem)
If f is analytic on Ω and γ is a closed curve in Ω such that $\gamma \sim 0$ in Ω, then $\int_{\gamma} f(z)\,dz = 0$.

Proof: As γ is closed and bounded subset of $\mathbb{C}$, it is compact. Since Ω^c is closed and $\Omega^c \cap \gamma = \varnothing$, if

$$r = d(\gamma, \Omega^c) = \inf\{|z - w| : z \in \gamma,\ w \in \Omega^c\},$$

then $r > 0$, by Theorem 1.4.41. If $z = \phi(t),\ t \in [a, b]$ is the parametric equation of γ, then ϕ is a continuous function on the compact set $[a, b]$, and hence, it is uniformly continuous on $[a, b]$, by Theorem 1.6.26. Then, there exists $\delta > 0$ such that

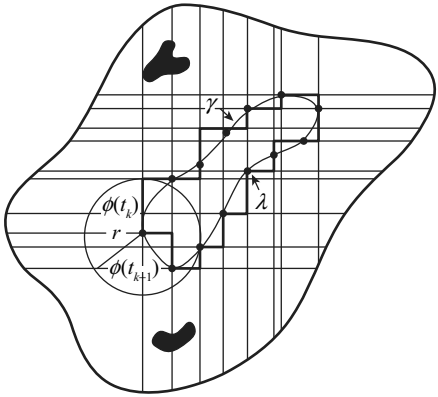
$$t, s \in [a, b] \text{ with } |t - s| < \delta \Rightarrow |z(t) - z(s)| < r.$$

If $\{t_0, t_1, t_2, \ldots, t_n\} \subseteq [a, b]$ such that $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ and $t_i - t_{i-1} < \delta$, then let $\gamma_k = \phi([t_{k-1}, t_k])$. Then, clearly,

$$\gamma = \bigcup_{k=1}^{n} \gamma_k \text{ and } \gamma_k \subset B(\phi(t_k), r),\ 1 \leq k \leq n.$$

Moreover, by the choice of r, we have $B(\phi(t_k), r)$, which does not intersect Ω^c, as $\phi(t_k) \in \gamma \subset \Omega$. Since $\phi(t_k), \phi(t_{k-1}) \in \gamma_k \subseteq B(\phi(t_k), r)$, and $B(\phi(t_k), r)$ is a convex set,[3] we can connect $\phi(t_k)$ and $\phi(t_{k-1})$ by a polygon $\lambda_k \subseteq B(\phi(t_k), r)$, which consists of line segments parallel to coordinate axes. Hence, $\gamma_k \cup (-\lambda_k)$ is a closed curve contained in the simply connected region $B(\phi(t_k), r)$, $\forall 1 \leq k \leq n$.

[3] A set C is said to be a convex set if for every pair of points of C, the line segment joining the two points is contained in C.



Hence, by Cauchy's theorem for simply connected region, we have

$$\int\limits_{\gamma_k \cup (-\lambda_k)} f(z)\,dz = 0 \Rightarrow \int\limits_{\gamma_k} f(z)\,dz = \int\limits_{\lambda_k} f(z)\,dz,\ 1 \le k \le n.$$

Therefore, if $\lambda = \bigcup\limits_{k=1}^{n} \lambda_k$, then

$$\int\limits_{\gamma} f(z)dz = \int\limits_{\bigcup_{k=1}^{n} \gamma_k} f(z)\,dz = \int\limits_{\bigcup_{k=1}^{n} \lambda_k} f(z)\,dz = \int\limits_{\lambda} f(z)\,dz. \tag{4.12}$$

Now, we subdivide the entire complex plane into finite number of bounded rectangles B_p, $1 \le p \le \mu$, and finite number of unbounded rectangles U_q, $1 \le q \le \nu$, by extending each line segment involved in the construction of λ into straight lines. Now, for each $p \in \{1, 2, \ldots, \mu\}$ and $q \in \{1, 2, \ldots, \nu\}$, we fix one point $b_p \in \text{Int}\, B_q$ and $u_q \in \text{Int}\, U_q$. If we denote $\bigcup\limits_{p=1}^{\mu} \text{WN}(\lambda, b_p)\partial B_p$ by χ, then we claim that after removing the line segments, which are common sides of two rectangles in χ, we get $\chi = \lambda$. First, we note that for $1 \le j \le \mu$ and $1 \le l \le \nu$, we have

$$\begin{aligned} \text{WN}(\chi, b_j) &= \text{WN}\left(\bigcup_{p=1}^{\mu} \text{WN}(\lambda, b_p)\partial B_p, b_j\right) \\ &= \sum_{p=1}^{\mu} \text{WN}(\lambda, b_p)\text{WN}(\partial B_p, b_j) \end{aligned}$$

$$= \text{WN}(\lambda, b_j), \quad \text{since WN}(\partial B_p, b_j) = \begin{cases} 1 & \text{if } p = j \\ 0 & \text{if } p \neq j \end{cases}, \tag{4.13}$$

and

$$\begin{aligned} \text{WN}(\chi, u_l) &= \text{WN}\left(\bigcup_{p=1}^{\mu} \text{WN}(\lambda, b_p)\partial B_p, u_l\right) \\ &= \sum_{p=1}^{\mu} \text{WN}(\lambda, b_p)\text{WN}(\partial B_p, u_l) \\ &= 0, \qquad \text{since WN}(\partial B_p, u_l) = 0. \end{aligned} \tag{4.14}$$

Next, we claim that the multiplicity m of any line segment $L_{p,\,j}$ in $\lambda \cup (-\chi)$, which is common to two bounded rectangles B_p and B_j, is zero for some $1 \leq p,j \leq \mu$. Since $\lambda \cup (-\chi) \cup (-m\partial B_p)$ does not have the line $L_{p,\,j}$, the points b_p and b_j belong to the same region determined by $\lambda \cup (-\chi) \cup (-m\partial B_p)$. Therefore,

$$\text{WN}(\lambda \cup (-\chi) \cup (-m\partial B_p), b_p) = \text{WN}(\lambda \cup (-\chi) \cup (-m\partial B_p), b_j).$$

Hence, by using equation(4.13), we get

$$\text{WN}(\lambda, b_p) - \text{WN}(\lambda, b_j) - m = \text{WN}(\lambda, b_p) - \text{WN}(\lambda, b_j) - 0 \Rightarrow m = 0.$$

Similarly, if $L_{p,q}$ is the common side of a bounded rectangle B_p and an unbounded rectangle U_q, then arguing as before by applying equation(4.14), we get the multiplicity of the line segment $L_{p,q}$ in $\lambda \cup (-\chi)$ as zero. Hence, our claim holds.

By Cauchy's theorem for rectangle, we have $\int\limits_{\partial B_p} f(z)\,dz = 0$, provided $B_p \subseteq \Omega$. If $B_p \not\subset \Omega$, then there exists $a \in \text{Int}\, B_p$ and $a \notin \Omega$. Since γ is homologous to zero in Ω, we get $\text{WN}(\gamma, a) = 0$. As $\gamma \cup (-\lambda)$ does not enclose a, we get

$$\text{WN}(\gamma \cup (-\lambda), a) = 0 \Rightarrow \text{WN}(\lambda, a) = \text{WN}(\gamma, a) = 0.$$

Since a and b_p belong to the same region determined by λ, we get

$$\text{WN}(\lambda, b_p) = \text{WN}(\lambda, a) = 0.$$

This implies that for each $p = 1, 2, \ldots, \mu$,

$$\text{either} \int\limits_{\partial B_p} f(z)\,dz = 0 \text{ or } \text{WN}(\lambda, b_p) = 0.$$

Hence, it follows that $\int_{\lambda} f(z)\,dz = \int_{\chi} f(z)\,dz = \sum_{p=1}^{\mu} \text{WN}(\lambda, b_p) \int_{\partial B_p} f(z)\,dz = 0.$

□

Definition 4.4.2 A cycle Γ is a finite union of closed curves $\bigcup_{j=1}^{m} \gamma_j$.

We define $\int_{\Gamma} f(z)\,dz = \sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz$, so that we get $\text{WN}(\Gamma, a) = \sum_{j=1}^{m} \text{WN}(\gamma_j, a)$.

Then, the general version of Cauchy's theorem can be further extended to the following form easily.

THEOREM 4.4.3 If f is analytic on a region Ω and Γ is a cycle, which is homologous to zero in Ω, then $\int_{\Gamma} f(z)\,dz = 0$.

4.5 LOCAL CORRESPONDENCE THEOREM AND ITS CONSEQUENCES

In this section, we shall use the following result, which will be proved in Chapter 5. See Result 5.2.3.

RESULT 4.5.1 If f is an analytic function on a region Ω with a zero of order k at $z_0 \in \Omega$, then there exists an analytic function g on Ω such that g is analytic on $\Omega, f(z) = (z - z_0)^k g(z)$ on Ω and $g(z_0) \neq 0$.

THEOREM 4.5.2 Let f be a non-zero analytic function on a region Ω with zeroes $z_1, z_2, \ldots, z_n$, including multiplicities. If γ is a closed curve in Ω such that $\gamma \sim 0$ in Ω and it is not passing through any z_j, then

$$\frac{1}{i2\pi} \int_{\gamma} \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \text{WN}(\gamma, z_j).$$

Proof: Let $\Omega_1 \subseteq \Omega$ be a region such that γ and each region determined by γ are contained in Ω_1. We also assume that the only zeroes of f in Ω_1 are $\{z_j : j = 1, 2, \ldots, n\}$. Since z_j's are the zeroes of f in Ω, by Result 4.5.1, we can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n) g(z),\ \forall z \in \Omega_1,$$

where g is analytic on Ω_1 and is nowhere zero on Ω_1. Hence,

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}, \ \forall z \text{ on } \gamma.$$

We note that $\frac{g'}{g}$ is analytic on Ω_1 because g and g' are analytic and g is nowhere zero on Ω_1. Therefore,

$$\frac{1}{i2\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \frac{1}{i2\pi}\int_\gamma \frac{dz}{z-z_j} + \frac{1}{i2\pi}\int_\gamma \frac{g'(z)}{g(z)}\,dz = \sum_{j=1}^{n} \text{WN}(\gamma, z_j),$$

since we have $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$, by general version of Cauchy's theorem.

Note that if we take γ as a simple closed curve in the above theorem, then $\frac{1}{i2\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ counts the number of zeroes of f inside γ. □

THEOREM 4.5.3 (Local correspondence theorem)
Let f be a non-constant analytic function on a region Ω, $z_0 \in \Omega$ and $f(z_0) = w_0$. For each $w \in f(\Omega)$, if $g_w(z) = f(z) - w_0$, $\forall z \in \Omega$ and the order of g_{w_0} at z_0 is n, then there exists $r > 0$ and $\rho > 0$ such that for each $w \in B(w_0, \rho)\backslash\{w_0\}$, there exists exactly n distinct roots for g_w in $B(z_0, r)\backslash\{z_0\}$.

Proof: Since f is non-constant, g_{w_0} is not identically zero, and hence, by the principle of analytic continuation, it follows that z_0 is isolated. Therefore, there exists $r > 0$ such that

1. Cl $B(z_0, r) \subseteq \Omega$.
2. $g_{w_0} \neq 0$ on $B(z_0, r)\backslash\{z_0\}$, since z_0 is an isolated zero of g_{w_0}.
3. $f' \neq 0$ on $B(z_0, r)\backslash\{z_0\}$. (This is possible because of the following reason. Since f is non-constant, f' is not identically 0. If $f'(z_0) = 0$, then z_0 must be an isolated zero of f'. Then, there exists $r > 0$. If $f'(z_0) \neq 0$, using the continuity of f', we can find such an $r > 0$.)

Let γ be the circle $|z - z_0| = r$. Since $z_0 \notin \gamma$, we have $w_0 = f(z_0) \notin f(\gamma)$. Hence, we can choose $\rho > 0$ such that $B(w_0, \rho) \cap f(\gamma) = \varnothing$. Then, for each $w \in B(w_0, \rho)$, we get that w and w_0 lie in a same region determined by $f(\gamma)$. Therefore, from a property of winding number (Thereom 4.2.5), it follows that

$$\text{WN}(f(\gamma), w) = \text{WN}(f(\gamma), w_0). \tag{4.15}$$

For each $w \in B(w_0, \rho)$, if n_w denotes the number of zeroes of g_w inside γ including the multiplicities and if $J_w = \{1, 2, \ldots, n_w\}$, then using Theorem 4.5.2 and using the change of variable $\zeta = f(z)$, we get

$$\begin{aligned}\sum_{j\in J_w} \text{WN}(\gamma, z_j) &= \frac{1}{i2\pi}\int_\gamma \frac{g_w'(z)}{g_w(z)}\,dz \\ &= \frac{1}{i2\pi}\int_\gamma \frac{f'(z)}{f(z)-w}\,dz \\ &= \frac{1}{i2\pi}\int_{f(\gamma)} \frac{d\zeta}{\zeta - w} \\ &= \text{WN}(f(\gamma), w).\end{aligned}$$

Since the only one zero of g_{w_0} inside γ is z_0 and its multiplicity is n, we have $\sum_{j\in J_{w_0}} \text{WN}(\gamma, z_j) = n$. Therefore, from equation (4.15), we get

$$n = \sum_{j\in J_{w_0}} \text{WN}(\gamma, z_j) = \text{WN}(f(\gamma), w_0) = \text{WN}(f(\gamma), w) = \sum_{j\in J_w} \text{WN}(\gamma, z_j).$$

In other words, the number of zeroes of g_w inside $B(z_0, r)$ is n. Furthermore, $g_w' = f' \neq 0$ on $B(z_0, r)\backslash\{z_0\}$ implies that every zero of g_w in $B(z_0, r) \setminus \{z_0\}$ is simple, and hence, for each $w \in B(w_0, \rho) \setminus \{w_0\}$, g_w has exactly n distinct roots inside $B(z_0, r)\backslash\{z_0\}$. □

THEOREM 4.5.4 (Open mapping theorem)
Every non-constant analytic function on a region is an open mapping.

Proof: Let f be a non-constant analytic function on a region Ω. To prove that f is an open mapping, we have to show that if U is open in Ω, then $f(U)$ is an open subset of $\mathbb{C}$. Let U be an open subset of Ω and let $w_0 \in f(U)$. Then, there exists $z_0 \in U$ such that $w_0 = f(z_0)$. Choose $\epsilon > 0$ such that $B(z_0, \epsilon) \subseteq U$. By local correspondence theorem, there exists $0 < r < \epsilon$ and $\rho > 0$ such that for every $w \in B(w_0, \rho)$, there exists $z \in B(z_0, r)$ such that $f(z) = w$. Therefore, we have proved that

$$B(w_0, \rho) \subseteq f(B(z_0, r)) \subseteq f(B(z_0, \epsilon)) \subseteq f(U).$$

Thus, $f(U)$ is an open set. □

COROLLARY 4.5.5 If $f : \Omega_1 \to \Omega_2$ is an analytic bijection, then $f^{-1} : \Omega_2 \to \Omega_1$ is also an analytic function and $(f^{-1})'(w_0) = \dfrac{1}{f'(z_0)}$, where $z_0 = f^{-1}(w_0)$, $\forall w_0 \in \Omega_2$.

Proof: First, we claim that f' is nowhere zero on Ω_1. Suppose $f'(z_0) = 0$ for some $z_0 \in \Omega_1$, then $f(z) - f(z_0)$ has at least two zeroes at z_0. Therefore, by local correspondence theorem, there exist $r > 0$ and $\rho > 0$ such that for each $w \in B(w_0, \rho) \setminus \{w_0\}$, there exist at least two distinct zeroes for $f(z) - w$ from $B(z_0, r)$, which is a contradiction to the injectivity of f. Therefore, our claim holds. If g is the inverse of f, then by open mapping theorem, g is a continuous map. In fact, g is analytic; indeed, if $w = f(z)$ and $f(z_0) = w_0$, then $w \to w_0$ whenever $z \to z_0$, and hence,

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \lim_{z \to z_0} \frac{1}{\dfrac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)}.$$

Thus, the theorem follows. □

THEOREM 4.5.6 (Maximum modulus principle)
If f is a non-constant analytic function on a region Ω, then $|f(z)|$ has no maximum in Ω.

Proof: To prove this theorem, we show that for each $z_0 \in \Omega$, there exists $z_1 \in \Omega$ such that $|f(z_1)| > |f(z_0)|$. Since f is a non-constant analytic function on Ω, then by open mapping theorem, f is an open map. Therefore, $f(\Omega)$ itself is an open subset of $\mathbb{C}$. If $w_0 = f(z_0)$, then $w_0 \in f(\Omega)$, and hence, there exists $\epsilon > 0$ such that $B(w_0, \epsilon) \subseteq f(\Omega)$. If $|w_0| < r < \epsilon$ and if

$$w_1 = \begin{cases} r \exp(i \arg w_0) & \text{if } w_0 \neq 0 \\ r & \text{if } w_0 = 0 \end{cases},$$

then $w_1 \in B(w_0, \epsilon) \subseteq f(\Omega)$ and $|w_1| > |w_0|$. Thus, there exists $z_1 \in \Omega$ such that $f(z_1) = w_1$, and hence, $|f(z_1)| = |w_1| > |w_0| = |f(z_0)|$. □

COROLLARY 4.5.7 [Minimum modulus principle]
Let f be a non-constant analytic function on a region Ω. If f is nowhere zero on Ω, then $|f|$ has no minimum in Ω.

Proof: Since f is nowhere zero on Ω, $\dfrac{1}{f}$ is also an analytic function on Ω. Hence, by maximum modulus principle, $\left|\dfrac{1}{f}\right|$ has no maximum in Ω. That is, for every $z \in \Omega$, there exists $z_1 \in \Omega$ such that $\left|\dfrac{1}{f(z)}\right| < \left|\dfrac{1}{f(z_1)}\right|$, equivalently, $|f(z_1)| < |f(z)|$. Thus, $|f|$ has no minimum on Ω. □

RESULT 4.5.8 (Schwarz lemma)
If $f : B(0,1) \to \text{Cl}\, B(0,1)$ is an analytic function such that $f(0) = 0$, then

1. $|f(z)| \leq |z|$ on $B(0,1)$ and $|f'(0)| \leq 1$.
2. If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ or if $|f'(0)| = 1$, then $f(z) = \exp(i\theta)z$, on $B(0,1)$ for some $\theta \in \mathbb{R}$.

Proof: If $g(z) = \dfrac{f(z)}{z}$ for $z \neq 0$, then g is analytic on $B(0,1)\backslash\{0\}$, and 0 is a removable singularity for g since

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$$

exists in $\mathbb{C}$. Hence, if we define $g(0) = f'(0)$, then g becomes analytic on $B(0,1)$, by Lemma 5.1.2. This implies that g is analytic on Cl $B(0,r)$ for every $0 < r < 1$. Since $|g|$ is a continuous function on the compact set Cl $B(0,r)$, there exists z_r with $|z_r| \leq 1$ such that $|g(z_r)| = \sup_{z\in \text{Cl}\, B(0,r)} |g(z)|$. If g is a constant function on $B(0,1)$, then we can choose this z_r with $|z_r| = 1$, and if g is non-constant, then by maximum modulus principle, $|g|$ does not attain its maximum on $|z| < r$, and hence, $|z_r| = r$. Since $0 < |z_r| < 1$, we have

$$\sup_{z\in \text{Cl}\, B(0,r)} |g(z)| = |g(z_r)| = \left|\frac{f(z_r)}{z_r}\right| = \frac{|f(z_r)|}{r} \leq \frac{1}{r},$$

and hence, by allowing $r \to 1$, we get $\sup_{|z|\leq 1} |g(z)| \leq 1$. Hence, by definition of g, we get $\left|\frac{f(z)}{z}\right| \leq 1$ on $B(0,1)\backslash\{0\}$ and $|f'(0)| \leq 1$. Thus, $|f(z)| \leq |z|$ on $B(0,1)$ and $|f'(0)| \leq 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 \in B(0,1)\backslash\{0\}$ or $|f'(0)| = 1$, then we get $|g(\zeta_0)| = 1$ for some $\zeta_0 (= z_0$ or 0) in $B(0,1)$. Hence, by maximum modulus principle, we conclude that g is a constant say c. Since $|g(\zeta_0)| = 1$, we obtain

$|c| = 1$. Therefore, using Corollary 2.4.25, there exists $\theta \in \mathbb{R}$ such that $c = \exp(i\theta)$, and hence, $f(z) = g(z) \cdot z = cz$, $\forall z$ in $B(0,1)$. □

RESULT 4.5.9 (Generalized Schwarz lemma)
Let f be analytic on $|z| < R$ such that $|f(z)| \leq S$, $\forall |z| < R$. If $|a| < R$ and $f(a) = b$, then $\left|\dfrac{S(f(z)-b)}{S^2 - \overline{b}f(z)}\right| \leq \left|\dfrac{R(z-a)}{R^2 - \overline{a}z}\right|$.

Proof: Consider the linear fractional transform

$$w = T(z) = \frac{R(z-a)}{R^2 - \overline{a}z},\ \forall w \in \mathbb{C}_\infty.$$

By Example 3.2.26, we get $T(B(0,R)) = B(0,1)$.

Similarly, if $\xi = \Lambda(\zeta) = \dfrac{S(\zeta - b)}{S^2 - \overline{b}\zeta}$, $\forall \zeta \in \mathbb{C}_\infty$, then we have $\Lambda(B(0,S)) = B(0,1)$. If $F = (\Lambda \circ f \circ T^{-1})$, then $F : B(0,1) \to B(0,1)$ is analytic and

$$F(0) = \Lambda(f(T^{-1}(0))) = \Lambda(f(a)) = \Lambda(b) = 0.$$

Hence, by Schwarz lemma, we get $|F(w)| \leq |w|$, $\forall w \in B(0,1)$. If $z \in B(0,R)$, then $|T(z)| < 1$, and hence,

$$|F(T(z))| \leq |T(z)| \Rightarrow |\Lambda(f(z))| \leq |T(z)| \Rightarrow \left|\frac{S(f(z)-b)}{S^2 - \overline{b}f(z)}\right| \leq \left|\frac{R(z-a)}{R^2 - \overline{a}z}\right|.$$

Thus, the result follows. □

Now, we can see some applications of Schwarz lemma, which are used to prove the uniqueness of Riemann mapping between a region and open unit disc, in Result 7.4.5.

COROLLARY 4.5.10 If $f : B(0,1) \to B(0,1)$ is an analytic bijection, satisfying $f(0) = 0$, then $f(z) = cz$, $\forall z \in B(0,1)$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Proof: Let $f : B(0,1) \to B(0,1)$ is the given bijection. Then, by Corollary 4.5.5, we get that $f^{-1} : B(0,1) \to B(0,1)$ is an analytic function and $(f^{-1})'(0) = \dfrac{1}{f'(0)}$. Applying Schwarz lemma to f and f^{-1}, we have

$$|f'(0)| \leq 1 \text{ and } \left|\frac{1}{f'(0)}\right| = |(f^{-1})'(0)| \leq 1.$$

The second inequality implies that $|f'(0)| \geq 1$. Therefore, $|f'(0)| = 1$. Then using the condition for equality in Schwarz lemma, we get $f(z) = cz, \forall z \in \mathbb{C}$ for some $c \in \mathbb{C}$. □

COROLLARY 4.5.11 If $f : B(0,1) \to B(0,1)$ is an analytic bijection, then $f(z) = c\dfrac{z-\alpha}{1-\overline{\alpha}z}$, $\forall z \in B(0,1)$ for some $c \in \mathbb{C}$ with $|c| = 1$, where $\alpha = f^{-1}(0) \in B(0,1)$.

Proof: If $T_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}, \forall z \in \mathbb{C}$, then we claim that $T_\alpha : B(0,1) \to B(0,1)$ is an analytic bijection such that $T_\alpha(\alpha) = 0$ and T_α^{-1} is also analytic.

If $|z| = 1$, then let $z = \exp(i\theta)$ for some $\theta \in \mathbb{R}$ so that

$$|T_\alpha(z)| = \left|\frac{\exp(i\theta)-\alpha}{1-\overline{\alpha}\exp(i\theta)}\right| = \left|\frac{\exp(i\theta)-\alpha}{\exp(i\theta)(\exp(-i\theta)-\overline{\alpha})}\right| = \left|\frac{\exp(i\theta)-\alpha}{(\exp(-i\theta)-\overline{\alpha})}\right| = 1.$$

As every linear fractional transform maps circles to circles, if C is the unit circle, then $T_\alpha(C) \subseteq C$, which implies that $T_\alpha(C) = C$. Since T_α has a unique singularity $\frac{1}{\overline{\alpha}}$, which is not in $\mathrm{Cl}\, B(0,1)$, T_α is an analytic function on $B(0,1)$, and hence, $T_\alpha(B(0,1))$ is a connected subset of $\{z \in \mathbb{C} : |z| \neq 1\}$. Therefore, either $T_\alpha(B(0,1)) = B(0,1)$ or $T_\alpha(B(0,1)) = \{z \in \mathbb{C} : |z| > 1\}$.

Since $T_\alpha(\alpha) = 0$ (by definition) and $\alpha \in B(0,1)$, we get $T_\alpha(B(0,1)) = B(0,1)$. As every linear fractional transform is a one-to-one map, $T_\alpha : B(0,1) \to B(0,1)$ is a bijection, satisfying $T_\alpha(\alpha) = 0$, and by Corollary 4.5.5, T_α^{-1} is also an analytic map on $B(0,1)$ satisfying $T_\alpha^{-1}(0) = \alpha$.

If $F = f \circ T_\alpha^{-1}$, then $F : B(0,1) \to B(0,1)$ is an analytic bijection and $F(0) = (f \circ T_\alpha^{-1})(0) = f(\alpha) = 0$. Therefore, by the previous corollary, we have $F(w) = cw, \forall w \in \mathbb{C}$, for some $c \in \mathbb{C}$ with $|c| = 1$. Thus, for an arbitrary $z \in B(0,1)$, $T_\alpha(z) \in B(0,1)$ so that $f(z) = f(T_\alpha^{-1}(T_\alpha(z))) = F(T_\alpha(z)) = cT_\alpha(z) = c\dfrac{z-\alpha}{1-\overline{\alpha}z}$. □

THEOREM 4.5.12 Let f be an analytic function on a simply connected region Ω. If f is nowhere zero, then $\log(f)$ can be suitably defined on Ω so that it is analytic on Ω.

Proof: Using Theorem 4.2.8, we get that every closed curve γ in Ω is homologous to zero in Ω. Since f is analytic and is nowhere zero, using

Theorem 4.5.5, we have

$$\frac{1}{i2\pi}\int_{\gamma}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta = 0,$$

for every closed curve γ is Ω. Then by Result 4.1.30, there exists an analytic function g on Ω, such that $g' = \dfrac{f'}{f}$. If $\log(f)$ is defined, then its derivative must be $\dfrac{f'}{f}$. Hence, we expect that $\log(f)$ would be defined by $g + c$ for some suitable constant c. We find c from

$$\log(f) = g + c \Leftrightarrow \exp c = f\exp(-g).$$

To conclude this proof, first we show that $f\exp(-g)$ is a constant. This can be obtained since

$$\begin{aligned}\frac{d}{dz}(f\exp(-g)) &= f'\exp(-g) - f\exp(-g)g' \\ &= f'\exp(-g) - f\exp(-g)\frac{f'}{f} \\ &= 0\end{aligned}$$

and by using Theorem 2.2.19. Hence, let $f\exp(-g) = \exp(c)$ for some constant c, and this is possible since $f\exp(-g)$ is nowhere zero. Then, for a fixed $z_0 \in \Omega$,

$$f(z_0)\exp(-g(z_0)) = \exp(c) \Rightarrow c = \log(f(z_0)) - g(z_0),$$

where $\log(f(z_0))$ is any one of the infinite number of its values. Thus, if $\log(f)$ is defined by $g + \log(f(z_0)) - g(z_0)$, then $\log(f)$ is differentiable. □

COROLLARY 4.5.13 If f and Ω are as in the previous theorem, then for every $n \in \mathbb{N}$, there exists an analytic function g on Ω such that $g^n = f$. In other words, we can define $\sqrt[n]{f}$ as an analytic function on Ω.

Proof: Using the previous theorem, we can define $\log(f)$ as an analytic function on Ω. If we define $g(z) = \exp\left(\frac{1}{n}\log(f(z)\right)$, $\forall z \in \Omega$, then clearly g is an analytic function on Ω and

$$g^n(z) = \left[\exp\left(\frac{1}{n}\log(f(z))\right)\right]^n = \exp\left(\log(f(z)\right) = f(z),\ \forall z \in \Omega,$$

and hence, g is the required analytic function such that $g = \sqrt[n]{f}$. □

THEOREM 4.5.14 (Hurwitz theorem)
Let (f_n) be a sequence of analytic functions such that f_n is nowhere zero for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on every compact subset of Ω as $n \to \infty$, then either f is identically zero or f is nowhere zero.

Proof: Assume that f is not identically zero. Let $z \in \Omega$ be arbitrary. Then, we claim that there exists $\delta > 0$ such that $f \neq 0$ on $\text{Cl}\, B(z, \delta)\backslash\{z\}$.
If $f(z) \neq 0$, then using the continuity of f at z, for $\epsilon = \dfrac{|f(z)|}{2}$, there exists $r > 0$ such that

$$|f(w) - f(z)| < \frac{|f(z)|}{2} \text{ whenever } |w - z| < r.$$

If $0 < \delta < r$ and $w \in \text{Cl}\, B(z, \delta)$, then

$$|f(z)| - |f(w)| \leq |f(w) - f(z)| < \frac{|f(z)|}{2} \Rightarrow |f(w)| > \frac{|f(z)|}{2} > 0.$$

Thus, our claim holds, in this case.
If $f(z) = 0$, then by Theorem 4.5.1, there exist $m \in \mathbb{N}$ and an analytic function g on Ω such that

$$f(w) = (w - z)^m g(w), \ \forall w \in \Omega \text{ and } g(z) \neq 0.$$

Then by the previous argument, there exists $\delta > 0$ such that $g \neq 0$ on $\text{Cl}\, B(z, \delta)$. This implies that $f \neq 0$ on $0 < |w - z| \leq \delta$. Thus, our claim holds.
Since $f_n \to f$ uniformly on every compact subset of Ω as $n \to \infty$, then by Theorem 4.3.4, we get $f_n' \to f'$ uniformly on every compact subset of Ω as $n \to \infty$. By a similar technique employed in Theorem 1.3.9, we obtain $\dfrac{f_n'}{f_n} \to \dfrac{f'}{f}$ uniformly on $|\zeta - z| = \delta$, as $n \to \infty$.

Indeed, since $|f|$ is real, continuous, and nowhere zero on $|\zeta - z| = \delta$, there exists $\mu > 0$ such that $\mu = \inf\{|f(\zeta)| : |\zeta - z| = \delta\}$. Let M_1 and M_2 be positive real numbers such that

$$M_1 > \sup\{|f(\zeta)| : |\zeta - z| = \delta\} \text{ and } M_2 > \sup\{|f'(\zeta)| : |\zeta - z| = \delta\}.$$

Then, for a given $\epsilon > 0$, there exist N_1 and $N_2 \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \min\left\{\frac{|\mu|}{2}, \frac{\epsilon|\mu|^2}{4M_2}\right\}, \ \forall n \geq N_1$$

and

$$|f_n'(\zeta) - f'(\zeta)| < \frac{\epsilon|\mu|^2}{4M_1},\ \forall n \geq N_2.$$

Therefore, for $n \geq N_1$, we have $|f| - |f_n| \leq |f_n - f| < \frac{|\mu|}{2}$, and hence,

$$\frac{|\mu|}{2} = |\mu| - \frac{|\mu|}{2} \leq |f| - \frac{|\mu|}{2} < |f_n|.$$

For every $\zeta \in \mathbb{C}$ with $|\zeta - z| = \delta$ and for every $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
&\left|\frac{f_n'(\zeta)}{f_n(\zeta)} - \frac{f'(\zeta)}{f(\zeta)}\right| \\
&= \left|\frac{f_n'(\zeta)f(\zeta) - f_n(\zeta)f'(\zeta)}{f_n(\zeta)f(\zeta)}\right| \\
&\leq \frac{1}{|f_n(\zeta)f(\zeta)|}\left[\left|f_n'(\zeta)f(\zeta) - f(\zeta)f'(\zeta)\right| + \left|f(\zeta)f'(\zeta) - f_n(\zeta)f'(\zeta)\right|\right] \\
&\leq \frac{2}{|\mu|^2}\left[M_1\left|f_n'(\zeta) - f'(\zeta)\right| + M_2\left|f(\zeta) - f_n(\zeta)\right|\right] \\
&\leq \frac{2}{|\mu|^2}\left[M_1\frac{\epsilon|\mu|^2}{4M_1} + M_2\frac{\epsilon|\mu|^2}{4M_2}\right] \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, by Theorem 4.5.2, we get that the number of zeroes of f in $B(z, \delta)$ is equal to

$$\begin{aligned}
\frac{1}{i2\pi}\int\limits_{|\zeta - z| = \delta}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \lim_{n\to\infty}\frac{1}{i2\pi}\int\limits_{|\zeta - z| = \delta}\frac{f_n'(\zeta)}{f_n(\zeta)}\,d\zeta \\
&= \frac{1}{i2\pi}\int\limits_{|\zeta - z| = \delta}\left(\lim_{n\to\infty}\frac{f_n'(\zeta)}{f_n(\zeta)}\right)d\zeta \\
&= 0,
\end{aligned}$$

since each f_n has no zeroes. Therefore, $f \neq 0$ on $B(z, \delta)$. In particular, $f(z) \neq 0$. Using that $z \in \Omega$ is arbitrary, we get that f is nowhere zero on Ω. □

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5

Series Developments and Infinite Products

5.1 TAYLOR SERIES AND LAURENT SERIES

Definition 5.1.1 Let f be an analytic function on $\Omega\backslash\{a\}$. Then, the point a is said to be a removable singularity of f if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$.

LEMMA 5.1.2 Let f be an analytic function on $\Omega\backslash\{a\}$ for some $a \in \Omega$. Then, the point a is a removable singularity of f if and only if there exists an analytic function g on Ω such that $f(z) = g(z),\ \forall z \in \Omega\backslash\{a\}$.

Proof: Assume that there exists an analytic function g on Ω such that $f(z) = g(z),\ \forall z \in \Omega\backslash\{a\}$. Then, g is continuous at a. Therefore, $\lim_{z\to a} f(z) = \lim_{z\to a} g(z) = g(a)$, which exists in $\mathbb{C}$.

Conversely, assume that a is a removable singularity. We choose $r > 0$ such that $B(a, r)$ is contained in Ω. Then, for each $z \in B(a, r)\backslash\{a\}$, we define

$$g_z(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z},\ \forall \zeta \in \Omega\backslash\{a, z\}.$$

Then, g_z is analytic on $\Omega\backslash\{a, z\}$,

$$\lim_{\zeta\to z} g_z(\zeta) = \lim_{\zeta\to z} \frac{f(\zeta) - f(z)}{\zeta - z} = f'(z)$$

and

$$\lim_{\zeta\to a} g_z(\zeta) \quad = \quad \lim_{\zeta\to a} \frac{f(\zeta) - f(z)}{\zeta - z} = \frac{\lim_{\zeta\to a} f(\zeta) - f(z)}{a - z}.$$

Therefore, g_z is bounded on some neighbourhoods of a and z. Using Cauchy's theorem for open disk with exceptional points, if $0 < s < r$ and $0 < |z-a| < s$, then we get

$$\int\limits_{|\zeta-a|=s} g_z(\zeta)\,d\zeta = 0 \Rightarrow \frac{1}{i2\pi}\int\limits_{|\zeta-a|=s} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{f(z)}{i2\pi}\int\limits_{|\zeta-a|=s} \frac{d\zeta}{\zeta - z} = f(z).$$

Since f is continuous on $|z-a| = s$, if $f_1(z) = \dfrac{1}{i2\pi}\displaystyle\int\limits_{|\zeta-a|=s} \frac{f(\zeta)}{\zeta - z}\,d\zeta$, then f_1 is analytic on $B(a,r)$ (including at a) and $f_1(z) = f(z), \forall z \in B(a,r)$. Hence, if $g : \Omega \to \mathbb{C}$ is defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \neq a \\ f_1(a) & \text{if } z = a \end{cases},$$

then g is analytic on Ω and $g(z) = f(z),\ \forall z \in \Omega\backslash\{a\}$. □

THEOREM 5.1.3 (Taylor's theorem)
Let f be an analytic function on Ω and $z_0 \in \Omega$. Then, for every $n \in \mathbb{N}$, there exists an analytic function ϕ_n on Ω such that

$$f(z) = \sum_{k=1}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \phi_n(z)(z - z_0)^n,\ \forall z \in \Omega.$$

Furthermore,

$$\phi_n(z) = \frac{1}{i2\pi}\int\limits_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n},\ d\zeta,\ \forall z \text{ inside } C,$$

where C is a circle $|z - z_0| = r$ contained in Ω.

Proof: If we define

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0},\ \forall z \in \Omega\backslash\{z_0\},$$

then F is analytic on $\Omega\backslash\{z_0\}$ and z_0 is a removable singularity for F, as

$$\lim_{z\to z_0} F(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Then, by Lemma 5.1.2, there exists an analytic function ϕ_1 on Ω such that $\phi_1(z) = F(z)$, $\forall z \in \Omega\backslash\{z_0\}$. Hence, for every $z \in \Omega\backslash\{z_0\}$, clearly we have $f(z)-f(z_0) = \phi_1(z)(z-z_0)$ and $f(z_0)-f(z_0) = 0 = \phi_1(z_0)(z_0-z_0)$. Therefore,

$$f(z) = f(z_0) + \phi_1(z)(z - z_0),\ \forall z \in \Omega.$$

Applying the same technique, we can write $\phi_1(z) = \phi_1(z_0) + \phi_2(z)(z - z_0)$ for some analytic function ϕ_2 on Ω. Proceeding like this, at the nth stage, we get $\phi_{n-1}(z) = \phi_{n-1}(z_0) + \phi_n(z)(z - z_0)$ for some analytic function ϕ_n on Ω. Therefore, for every $z \in \Omega$,

$$\begin{aligned} f(z) &= f(z_0) + \phi_1(z)(z - z_0) \\ &= f(z_0) + (\phi_1(z_0) + \phi_2(z)(z - z_0))\,(z - z_0) \\ &= f(z_0) + \phi_1(z_0)(z - z_0) + \phi_2(z)(z - z_0)^2 \\ &= f(z_0) + \phi_1(z_0)(z - z_0) + \phi_2(z_0)(z - z_0)^2 + \phi_3(z)(z - z_0)^3 \\ &\vdots \\ &= f(z_0) + \sum_{k=1}^{n-1} \phi_k(z_0)(z - z_0)^k + \phi_n(z)(z - z_0)^n. \end{aligned}$$

Differentiating n times, we get

$$f^{(n)}(z) = \sum_{j=0}^{n} {}^nC_j \frac{d^j}{dz^j}((z - z_0)^n)\phi_n^{(n-j)}(z),\ \forall z \in \Omega.$$

For each $j = 0, 1, 2, \ldots, n-1$, $\frac{d^j}{dz^j}((z - z_0)^n)$ has the factor $(z - z_0)$, and $\frac{d^n}{dz^n}((z - z_0)^n) = n!$, we get $f^{(n)}(z_0) = n!\phi_n(z_0)$. Hence, $\phi_n(z_0) = \dfrac{f^{(n)}(z_0)}{n!}$, $\forall n \in \mathbb{N}$. Therefore, we have

$$f(z) = f(z_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \phi_n(z)(z - z_0)^n,\ \forall z \in \Omega.$$

Next, we choose $r > 0$ such that $ClB(z_0, r) \subseteq \Omega$. If C is the circle with centre z_0 and radius $r > 0$, then by Cauchy's integral formula, we have

$$\phi_n(z) = \frac{1}{i2\pi} \int_C \frac{\phi_n(\zeta)}{\zeta - z}\, d\zeta,\ \forall z \in B(z_0, r).$$

Since

$$\phi_n(z) = \frac{1}{(z-z_0)^n}\left(f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right),$$

we have

$$\phi_n(z) = \frac{1}{i2\pi}\int_C \left[\frac{f(\zeta)}{(\zeta-z_0)^n(\zeta-z)} - \sum_{k=0}^{n-1}\frac{f^{(k)}(z_0)}{k!(\zeta-z_0)^{n-k}(\zeta-z)}\right] d\zeta. \quad (5.1)$$

Now, we claim that $\int_C \frac{d\zeta}{(\zeta-z_0)^j(\zeta-z)} = 0$, $\forall j \in \mathbb{N}$ and $\forall z \in B(z_0, r)$. For each $j \in \mathbb{R}$ and $z \in B(z_0, r)$, define

$$F_{j,z}(w) = \int_C \frac{d\zeta}{(\zeta-w)^j(\zeta-z)}, \ \forall w \in B(z_0, r).$$

Applying partial fraction technique, we get

$$\frac{1}{(\zeta-w)(\zeta-z)} = \frac{1}{z-w}\left(\frac{1}{\zeta-z} - \frac{1}{\zeta-w}\right),$$

and hence, using Example 4.2.6, for every $w \in B(z_0, r)$, we have

$$\begin{aligned} F_{1,z}(w) &= \int_C \frac{d\zeta}{(\zeta-w)(\zeta-z)} = \frac{1}{z-w}\int_C \left(\frac{1}{\zeta-z} - \frac{1}{\zeta-w}\right) d\zeta \\ &= \frac{i2\pi}{z-w}\left(\text{WN}(C,z) - \text{WN}(C,w)\right) = 1 - 1 = 0. \end{aligned}$$

Using Theorem 4.3.2, we get

$$\begin{aligned} F_{j,z}(w) &= \frac{F^{(1)}_{j-1,z}(w)}{j} = \frac{F^{(2)}_{j-2,z}(w)}{j(j-1)} \\ &= \frac{F^{(3)}_{j-3,z}(w)}{j(j-1)(j-2)} = \cdots = \frac{F^{(j-1)}_{1,z}(w)}{j!} = 0, \end{aligned}$$

for all $j > 1$. Thus, $F_{j,z}(w) = 0$, $\forall j > 0$ and $\forall w \in B(z_0, r)$. In particular,

$$\int_C \frac{d\zeta}{(\zeta-z_0)^j(\zeta-z)} = F_{j,z}(z_0) = 0, \ \forall j = 1, 2, 3, \ldots, n-1.$$

Using this observation in equation (5.1), we get

$$\phi_n(z) = \frac{1}{i2\pi}\int\limits_C \frac{f(\zeta)}{(\zeta - z_0)^n(\zeta - z)}\,d\zeta,\ \forall z \in B(z_0, r).$$

Hence, the theorem follows. □

COROLLARY 5.1.4 Let P be a non-constant polynomial and $z_0 \in \mathbb{C}$. If $m \in \mathbb{N}$ such that $P^{(k)}(z_0) = 0, \forall k = 0, 1, 2 \ldots, m-1$ and $P^{(m)}(z_0) \neq 0$, then m is the order of the zero z_0 for P.

Proof: By Taylor's theorem, we have $P(z) = (z - z_0)^m\phi(z)$ for some analytic function ϕ on $\mathbb{C}$ such that $\phi(z_0) \neq 0$. Then, $\phi(z) = \frac{P(z)}{(z-z_0)^m}, \forall z \in \mathbb{C}$, which is a rational function and has no pole in $\mathbb{C}$. Hence, using Lemma 3.1.16, we get ϕ, which is a polynomial. Therefore, m is the order of the zero of P at z_0. □

THEOREM 5.1.5 (Infinite Taylor series)
Let f be analytic in a region Ω and $z_0 \in \Omega$. Then, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,\ \forall z \in B(z_0, r),$$

where r is a positive real number such that $ClB(z_0, r) \subseteq \Omega$.

Proof: For every $n \in \mathbb{N}$, from Taylor theorem, we get

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \phi_n(z)(z - z_0)^n,\ \forall z \in \Omega,$$

where

$$\phi_n(z) = \frac{1}{i2\pi}\int\limits_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n}\,d\zeta,\ \forall z \in B(z_0, r),$$

where C is the circle $|\zeta - z_0| = r$. If $M = \sup\limits_{\zeta \in C} |f(\zeta)|$, then for $z \in B(z_0, r)$,

$$\begin{aligned}\left| f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \right| &= |z - z_0|^n |\phi_n(z)| \\ &\leq \frac{|z - z_0|^n}{2\pi}\int\limits_C \frac{|f(\zeta)|}{|\zeta - z||\zeta - z_0|^n}\,|d\zeta|\end{aligned}$$

$$\leq \frac{|z-z_0|^n}{2\pi}\int\limits_C \frac{M}{r^n(r-|z-z_0|)}\,|d\zeta|$$

$$\leq \frac{Mr}{r-|z-z_0|}\left(\frac{|z-z_0|}{r}\right)^n \to 0 \text{ as } n\to\infty$$

$$(\text{as } \int\limits_C |d\zeta| = 2\pi r).$$

Thus, $f(z) = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum\limits_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,\ \forall z\in B(z_0,r).$

□

Remark 5.1.6: At this juncture, it should be noted that finite Taylor series expansion $f(z) = \sum\limits_{k=0}^{n-1}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + \phi_n(z)(z-z_0)^n$ of an analytic function f on Ω is valid on the entire region Ω, but the infinite Taylor series expansion $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ is valid only in a largest possible neighbourhood $B(z_0,r)\subseteq\Omega$.

Definition 5.1.7 Let the power series $\sum\limits_{n=0}^{\infty}a_n(z-a)^n$ and $\sum\limits_{n=0}^{\infty}b_n(z-a)^n$ converge on $B(a,R)$. Then, the Cauchy product of the two power series is defined by

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)(z-a)^n,\ \forall z\in B(a,R).$$

THEOREM 5.1.8 If $f(z) = \sum\limits_{n=0}^{\infty}a_n(z-a)^n$ and $g(z) = \sum\limits_{n=0}^{\infty}b_n(z-a)^n$,$\forall z \in B(a,R)$, then $(f\cdot g)(z) = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{k=0}^{n}a_kb_{n-k}\right)(z-a)^n,\ \forall z\in B(a,R)$.

Proof: We know that f and g are analytic on $B(a,R)$, and hence, $f\cdot g$ is also analytic on $B(a,R)$. By Taylor's theorem, we have

$$(f\cdot g)(z) = \sum_{n=0}^{\infty}\frac{(f\cdot g)^{(n)}(a)}{n!}(z-a)^n,\ \forall z\in B(a,R).$$

Now using Leibniz rule (Result 2.1.13) and Corollary 2.3.4, we have

$$\begin{aligned}\frac{(f\cdot g)^{(n)}(a)}{n!} &= \frac{1}{n!}\sum_{k=0}^{n} {}^{n}C_k f^{(k)}(a)\cdot g^{(n-k)}(a)\\ &= \frac{1}{n!}\sum_{k=0}^{n} {}^{n}C_k f^{(k)}(a)\cdot g^{(n-k)}(a)\\ &= \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(a_k k!)\cdot(b_{n-k}(n-k)!)\\ &= \sum_{k=0}^{n} a_k b_{n-k}.\end{aligned}$$

Hence, for every $z \in B(a,R)$, using Corollary 2.3.5, we get $(f\cdot g)(z) = \sum_{n=0}^{\infty}\frac{(f\cdot g)^{(n)}(a)}{n!}(z-a)^n$. □

COROLLARY 5.1.9 If $\sum_{n=0}^{\infty} a_n(z-a)^n$ and $\sum_{n=0}^{\infty} b_n(z-a)^n$ converges on $B(a,R)$, then the Cauchy product of the two power series converge on $B(a,R)$.

The proof of this corollary follows immediately from the previous theorem.

Definition 5.1.10 Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ and $0 \neq \sum_{n=0}^{\infty} b_n(z-a)^n$ converge $\forall z \in B(a,R)$. Then, the quotient

$$\sum_{n=0}^{\infty} d_n(z-a)^n = \frac{\sum_{n=0}^{\infty} a_n(z-a)^n}{\sum_{n=0}^{\infty} b_n(z-a)^n}$$

of the power series is defined as the Taylor series of $\frac{f}{g}$, where

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \text{ and } 0 \neq g(z) = \sum_{n=0}^{\infty} b_n(z-a)^n,\ \forall z \in B(a,R).$$

Finding the coefficients of the Taylor series of $\frac{f}{g}$ in terms of a_n and b_n are not easy as that of $f \cdot g$. However, the first few coefficients of the Taylor series of $\frac{f}{g}$ can be calculated successively.

In the following sequel, we use the notation $\mathrm{AN}_r^s(a)$ to denote the annulus $\{z \in \mathbb{C} : r < |z - a| < s\}$.

Definition 5.1.11 We say that $\sum_{n=-\infty}^{\infty} a_n$ is convergent if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are convergent; in this case, we write $\sum_{n=-\infty}^{\infty} a_n = \sum_{n=1}^{\infty} a_{-n} + \sum_{n=0}^{\infty} a_n$.

THEOREM 5.1.12 For a given $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, if $r = \limsup_{n\to\infty} |a_{-n}|^{1/n}$, $s = \limsup_{n\to\infty} |a_n|^{1/n}$ in $[-\infty, +\infty]$, and $r < s$, then

1. $\sum_{n=-\infty}^{\infty} a_n z^n$ converges on $r < |z - z_0| < s$,
2. $\sum_{n=-\infty}^{\infty} a_n z^n$ converges uniformly on $r_1 \le |z - z_0| \le s_1$, if $r < r_1 < s_1 < s$,
3. $\sum_{n=-\infty}^{\infty} a_n z^n$ diverges on $|z - z_0| < r$ or $|z - z_0| > s$.

Proof of this theorem follows immediately from Abel's theorem on convergence of power series (Theorem 2.3.2).

THEOREM 5.1.13 (Laurent's Theorem)
Let f be analytic on an annulus $\mathrm{AN}_{R_1}^{R_2}(z_0) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. Then, f can be written as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k, \ \forall z \in \mathrm{AN}_{R_1}^{R_2}(z_0).$$

Proof: Choose positive real numbers r_1 and r_2 such that $R_1 < r_1 < r_2 < R_2$. If C_j is the circle with centre z_0 and radius r_j, for $j = 1, 2$, then we claim that $C_2 \cup (-C_1) \sim 0$ in $\mathrm{AN}_{R_1}^{R_2}(z_0)$. Let $w \notin \mathrm{AN}_{R_1}^{R_2}(z_0)$.

Case 1: $|w - z_0| > R_2$.
Then, $|w - z_0| > r_1$ and $|w - z_0| > r_2$, which implies $\text{WN}(C_2 \cup (-C_1), w) = \text{WN}(C_2, w) - \text{WN}(C_1, w) = 0$.

Case 2: $|w - z_0| < R_1$. In this case, $|w - z_0| < r_1$, $|w - z_0| < r_2$, and hence, $\text{WN}(C_2 \cup (-C_1), w) = \text{WN}(C_2, w) - \text{WN}(C_1, w) = 1 - 1 = 0$.

Hence, our claim holds. Then, by the general version of Cauchy's integral formula, we have

$$\text{WN}(C_2 \cup (-C_1), z) f(z) = \frac{1}{i2\pi} \int\limits_{C_2 \cup (-C_1)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \ \forall z \notin C_2 \cup (-C_1).$$

In particular, as $\text{WN}(C_2 \cup (-C_1), z) = 1$ whenever $r_1 < |z - z_0| < r_2$, we have

$$f(z) = \frac{1}{i2\pi} \int\limits_{C_2 \cup (-C_1)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{i2\pi} \int\limits_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{i2\pi} \int\limits_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Since $\dfrac{|\zeta - z_0|}{|z - z_0|} = \dfrac{r_1}{|z - z_0|} < 1, \forall \zeta \in C_1$, we get

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \left(\frac{-1}{z - z_0}\right)\left(\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}\right) = \left(\frac{-1}{z - z_0}\right) \sum_{k=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^k. \end{aligned}$$

We note that the power series on the right-hand side converges uniformly on C_1. Similarly, using $\dfrac{|z - z_0|}{|\zeta - z_0|} < 1, \forall \zeta \in C_2$, we get

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \left(\frac{1}{\zeta - z_0}\right)\left(\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\right) = \left(\frac{1}{\zeta - z_0}\right) \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k, \end{aligned}$$

and the series uniformly converges on C_2, $\forall z \in \text{AN}_{r_1}^{r_2}(z_0)$.

Therefore, we get

$$f(z) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{i2\pi} \int\limits_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta$$

$$
\begin{aligned}
&+\sum_{k=0}^{\infty}\frac{(z-z_0)^{-k-1}}{i2\pi}\int\limits_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-k}}\,d\zeta\\
=\;&\sum_{k=0}^{\infty}\frac{(z-z_0)^{k}}{i2\pi}\int\limits_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta\\
&+\sum_{j=1}^{\infty}\frac{(z-z_0)^{-j}}{i2\pi}\int\limits_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-j+1}}\,d\zeta\\
=\;&\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k,
\end{aligned}
$$

where $a_k=\begin{cases}\dfrac{1}{i2\pi}\displaystyle\int\limits_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta & \text{if } k\geq 0\\ \dfrac{1}{i2\pi}\displaystyle\int\limits_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta & \text{if } k<0\end{cases}, \forall k\in\mathbb{Z}.$ □

RESULT 5.1.14 (Uniqueness)

If $f(z)=\sum\limits_{k=-\infty}^{\infty}c_k(z-z_0)^k$, $\forall z\in \mathrm{AN}_r^s(z_0)$ and if the Laurent series of f is $f(z)=\sum\limits_{k=-\infty}^{\infty}a_k(z-z_0)^k$, $\forall z\in \mathrm{AN}_r^s(z_0)$, then $c_k=a_k,\ \forall k\in\mathbb{Z}$.

Proof: Let n be a non-negative integer. Then, by definition, we have

$$
\begin{aligned}
a_n &= \frac{1}{i2\pi}\int\limits_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\\
&= \frac{1}{i2\pi}\int\limits_{C_2}\frac{\sum\limits_{k=-\infty}^{\infty}c_k(\zeta-z_0)^k}{(\zeta-z_0)^{n+1}}\,d\zeta\\
&= \sum_{k=-\infty}^{\infty}c_k\frac{1}{i2\pi}\int\limits_{C_2}(\zeta-z_0)^{k-n-1}\,d\zeta\\
&\qquad\text{(as the series converges uniformly on } C_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} c_k \frac{1}{i2\pi} \int_0^{2\pi} (r_2 \exp(it))^{k-n-1} \, ir_2 \exp(it)\, dt \\
&\quad \text{(as } C_2 \text{ is given by } \zeta(t) = z_0 + r_2 \exp(it),\ t \in [0, 2\pi]) \\
&= \sum_{k=-\infty}^{\infty} c_k i r_2^{k-n} \frac{1}{i2\pi} \int_0^{2\pi} \exp(i(k-n)t)\, dt \\
&= c_n,
\end{aligned}
$$

since $\frac{1}{2\pi} \int_0^{2\pi} \exp(i(k-n)t)\, dt = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$. By a similar argument, we get $a_n = c_n$ for every negative integer also. Hence, the result follows. □

Algorithm 5.1.15 (To find Laurent series of rational functions)
Algorithm to find Laurent series

Step 1. If the given rational function is $\frac{P(z)}{Q(z)}$, where $P(z)$, $Q(z)$ are polynomials, then find the partial fraction expansion of $\frac{1}{Q(z)}$ if $Q(z)$ has more than one root. In this case, a typical summand in the partial fraction expansion is of the form $\frac{A}{(z \pm a)^m}$ for some $A, a \in \mathbb{C}$.

Step 2(a). To expand $\frac{1}{(z \pm a)^m}$, in $|z - z_0| < R$ for some $R > 0$ and for some $a \in \mathbb{C}$ with $R < |b|$, where $b = -(z_0 \pm a)$.
Then write

$$
\begin{aligned}
\frac{1}{(z \pm a)^m} &= \frac{1}{((z - z_0) - b)^m} \\
&= \frac{1}{|b|^m} \frac{1}{\left(1 - \frac{(z - z_0)}{b}\right)^m} \\
&= \frac{1}{|b|^m} \frac{1}{m!} \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1) \left(\frac{(z - z_0)}{b}\right)^{k-m}.
\end{aligned}
$$

Step 2(b). To expand $\frac{1}{(z \pm a)}$, in $|z - z_0| > R$ for some $R > 0$ and for some $a \in \mathbb{C}$ with $|b| < R$, where $b = -(z_0 \pm a)$.

Then write

$$\begin{aligned}\frac{1}{(z \pm a)^m} &= \frac{1}{((z-z_0)-b)^m} \\ &= \frac{1}{(z-z_0)^m} \frac{1}{\left(1-\frac{b}{(z-z_0)}\right)^m} \\ &= \frac{1}{(z-z_0)^m} \frac{1}{m!} \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1) \left(\frac{b}{(z-z_0)}\right)^{k-m}.\end{aligned}$$

Step 3. Finally, replace all the summands by the corresponding series and simplify.

From the following examples, one can understand the algorithm still better.

Example 5.1.16 Find the series development of the following functions in the specified regions:

1. $\dfrac{1}{z^2-5z+4}$ in (i) $0 < |z| < 1$; (ii) $1 < |z| < 4$; (iii) $|z| > 4$.
2. $\dfrac{z+1}{z^2-z-6}$ in (i) $0 < |z| < 2$; (ii) $2 < |z| < 3$; (iii) $|z| > 3$.
3. $\dfrac{z}{z^2-8z+15}$ in (i) $0 < |z-2| < 1$; (ii) $1 < |z-2| < 3$; (iii) $|z-2| > 3$.
4. $\dfrac{z+5}{z^2-6z+8}$ in $1 < |z+1| < 3$.

Solution:

1. Applying partial fraction technique, we write

$$\frac{1}{z^2-5z+4} = \frac{1/3}{z-4} - \frac{1/3}{z-1}.$$

(a) If $0 < |z| < 1$, then we have[1]

$$\frac{1/3}{z-4} = -\frac{1}{12\left(1-\frac{z}{4}\right)} = -\frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k \tag{5.2}$$

[1]Here note that $|z| < 1 < 4$ so we have taken -4 and -1 as the common factors in equations (5.2) and (5.3), respectively.

and

$$\frac{-1/3}{z-1} = \frac{1}{3\,(1-z)} = \frac{1}{3}\sum_{k=0}^{\infty} z^k. \tag{5.3}$$

Thus, we get

$$\begin{aligned}\frac{1}{z^2-5z+4} &= -\frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k + \frac{1}{3}\sum_{k=0}^{\infty} z^k \\ &= \frac{1}{3}\sum_{k=0}^{\infty}\left(1-\frac{1}{4^{k+1}}\right)z^k,\ 0<|z|<1.\end{aligned}$$

(b) If $1<|z|<4$, then we have[2]

$$\frac{1/3}{z-4} = -\frac{1}{12\left(1-\frac{z}{4}\right)} = -\frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k \tag{5.4}$$

and

$$\frac{-1/3}{z-1} = -\frac{1}{3z\left(1-\frac{1}{z}\right)} = -\frac{1}{3z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k+1}. \tag{5.5}$$

Thus, we get

$$\begin{aligned}\frac{1}{z^2-5z+4} &= -\frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k+1} \\ &= -\frac{1}{3}\left[\sum_{k=0}^{\infty}\frac{1}{4^{k+1}}z^k + \sum_{k=1}^{\infty} z^{-k}\right],\ 1<|z|<4.\end{aligned}$$

(c) If $|z|>4$, then we have

$$\frac{1/3}{z-4} = \frac{1}{3z\left(1-\frac{4}{z}\right)} = \frac{1}{3z}\sum_{k=0}^{\infty}\left(\frac{4}{z}\right)^k = \frac{1}{3}\sum_{k=0}^{\infty}\frac{4^k}{z^{k+1}}$$

[2]Here note that $1<|z|<4$ so we have taken -4 and z as the common factors in equations (5.4) and (5.5), respectively.

and

$$\frac{-1/3}{z-1} = -\frac{1}{3z\left(1-\frac{1}{z}\right)} = -\frac{1}{3z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k+1}.$$

Thus, we get

$$\begin{aligned}\frac{1}{z^2-5z+4} &= \frac{1}{3}\sum_{k=0}^{\infty}\frac{4^k}{z^{k+1}} - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k+1} \\ &= -\frac{1}{3}\sum_{k=1}^{\infty}(4^k-1)z^{-(k+1)},\ 0<|z|<1.\end{aligned}$$

2. Applying partial fraction technique, we write

$$\frac{1}{z^2-z-6} = \frac{1/5}{z-3} - \frac{1/5}{z+2}.$$

(a) If $0<|z|<2$, then we have

$$\begin{aligned}\frac{z+1}{5\,(z-3)} &= -\frac{z+1}{15\left(1-\frac{z}{3}\right)} \\ &= -\frac{z+1}{15}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k \\ &= -\frac{1}{15}\left[\sum_{k=0}^{\infty}3\left(\frac{z}{3}\right)^{k+1} + \sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k\right] \\ &= -\frac{1}{15}\left[\sum_{k=0}^{\infty}3\left(\frac{z}{3}\right)^{k+1} + 1 + \sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k+1}\right] \\ &= -\frac{1}{15} + \frac{4}{15}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k+1}\end{aligned}$$

and

$$\begin{aligned}\frac{-(z+1)}{5(z+2)} &= \frac{-(z+1)}{10\left(1+\frac{z}{2}\right)} \\ &= -\frac{z+1}{10}\sum_{k=0}^{\infty}\left(\frac{z}{-2}\right)^k\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{10}\left[\sum_{k=0}^{\infty}(-2)\left(\frac{z}{-2}\right)^{k+1}+\sum_{k=0}^{\infty}\left(\frac{z}{-2}\right)^{k}\right]\\
&= -\frac{1}{10}\left[\sum_{k=0}^{\infty}(-2)\left(\frac{z}{-2}\right)^{k+1}+1+\sum_{k=0}^{\infty}\left(\frac{z}{-2}\right)^{k+1}\right]\\
&= \frac{-1}{10}+\frac{1}{10}\sum_{k=0}^{\infty}\left(\frac{z}{-2}\right)^{k+1}.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\frac{z+1}{z^2-z-6} &= -\frac{1}{15}+\frac{4}{15}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k+1}+\frac{-1}{10}+\frac{1}{10}\sum_{k=0}^{\infty}\left(\frac{z}{-2}\right)^{k+1}\\
&= \frac{-1}{6}+\sum_{k=1}^{\infty}\left[\frac{4}{15}3^{-k}+\frac{1}{10}(-2)^{k}\right]z^{k}, 0<|z|<2.
\end{aligned}
$$

(b) If $2<|z|<3$, then using (i), we have

$$
\frac{z+1}{5\,(z-3)} = -\frac{z+1}{15\left(1-\frac{z}{3}\right)} = -\frac{1}{15}+\frac{4}{15}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k+1}
$$

and

$$
\begin{aligned}
-\frac{z+1}{5(z+2)} &= -\frac{z+1}{5z\left(1+\frac{2}{z}\right)}\\
&= -\frac{z+1}{5z}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k}\\
&= -\frac{1}{5}\left[\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k}-\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}\right]\\
&= -\frac{1}{5}\left[1+\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}-\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}\right]\\
&= -\frac{1}{5}+\frac{1}{10}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}.
\end{aligned}
$$

Thus, we get

$$\begin{aligned}\frac{z+1}{z^2-z-6} &= -\frac{1}{15}+\frac{4}{15}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^{k+1}-\frac{1}{5}+\frac{1}{10}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}\\ &= -\frac{4}{15}+\frac{4}{15}\sum_{k=1}^{\infty}3^{-k}z^k\\ &\quad +\frac{1}{10}\sum_{k=1}^{\infty}(-2)^k z^{-k},\ 2<|z|<3.\end{aligned}$$

(c) If $|z|>3$, then we have

$$\begin{aligned}\frac{z+1}{5(z-3)} &= \frac{z+1}{5z\left(1-\frac{3}{z}\right)}\\ &= \frac{z+1}{5z}\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k\\ &= \frac{1}{5}\left[\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k+\sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{3}{z}\right)^{k+1}\right]\\ &= \frac{1}{5}+\left[1+\sum_{k=1}^{\infty}\left(\frac{3}{z}\right)^k+\sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{3}{z}\right)^k\right]\\ &= \frac{1}{5}+\frac{4}{15}\sum_{k=1}^{\infty}\left(\frac{3}{z}\right)^k,\end{aligned}$$

and from case (ii), we have

$$-\frac{z+1}{5(z+2)}=-\frac{z+1}{5z\left(1+\frac{2}{z}\right)}=-\frac{1}{5}+\frac{1}{10}\sum_{k=0}^{\infty}\left(-\frac{2}{z}\right)^{k+1}.$$

Thus, we get

$$\begin{aligned}\frac{z+1}{z^2-z-6} &= \frac{4}{15}\sum_{k=1}^{\infty}3^k z^{-k}+\frac{1}{10}\sum_{k=1}^{\infty}(-2)^k z^{-k}\\ &= \sum_{k=1}^{\infty}\left(\frac{4}{5}3^{k-1}+\frac{1}{10}(-2)^k\right)z^{-k},\ |z|>3.\end{aligned}$$

3. Applying partial fraction technique, we write

$$\frac{1}{z^2-8z+15}=\frac{1/2}{z-5}-\frac{1/2}{z-3}.$$

(a) If $0<|z-2|<1$, then we have

$$\begin{aligned}
\frac{z}{2(z-5)} &= \frac{z}{2(z-2-3)}\\
&= \frac{z-2+2}{-6}\frac{1}{\left(1-\frac{z-2}{3}\right)}\\
&= -\left(\frac{z-2}{6}+\frac{1}{3}\right)\frac{1}{\left(1-\frac{z-2}{3}\right)}\\
&= -\left(\frac{z-2}{6}+\frac{1}{3}\right)\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^k\\
&= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^{k+1}-\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^k\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{z-2}{3}\right)^k-\frac{1}{3}-\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{z-2}{3}\right)^k\\
&= -\frac{1}{3}-\frac{5}{6}\sum_{k=1}^{\infty}3^{-k}(z-2)^k
\end{aligned}$$

and

$$\begin{aligned}
-\frac{z}{2(z-3)} &= -\frac{z}{2(z-2-1)}\\
&= \frac{z-2+2}{2}\frac{1}{(1-(z-2))}\\
&= \left(\frac{z-2}{2}+1\right)\frac{1}{(1-(z-2))}\\
&= \left(\frac{z-2}{2}+1\right)\sum_{k=0}^{\infty}(z-2)^k\\
&= \frac{1}{2}\sum_{k=0}^{\infty}(z-2)^{k+1}+\sum_{k=0}^{\infty}(z-2)^k\\
&= \frac{1}{2}\sum_{k=1}^{\infty}(z-2)^k+1+\sum_{k=1}^{\infty}(z-2)^k
\end{aligned}$$

$$= 1 + \frac{3}{2}\sum_{k=1}^{\infty}(z-2)^k.$$

Therefore,

$$\begin{aligned}\frac{1}{z^2-8z+15} &= -\frac{1}{3} - \frac{5}{6}\sum_{k=1}^{\infty}3^{-k}(z-2)^k + 1 \\ &\quad + \frac{3}{2}\sum_{k=1}^{\infty}(z-2)^k \\ &= \frac{2}{3} + \sum_{k=1}^{\infty}\left(\frac{3}{2} - \frac{5}{6}3^{-k}\right)(z-2)^k.\end{aligned}$$

(b) If $1 < |z-2| < 3$, then we have

$$\begin{aligned}\frac{z}{2(z-5)} &= \frac{z}{2(z-2-3)} \\ &= \frac{z-2+2}{-6}\frac{1}{\left(1-\frac{z-2}{3}\right)} \\ &= -\left(\frac{z-2}{6}+\frac{1}{3}\right)\frac{1}{\left(1-\frac{z-2}{3}\right)} \\ &= -\left(\frac{z-2}{6}+\frac{1}{3}\right)\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^k \\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^{k+1} - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z-2}{3}\right)^k \\ &= -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{z-2}{3}\right)^k - \frac{1}{3} - \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{z-2}{3}\right)^k \\ &= -\frac{1}{3} - \frac{5}{6}\sum_{k=1}^{\infty}3^{-k}(z-2)^k\end{aligned}$$

and

$$\begin{aligned}-\frac{z}{2(z-3)} &= -\frac{z}{2(z-2-1)} \\ &= -\frac{z-2+2}{2(z-2)}\frac{1}{\left(1-\frac{1}{z-2}\right)}\end{aligned}$$

$$\begin{aligned} &= -\left(\frac{1}{2} - \frac{1}{z-2}\right) \frac{1}{\left(1 - \frac{1}{z-2}\right)} \\ &= \frac{-1}{2} \sum_{k=0}^{\infty} (z-2)^{-k} + \sum_{k=1}^{\infty} (z-2)^{-k} \\ &= -\frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} (z-2)^{-k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{z^2 - 8z + 15} &= -\frac{1}{3} - \frac{5}{6} \sum_{k=1}^{\infty} 3^{-k} (z-2)^k - \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} (z-2)^{-k} \\ &= -\frac{5}{6} - \frac{5}{6} \sum_{k=1}^{\infty} 3^{-k} (z-2)^k + \frac{1}{2} \sum_{k=1}^{\infty} (z-2)^{-k}. \end{aligned}$$

(c) If $|z-2| > 3$, then we have

$$\begin{aligned} \frac{z}{2(z-5)} &= \frac{z}{2(z-2-3)} \\ &= \frac{z-2+2}{2(z-2)} \frac{1}{\left(1 - \frac{3}{z-2}\right)} \\ &= \left(\frac{1}{2} + \frac{1}{z-2}\right) \frac{1}{\left(1 - \frac{3}{z-2}\right)} \\ &= \left(\frac{1}{2} + \frac{1}{z-2}\right) \sum_{k=0}^{\infty} \left(\frac{3}{z-2}\right)^k \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{3}{z-2}\right)^k + \frac{1}{3(z-2)} \sum_{k=0}^{\infty} \left(\frac{3}{z-2}\right)^k \\ &= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{3}{z-2}\right)^k + \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{3}{z-2}\right)^k \\ &= \frac{1}{2} + \frac{5}{6} \sum_{k=1}^{\infty} 3^k (z-2)^{-k} \end{aligned}$$

and

$$\begin{aligned}
-\frac{z}{2(z-3)} &= -\frac{z}{2(z-2-1)} \\
&= -\frac{z-2+2}{2(z-2)}\frac{1}{\left(1-\frac{1}{z-2}\right)} \\
&= -\left(\frac{1}{2}-\frac{1}{z-2}\right)\frac{1}{\left(1-\frac{1}{z-2}\right)} \\
&= -\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{\infty}(z-2)^{-k}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{z^2-8z+15} &= \frac{1}{2}+\frac{5}{6}\sum_{k=1}^{\infty}3^k(z-2)^{-k}-\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{\infty}(z-2)^{-k} \\
&= \frac{5}{6}\sum_{k=1}^{\infty}3^{-k}(z-2)^k+\frac{1}{2}\sum_{k=1}^{\infty}(z-2)^{-k}.
\end{aligned}$$

4. Applying partial fraction technique, we write

$$\frac{1}{z^2-6z+8}=\frac{1/2}{z-4}-\frac{1/2}{z-2}.$$

(a) If $1<|z+1|<3$, then we have

$$\begin{aligned}
\frac{z+5}{2(z-4)} &= \frac{z+5}{2(z+1-5)} \\
&= -\frac{z+5}{10}\frac{1}{\left(1-\frac{z+1}{5}\right)} \\
&= -\frac{z+1+4}{10}\sum_{k=0}^{\infty}\left(\frac{z+1}{5}\right)^k \\
&= -\frac{1}{10}\left[5\sum_{k=0}^{\infty}\left(\frac{z+1}{5}\right)^{k+1}+4\sum_{k=0}^{\infty}\left(\frac{z+1}{5}\right)^k\right] \\
&= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z+1}{5}\right)^{k+1}-\frac{2}{5}\sum_{k=0}^{\infty}\left(\frac{z+1}{5}\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{z+1}{5}\right)^k - \frac{2}{5} - \frac{2}{5}\sum_{k=1}^{\infty}\left(\frac{z+1}{5}\right)^k \\
&= -\frac{2}{5} - \frac{7}{10}\sum_{k=1}^{\infty} 5^{-k}(z+1)^k
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{z+5}{2(z+2)} &= -\frac{z+5}{2(z+1+1)} \\
&= -\frac{z+1+4}{2(z+1)\left(1+\left(\frac{1}{z+1}\right)\right)} \\
&= -\left(\frac{1}{2}+\frac{2}{z+1}\right)\sum_{k=0}^{\infty}\left(\frac{-1}{z+1}\right)^k \\
&= -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-1}{z+1}\right)^k + 2\sum_{k=0}^{\infty}\left(\frac{-1}{z+1}\right)^{k+1} \\
&= -\frac{1}{2} - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{-1}{z+1}\right)^k + 2\sum_{k=1}^{\infty}\left(\frac{-1}{z+1}\right)^k \\
&= -\frac{1}{2} + \frac{3}{2}\sum_{k=1}^{\infty}(-1)^k\,(z+1)^{-k}\,.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\frac{1}{z^2-5z+4} &= -\frac{1}{2} - \frac{7}{10}\sum_{k=1}^{\infty} 5^{-k}(z+1)^k + 2 \\
&\quad -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\,(z+1)^{-k} \\
&= -1 - \frac{7}{10}\sum_{k=1}^{\infty} 5^{-k}(z+1)^k \\
&\quad +\frac{3}{2}\sum_{k=1}^{\infty}(-1)^k\,(z+1)^{-k}, \forall z \text{ with } 1<|z+1|<3.
\end{aligned}
$$

Exercise 5.1.17 Find the Laurent series expansions of the given functions in the specified regions.

1. $f(z) = \dfrac{1}{z^2(1-z)}$ in $0 < |z| < 1$.
2. $f(z) = \dfrac{1}{1+z}$ in (i) $0 < |z-i| < \sqrt{2}$, (ii) $|z-i| > \sqrt{2}$.
3. $f(z) = \dfrac{1}{z^2 - 5z + 6}$ in $0 < |z| < 2$.
4. $f(z) = \dfrac{(1-z)^3}{z-2}$ in $|z-1| > 1$.
5. $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ in (i) $0 < |z-1| < 2$, (ii) $0 < |z-3| < 2$.

Answers:

1. $\displaystyle\sum_{n=-2}^{\infty} \frac{1}{z^n}$.
2. (i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{1+i}\right)^n$, (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1+i}{z-i}\right)^n$.
3. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) z^{n-1}$.
4. $\displaystyle -(z-1)^3 \sum_{n=1}^{\infty} \frac{1}{(z-1)^n}$.
5. (i) $\displaystyle -\frac{1}{2(z-1)^2} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$, (ii) $\displaystyle \frac{1}{4(z-3)} \sum_{n=1}^{\infty} n \left(-\left[\frac{z-3}{2}\right]\right)^{n-1}$.

5.2 ZEROES, POLES, AND SINGULARITIES

Definition 5.2.1 If f is an analytic function on a region Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$, then z_0 is called a zero of f. If there exists $k \in \mathbb{N}$, such that

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0 \text{ and } f^{(k)}(z_0) \neq 0,$$

then k is called the order of the zero of f at z_0.

THEOREM 5.2.2 Let f be an analytic function, which is not identically zero on a region Ω. If $f(z_0) = 0$ for some $z_0 \in \Omega$, then order of zero of f at z_0 exists.

Proof: Let $A = \left\{z \in \Omega : f(z) = 0 \text{ and } f^{(k)}(z) = 0,\ \forall k \in \mathbb{N}\right\}$ and $B = \Omega \backslash A$. That is, $B = \left\{z \in \Omega : f(z) \neq 0 \text{ or } f^{(k)}(z) \neq 0 \text{ for some } k \in \mathbb{N}\right\}$. Then clearly, $\Omega = A \cup B$ and $A \cap B = \varnothing$. Since $f^{(k)}$ is analytic (see Corollary 4.3.3), $f^{(k)}$ is continuous for every $k = 0, 1, 2, 3, \ldots$. Therefore,

$$\left\{z \in \Omega : f^{(k)}(z) = 0\right\} = \left(f^{(k)}\right)^{-1}(\{0\})$$

is a closed set, and hence, $\left\{z \in \Omega : f^{(k)}(z) \neq 0\right\}$ is an open set for every $k = 0, 1, 2, 3, \ldots$. Therefore, $B = \bigcup_{k=0}^{\infty} \left\{z \in \Omega : f^{(k)}(z) \neq 0\right\}$ is an open set.
Next, we show that A is also an open set. If $a \in A$ is arbitrary, then for every $n \in \mathbb{N}$, by Taylor's theorem, we have

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(z-a)^k + \phi_n(z)(z-a)^n = \phi_n(z)(z-a)^n,\ \forall z \in \Omega,$$

where

$$\phi_n(z) = \frac{1}{i2\pi} \int\limits_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^n(\zeta - z)}\, d\zeta,\ \forall z \text{ inside } C,$$

where $r > 0$ is such that $ClB(a, r) \subseteq \Omega$. If $M = \sup\limits_{\zeta \in C} |f(\zeta)|$, then $M < \infty$ and for every $z \in B(a, r)$,

$$\begin{aligned}
|f(z)| &\leq \frac{|z-a|^n}{2\pi} \int\limits_{|\zeta - a| = r} \frac{|f(\zeta)|}{|\zeta - a|^n |\zeta - z|}\, |d\zeta| \\
&\leq \frac{M|z-a|^n}{2\pi} \int\limits_{|\zeta - a| = r} \frac{|d\zeta|}{r^n(r - |z-a|)} \\
&\quad (\text{as } |\zeta - a| \leq |z - \zeta| + |z - a| \Rightarrow r - |z-a| \leq |z - \zeta|) \\
&\leq \frac{M|z-a|^n}{2\pi} \frac{2\pi r}{r^n(r - |z-a|)} \\
&\leq \left(\frac{|z-a|}{r}\right)^n \frac{Mr}{r - |z-a|} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $f = 0$ on $B(a, r)$, and hence, $f^{(k)}(z) = 0$, for all $k = 0, 1, 2, 3, \ldots$. Thus, $B(a, r) \subseteq A$. Given that $z_0 \in \Omega$ such that $f(z_0) = 0$. Suppose $f^{(k)}(z_0) = 0$, $\forall k \in \mathbb{N}$, then $A \neq \varnothing$. By using the connectedness of Ω, we get

$$B = \varnothing \Rightarrow \Omega = A \Rightarrow f = 0 \text{ on } \Omega,$$

which is a contradiction. Hence, there exists a least positive integer k such that $f^{(k)}(z_0) \neq 0$. In other words, order of the zero at z_0 exists for f. □

RESULT 5.2.3 If f is an analytic function on a region Ω with a zero of order k at $z_0 \in \Omega$, then there exists an analytic function g on Ω such that g is analytic on Ω, $f(z) = (z - z_0)^k g(z)$ on Ω and $g(z_0) \neq 0$.

Proof: By Taylor's theorem, there exists an analytic function g on Ω such that

$$f(z) = \sum_{j=0}^{k-1} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j + g(z)(z - z_0)^k = g(z)(z - z_0)^k, \ \forall z \in \Omega.$$

Suppose $g(z_0) = 0$, then using

$$f^{(k)}(z) = \sum_{\nu=0}^{k} kC_\nu g^{(k-\nu)}(z) \frac{d^\nu}{dz^\nu}\left((z - a)^k\right)$$

and

$$\frac{d^\nu}{dz^\nu}\left((z - a)^k\right)\Big|_{z=a} = \begin{cases} 0 & \nu < k \\ k! & \nu = k \end{cases},$$

we obtain $f^{(k)}(a) = g(a)k! = 0$, which contradicts the fact that the order of the zero of f at z_0 is k. Thus, $g(a) \neq 0$. □

THEOREM 5.2.4 (Principle of analytic continuation)
Let f and g be analytic functions on a region Ω. If $\{z \in \Omega : f(z) = g(z)\}$ has a limit point in Ω, then $f(z) = g(z)$, $\forall z \in \Omega$.

Proof: Since f and g are analytic, clearly $f - g$ is an analytic function. If z_0 is a limit point of $\{z \in \Omega : f(z) = g(z)\}$, then there exists a sequence (z_n) from $\Omega \backslash \{z_0\}$ such that

$$f(z_n) - g(z_n) = 0, \ \forall n \in \mathbb{N} \text{ and } z_n \to z_0 \text{ as } n \to \infty.$$

Therefore, $(f-g)(z_0) = \lim_{n\to\infty}(f-g)(z_n) = 0$. If $f(w) \neq g(w)$ for some $w \in \Omega$, then $f-g$ is not identically zero. Then, by the previous result, there exists $k \in \mathbb{N}$ such that

$$(f-g)(z) = (z-z_0)^k \phi(z),\ \forall z \in B(z_0, r),$$

for some analytic function ϕ on Ω such that $\phi(z_0) \neq 0$. Since ϕ is continuous at z_0, for $\dfrac{|\phi(z_0)|}{2} > 0$, we find $0 < \delta < r$ such that

$$\begin{aligned} |z-z_0| < \delta \quad &\Rightarrow \quad |\phi(z) - \phi(z_0)| < \frac{|\phi(z_0)|}{2} \\ &\Rightarrow \quad |\phi(z_0)| - |\phi(z)| \leq |\phi(z) - \phi(z_0)| < \frac{|\phi(z_0)|}{2} \\ &\Rightarrow \quad |\phi(z)| \geq \frac{|\phi(z_0)|}{2}. \end{aligned}$$

Therefore, ϕ is nowhere zero on $B(z_0, \delta)$. Since $z_n \to z_0$ as $n \to \infty$, for this $\delta > 0$, we find an $m \in \mathbb{N}$ such that

$$\begin{aligned} |z_m - z_0| < \delta \quad &\Rightarrow \quad z_m \in B(z_0, \delta) \\ &\Rightarrow \quad (z_m - z_0)^k \phi(z_m) = (f-g)(z_m) = 0 \\ &\Rightarrow \quad \phi(z_m) = 0. \end{aligned}$$

which is a contradiction. Hence, $f(z) = g(z)$, $\forall z \in \Omega$. □

COROLLARY 5.2.5 If f is a non-constant analytic function on a region Ω, then f cannot have uncountable number of distinct zeroes.

Proof: Let $Z(f) = \{z \in \Omega : f(z) = 0\}$. By principle of analytic continuation, $Z(f)$ has no limit point in Ω. Hence, for every $z \in Z(f)$, there exists $r_z > 0$ such that $B(z, r_z) \cap Z(f) = \{z\}$. From Theorem 2.4.34, there exists a countable collection $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$ of open balls such that for every $z \in Z(f)$, there exists $B_{n_z} \in \mathscr{B}$ such that $z \in B_{n_z} \subseteq B(z, r_z)$. Therefore, it follows that

$$\{z\} \subseteq B_{n_z} \cap Z(f) \subseteq B(z, r_z) \cap Z(f) = \{z\},$$

and hence, $B_{n_z} \cap Z(f) = \{z\}$, $\forall z \in Z(f)$. If we define

$$\phi : Z(f) \to \mathbb{N} \text{ by } \phi(z) = n_z,\ \forall z \in Z(f),$$

where $n_z \in \mathbb{N}$ is the least positive integer such that $z \in B_{n_z} \subseteq B(z, r_z)$, then claim that ϕ is one-to-one. If $z, w \in Z(f)$ be such that $\phi(z) = \phi(w)$, then

$$B_{n_z} = B_{n_w} \Rightarrow \{z\} = B_{n_z} \cap Z(f) = B_{n_w} \cap Z(f) = \{w\} \Rightarrow z = w.$$

Thus, ϕ is an injection. Hence, $Z(f)$ is at most countable. □

LEMMA 5.2.6 Let f be a non-zero analytic function on a region Ω and γ be a closed curve not passing through any zero of f. Then, the number of zeroes of f enclosed by γ is finite.

Proof: We note that the regions enclosed by a closed curve are bounded sets. Suppose, there are infinitely many zeroes of f, which are enclosed by γ, then the set $\{z \in \Omega : f(z) = 0\}$ has a limit point, by Bolzano–Weierstrass property (Theorem 1.4.47). Hence, by the principle of analytic continuation (Theorem 5.2.4), it would follow that f is identically zero. Hence, there must be only finitely many zeroes of f enclosed by γ. □

PROBLEM 5.2.7 If f is an entire function such that $|f| = 1$ on $\{z \in \mathbb{C} : |z| = 1\}$, then $f = cz^n$, $\forall z \in \mathbb{C}$, for some constant $c \in \mathbb{C}$ with $|c| = 1$ and for some $n \in \{0, 1, 2, \ldots\}$.

Solution: If f is a constant function, then the proof is obvious. Hence, assume that f is not a constant. Let $z_1, z_2, \ldots, z_n$ be the zeroes of f including multiplicities on Cl $B(0, 1)$. (Suppose f has infinite number of zeroes in Cl $B(0, 1)$, being a compact set, then by Theorem 1.4.47, the zeroes of f can have limit point in Cl $B(0, 1)$. Thus, by the principle of analytic continuation (Theorem 5.2.4), f becomes identically zero on $\mathbb{C}$, which is not possible.) Moreover, note that $|z_k| < 1$, $\forall k \in \{1, 2, \ldots, n\}$, as $|z_k| \leq 1$ and $|f(z)| = 1$ on $|z| = 1$.

Consider the rational function $R(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}$, whose finite zeroes are $z_1, z_2, \ldots, z_n$ and its finite poles are $\frac{1}{\overline{z_k}}$, $k = 1, 2, 3, \ldots, n$, with $z_k \neq 0$. If $F(z) = \frac{f(z)}{R(z)}$, then after removing the common zeroes $z_1, z_2, \ldots, z_n$ (which are removable singularities for F), F becomes an entire function with zeroes at the poles of R. Since $\left|\frac{1}{\overline{z_k}}\right| > 1$, F is nowhere zero on Cl $B(0, 1)$. As $|F(z)|$ is continuous real-valued function on the compact set Cl $B(0, 1)$, there exist $w_1, w_2 \in$ Cl $B(0, 1)$ such that

$$|F(w_1)| \leq |F(z)| \leq |F(w_2)|, \ \forall z \in \text{Cl } B(0, 1).$$

Hence, applying maximum and minimum principle for analytic functions, we see $|w_1| = |w_2| = 1$, and hence, by hypothesis,

$$1 = |F(w_1)| \leq |F(z)| \leq |F(w_2)| = 1 \text{ on Cl } B(0, 1).$$

In particular, maximum of $|F|$ is attained at all points of $B(0,1)$; hence, again by using maximum modulus principle, we get $F = c$ on Cl $B(0,1)$ for some $c \in \mathbb{C}$ with $|c| = 1$. Then, by principle of analytic continuation, we get $F = c$ on $\mathbb{C} \setminus \left\{ \frac{1}{\overline{z_k}} : k = 1, 2, \ldots n \text{ with } z_k \neq 0 \right\}$. Therefore,

$$f(z) = c \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \forall z \in \mathbb{C} \setminus \left\{ \frac{1}{\overline{z_k}} : k = 1, 2, \ldots n \text{ with } z_k \neq 0 \right\}.$$

If $z_k \neq 0$ for some k, then f should have a pole at z_k, which is a contradiction to the assumption on f. Hence, $z_k = 0$, $\forall k = 1, 2, 3, \ldots, n$. In other words, $f(z) = cz^n$ for all $z \in \mathbb{C}$.

Definition 5.2.8 Let f be a complex-valued function on a region Ω and $a \in \Omega$. We say that a is a singularity of f if f is not differentiable at a. A singularity a of f is called

1. a removable singularity if $\lim_{z \to a} f(z)$ exists in $\mathbb{C}$.
2. a pole if $\lim_{z \to a} f(z) = \infty$.
3. an essential singularity if $\lim_{z \to a} f(z)$ does not exist in $\mathbb{C}_\infty$. That is, a is neither a removable singularity nor a pole.
4. an isolated singularity if there exists $r > 0$ such that f is analytic in $B(a, r) \setminus \{a\}$.

RESULT 5.2.9 Let f be an analytic function on $\Omega \setminus \{a\}$. Then, the following statements are equivalent:

1. a is a removable singularity for f,
2. f is bounded on $B(a, r) \setminus \{a\}$ for some $r > 0$,
3. $\lim_{z \to a} (z - a) f(z) = 0$,
4. There exists an analytic function ϕ on Ω such that $\phi(z) = f(z)$, $\forall z \in \Omega \setminus \{a\}$.

Proof:

(1) $\Rightarrow$ (2) Since a is a removable singularity of f, we have $\lim_{z \to a} f(z) = \ell$ for some $\ell \in \mathbb{C}$. Therefore, for $\epsilon = 1$, there exists $r > 0$ such that

$$0 < |z - a| < r \Rightarrow |f(z) - \ell| < 1.$$

Therefore, $|f(z)| \leq 1 + |\ell|$, $\forall z \in B(a, r) \setminus \{a\}$. Hence, (2) follows.

(2) $\Rightarrow$ (3) By assumption, there exists $r > 0$ and $M > 0$ such that

$$|f(z)| \leq M, \ \forall z \in B(a,r)\backslash\{a\}.$$

For a given $\epsilon > 0$, choose $\delta = \min\left\{\frac{\epsilon}{M}, r\right\}$ so that $\delta > 0$ and

$$0 < |z-a| < \delta \Rightarrow |z-a|\,|f(z)| \leq \delta M \leq \frac{\epsilon}{M} M = \epsilon.$$

Therefore, (3) holds.

(3) $\Rightarrow$ (4) Assume that $\lim_{z\to a}(z-a)f(z) = 0$. Choose $r > 0$ such that $ClB(a,r) \subseteq \Omega$. Define

$$\phi(z) = \begin{cases} \int\limits_{|\zeta - a| = r} \dfrac{f(\zeta)}{\zeta - a}\, d\zeta, & \forall z \in B(a,r) \\ f(z) & \forall z \in \Omega\backslash B(a,r) \end{cases}.$$

Then, by Theorem 4.3.2 and by Cauchy's integral formula (Theorem 4.3.1), we get ϕ, which is analytic on Ω and $\phi(z) = f(z)$, $\forall z \in \Omega\backslash\{a\}$.

(4) $\Rightarrow$ (1) In particular, ϕ is continuous at a, and hence,

$$\phi(a) = \lim_{z\to a} \phi(z) = \lim_{z\to a} f(z).$$

Thus, proof (1) follows.

Hence, the result follows.

Definition 5.2.10 Let f be a complex-valued function on a region Ω. f is said to be a meromorphic function if for each $z \in \Omega$, either f is differentiable at z or f has a pole at z.

RESULT 5.2.11 Let f be an analytic function on a region Ω except at a pole a. Then, there exist $\delta > 0$ and $p \in \mathbb{N}$ such that

$$f(z) = (z-a)^{-p}\psi(z), \ \forall z \in B(a,\delta)\backslash\{a\},$$

where ψ is analytic on $B(a,\delta)$ and $\psi(a) \neq 0$.

Proof: Since $\lim_{z\to a} f(z) = \infty$, there exists $\epsilon > 0$ such that $B(a,\epsilon) \subseteq \Omega$ and

$$0 < |z-a| < \epsilon \Rightarrow |f(z)| > 1.$$

Hence, $f(z) \neq 0, \forall z \in B(a,\epsilon)\backslash\{a\}$. If

$$g(z) = \frac{1}{f(z)},\ \forall z \in B(a,\epsilon)\backslash\{a\},$$

then g is analytic and nowhere zero on $B(a,\epsilon)\backslash\{a\}$. Since

$$\lim_{z\to a} g(z) = \lim_{z\to a} \frac{1}{f(z)} = 0,$$

a is a removable singularity for g. Therefore, if we define

$$g(a) = \lim_{z\to a} g(z) = \frac{1}{f(z)} = 0,$$

then g becomes analytic on $B(a,\epsilon)$, and hence, there exist $p \in \mathbb{N}$ and an analytic function φ on $B(a,\epsilon)$ such that

$$g(z) = (z-a)^p \varphi(z),\ \forall z \in B(a,\epsilon) \text{ and } \varphi(a) \neq 0.$$

Since φ is non-zero at a and g is nowhere zero on $B(a,\delta) \setminus \{a\}$, it follows that φ is nowhere zero on $B(a,r)$. If we define

$$\psi(z) = \frac{1}{\varphi(z)},\ \forall z \in B(a,\delta),$$

then ψ is a nowhere zero analytic function on $B(a,\delta)$, and

$$f(z) = (z-a)^{-p}\psi(z),\ \forall z \in B(a,\delta).$$

Hence, the result follows. □

COROLLARY 5.2.12 If f is a meromorphic function on Ω, then f has at most countable number of poles.

Proof: From Result 5.2.11, every pole of f is isolated. Hence, by a similar argument employed in the proof of Corollary 5.2.5, we can prove this corollary. □

Definition 5.2.13 If a is a pole of f, then the order of the pole at a is defined by the natural number p such that

$$f(z) = (z-a)^{-p}\psi(z),\ \forall z \in B(a,\delta)\backslash\{a\},$$

for some $\delta > 0$ and for some analytic function ψ on $B(a,\delta)$ with $\psi(a) \neq 0$.

If f is analytic on Ω except at an isolated singularity a then the nature of singularity at a can be described easily from the Laurent series expansion of f in $B(a,r)\backslash\{a\}$ as follows, where $r > 0$ is such that $ClB(a,r) \subseteq \Omega$.

RESULT 5.2.14 If f has an isolated singularity at a and

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n, \text{ in } B(a,r)\backslash\{a\},$$

for some $r > 0$, then a is

1. a removable singularity iff $c_n = 0, \ \forall n < 0$.
2. a pole iff there exists $m \in \mathbb{N}$ such that $c_{-m} \neq 0$ and $c_n = 0, \ \forall n < -m$. Furthermore, this m is the order of the pole of f at a.
3. an essential singularity iff $c_n \neq 0$ for infinitely many $n < 0$.

Proof: Let a be an isolated singularity of $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ in $0 < |z-a| < r$.

1. Assume that a is a removable singularity. Suppose there exists at least one $k < 0$ such that $c_k \neq 0$, then

   $$\lim_{z\to a} f(z) = \lim_{z\to a} \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

   does not exist in $\mathbb{C}$, which is a contradiction. Hence, $c_n = 0, \forall n < 0$. Conversely, assume that $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then, $\lim_{z\to a} f(z) = f(a)$ exists in $\mathbb{C}$. Hence, f has a removable singularity at a.

2. Assume that a is a pole of order m. Then, using Theorem 5.2.11, we get

   $$f(z) = (z-a)^{-m}\psi(z), \ \forall z \in B(a,r)\backslash\{a\},$$

   where ψ is analytic on $B(a,\delta)$ and $\psi(a) \neq 0$. As ψ is analytic on $B(a,\delta)$, we have the Taylor series expansion

   $$\psi(z) = \sum_{n=0}^{\infty} b_n(z-a)^n, \ \forall z \in B(a,r).$$

   Hence, we have

   $$f(z) = (z-a)^{-m}\sum_{n=0}^{\infty} b_n(z-a)^n = \sum_{n=0}^{\infty} b_n(z-a)^{n-m}, \ \forall z \in B(a,r)\backslash\{a\}.$$

Since the coefficient of $(z-a)^{-m}$ is $b_0 = \psi(a) \neq 0$, the proof of this part follows. Conversely, assume that $c_{-m} \neq 0$ and $c_n = 0,\ \forall n < -m$ for some $m \in \mathbb{N}$. Then, let

$$\psi(z) = \begin{cases} (z-a)^m f(z), & \text{if } z \neq a \\ c_{-m} & \text{if } z = a, \end{cases} \quad \forall z \in B(a,r).$$

Since

$$\begin{aligned} \psi(z) &= (z-a)^m \sum_{n=-m}^{\infty} c_n (z-a)^n \\ &= \sum_{n=-m}^{\infty} c_n (z-a)^{n+m} \\ &= \sum_{n=0}^{\infty} c_{n-m} (z-a)^n,\ |z-a| < r \end{aligned}$$

and the power series $\sum_{n=0}^{\infty} c_{n-m}(z-a)^n$ is analytic inside $B(a,r)$, we get that ψ is an analytic function in $B(a,r)$ with

$$f(z) = (z-a)^{-m}\psi(z),\ \forall z \in B(a,r)\backslash\{a\} \text{ and } \psi(a) = c_{-m} \neq 0.$$

Hence, f has a pole at a of order m.

3. a is essential singularity iff a is neither a removable singularity nor a pole iff $c_n \neq 0$ for some $n < 0$ and for every $m \in \mathbb{N}$, there exists $n \in \mathbb{Z}$ such that $n < -m$ and $c_n \neq 0$ iff $c_n \neq 0$ for infinitely many $n < 0$.

Hence, the result follows. □

RESULT 5.2.15 Let f be an analytic function on a region Ω except at an isolated singularity a.

1. If $\lim_{z \to a} |z-a|^r |f(z)| = 0$ for some $r \in \mathbb{R}$

   or

2. if $\lim_{z \to a} |z-a|^s |f(z)| = \infty$ for some $s \in \mathbb{R}$,

then there exists an integer m such that

$$\lim_{z \to a} |z-a|^j |f(z)| = 0,\ \forall j > m \text{ and } \lim_{z \to a} |z-a|^k |f(z)| = \infty,\ \forall k < m.$$

Proof: Let (1) hold. Then, choose an integer $n \geq r$. Now,

$$\lim_{z\to a} |z-a|^{n+1}|f(z)| = \lim_{z\to a} |z-a|^{n-r+1} \lim_{z\to a} |z-a|^{r}|f(z)| = 0.$$

Hence, if $\phi(z) = (z-a)^n f(z)$, $\forall z \in \Omega\backslash\{a\}$, then ϕ is an analytic function except at a, and it has removable singularity at a and

$$\begin{aligned} |\phi(a)| &= \lim_{z\to a} |\phi(z)| \\ &= \lim_{z\to a} |z-a|^{n}|f(z)| \\ &= \lim_{z\to a} |z-a|^{n-r} \lim_{z\to a} |z-a|^{r}|f(z)| \\ &= 0. \end{aligned}$$

Hence, $\phi(a) = 0$. Since ϕ is not identically zero[3] using Result 5.2.3, we have

$$\phi(z) = (z-a)^{\nu}\phi_1(z),\ \forall z \in B(a,r) \subseteq \Omega,$$

for some $\nu \in \mathbb{N}$, for some $r > 0$ and for an analytic function ϕ_1 on $B(a,r)$ with $\phi_1(a) \neq 0$. Now, let $m = n - \nu$. Then,

$$\lim_{z\to a} |z-a|^{m}|f(z)| = \lim_{z\to a} |\phi_1(z)| = |\phi_1(a)|,$$

which is neither 0 nor ∞. Hence, for $j > m$,

$$\lim_{z\to a} |z-a|^{j}|f(z)| = \lim_{z\to a} |z-a|^{j-m} \lim_{z\to a} |z-a|^{m}|f(z)| = 0 \cdot |\phi_1(a)| = 0,$$

and for $k < m$,

$$\lim_{z\to a} |z-a|^{k}|f(z)| = \lim_{z\to a} |z-a|^{k-m} \lim_{z\to a} |z-a|^{m}|f(z)| = \infty \cdot |\phi_1(a)| = \infty.$$

Next, assume that (2) holds. Then, choose an integer l such that $l < s$. Then,

$$\lim_{z\to a} |z-a|^{l}|f(z)| = \lim_{z\to a} |z-a|^{l-s} \lim_{z\to a} |z-a|^{s}|f(z)| = \infty,$$

and hence, a is a pole for ψ, where

$$\psi(z) = (z-a)^{l}f(z),\ \forall z \in \Omega\backslash\{a\}.$$

Hence, by using Result 5.2.11, there exist a negative integer λ, $\delta > 0$ and an analytic function ψ_1 on $B(a,\delta) \subseteq \Omega$ such that

$$\psi(z) = (z-a)^{\lambda}\psi_1(z),\ \forall z \in B(a,\delta)\backslash\{a\}.$$

[3] As f has a singularity, f should not be identically zero, and hence, ϕ is not identically zero.

Therefore,

$$f(z) = (z-a)^{-l}\psi(z) = (z-a)^{\lambda-l}\psi_1(z), \quad \forall z \in B(a,\delta) \setminus \{a\}.$$

If $m = l - \lambda$, then

$$\lim_{z\to a} |z-a|^m |f(z)| = \lim_{z\to a} |\psi_1(z)| = |\psi_1(a)|,$$

which is neither 0 nor ∞. Thus, by a similar argument, we get the result.

Note that if there exist two integers m_1 and m_2 such that

$$\lim_{z\to a} |z-a|^j |f(z)| = 0, \ \forall j > m_i$$

and

$$\lim_{z\to a} |z-a|^k |f(z)| = \infty, \ \forall k < m_i, \text{ for } i = 1, 2,$$

then obviously, $m_1 = m_2$. □

Definition 5.2.16 Let f be an analytic function on a region Ω except at an isolated singularity a. If there exists an integer m such that $\lim_{z\to a} |z-a|^j |f(z)| = 0, \ \forall j > m$ and $\lim_{z\to a} |z-a|^k |f(z)| = \infty, \ \forall \ k < m$, then m is called the algebraic order of f at a. It is easy to see that algebraic order of f at a is m iff $\lim_{z\to a} |z-a|^m |f(z)|$ exists and is neither 0 nor ∞.

The following lemma characterizes the singularity in terms of algebraic order.

LEMMA 5.2.17 Let f be an analytic function on a region Ω except at an isolated singularity a, and the algebraic order of f at a is m. Then,

1. $m < 0$ iff a is a removable singularity of f and $f(a) = 0$,
2. $m = 0$ iff a is a removable singularity of f and $f(a) \neq 0$,
3. $m > 0$ iff a is a pole of f.

Proof: Since m is the algebraic order of f at a, we have

$$\lim_{z\to a} |z-a|^j |f(z)| = 0, \ \forall j > m \text{ and } \lim_{z\to a} |z-a|^k |f(z)| = \infty, \ \forall k < m.$$

1. $m < 0 \Rightarrow \lim_{z\to a} |z-a|^0 |f(z)| = 0$. Hence, a is a removable singularity of f and $f(a) = 0$. Conversely, assume that a is a removable singularity

of f and $f(a) = 0$. Then, $f(z) = (z-a)^n g(z)$ in a neighbourood of a for some $n \in \mathbb{N}$ and for some analytic function g on the neighbourhood of a such that $g(a) \neq 0$. Therefore,

$$\lim_{z \to a} |z-a|^{-n} |f(z)| = \lim_{z \to a} |f(z)| = |g(a)|,$$

which is neither 0 nor ∞; hence, $-n$ is the algebraic order of f at a.

2. $m = 0 \Rightarrow \lim_{z \to a} |z-a|^0 |f(z)| \neq 0$. Hence, a is a removable singularity of f and $f(a) \neq 0$. Conversely, assume that a is a removable singularity of f and $f(a) \neq 0$. Then, $\lim_{z \to a} |z-a|^0 |f(z)| = |f(a)|$, which is neither 0 nor ∞, and hence, 0 is the algebraic order of f at a.

3. If $m > 0$, then $\lim_{z \to a} |z-a|^0 |f(z)| = \infty \Rightarrow f$ has a pole at a. Conversely, assume that a is a pole for f. Then, we have $f(z) = (z-a)^{-n} g(z)$ in a neighbourhood of a for some $n \in \mathbb{N}$ and for an analytic function g on the neighbourhood of a such that $g(a) \neq 0$. Therefore,

$$\lim_{z \to a} |z-a|^n |f(z)| = \lim_{z \to a} |g(z)| = |g(a)|,$$

which is neither 0 nor ∞. Hence, the algebraic order of f at a is $n > 0$. □

COROLLARY 5.2.18 If f is an analytic function on a region Ω except at an isolated singularity a, then the algebraic order of f at a does not exist iff a is an essential singularity of f.

Proof: The algebraic order m of f at a exists iff $m < 0$ or $m = 0$ or $m > 0$ iff a is a removable singularity or a pole iff a is not an essential singularity of f. Thus, the corollary follows. □

THEOREM 5.2.19 (Weierstrass theorem for essential singularity) Let f be an analytic function on a region $\Omega \backslash \{a\}$. If f has an essential singularity at a, then given $c \in \mathbb{C}$, given $\epsilon > 0$, and given $\delta > 0$, there exists $z \in B(a, \delta)$ such that $|f(z) - c| < \epsilon$.

Proof: Suppose that this theorem is not true. That is, there exist $c \in \mathbb{C}$, $\epsilon > 0$ and $\delta > 0$ such that

$$|z-a| < \delta \Rightarrow |f(z) - c| \geq \epsilon.$$

Clearly,

$$\lim_{z\to a}|z-a|^{-1}|f(z)-c| \geq \lim_{z\to a}|z-a|^{-1}\epsilon = \infty.$$

Hence, by Result 5.2.15, there exists $r > 0$, such that

$$\lim_{z\to a}|z-a|^{r}|f(z)-c| = 0.$$

Therefore,

$$\lim_{z\to a}|z-a|^{r}|f(z)| \leq \lim_{z\to a}|z-a|^{r}|f(z)-c| + \lim_{z\to a}|z-a|^{r}|c| = 0.$$

It follows that f has algebraic order at a. Then, a is a pole of f or a removable singularity of f. This is a contradiction to the assumption that a is an essential singularity of f. Hence, the theorem follows. □

Exercise 5.2.20 Prove that the algebraic order of f at a is the least integer $m \in \mathbb{Z}$ such that $c_m \neq 0$ in the Laurent series expansion of $f(z) = \sum_{k=-\infty}^{\infty} c_k(z-z_0)^k$, $0 < |z-z_0| < \delta$ for some $\delta > 0$.

5.3 PARTIAL FRACTION OF ENTIRE FUNCTIONS

Definition 5.3.1 Let f be a meromorphic function on $\mathbb{C}$. If $\{a_n : n \in \mathbb{N}\}$ is the set of all poles of f, then the following representation of f is called the partial fraction expansion of f.

$$f(z) = \sum_{n=-\infty}^{\infty}\left(P_n\left(\frac{1}{z-a_n}\right) - Q_n(z)\right) + \phi(z),\ \forall z \in \mathbb{C}\backslash\{a_n : n \in \mathbb{N}\},$$

where P_n is a polynomial without constant term, Q_n is a polynomial $\forall n \in \mathbb{N}$, and ϕ is an entire function.

In the partial fraction of f, $P_n\left(\frac{1}{z-a_n}\right)$ is called the singular part of f at a_n.

THEOREM 5.3.2 (Mittag-Leffler theorem)
Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \to \infty$ as $n \to \infty$ and let $\{P_n\}$ be an arbitrary sequence of polynomials without constant term. Then, there exists a meromorphic function such that whose poles are precisely $\{a_n : n \in \mathbb{N}\}$ and the singular part of the function at a_n is $P_n\left(\frac{1}{z-a_n}\right)$.

Proof: We shall construct a meromorphic function f such that

$$f(z) = \sum_{n=0}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - Q_n(z) \right),\ \forall z \in \mathbb{C}\backslash\{a_n : n = 0, 1, 2, \ldots\},$$

for some suitable polynomial Q_n such that the above series converges uniformly on every compact subset of $\mathbb{C}\backslash\{a_n : n = 0, 1, 2, \ldots\}$.

Let $M > 0$ be arbitrary. Since $a_n \to \infty$ as $n \to \infty$, we choose $N \in \mathbb{N}$ such that $4|a_n| > M, \forall n \geq N$. For $n \geq N$, if $g_n(z) = P_n\left(\frac{1}{z-a_n}\right),\ \forall z \in B(0, |a_n|)$, then g_n is analytic on $B(0, |a_n|)$. Then, the finite Taylor series expansion of g_n is given by

$$g_n(z) = P_n\left(\frac{1}{z-a_n}\right) = \sum_{k=0}^{m_n} c_k z^k + \phi_{m_n+1}(z) z^{m_n+1},\ \forall z \in B(0, |a_n|),$$

where $m_n \in \mathbb{N}$ is such that

$$2^{m_n - n} > M_n = \sup_{|z| \leq \frac{|a_n|}{2}} \left| P_n\left(\frac{1}{z-a_n}\right) \right| < +\infty$$

and

$$\phi_{m_n+1}(z) = \frac{1}{i2\pi} \int\limits_{|\zeta|=\frac{|a_n|}{2}} \frac{P_n\left(\frac{1}{z-a_n}\right)}{\zeta^{m_n+1}(\zeta - z)}\, d\zeta,\ \forall z \in B\left(0, \frac{|a_n|}{2}\right).$$

Let $Q_n(z) = \sum\limits_{k=0}^{m_n} c_k z^k,\ \forall z \in \mathbb{C}$. Now for $z \in B\left(0, \frac{|a_n|}{4}\right)$,

$$\begin{aligned}
\left| P_n\left(\frac{1}{z-a_n}\right) - Q_n(z) \right| &= \left| \phi_{m_n+1}(z) z^{m_n+1} \right| \\
&\leq |z|^{m_n+1} \frac{M_n}{2\pi} \int\limits_{|\zeta|=\frac{|a_n|}{2}} \frac{|d\zeta|}{\left(\frac{|a_n|}{2}\right)^{m_n+1} |\zeta - z|} \\
&\leq |z|^{m_n+1} \frac{M_n}{2\pi} \int\limits_{|\zeta|=\frac{|a_n|}{2}} \frac{|d\zeta|}{\left(\frac{|a_n|}{2}\right)^{m_n+1} \left(\frac{|a_n|}{2} - \frac{|a_n|}{4}\right)}
\end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{|a_n|}{4}\right)^{m_n+1} \frac{M_n}{2\pi} 2\pi \frac{|a_n|}{2} \left(\frac{2}{|a_n|}\right)^{m_n+1} \frac{4}{|a_n|} \\ &= 2^{-m_n} M_n < 2^{-n}. \end{aligned}$$

Hence, by comparison test, $\sum_{n=N}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - Q_n(z)\right)$ converges uniformly on $|z| \leq M$. As every compact subset of $\mathbb{C}\backslash\{a_n : n = 0, 1, 2, \ldots\}$ is contained in $B(0, M)$ for some $M > 0$, we conclude that $\sum_{n=N}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - Q_n(z)\right)$ converges uniformly on every compact subset of $\mathbb{C}\backslash\{a_n : n = 0, 1, 2, \ldots\}$. Thus, there exists a meromorphic function

$$f(z) = \sum_{n=0}^{\infty} \left(P_n\left(\frac{1}{z-a_n}\right) - Q_n(z)\right), +\phi(z), \forall z \in \mathbb{C}\backslash\{a_n : n = 0, 1, 2, \ldots\},$$

whose singular part at a_n is $P_n\left(\frac{1}{z-a_n}\right)$, $\forall n = 0, 1, 2, \ldots$, where ϕ is an arbitrary entire function. □

Example 5.3.3 Prove the following:

1. $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}, \forall z \in \mathbb{C}\backslash\mathbb{Z}.$

2. $\pi \cot(\pi z) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{z-k}, \forall z \in \mathbb{C}\backslash\mathbb{Z}.$

3. $\frac{\pi}{\sin(\pi z)} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k}, \forall z \in \mathbb{C}\backslash\mathbb{Z}.$

Let $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$, $\forall z \in \mathbb{C}\backslash\mathbb{N}$. Clearly, f has double pole at every integer. Therefore, the Laurent series expansion of f around 0 is of the form

$$f(z) = \frac{A_2}{z^2} + \frac{A_1}{z} + \sum_{n=0}^{\infty} A_n z^n, \ 0 < |z| < \delta \text{ for some } \delta > 0,$$

and hence, the singular part of f at 0 is $\frac{A_2}{z^2} + \frac{A_1}{z}$, where

$$A_{-2} = \lim_{z\to 0} \left(z^2 \frac{\pi^2}{\sin^2(\pi z)}\right) = 1 \text{ and } A_{-1} = 0,$$

as f is an even function, and hence, all odd coefficients in the Laurent series expansion of f around 0 are zeroes. Thus, the singular part of f at 0 is $\frac{1}{z^2}$. Since $\sin^2(\pi(z-n)) = \sin^2(\pi z)$, the singular part of $\sin^2(\pi z)$ at n is $\frac{1}{(z-n)^2}$. Therefore, by Mittag-Leffler theorem, we have

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + \phi(z),\ \forall z \in \mathbb{C}\backslash\mathbb{N},$$

for some entire function ϕ. (As $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ itself is convergent, we have taken $Q_n(z) = 0$, $\forall z \in \mathbb{C}$ and $n \in \mathbb{N}$.) We claim that ϕ is identically zero. First, we note that as $\frac{\pi^2}{\sin^2(\pi z)}$ and $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ are periodic functions of period 1, ϕ is also a periodic function of period 1. Now, writing $z = x + iy$, we get

$$\begin{aligned}
|\sin(\pi z)|^2 &= |\sin(\pi(x+iy))|^2 \\
&= |\sin(\pi x)\cosh(\pi y) + i\sinh(\pi y)\cos(\pi x)|^2 \\
&= \sin^2(\pi x)\cosh^2(\pi y) + \sinh^2(\pi y)\cos^2(\pi x) \\
&= (1-\cos^2(\pi x))\cosh^2(\pi y) + \sinh^2(\pi y)\cos^2(\pi x) \\
&= \cosh^2(\pi y) - \cos^2(\pi x)(\cosh^2(\pi y) - \sinh^2(\pi y)) \\
&= \cosh^2(\pi y) - \cos^2(\pi x) \to \infty \text{ as } |y| \to +\infty \text{ uniformly in } x.
\end{aligned}$$

Therefore,

$$g(x+iy) \to 0 \text{ as } |y| \to +\infty \text{ uniformly on } \mathbb{C}\backslash\mathbb{Z}. \tag{5.6}$$

In particular, g is bounded in the strip $\{(x,y) : 0 \le x \le 1,\ y \in \mathbb{R}\}$. Since g is a periodic function of period 1, g is bounded on $\mathbb{C}$. Thus, by Liouville's theorem (Theorem 4.3.11), g is a constant function. Moreover, using equation (5.6), we conclude that g is identically 0. Thus,

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{(z-k)^2}.$$

Integrating the above equation, we get

$$\pi^2 \left(\frac{-\cot(\pi z)}{\pi}\right) = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(\frac{-1}{z-k}\right),$$

which implies the required series representation of $\pi\cot(\pi z)$. Next, consider

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{(-1)^k}{z-k} &= \lim_{j\to\infty}\sum_{k=-(2j+1)}^{(2j+1)}\frac{(-1)^k}{z-k}\\
&= \lim_{j\to\infty}\left(\sum_{k=-j}^{j}\frac{1}{z-2k}-\sum_{k=-j-1}^{j}\frac{1}{z-(2k+1)}\right)\\
&= \frac{1}{2}\lim_{j\to\infty}\left(\sum_{k=-j}^{j}\frac{1}{\frac{z}{2}-k}-\sum_{k=-j-1}^{j}\frac{1}{\frac{z-1}{2}-k}\right)\\
&= \frac{\pi}{2}\left(\cot\left(\frac{\pi z}{2}\right)-\cot\left(\frac{\pi(z-1)}{2}\right)\right)\\
&= \frac{\pi}{2}\left(\frac{\cos\left(\frac{\pi z}{2}\right)}{\sin\left(\frac{\pi z}{2}\right)}-\frac{\cos\left(\frac{\pi(z-1)}{2}\right)}{\sin\left(\frac{\pi(z-1)}{2}\right)}\right)\\
&= \frac{\pi}{2}\left(\frac{\cos\left(\frac{\pi z}{2}\right)}{\sin\left(\frac{\pi z}{2}\right)}+\frac{\sin\left(\frac{\pi z}{2}\right)}{\cos\left(\frac{\pi z}{2}\right)}\right)\\
&= \frac{\pi}{2}\frac{1}{\sin\left(\frac{\pi z}{2}\right)\cos\left(\frac{\pi z}{2}\right)}\\
&= \frac{\pi}{\sin(\pi z)}.
\end{aligned}$$

PROBLEM 5.3.4 Find the partial fraction expansion for sec and deduce the Gregory–Leibniz–Madhava series, which is given by $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$.

Solution:
From Example 5.3.3(3), we have

$$\frac{\pi}{\cos\left(\frac{\pi}{2}-\pi z\right)}=\frac{\pi}{\sin(\pi z)}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{z-n}.$$

By substituting $w=\frac{\pi}{2}-\pi z$, we get

$$\frac{\pi}{\cos(w)}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{-\frac{w}{\pi}+\frac{1}{2}-n},$$

which implies that

$$\begin{aligned}
\sec(w) &= -\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{w+\left(n-\frac{1}{2}\right)\pi} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{w+\left(n-\frac{1}{2}\right)\pi} - \sum_{n=0}^{\infty} \frac{(-1)^n}{w+\left(-n-\frac{1}{2}\right)} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{w+\left(n-\frac{1}{2}\right)\pi} + \sum_{m=1}^{\infty} \frac{(-1)^m}{w-\left(m-\frac{1}{2}\right)} \\
&\quad \text{(using the change of variable } m = n+1 \text{ in the second sum)} \\
&= \sum_{n=1}^{\infty}(-1)^n \left[-\frac{1}{w+\left(n-\frac{1}{2}\right)\pi} + \frac{1}{w-\left(n-\frac{1}{2}\right)\pi} \right] \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{2\left(n-\frac{1}{2}\right)\pi}{w^2-\left(n-\frac{1}{2}\right)^2\pi^2} \\
&= 4\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)\pi}{4w^2-(2n-1)^2\pi^2}.
\end{aligned}$$

Substituting $w = 0$ in the above series representation of $\sec(w)$, we get

$$1 = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\pi} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4}.$$

Exercise 5.3.5

1. Prove that $\pi \tan\left(\frac{\pi z}{2}\right) = 4z\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-4z^2}$. Hint: Use the identity $\tan\left(\frac{\pi z}{2}\right) = \cot\left(\frac{\pi z}{2}\right) - 2\cot(\pi z)$ and the partial fraction expansion of $\pi\cot(\pi z)$.
2. Obtain the value of Gregory–Leibniz–Madhava series from the partial fraction expansions of $\pi z\cot(\pi z)$ and $\frac{\pi}{\sin(\pi z)}$. Hint: Substitute $z = \frac{1}{4}$ in both expansions.

PROBLEM 5.3.6 Prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Solution:
From Example 5.3.3, we have

$$\begin{aligned}\frac{\pi^2}{\sin^2(\pi z)} &= \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2} \\ &= \sum_{0\neq k\in\mathbb{Z}} \frac{1}{k^2} \frac{1}{\left(1-\frac{z}{k}\right)^2} + \frac{1}{z^2} \\ &= \sum_{0\neq k\in\mathbb{Z}} \frac{1}{k^2} \sum_{m=1}^{\infty} m \left(\frac{z}{k}\right)^{m-1} + \frac{1}{z^2}. \end{aligned} \tag{5.7}$$

Since the Laurent series of $\dfrac{1}{\sin^2(\pi z)}$ is given by

$$\begin{aligned}\frac{1}{\sin^2(\pi z)} &= \left[\frac{1}{\pi z} \frac{1}{1-\left(\frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \cdots\right)}\right]^2 \\ &= \left[\frac{1}{\pi z} \sum_{n=0}^{\infty} \left(\frac{(\pi z)^2}{3} + \frac{(\pi z)^4}{5!} - \cdots\right)^n\right]^2 \\ &= \left[\frac{1}{\pi z} \sum_{n=0}^{\infty} \left(\frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \cdots\right)^n\right]^2 \\ &= \frac{1}{(\pi z)^2} + \frac{2}{3!} + \sum_{m=1}^{\infty} A_m z^m, \end{aligned}$$

for some $A_m \in \mathbb{C}$, $m \in \mathbb{N}$. Hence,

$$\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2} + \frac{2\pi^2}{3!} + \sum_{m=1}^{\infty} A_m z^m. \tag{5.8}$$

Equating the constant terms of the expansions of $\dfrac{\pi^2}{\sin^2(\pi z)}$ in equations (5.7) and (5.8), we get

$$\frac{2\pi^2}{6} = \sum_{0\neq k\in\mathbb{Z}} \frac{1}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

5.4 INFINITE PRODUCT

Definition 5.4.1 (Rigorous definition) A sequence (ζ_k) of non-zero complex numbers is said to be multiplyable if $\left(\prod_{k=1}^{n} \zeta_k\right)$ converges in $\mathbb{C}\backslash\{0\}$. Then, the product of (ζ_k) is denoted by $\prod_{k=1}^{n} \zeta_k$.

Although the above definition is rigorous, it is customary to define the infinite product of complex numbers as follows.

Definition 5.4.2 (Customary definition) $\prod_{k=1}^{n} \zeta_k$ is said to be convergent if $\lim_{n\to\infty} \left(\prod_{k=1}^{n} \zeta_k\right)$ exists in $\mathbb{C}\backslash\{0\}$.

Note that according to this definition, $\prod_{k=1}^{\infty} \frac{1}{2}$ does not converge, as $\prod_{k=1}^{n} \frac{1}{2} = \frac{1}{2^n} \to 0$ as $n \to \infty$.

LEMMA 5.4.3 If $\prod_{k=1}^{n} \zeta_k$ converges, then $\zeta_k \to 1$ as $k \to \infty$.

Proof: Let $\Phi_n = \prod_{k=1}^{n} \zeta_k$, $\forall n \in \mathbb{N}$. Then, by assumption, we have $\lim_{n\to\infty} \Phi_n = A$ for some $A \in \mathbb{C}\backslash\{0\}$. As $\zeta_k = \frac{\Phi_k}{\Phi_{k-1}}$, $\forall k > 1$, allowing $k \to \infty$, we get $\lim_{k\to\infty} \zeta_k = \frac{A}{A} = 1$. □

LEMMA 5.4.4 $\prod_{k=1}^{\infty} \zeta_k$ converges iff $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges, for a suitable branch of log.

Proof: Using Corollary 2.4.25, for every ζ_k, there exists unique $\theta_k \in (0, 2\pi]$ such that $\arg \zeta_k = \theta_k$. Since $\{\theta_k : k \in \mathbb{N}\}$ is countable, there exists $\theta \in (0, 2\pi)$ such that $\arg \zeta_k \neq \theta$, $\forall k \in \mathbb{N}$. Now, we define a branch of log as follows. Let $\Omega_\theta = \mathbb{C}\backslash(\{0\} \cup \{r\exp(i\theta) : r > 0\})$ and define

$$\log z = \ln|z| + i \arg z \text{ with } \arg z \in (\theta - 2\pi, \theta),$$

so that Ω_θ contains positive real axis and $\arg 1 = 0$. Then, as in the proof of Result 3.3.9, we can prove that log is analytic on Ω_θ. Now, assume that $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges. Then, using the continuity of exp, we get

$$\begin{aligned}
\exp\left(\sum_{k=1}^{\infty} \log(\zeta_k)\right) &= \exp\left(\lim_{n\to\infty} \sum_{k=1}^{n} \log(\zeta_k)\right) \\
&= \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \log(\zeta_k)\right) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(\log(\zeta_k)\right) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \zeta_k.
\end{aligned}$$

Therefore, $\prod_{k=1}^{\infty} \zeta_k$ converges.

Conversely, assume that $\prod_{k=1}^{\infty} \zeta_k$ converges to A. If $\Phi_n = \prod_{k=1}^{n} \zeta_k$, $\forall n \in \mathbb{N}$, then we have

$$\log\left(\frac{\Phi_n}{A}\right) = \sum_{k=1}^{n} \log(\zeta_k) - \log A + i2\pi\nu_n, \text{ where } \nu_n \in \mathbb{Z},\ \forall n \in \mathbb{N}.$$

Using the continuity of log at 1, we get $\log\left(\frac{\Phi_n}{A}\right) \to \log(1) = 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \left[\log\left(\frac{\Phi_{n+1}}{A}\right) - \log\left(\frac{\Phi_n}{A}\right)\right] \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n+1} \log(\zeta_k) - \sum_{k=1}^{n} \log(\zeta_k) + i2\pi(\nu_{n+1} - \nu_n)\right] \\
&= \lim_{n\to\infty} \left[\log(\zeta_{n+1}) + i2\pi(\nu_{n+1} - \nu_n)\right] \\
&= \lim_{n\to\infty} i2\pi(\nu_{n+1} - \nu_n) \text{ (by Lemma 5.4.3).}
\end{aligned}$$

Using $\frac{\Phi_n}{A} \to 1$ and $\zeta_n \to 1$ as $n \to \infty$, we obtain

$$\arg\left(\frac{\Phi_{n+1}}{A}\right) - \arg\left(\frac{\Phi_n}{A}\right) \to 0 \text{ and } \arg\zeta_{n+1} \to 0 \text{ as } n \to \infty.$$

Since

$$(\nu_{n+1} - \nu_n)2\pi = \arg\left(\frac{\Phi_{n+1}}{A}\right) - \arg\left(\frac{\Phi_n}{A}\right) + \arg(1 + \zeta_{n+1}),$$

we get $\nu_{n+1} - \nu_n \to 0$ as $n \to \infty$. Therefore, by definition of a convergent sequence, for $\epsilon = \frac{1}{2} > 0$, there exists $N \in \mathbb{N}$ such that $|\nu_{n+1} - \nu_n| < \frac{1}{2}$. Since $\nu_{n+1} - \nu_n$ is an integer, for each $n \in \mathbb{N}$, we have

$$\nu_n = \nu_{n+1} = \nu(\text{say}),\ \forall n \geq N.$$

Therefore, for $n \geq N$, we get

$$\sum_{k=1}^{n} \log(\zeta_k) = \log\left(\frac{\Phi_n}{A}\right) + \log A - i2\pi\nu \to \log A - i2\pi\nu \text{ as } n \to \infty.$$

Thus, $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges. □

It is interesting to note that we can use any branch of log which is analytic on the positive real axis, in particular the principal logarithm.

Indeed, $\zeta_k \to 1$ iff $\log(\zeta_k) \to 0$ implies that both $\prod_{k=1}^{\infty} \zeta_k$ and $\sum_{k=1}^{\infty} \log(\zeta_k)$ do not converge whenever $\zeta_k \not\to 1$ (or equivalently $\log(\zeta_k) \not\to 0$). In the other case, as $\zeta_k \to 1$ as $k \to \infty$, we can very well define principal logarithm of ζ_k by $\log(\zeta_k) = \text{In}(|\zeta_k|) + i\arg(\zeta_k)$ where $\zeta_k \in (-\pi, \pi]$. However, we can find $N \in \mathbb{N}$ such that $\zeta_k \in \mathbb{C}\backslash\{(x, 0) : x \leq 0\}\ \forall k \geq N$, using $\zeta_k \to 1$ as $k \to \infty$. This is sufficient to get the proof of the previous lemma for principal logarithm.

Definition 5.4.5 An infinite product $\prod_{k=1}^{\infty} \zeta_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges absolutely.

As in the case of infinite series, we can define the concept of rearrangement of an infinite product as follows.

Definition 5.4.6 If $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then $\prod_{k=1}^{\infty} \zeta_{f(k)}$ is called an rearrangement of $\prod_{k=1}^{\infty} \zeta_k$.

RESULT 5.4.7 If $\prod_{k=1}^{\infty} \zeta_k$ converges absolutely, then every rearrangement of $\prod_{k=1}^{\infty} \zeta_k$ converges.

Proof: By definition, $\prod_{k=1}^{\infty} \zeta_k$ converges absolutely means that $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges absolutely. Then, using Result 1.3.29, we get every rearrangement of $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges, and hence, every rearrangement of $\prod_{k=1}^{\infty} \zeta_k$ converges. □

THEOREM 5.4.8 $\prod_{k=1}^{\infty} \zeta_k$ converges absolutely iff $\sum_{k=1}^{\infty} (\zeta_k - 1)$ converges absolutely.

Proof: First, we note that if $\prod_{k=1}^{\infty} \zeta_k$ converges absolutely or $\sum_{k=1}^{\infty} (\zeta_k - 1)$ converges absolutely, then by Lemma 5.4.3 or by Result 1.3.27, we have $\zeta_k \to 1$ as $k \to \infty$.

We claim that $\sum_{k=1}^{\infty} (\zeta_k - 1)$ converges absolutely iff $\sum_{k=1}^{\infty} \log(\zeta_k)$ converges absolutely. We note that Taylor series expansion of $\log(\zeta_k)$ around 1 is given by

$$\log(\zeta_k) = \sum_{m=0}^{\infty} \frac{\log^{(m)}(1)}{m!} (\zeta_k - 1)^m = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\zeta_k - 1)^m.$$

Therefore,

$$\lim_{k \to \infty} \left| \frac{\log(\zeta_k)}{\zeta_k - 1} \right| = \lim_{k \to \infty} \left| \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\zeta_k - 1)^{m-1} \right| = 1.$$

Therefore, for a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} k \geq N \quad &\Rightarrow \quad \left| \frac{\log(\zeta_k)}{\zeta_k - 1} - 1 \right| < \epsilon \\ &\Rightarrow \quad 1 - \epsilon < \left| \frac{\log(\zeta_k)}{\zeta_k - 1} \right| < 1 + \epsilon \\ &\Rightarrow \quad (1 - \epsilon)(\zeta_k - 1) < |\log(\zeta_k)| < (1 + \epsilon)(\zeta_k - 1). \end{aligned}$$

Therefore, by comparison test (Theorem 1.3.30), our claim holds.

Thus, by definition of absolute convergence of infinite product and by our claim, we have

$$\prod_{k=1}^{\infty} \zeta_k \text{ converges absolutely} \quad \text{iff} \quad \sum_{k=1}^{\infty} \log(\zeta_k) \text{ converges absolutely}$$
$$\text{iff} \quad \sum_{k=1}^{\infty} (\zeta_k - 1) \text{ converges absolutely.} \qquad \square$$

PROBLEM 5.4.9 Prove that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$ converges and find its value.

Solution:
Let $P_m = \prod_{n=2}^{m} \left(1 - \frac{1}{n^2}\right)$, $\forall m \in \mathbb{N}$ with $m \geq 2$. Therefore,

$$\begin{aligned} P_m &= \prod_{n=2}^{m} \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) \\ &= \prod_{n=2}^{m} \frac{n-1}{n} \, \frac{n+1}{n} \\ &= \left(\frac{1}{2} \, \frac{3}{2}\right) \left(\frac{2}{3} \, \frac{4}{3}\right) \cdots \left(\frac{m-1}{m} \, \frac{m+1}{m}\right) \\ &= \frac{1}{2} \, \frac{m+1}{m} \to \frac{1}{2} \neq 0 \text{ as } m \to \infty. \end{aligned}$$

Hence, $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$ converges to $\frac{1}{2}$.

PROBLEM 5.4.10 Prove that $\prod_{n=1}^{\infty} (1 + z^{2^n}) = \frac{1}{1-z}$, when $|z| < 1$.

Solution:
If $P_m(z) = \prod_{n=1}^{m-1} (1 + z^{2^n})$, $\forall m \in \mathbb{N}$, then we claim that $P_m(z) = \sum_{k=0}^{2^m - 1} z^k$, $\forall m \in \mathbb{N}$. For $m = 1$, our claim is true. Assume that our claim is true for some $m \in \mathbb{N}$, then for

$$\begin{aligned} P_{m+1}(z) &= \prod_{n=1}^{m} (1 + z^{2^n}) \\ &= P_m(z) \, (1 + z^{2^m}) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{2^m-1} z^k\right)(1+z^{2^m}) \quad \text{(by assumption)}\\
&= \sum_{k=0}^{2^m-1} z^k + \sum_{k=0}^{2^m-1} z^{k+2^m}\\
&= \sum_{k=0}^{2^m-1} z^k + \sum_{j=2^m}^{2^{m+1}-1} z^j \text{ (using the change of variable } j = k+2^m)\\
&= \sum_{j=0}^{2^{m+1}-1} z^j.
\end{aligned}
$$

Hence, by induction, our claim holds. Since $|z| < 1$,

$$\lim_{m\to\infty} P_m(z) = \lim_{m\to\infty} \sum_{k=0}^{2^m-1} z^k = \frac{1}{1-z}$$

using Example 1.3.31.

PROBLEM 5.4.11 Prove that $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}$ converges absolutely and uniformly on every compact subset of $\mathbb{C}$.

Solution:

To prove this statement, we shall obtain that $\sum_{n=1}^{\infty}\log\left(\left[1+\frac{z}{n}\right]e^{\frac{-z}{n}}\right)$ converges absolutely and uniformly on $\{z \in \mathbb{C} : |z| \le R\}$, $\forall R > 0$ with respect to the principal branch of log. As principal logarithm is analytic at 1, we can expand the Taylor series expansion of $\log(1+w) = -\sum_{k=1}^{\infty}\frac{1}{k}w^k$, $\forall w \in B(0,1)$. Therefore, for each $|z| \le R$ and each $n > R$, we have

$$
\begin{aligned}
\left|\log\left(\left[1+\frac{z}{n}\right]e^{\frac{-z}{n}}\right)\right| &= \left|\log\left(1-\frac{z}{n}\right)-\frac{z}{n}\right|\\
&= \left|\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k - \frac{z}{n}\right|\\
&= \left|\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\left(\frac{z}{n}\right)^k\right|
\end{aligned}
$$

$$\begin{aligned} &\leq \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{R}{n}\right)^k \\ &\leq \frac{1}{2}\frac{R^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{R}{n}\right)^k \\ &\leq \frac{1}{2}\frac{R^2}{n^2} \left(\frac{1}{1-\frac{R}{n}}\right) \\ &= \frac{R^2}{2n(n-R)}, \end{aligned}$$

which is the nth term of a convergent series of real numbers (i.e., this term is independent of z). Hence, by comparison test, $\sum_{n=1}^{\infty} \log\left(\left[1+\frac{z}{n}\right]e^{\frac{-z}{n}}\right)$ converges absolutely and uniformly on $\{z \in \mathbb{C} : |z| \leq R\}$.

Exercise 5.4.12 Prove the following:

1. $\prod_{n=2}^{\infty} \left(1 + \frac{2n+1}{n^2-1}\right) = \frac{1}{3}.$
2. $\prod_{n=2}^{\infty} \left(1 - \frac{2n+1}{n(n+2)}\right) = 3.$
3. $\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n}\right) = 1.$

Definition 5.4.13 Let f be an entire function with zeroes $\{a_n : n \in \mathbb{N}\}$. Then, the representation

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp(P_n(z)),\ \forall z \in \mathbb{C}$$

is called the *canonical product* representation of f, where g is an entire function, P_n are some polynomials, and $m \in \{0, 1, 2, 3, \ldots\}$.

THEOREM 5.4.14 (Weierstrass theorem on existence of entire functions with prescribed zeroes)
Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \to \infty$ as $n \to \infty$. Then, there exists an entire function f whose zeroes are $\{a_n : n \in \mathbb{N}\}$.

Proof:

Case (i) Let f be an entire function without any zeroes in $\mathbb{C}$. In view of Theorem 4.5.12, we observe that if f is an entire function and has no zeroes, then there exists an entire function g such that $g(z) = \log(f(z))$, $\forall z \in \mathbb{C}$ and $g'(z) = \dfrac{f'(z)}{f(z)}$, $\forall z \in \mathbb{C}$. Now, we claim that $f\exp(-g)$ is a constant function on $\mathbb{C}$.

$$\begin{aligned}\frac{d}{dz}(f(z)\exp(-g(z))) &= f'(z)\exp(-g(z)) - \exp(-g(z))g'(z)f(z)\\ &= f'(z)\exp(-g(z)) - \exp(-g(z))\frac{f'(z)}{f(z)}f(z)\\ &= f'(z)\exp(-g(z)) - \exp(-g(z))f'(z)\\ &= 0.\end{aligned}$$

Hence, $f\exp(-g) = k$ for some $0 \neq k \in \mathbb{C}$. Then, choose $c \in \mathbb{C}$ such that $\exp(c) = k$. Therefore, $f(z)\exp(-g(z)) = \exp(c)$, $\forall z \in \mathbb{C}$. Thus, $g + c$ is an entire function such that $f = \exp(g + c)$.

Case (ii) If f has finite number of zeroes, say for example, and $a_1, a_2, a_3, \ldots, a_n \in \mathbb{C}\backslash\{0\}$ including multiplicities and 0 is a zero of order m (where $m \geq 0$), then applying Result 5.2.3 repeatedly, we get an entire function h, which has no zeroes in $\mathbb{C}$, such that

$$\begin{aligned}f(z) &= z^m(z - a_1)(z - a_2)\cdots(z - a_n)h(z)\\ &= z^m\prod_{k=1}^{n}\left(1 - \frac{z}{a_k}\right) \times (-1)^n\prod_{k=1}^{n}a_k \times h(z),\ \forall z \in \mathbb{C}.\end{aligned}$$

Therefore, by Case (i), there exists an entire function g such that

$$(-1)^n\prod_{k=1}^{n}a_k h(z) = \exp(g(z)),\ \forall z \in \mathbb{C}.$$

Thus, we can write

$$f(z) = z^m \exp(g(z)) \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right), \ \forall z \in \mathbb{C}.$$

Case (iii) Let f has countably infinite number of zeroes $\{a_n : n \in \mathbb{N}\}$ such that $a_n \to \infty$ as $n \to \infty$. In this case, with the motivation from Case (ii), we shall find an entire function f such that

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \ \forall z \in \mathbb{C}.$$

However, the convergence of $\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$ is not guaranteed for an arbitrary sequence of complex numbers with $a_n \to \infty$ as $n \to \infty$. Thus, we have to find a suitable sequence of polynomials (P_n) such that $\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp(P_n(z))$ converges. We know that $\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp(P_n(z))$ converges iff $\sum_{n=1}^{\infty} \left[\log\left(1 - \frac{z}{a_k}\right) + P_n(z)\right]$ converges with respect to a suitable branch of log. Let $R > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $|a_n| > 2R$, $\forall n \geq n_0$. If $|z| \leq R$ and $n \geq n_0$, then $\log\left(1 - \frac{z}{a_n}\right)$ is analytic and its Taylor series is given by

$$\begin{aligned} \log\left(1 - \frac{z}{a_n}\right) &= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{a_n}\right)^k \\ &= -\sum_{k=1}^{n} \frac{1}{k} \left(\frac{z}{a_n}\right)^k - \sum_{k=n+1}^{\infty} \frac{1}{k} \left(\frac{z}{a_n}\right)^k. \end{aligned}$$

Therefore, for $n \geq n_0$, if $P_n(z) = \sum_{k=1}^{n} \frac{1}{k} \left(\frac{z}{a_n}\right)^k$, $\forall z \in B(0, R)$ then

$$\begin{aligned} \left|\log\left(1 - \frac{z}{a_n}\right) + P_n(z)\right| &\leq \sum_{k=n+1}^{\infty} \left|\frac{1}{k} \left(\frac{z}{a_n}\right)^k\right| \\ &\leq \frac{1}{n+1} \sum_{k=n+1}^{\infty} \left(\frac{R}{|a_n|}\right)^k \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{R}{|a_n|}\right)^{n+1} \sum_{k=0}^{\infty} \left(\frac{R}{|a_n|}\right)^{k} \quad \left(\text{since } \frac{1}{n+1} \leq 1\right) \\ &= \left(\frac{R}{|a_n|}\right)^{n+1} \left(\frac{1}{1-\frac{R}{|a_n|}}\right) \\ &\leq M\left(\frac{1}{2}\right)^{n+1} \quad \text{for some } M > 0. \end{aligned}$$

Since $\left(\frac{1}{1-\frac{R}{|a_n|}}\right)$ converges to 1, such an $M > 0$ exists, by Corollary 1.3.4. Hence, by comparison test, $\sum_{n=1}^{\infty} \log\left(1-\frac{z}{a_n}\right) + P_n(z)$ converges absolutely on $B(0,R)$. Therefore, a most general entire function that has the zeroes $\{a_n : n \in \mathbb{N}\}$, is of the form

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} \left(1-\frac{z}{a_k}\right) \exp(P_n(z)),\ \forall z \in \mathbb{C},$$

for some entire function g. □

COROLLARY 5.4.15 Every meromorphic function on $\mathbb{C}$ is a quotient of two entire functions at all points of $\mathbb{C}$ except at the zeroes of f.

Proof: Let f be an arbitrary meromorphic function. From Corollary 5.2.12, we observe that f has at most countable number of poles. If there are countably infinite number of poles, then they must tend to ∞, otherwise, it should have a finite limit point, which is a contradiction to the fact that the poles are isolated. Hence, we can apply the previous theorem, and we can find an entire function G such that its zeroes are the poles of f. Hence, $f \cdot G$ has removable singularity at every pole of f, and it is analytic at every other points of $\mathbb{C}$. If F is an entire function such that $(f \cdot G)(z) = F(z)$, for all point of $\mathbb{C}$ except at the poles of f, then clearly, $f = \frac{F}{G}$ at all points of $\mathbb{C}$ except at the poles of f. □

THEOREM 5.4.16 (Euler's product for sin)

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right).$$

Proof: As $\sin(\pi n) = 0$, for every $n \in \mathbb{Z}$, from Theorem 5.4.14, we write

$$\sin(\pi z) = z \exp(g(z)) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n}\right), \forall z \in \mathbb{C},$$

for some entire function g. Applying logarithmic derivative on both sides, we get

$$\frac{\pi \cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + g'(z) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

From Example 5.3.3, we have

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Hence, it follows that $g'(z) = 0$, $\forall z \in \mathbb{C} \Rightarrow g(z) = c$, $\forall z \in \mathbb{C}$ for some constant $c \in \mathbb{C}$. Therefore,

$$\sin(\pi z) = z \exp(c) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n}\right), \forall z \in \mathbb{C}.$$

Using the easy fact that $\dfrac{\sin(\pi z)}{\pi z} \to 1$ as $z \to 0$, we observe that

$$\begin{aligned}
1 &= \lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} \\
&= \lim_{z\to 0} \frac{z \exp(c) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n}\right)}{\pi z} = 1 \\
&\Rightarrow \frac{\exp(c)}{\pi} = 1 \\
&\Rightarrow \exp(c) = \pi \\
&\Rightarrow \sin(\pi z) = \pi z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n}\right) \\
&\Rightarrow \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) \exp\left(\frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \\
&\Rightarrow \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\end{aligned}$$

□

COROLLARY 5.4.17 $\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$

Proof: Using $\sin(2\pi z) = 2\sin(\pi z)\cos(\pi z)$, we have

$$\begin{aligned}
\cos(\pi z) &= \frac{\sin(2\pi z)}{2\sin(\pi z)} \\
&= \frac{2\pi z \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{2\pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)} \\
&= \frac{\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}. \qquad (5.9)
\end{aligned}$$

Applying the following observation

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right) &= \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right) \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k)^2}\right) \\
&= \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)
\end{aligned}$$

in equation (5.9), we get

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

□

COROLLARY 5.4.18 (Wallis' formulae)

$$\begin{aligned}
\frac{\pi}{2} &= \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}. \\
\sqrt{\pi} &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \prod_{j=1}^{n} \frac{2j}{2j-1}.
\end{aligned}$$

Proof: Substituting $z = \frac{1}{2}$ in the Euler's product for sin, we get

$$\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) \Rightarrow 1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right),$$

which implies the first formula.

We rewrite the first formula as

$$\begin{aligned}
\frac{\pi}{2} &= \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \\
&= \lim_{n\to\infty}\prod_{j=1}^{n}\frac{(2j)^2}{(2j-1)^2}\frac{2j-1}{2j+1} \\
&= \lim_{n\to\infty}\left[\prod_{j=1}^{n}\frac{(2j)^2}{(2j-1)^2} \times \left(\frac{1}{3}\cdot\frac{3}{5}\cdots\frac{2n-1}{2n+1}\right)\right] \\
&= \lim_{n\to\infty}\left[\prod_{j=1}^{n}\frac{(2j)^2}{(2j-1)^2} \times \frac{1}{2n+1}\right] \\
&= \lim_{n\to\infty}\frac{1}{2n}\prod_{j=1}^{n}\frac{(2j)^2}{(2j-1)^2} \quad \left(\text{as } \lim_{n\to\infty}\frac{1}{1+\frac{1}{2n}} = 1\right).
\end{aligned}$$

This implies that $\sqrt{\pi} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\prod_{j=1}^{n}\frac{2j}{2j-1}$. □

Exercise 5.4.19

1. Prove the following:
   (a) $\sinh(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right)$.
   (b) $\cosh(\pi z) = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{(2n-1)^2}\right)$.
2. Prove that $\sqrt{2} = \prod_{n=1}^{\infty}\left(1 + \frac{(-1)^{n+1}}{2n-1}\right)$. Hint: Substitute $z = \frac{\pi}{4}$ in the expansion of $\cos(\pi z)$.
3. Prove that $\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$. Hint: Substitute $z = \frac{1}{2}$ in the expansion of $\frac{\sin(\pi z)}{\pi z}$.

RESULT 5.4.20 $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Proof: [4]Recall that the Euler's product for sin is given by

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), \ \forall x \in [0, 1).$$

If $g(x) = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2}\right)$, then from the proof of Lemma 5.4.4, we see that $\frac{\sin(\pi x)}{\pi x} = \exp(g(x))$. Therefore,

$$\log\left(\frac{\sin(\pi x)}{\pi x}\right) = g(x) = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\left(\frac{x^2}{n^2}\right)^k}{k}, \ \forall x \in [0, 1).$$

Since

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \left|\frac{\left(\frac{x^2}{n^2}\right)^k}{k}\right| = \sum_{n=1}^{\infty} \log\left(1 - \frac{|x|^2}{n^2}\right) = g(|x|) < +\infty,$$

we can apply Fubini's theorem[5] to get

$$\log\left(\frac{\sin(\pi x)}{\pi x}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right)\frac{x^{2k}}{k} = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right)\frac{x^{2k}}{k}, \ \forall x \in [0, 1). \tag{5.10}$$

On the other hand, we have

$$\begin{aligned}
\log\left(\frac{\sin(\pi x)}{\pi z}\right) &= \log\left(1 - \left[\frac{(\pi x)^2}{3!} - \frac{(\pi x)^4}{5!} + \cdots\right]\right) \\
&= \left[\frac{(\pi x)^2}{3!} - \frac{(\pi x)^4}{5!} + \cdots\right] + \frac{1}{2}\left[\frac{(\pi x)^2}{3!} - \frac{(\pi x)^4}{5!} + \cdots\right]^2 + \cdots \\
&= \frac{1}{3!}(\pi x)^2 + \left(-\frac{1}{5!} + \frac{1}{2 \times (3!)^2}\right)(\pi x)^4
\end{aligned}$$

[4]There are plenty of proofs for these identities. We present here the Euler's proof.

[5]Fubini's theorem is a well-known big theorem on integral over a product measure space. As it is beyond the scope of this book, we have not even stated this theorem. Interested readers can refer to any book on measure and integration to see this theorem.

$$+\left(\frac{1}{7!}-\frac{1}{3!\times 5!}+\frac{1}{3\times(3!)^3}\right)(\pi x)^6+\cdots. \qquad (5.11)$$

Comparing the coefficients of x^2 and x^4 from equations (5.10) and (5.11), we get $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\pi^4\left(-\frac{1}{120}+\frac{1}{72}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Thus, we get $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\frac{1}{n^4}$. □

Exercise 5.4.21 Prove that $\frac{\pi^6}{945}$. Hint: Compare the coefficients of x^6 from equations (5.10) and (5.11).

5.5 GAMMA FUNCTION AND ITS PROPERTIES

Definition 5.5.1 The gamma function Γ is defined by

$$\Gamma(z)=\frac{\exp(-\gamma z)}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right),\ \forall z\in\mathbb{C}\backslash\{-n:n\in\mathbb{N}\},$$

where $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n)\right)$, which is called the Euler's constant.

First, we note that the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)$ converges uniformly on the compact subsets of $\mathbb{C}$. If $|z|\le r$ and $n>2r$, then

$$\begin{aligned}\left|\log\left(1+\frac{z}{n}\right)-\frac{z}{n}\right| &= \left|\left(\frac{z}{n}-\frac{z^2}{2n^2}+\frac{z^3}{3n^3}-\cdots\right)-\frac{z}{n}\right|\\ &\le \sum_{k=2}^{\infty}\frac{r^k}{kn^k}\\ &< \sum_{k=2}^{\infty}\frac{r^k}{n^k}\\ &= \frac{r^2}{n^2}\sum_{k=0}^{\infty}\frac{r^k}{n^k}\end{aligned}$$

$$= \frac{r^2}{n^2}\left(1-\frac{r}{n}\right)^{-1} \text{ (as } 0 < \frac{r}{n} < \frac{1}{2} < 1)$$
$$= 2\frac{r^2}{n^2} \text{ (as } \left(1-\frac{r}{n}\right)^{-1} < 2).$$

Hence, $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) - \frac{z}{n}$ converges uniformly on compact subsets of $\mathbb{C}$. Equivalently, the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)$ converges uniformly on the compact subsets of $\mathbb{C}$. If $G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)$, then $\Gamma(z) = \frac{1}{z\exp(\gamma z)G(z)}$, and hence, Γ is well defined on $\mathbb{C}\setminus\{-n : n \in \mathbb{N}\}$.

THEOREM 5.5.2 (Properties of Γ)
The Γ function satisfies the following properties:

1. $\Gamma(z+1) = z\Gamma(z),\ \forall z \in \mathbb{C}$,
2. $\Gamma(n) = (n-1)!,\ \forall n \in \mathbb{N}$,
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,
4. **Legendre's duplication formula**
   $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \forall z \in \mathbb{C}$.

Proof:

1. Let $G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right),\ \forall z \in \mathbb{C}$. Hence, the zeroes of G are $-1,-2,-3,\ldots$. This implies that the zeroes of $G(z-1)$ are $0,-1,-2,-3,\ldots$. Therefore, from Theorem 5.4.14, we have

   $$G(z-1) = z\exp(\phi(z))G(z),\ \forall z \in \mathbb{C},$$

   for some entire function ϕ. That is, for each $z \in \mathbb{C}$, we have

   $$\prod_{n=1}^{\infty}\left(1+\frac{z-1}{n}\right)\exp\left(-\frac{(z-1)}{n}\right) = z\exp(\phi(z))\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right).$$

Finding the logarithmic derivative of the above equation, we get

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)=\frac{1}{z}+\phi'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right). \quad (5.12)$$

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) &= \frac{1}{z}-1+\sum_{n=2}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)\\
&= \frac{1}{z}-1+\sum_{m=1}^{\infty}\left(\frac{1}{z+m}-\frac{1}{m+1}\right)\\
&= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)\\
&\quad +\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)\\
&\quad +\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)\\
&= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).
\end{aligned}$$

Applying the last observation in equation (5.12), we get $\phi'(z) = 0$, $\forall z \in \mathbb{C}$, and hence, it is a constant, say, C. To find the value of C, substitute $z = 1$ in $G(z-1) = z\exp(\phi(z))G(z)$ to get

$$G(0)=\exp(C)G(1)\Rightarrow 1=\exp(C)\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\exp\left(-\frac{1}{n}\right),$$

which implies that

$$\begin{aligned}
-C &= \sum_{n=1}^{\infty}\left(\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)\\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\left(\ln(N+1)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)\right)\\
&= \lim_{N\to\infty}\left(\ln(N)+\ln\left(\frac{N+1}{N}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)\right)\\
&= \lim_{N\to\infty}\left(\ln(N)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)\right),
\end{aligned}
$$

as $\lim_{N\to\infty}\ln\left(\frac{N+1}{N}\right)=\ln(1)=0$. Thus, $C=\gamma$, the Euler's constant. Thus, we have

$$G(z-1)=z\exp(\gamma)G(z),\ \forall z\in\mathbb{C}.$$

From the definition of Γ and G, we can easily write that

$$
\begin{aligned}
\Gamma(z) &= \frac{1}{z\exp(\gamma z)G(z)}\\
\Gamma(z+1) &= \frac{1}{(z+1)\exp(\gamma(z+1))G(z+1)}\\
&= \frac{1}{(z+1)\exp(\gamma(z+1))G(z)(z+1)^{-1}\exp(-\gamma)}\\
&= \frac{1}{\exp(\gamma z)G(z)}\\
&= z\Gamma(z).
\end{aligned}
$$

2. We prove this identity by induction on n.
   $\Gamma(1)=\dfrac{1}{\exp(\gamma)G(1)}=1=0!$, as $1=G(0)=\exp(\gamma)G(1)$.
   Assume that this identity holds for some $n\in\mathbb{N}$. Now, by using (1) and induction hypothesis, we get $\Gamma(n+1)=n\Gamma(n)=n(n-1)!=n!$ thus, $\Gamma(n)=n!,\ \forall n\in\mathbb{N}$.

3. From the definition of G and Example 5.3.3, we have $zG(z)G(-z)=z\prod_{n=1}^{\infty}\left(1-\dfrac{z^2}{n^2}\right)=\dfrac{\sin(\pi z)}{\pi}$. Therefore, for every $z\in\mathbb{C}$,

$$
\begin{aligned}
\Gamma(1-z)\Gamma(z) &= (-z)\Gamma(-z)\Gamma(z)\\
&= (-z)\frac{1}{(-z)\exp(-\gamma z)G(-z)}\frac{1}{z\exp(\gamma z)G(z)}\\
&= \frac{1}{zG(-z)G(z)}=\frac{\pi}{\sin(\pi z)}.
\end{aligned}
$$

Substituting $z = \dfrac{1}{2}$, we get

$$\Gamma\left(1-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{1} \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

4. Finding the logarithmic derivative on both sides of

$$\Gamma(z) = \frac{\exp(-\gamma z)}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right),$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{-1}{z} - \gamma + \sum_{n=1}^{\infty}\left(\frac{-1}{z+n}+\frac{1}{n}\right).$$

Finding the derivative on both sides of the last equation, we get

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(z+n)^2} = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

Therefore,

$$\begin{aligned}
&\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}\right)\\
&\quad= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} + \sum_{n=0}^{\infty}\frac{1}{\left(z+\frac{1}{2}+n\right)^2}\\
&\quad= 4\left(\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2z+2n+1)^2}\right)\\
&\quad= 4\sum_{m=0}^{\infty}\frac{1}{(2z+m)^2}\\
&\quad= 4\times\frac{1}{2}\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)\\
&\quad= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right).
\end{aligned}$$

Integrating on both sides, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'\left(z+\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} + a,$$

for some constant $a \in \mathbb{C}$ Again integrating on both sides, we get

$$\log\left(\Gamma(z)\right) + \log\left(\Gamma\left(z+\frac{1}{2}\right)\right) = \log\left(\Gamma(2z)\right) + az + b,$$

which implies the following equation.

$$\Gamma(z) \times \Gamma\left(z+\frac{1}{2}\right) = \exp(az+b)\Gamma(2z). \tag{5.13}$$

Substituting $z = \frac{1}{2}$ and $z = 1$ in equation (5.13), we get

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1) = \exp\left(\frac{a}{2}+b\right)\Gamma(1) \Rightarrow \frac{a}{2} + b = \log(\sqrt{\pi}), \tag{5.14}$$

$$\Gamma(1)\Gamma\left(\frac{3}{2}\right) = \exp(a+b)\Gamma(2) \Rightarrow a + b = \log\left(\frac{\sqrt{\pi}}{2}\right). \tag{5.15}$$

$$\begin{aligned}\text{Equation (5.15)}-\text{equation (5.14)} \Rightarrow \frac{a}{2} &= \log\left(\frac{\sqrt{\pi}}{2}\right) - \log(\sqrt{\pi}) \\ &= -\log(2) \Rightarrow a = -2\log 2.\end{aligned}$$

Using this value in equation (5.15), we get $b = \log(2) + \frac{1}{2}\log(\pi)$. Therefore, from equation (5.13), we get

$$\begin{aligned}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= \exp\left((-2\log(2))z + \log(2) + \frac{1}{2}\log(\pi)\right)\Gamma(2z) \\ &= 2^{-2z+1}\sqrt{\pi}\Gamma(2z).\end{aligned}$$

Thus, we get

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

□

Example 5.5.3 Prove that $\Gamma(z) = \lim_{N\to\infty} \frac{N^z N!}{z(z+1)\cdots(z+N)}$, $\forall z \in \mathbb{C}\backslash\{-1,-2,-3,\ldots\}$.

Solution:

For $z \in \mathbb{C}\backslash\{0,-1,-2,-3,\ldots\}$, we have

$$\begin{aligned}
\Gamma(z) &= \frac{\exp(-\gamma z)}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\\
&= \lim_{N\to\infty}\frac{\exp\left(-\left[\sum_{n=1}^{N}\frac{1}{n}-\ln(N)\right]z\right)}{z}\cdot\lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\\
&= \lim_{N\to\infty}\left(\frac{\exp\left(-\left[\sum_{n=1}^{N}\frac{1}{n}-\ln(N)\right]z\right)}{z}\cdot\prod_{n=1}^{N}\left(\frac{z+n}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\right)\\
&= \lim_{N\to\infty}\prod_{n=1}^{N}\exp\left(-\frac{z}{n}\right)\frac{\exp\left(z\ln(N)\right)}{z}\cdot\prod_{n=1}^{N}\left(\frac{n}{z+n}\right)\exp\left(\frac{z}{n}\right)\\
&= \lim_{N\to\infty}\frac{N^z}{z}\cdot\prod_{n=1}^{N}\left(\frac{n}{z+n}\right)\\
&= \lim_{N\to\infty}\frac{N^z N!}{z(z+1)\cdots(z+N)}.
\end{aligned}$$

In the following example, we justify that Γ has some other integral representation. However, this justification depends on dominated convergence theorem. Hence, the following example is for those who are familiar with the dominated convergence theorem.

Example 5.5.4 Prove that $\Gamma(z) = \int_0^{\infty} t^{z-1}\exp(-t)\,dt$, $\forall z = (x,y) \in \mathbb{C}$ with $x > 0$.

Solution:

First, we claim that the above integral exists for all $z = (x,y) \in \mathbb{C}$ with $x > 0$. Since $\exp(-t) \le 1, \forall t \in [0,\infty]$, we have

$$\left|t^{z-1}\exp(-t)\right| < |\exp((z-1)\log t)|$$

$$\begin{aligned} &= |\exp((x-1)\log t)\exp(iy\log t)| \\ &= |\exp((x-1)\log t)|\,. \end{aligned}$$

Since for $x > 0$,

$$\begin{aligned} \int_0^\infty t^{x-1}\exp(-t)\,dt &= \int_0^1 t^{x-1}\exp(-t)\,dt + \int_1^\infty t^{x-1}\exp(-t)\,dt \\ &= \lim_{\epsilon\to 0}\int_\epsilon^1 t^{x-1}\exp(-t)\,dt + \int_1^\infty t^{x-1}\exp(-t)\,dt \\ &= \lim_{\epsilon\to 0}\frac{1-\epsilon^x}{x} + \int_1^\infty t^{x-1}\exp(-t)\,dt \\ &< \frac{1}{x} + \int_1^\infty \exp(-t/2)\,dt < +\infty, \end{aligned}$$

our claim holds. Next, we claim that if $G(z) = \int_0^\infty t^{z-1}\exp(-t)\,dt$, $\forall z \in \{z \in \mathbb{C} : Re\, z > 0\}$, then G is an analytic function. Fix $z_0 \in \mathbb{C}$ such that $Re\, z > 0$. Now, for $z_0 \in \mathbb{C}$ such that $Re\, z > 0$, by applying dominated convergence theorem, we get

$$\begin{aligned} &\lim_{z\to z_0}\left|\frac{G(z)-G(z_0)}{z-z_0} - \int_0^\infty t^{z-1}\log(t)\exp(-t)\,dt\right| \\ &= \lim_{z\to z_0}\int_0^\infty \left(\frac{t^{z-1}-t^{z_0-1}}{z-z_0} - t^{z_0-1}\log(t)\right)\exp(-t)\,dt \\ &= \int_0^\infty \lim_{z\to z_0}\left(\frac{t^{z-1}-t^{z_0-1}}{z-z_0} - t^{z_0-1}\log(t)\right)\exp(-t)\,dt \\ &= 0. \end{aligned}$$

Hence, G is analytic on $\{z \in \mathbb{C} : Re\, z > 0\}$.

Using $\exp(-t) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$, $\forall t > 0$, we get $G(z) = \lim_{n\to\infty} G_n(z)$, $\forall z \in \{z \in \mathbb{C} : Re\, z > 0\}$, where $G_n(z) = \int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt$, $\forall z \in \mathbb{C}$, with $Re\, z > 0$, $\forall n \in \mathbb{N}$. We note that

$$\begin{aligned} G_n(z) &= \int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt \\ &= n^z \int_0^1 s^{z-1}(1-s)^n\, ds \\ &\qquad \text{(by using the change of variable } t = ns\text{)} \\ &= \begin{cases} \frac{1}{z}\left(\frac{n}{n-1}\right)^{z+1} G_{n+1}(z+1), & n \geq 2 \\ \frac{1}{z(z+1)}, & n = 1 \end{cases}. \\ &\qquad \text{(by applying integration by parts)} \end{aligned}$$

By iterating $n-1$ times, we get

$$G_n(z) = \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)},\ \forall z \in \mathbb{C} \text{ with } Re\, z > 0,\ \forall n \geq 1,$$

and hence,

$$G(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)},\ \forall z \in \mathbb{C} \text{ with } Re\, z > 0.$$

Thus, by using the previous example, we get

$$G(z) = \Gamma(z),\ z \in \mathbb{C} \text{ with } Re\, z > 0.$$

Remark 5.5.5: The integral $\int_0^\infty t^{z-1}\exp(-t)\, dt$ does not exist for $z \in \mathbb{C}$ with $Re\, z < 0$.
Indeed, if $z = x + iy$ with $x < 0$ and $\epsilon > 0$, then

$$\int_0^\infty |t^{z-1}\exp(-t)|\, dt \geq \int_\epsilon^1 t^{x-1}\exp(-t)\, dt = \exp(-1)\frac{1-\epsilon^x}{x} \to \infty \text{ as } \epsilon \to 0.$$

6

Residue Calculus

6.1 RESIDUE

Definition 6.1.1 Let f be an analytic function except at a singularity 'a'. The residue of f at 'a' is defined to be a complex number R, which makes $f(z) - \dfrac{R}{z-a}$ as the derivative of an analytic function on $B(a,\delta)\backslash\{a\}$ for some $\delta > 0$.

Residue of f at a is unique, and hence, it is denoted by $(\text{Res}\, f)(a)$.

LEMMA 6.1.2 If f is an analytic function on Ω except at an isolated singularity a, then residue of f at a exists and is unique.

Proof: Since a is an isolated singularity, then there exists $\delta > 0$ such that f is analytic on $(\text{Cl}\, B(a,\delta))\backslash\{a\}$. For $0 < r < \delta$, let C denote the circle $|z-a| = r$ and $R = \dfrac{1}{i2\pi}\displaystyle\int_C f(z)\,dz$. We claim that $\displaystyle\int_\gamma \left(f(z) - \frac{R}{z-a}\right) dz = 0$ for every closed curve γ in $B(a,\delta)\backslash\{a\}$.

Case 1: γ is homologous to 0 in $B(a,\delta)\backslash\{a\}$.
If

$$F(z) = f(z) - \frac{R}{z-a},\ \forall z \in \text{Cl}\, B(a,\delta)\backslash\{a\}.$$

then F is analytic on $\text{Cl}\ B(a,\delta)\backslash\{a\}$, and by general version of Cauchy's theorem, we get $\displaystyle\int_\gamma F(z)\,dz = 0$.

Case 2: γ is not homologous to 0 in $B(a,\delta)\backslash\{a\}$.
Then, $\text{WN}(\gamma, a) = m$ for some $m \in \mathbb{Z}\backslash\{0\}$. We claim that $\gamma - mC \sim 0$

in $B(a,\delta)\backslash\{a\}$. If $z_0 \notin B(a,\delta)\backslash\{a\}$, then either $z_0 = a$ or $z_0 \notin B(a,\delta)$. If $z_0 = a$, then

$$\text{WN}(\gamma - mC, z_0) = \text{WN}(\gamma, a) - m\text{WN}(C, a) = m - m \times 1 = 0.$$

If $z_0 \notin B(a,\delta)$, then

$$\text{WN}(\gamma, z_0) = 0 \text{ and } \text{WN}(C, z_0) = 0 \text{ and } \text{WN}(\gamma - mC, z_0) = 0.$$

Hence, by general version of Cauchy's theorem, we get

$$\begin{aligned} 0 &= \int\limits_{\gamma - mC} \left(f(z) - \frac{R}{z-a} \right) dz \\ &= \int\limits_{\gamma - mC} \left(f(z) - \frac{R}{z-a} \right) dz \\ &= \int\limits_{\gamma} \left(f(z) - \frac{R}{z-a} \right) dz - m \int\limits_{C} \left(f(z) - \frac{R}{z-a} \right) dz, \end{aligned}$$

which implies

$$\begin{aligned} \int\limits_{\gamma} \left(f(z) - \frac{R}{z-a} \right) dz &= m \left(\int\limits_{C} f(z)\, dz - R \int\limits_{C} \frac{dz}{z-a} \right) \\ &= m \left(i2\pi R - R \int\limits_{C} \frac{dz}{z-a} \right) \\ &= m\,(i2\pi R - Ri2\pi) = 0. \end{aligned}$$

Hence, by Result 4.1.30, F is the derivative of an analytic function. Thus, residue of f at a exists, and $(\text{Res}\, f)(a) = R = \dfrac{1}{i2\pi} \int\limits_{C} f(z)\, dz$.

If there are $R, S \in \mathbb{C}$ such that $f(z) - \dfrac{R}{z-a}$ and $f(z) - \dfrac{S}{z-a}$ are derivatives of the analytic functions Φ and Ψ on $B(a,\delta)\backslash\{a\}$ and $B(a,\epsilon)\backslash\{a\}$, respectively, then obviously, $R - S \in \mathbb{C}$ and $(\Phi - \Psi)'(z) = \dfrac{S-R}{z-a}$ in $B(a, \min\{\delta,\epsilon\}) \setminus \{a\}$. Hence, by Corollary 4.1.29, we get

$$\int\limits_{C} \frac{S-R}{z-a}\, dz = 0 \Rightarrow (R-S)i2\pi = 0 \Rightarrow R = S.$$

Thus, the lemma follows. □

LEMMA 6.1.3 If 'a' is an isolated singularity of f and if the Laurent series expansion of f in $B(a,r)\backslash\{a\}$ (for some $r > 0$) is $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$, then $(\text{Res}\, f)(a) = c_{-1}$.

Proof: We know that $(z-a)^n$ is the derivative of $\dfrac{(z-a)^{n+1}}{n+1}$ inside $B(a,r)\backslash\{a\}$. Using the fact that

$$\limsup_{n\to\infty}\left|\frac{c_n}{n+1}\right|^{\frac{1}{n}} = \limsup_{n\to\infty}|c_n|^{\frac{1}{n}} \text{ and } \limsup_{n\to\infty}\left|\frac{c_{-n}}{-n+1}\right|^{\frac{1}{n}} = \limsup_{n\to\infty}|c_{-n}|^{\frac{1}{n}},$$

we conclude that $\sum\limits_{\substack{n=-\infty\\ n\neq -1}}^{\infty}\dfrac{c_n}{n+1}(z-a)^n$ converges in the neighbourhood of a (except at a) on which $\sum\limits_{\substack{n=-\infty\\ n\neq -1}}^{\infty} c_n(z-a)^{n+1}$ converges (cf. the proof of Theorem 2.3.2). Hence,

$$f(z) - \frac{c_1}{z-a} = \sum_{\substack{n=-\infty\\ n\neq -1}}^{\infty} c_n(z-a)^n = \frac{d}{dz}\left(\sum_{\substack{n=-\infty\\ n\neq -1}}^{\infty}\frac{c_n}{n+1}(z-a)^{n+1}\right),$$

for every $0 < |z-a| < r$. Thus, $(\text{Res}\, f)(a) = c_{-1}$. □

RESULT 6.1.4 If f is a meromorphic function with a pole 'a' of order m, then $(\text{Res}\, f)\,(a) = \dfrac{1}{(m-1)!}\lim\limits_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)$.

Proof: If f has a pole of order m at a, then by Result 5.2.14, we have

$$f(z) = \sum_{n=-m}^{\infty} c_n(z-a)^n,\ \forall z \in B(a,r)\backslash\{a\},$$

for some $r > 0$. From the previous lemma, we get

$$\begin{aligned}&\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)\\ &\quad= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m\sum_{n=-m}^{\infty} c_n(z-a)^n\right)\end{aligned}$$

$$= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left(\sum_{n=0}^{\infty} c_{n-m}(z-a)^n\right) \text{ which is equal to } c_{-1} \text{ if } m = 1.$$

$$= \frac{1}{(m-1)!}\lim_{z\to a}\sum_{n=m-1}^{\infty} c_{n-m}\left(\prod_{\nu=0}^{m-2}(n-\nu)\right)(z-a)^{n-m+1} \text{ if } m \geq 2$$

$$= \frac{1}{(m-1)!}(m-1)!c_{-1}$$

$$= c_{-1}$$

$$= (\text{Res}\, f)(a).$$

Thus, the result follows. □

Example 6.1.5 Find the residue of f at all poles.

(1) $f(z) = \dfrac{z^2+3}{z-1}$; (2) $f(z) = \dfrac{z}{(z+1)(z-2)}$;

(3) $f(z) = \dfrac{2z+3}{(z+2)^2(z-3)}$; (4) $f(z) = \dfrac{z^2+2}{(z+2)^3}$.

Solution:

1. If $f(z) = \dfrac{z^2+3}{z-1}$, then 1 is the only simple pole of f. Then,
   $(\text{Res}\, f)(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z^2+3) = 4.$

2. If $f(z) = \dfrac{z}{(z+1)(z-2)}$, then -1 and 2 are the simple poles of f. Then,
   $(\text{Res}\, f)(-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\dfrac{z}{z-2} = \dfrac{1}{3}$
   and $(\text{Res}\, f)(2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\dfrac{z}{z+1} = \dfrac{2}{3}.$

3. If $f(z) = \dfrac{2z+3}{(z+2)^2(z-3)}$, then -2 is a double pole of f, and 3 is a simple pole of f. Then,
   $$(\text{Res}\, f)(-2) = \lim_{z\to -2}\frac{d}{dz}[(z+2)^2 f(z)] = \lim_{z\to -2}\frac{d}{dz}\left(\frac{2z+3}{z-3}\right)$$
   $$= \lim_{z\to -2}\frac{-9}{(z-3)^2} = \frac{-9}{25}$$
   and $(\text{Res}\, f)(3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\dfrac{2z+3}{(z+2)^2} = \dfrac{9}{25}.$

4. If $f(z) = \dfrac{z^2+2}{(z+2)^3}$, then -2 is the only pole of f of order 3. Then,

$$\begin{aligned}(\operatorname{Res} f)(-2) &= \frac{1}{2}\lim_{z\to -2}\frac{d^2}{dz^2}[(z+2)^3 f(z)] \\ &= \frac{1}{2}\lim_{z\to -2}\frac{d^2}{dz^2}\left(z^2+2\right) \\ &= \frac{1}{2}\times 2 = 1.\end{aligned}$$

Example 6.1.6 Evaluate the residues of the following functions at their poles:

(1) $\dfrac{z^2+1}{z^2+5z+6}$; (2) $\dfrac{z+1}{(z^2+1)^2}$; (3) $\dfrac{\exp(z)}{(z+3)(z-2)^2}$.

Solution:

1. Let $f(z) = \dfrac{z^2+1}{z^2+5z+6}$, $\forall z \in \mathbb{C}$. The poles of f are the zeroes of the polynomial z^2+5z+6. Therefore, $z=-2$ and $z=-3$ are the simple poles of f. Hence,

$$\begin{aligned}(\operatorname{Res} f)(-2) &= \lim_{z\to -2}(z+2)f(z) \\ &= \lim_{z\to -2}(z+2)\frac{z^2+1}{z^2+5z+6} \\ &= \lim_{z\to -2}\frac{z^2+1}{z+3} \\ &= 5. \\ (\operatorname{Res} f)(-3) &= \lim_{z\to -3}(z+3)f(z) \\ &= \lim_{z\to -3}(z+3)\frac{z^2+1}{z^2+5z+6} \\ &= \lim_{z\to -3}\frac{z^2+1}{z+2} \\ &= -10.\end{aligned}$$

2. Let $f(z) = \dfrac{z+1}{(z^2+1)^2}$, $\forall z \in \mathbb{C}$. $z=i$ and $z=-i$ are the poles of the order 2 for f.

$$(\operatorname{Res} f)(i) = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{z+1}{(z^2+1)^2}\right]$$

$$
\begin{aligned}
&= \lim_{z\to i}\frac{d}{dz}\left[\frac{z+1}{(z+i)^2}\right]\\
&= \lim_{z\to i}\left[\frac{(z+i)^2-(z+1)2(z+i)}{(z+i)^4}\right]\\
&= \frac{(i2)^2-(i+1)2(i2)}{(i2)^4}\\
&= \frac{-4-i4+4}{16}\\
&= -\frac{i}{4}.\\
(\operatorname{Res} f)(-i) &= \frac{1}{(2-1)!}\lim_{z\to -i}\frac{d}{dz}\left[(z+i)^2\frac{z+1}{(z^2+1)^2}\right]\\
&= \lim_{z\to -i}\frac{d}{dz}\left[\frac{z+1}{(z-i)^2}\right]\\
&= \lim_{z\to -i}\left[\frac{(z-i)^2-(z+1)2(z-i)}{(z-i)^4}\right]\\
&= \frac{i}{4}.
\end{aligned}
$$

3. Let $f(z)=\dfrac{\exp(z)}{(z+3)(z-2)^2}$, $\forall z\in\mathbb{C}$. $z=-3$ is a simple pole, and $z=2$ is a pole of the order 3 for f.

$$
\begin{aligned}
(\operatorname{Res} f)(-3) &= \lim_{z\to -3}\left[(z+3)\frac{\exp(z)}{(z+3)(z-2)^2}\right]\\
&= \lim_{z\to -3}\frac{\exp(z)}{(z+3)(z-2)^2}\\
&= \frac{\exp(-3)}{25}.\\
(\operatorname{Res} f)(2) &= \frac{1}{(3-1)!}\lim_{z\to 2}\frac{d^2}{dz^2}\left[(z-2)^3\frac{\exp(z)}{(z+3)(z-2)^2}\right]\\
&= \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left[\frac{\exp(z)}{z+3}\right]\\
&= \frac{1}{2}\lim_{z\to 2}\frac{d}{dz}\left[\frac{(z+2)\exp(z)}{(z+3)^2}\right]
\end{aligned}
$$

$$= \frac{1}{2}\lim_{z\to 2}\frac{\exp(z)[(z+3)^2-2(z+2)]}{(z+3)^3}$$
$$= \frac{17}{250}\exp(2).$$

Example 6.1.7 Find the residues of f at the given singularities.
(1) $\dfrac{\sin(3z)}{z}$ at 0; (2) $\dfrac{\cos(z)}{2z}$ at 0; (3) $\exp\left(\dfrac{2}{3z}\right)$ at 0; (4) $\dfrac{1}{\cos(z)}$ at $\dfrac{\pi}{2}$.

Solution:

1. Let $f(z) = \dfrac{\sin(3z)}{z}$, $\forall z \neq 0$. Since $\sin(3z) = \sum_{k=0}^{\infty}\dfrac{(-1)^k(3z)^{2k-1}}{(2k-1)!}$, we have $\lim_{z\to 0}\dfrac{\sin(z)}{z} = \lim_{z\to 0}\left(3+\sum_{k=1}^{\infty}\dfrac{(-1)^k(3z)^{2k}}{3(2k+1)!}\right) = 3$ in $\mathbb{C}$. Hence, 0 is a removable singularity for f, and hence, $(\mathrm{Res}\, f)(0) = 0$.

2. Let $f(z) = \dfrac{\cos(z)}{2z}$, $\forall z \neq 0$. By a similar argument we get $\dfrac{\cos 3z}{z} = \dfrac{1}{2z} + \sum_{k=1}^{\infty}\dfrac{(-1)^k(z)^{2k-1}}{2(2k)!}$, which is the Larurent series expansion of f in $0 < |z| < \infty$. Hence, $(\mathrm{Res}\, f)(0) = \dfrac{1}{2}$ (the coefficient of $\dfrac{1}{z}$).

3. If $f(z) = \exp\left(\dfrac{2}{3z}\right)$, then $f(z) = \sum_{k=0}^{\infty}\dfrac{\left(\frac{2}{3z}\right)^k}{k!}$. Hence, 0 is an essential singularity of f, and hence, $(\mathrm{Res}\, f)(0) = \dfrac{2}{3}$.

4. Let $f(z) = \dfrac{1}{\cos(z)}$, $\forall z \neq (2k-1)\dfrac{\pi}{2}$, $k \in \mathbb{Z}$. Clearly, $z = \dfrac{\pi}{2}$ is a simple pole. Hence, $(\mathrm{Res}\, f)(0) = \lim_{z\to\frac{\pi}{2}}\left(z-\dfrac{\pi}{2}\right)\dfrac{1}{\cos(z)} = \lim_{z\to\frac{\pi}{2}}\left(z-\dfrac{\pi}{2}\right)\dfrac{-1}{\sin(z-\frac{\pi}{2})} = -1$.

Exercise 6.1.8 Evaluate the residues of f at its singularities.
(1) $\dfrac{\sin(z)}{z^2}$ at 0; (2) $\dfrac{1}{\sin^2(z)}$ at π; (3) $\dfrac{\exp(z)-1}{z^4}$ at $z=0$.

Answers: (1) 1; (2) 0; (3) $\dfrac{1}{6}$.

RESULT 6.1.9 If f and g are analytic functions on a region Ω such that $f(z_0) \neq 0$ and g has a simple zero at z_0, then $\left(\text{Res}\,\dfrac{f}{g}\right)(z_0) = \dfrac{f(z_0)}{g'(z_0)}$.

Proof: By assumption, it is clear that $\dfrac{f}{g}$ has a simple pole at z_0. Hence, by Lemma 6.1.4,

$$\begin{aligned}
\left(\text{Res}\,\frac{f}{g}\right)(z_0) &= \lim_{z\to z_0}(z-z_0)\frac{f(z)}{g(z)} \\
&= \lim_{z\to z_0}\frac{f(z)}{\frac{g(z)}{z-z_0}} \\
&= \lim_{z\to z_0}\frac{f(z)}{\frac{g(z)-g(z_0)}{z-z_0}} \\
&= \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}} \\
&= \frac{f(z_0)}{g'(z_0)}.
\end{aligned}$$

Hence, the result follows. □

6.2 CAUCHY'S RESIDUE THEOREM

THEOREM 6.2.1 (Cauchy's residue theorem)
Let f be an analytic function on a region Ω except at a finite number of singularities. Then,

$$\int_\gamma f(z)\,dz = i2\pi \sum_{j=1}^{n} \text{WN}(\gamma, a_j)(\text{Res}\, f)(a_j),$$

for every closed curve γ in $\Omega \setminus \{a_1, a_2, \ldots, a_n\}$ such that $\gamma \sim 0$ in Ω.

Proof: Let $a_1, a_2, a_3, \ldots, a_n$ be the isolated singularities of f such that $a_j \neq a_k$ if $j \neq k$. Choose $\delta_j > 0$ such that

1. $\text{Cl}\, B(a_j, \delta_j) \subseteq \Omega \setminus \gamma$ and
2. $\text{Cl}\, B(a_j, \delta_j) \cap \text{Cl}\, B(a_k, \delta_k) = \varnothing$ for every $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$.

In view of proof of Lemma 6.1.2, if C_j is the circle $|z - a_j| = \delta_j$, then we have $(\text{Res}\, f)(a_j) = \dfrac{1}{i2\pi} \displaystyle\int_{C_j} f(z)\, dz$. We claim that $\gamma - \displaystyle\sum_{j=1}^{n} \text{WN}(\gamma, a_j) C_j \sim 0$ in $\Omega^* = \Omega \setminus \{a_1, a_2, a_3, \ldots, a_n\}$. If $z_0 \notin \Omega^*$, then either $z_0 \notin \Omega$ or $z_0 = a_j$ for some $j \in \{1, 2, 3, \ldots, n\}$. If $z_0 \notin \Omega$, then $\text{WN}(\gamma, z_0) = \text{WN}(C_j, z_0) = 0$ for all $j \in \{1, 2, 3, \ldots, n\}$, and hence, $\text{WN}\left(\gamma - \displaystyle\sum_{j=1}^{n} \text{WN}(\gamma, a_j) C_j\right) = 0$. If $z_0 = a_k$, then $\text{WN}(C_j, z_0) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ for all $j \in \{1, 2, 3, \ldots, n\}$, and hence, $\text{WN}\left(\gamma - \displaystyle\sum_{j=1}^{n} \text{WN}(\gamma, a_j) C_j\right) = \text{WN}(\gamma, a_k) - \text{WN}(\gamma, a_k) = 0$.
Hence, our claim holds. Therefore, by general version of Cauchy's theorem, we get

$$0 = \int_{\gamma - \sum_{j=1}^{n} \text{WN}(\gamma, a_j) C_j} f(z)\, dz = \int_{\gamma} f(z)\, dz - \sum_{j=1}^{n} \text{WN}(\gamma, a_j) \int_{C_j} f(z)\, dz,$$

which implies that $\displaystyle\int_{\gamma} f(z)\, dz = i2\pi \sum_{j=1}^{n} \text{WN}(\gamma, a_j)(\text{Res}\, f)(a_j)$. □

Example 6.2.2 Evaluate the following integrals using Cauchy's residue theorem:

1. $\displaystyle\int_{|z|=3} \frac{z+1}{z^2 + 7z + 10}\, dz.$
2. $\displaystyle\int_{|z|=4} \frac{3z^2 - 2z + 1}{z^2 - 2z}\, dz.$
3. $\displaystyle\int_{|z|=3} \frac{z+3}{(z^2 + 6z + 8)(z^2 + 1)}\, dz.$

4. $\int\limits_{|z-1|=3} \dfrac{z^2+2}{(z-2)^2(z+4)}\,dz.$

5. $\int\limits_{|z+1-i|=6} \dfrac{z}{(z^2-2z-15)(z^2+1)}\,dz.$

6. $\int\limits_{\gamma} \dfrac{\sin(z)}{(z^2+1)^2}\,dz$, where γ is the rectangle with vertices $(-1,0)$, $(1,0)$, $(1,2)$, and $(-1,2)$.

7. $\int\limits_{|z|=1} \dfrac{4z+1}{(z^2+1)(z^2+2)}\,dz.$

8. $\int\limits_{|z|=4} \dfrac{z^3+1}{z(z+1)(z+2)}\,dz.$

9. $\int\limits_{|z-1|=5} \dfrac{\exp(z)-1}{\sin(z)}\,dz.$

10. $\int\limits_{|z|=1} \dfrac{\exp(z)}{\sin(z)}\,dz.$

Solution:

1. Let $f(z) = \dfrac{z+1}{z^2+7z+10}$, $\forall z \in \mathbb{C}\backslash\{-2,-5\}$. Clearly, -2 and -5 are the simple poles of f. We note that -2 is enclosed by $|z|=3$ and -5 is not enclosed by $|z|=3$. Therefore, by Cauchy's residue theorem, we get

$$\begin{aligned}\int\limits_{|z|=3} \frac{z+1}{z^2+7z+10}\,dz &= i2\pi(\operatorname{Res} f)(-2)\\ &= i2\pi \lim_{z\to -2}(z+2)f(z)\\ &= i2\pi \lim_{z\to -2}\frac{z+1}{z+5}\\ &= \frac{-i2\pi}{3}.\end{aligned}$$

2. Let $f(z) = \dfrac{3z^2-2z+1}{z^2-2z}\,dz$, $\forall z \in \mathbb{C}\backslash\{0,2\}$. Clearly, 0 and 2 are the simple poles of f and both are enclosed by $|z|=4$. Therefore, by Cauchy's residue theorem, we get

$$\int\limits_{|z|=4} \frac{3z^2-2z+1}{z^2-2z}\,dz = i2\pi\left[(\mathrm{Res}\,f)(0)+(\mathrm{Res}\,f)(2)\right].$$

Since

$$\begin{aligned}
(\mathrm{Res}\,f)(0) &= \lim_{z\to 0} zf(z)\\
&= \lim_{z\to 0}\frac{3z^2-2z+1}{z-2}\\
&= -\frac{1}{2}\\
(\mathrm{Res}\,f)(2) &= \lim_{z\to 2}(z-2)f(z)\\
&= \lim_{z\to 2}\frac{3z^2-2z+1}{z}\\
&= \frac{9}{2},
\end{aligned}$$

we get

$$\int\limits_{|z|=4} \frac{3z^2-2z+1}{z^2-2z}\,dz = i2\pi\left[\frac{-1}{2}+\frac{9}{2}\right] = i8\pi.$$

3. Let $f(z) = \dfrac{z+3}{(z^2+6z+8)(z^2+1)}, \forall z \in \mathbb{C}\backslash\{-2,-4,i,-i\}$. Clearly, $-2,-4,i$, and $-i$ are simple poles of f, and $-2,i$, and $-i$ are enclosed by $|z|=3$, but -4 is not enclosed by $|z|=3$. Therefore, by Cauchy's residue theorem,

$$\begin{aligned}
&\int\limits_{|z|=3} \frac{z+3}{(z^2+6z+8)(z^2+1)}\,dz\\
&\quad = i2\pi\left[(\mathrm{Res}\,f)(-2)+(\mathrm{Res}\,f)(-i)+(\mathrm{Res}\,f)(i)\right].
\end{aligned}$$

Since

$$\begin{aligned}
(\mathrm{Res}\,f)(-2) &= \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\frac{z+3}{(z+4)(z^2+1)} = \frac{-1}{10},\\
(\mathrm{Res}\,f)(-i) &= \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\frac{z+3}{(z^2+6z+8)(z-i)}\\
&= \frac{-i+3}{(7-i6)(-i2)} = \frac{-11+i27}{170},
\end{aligned}$$

and

$$\begin{aligned}(\operatorname{Res} f)(i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{z+3}{(z^2+6z+8)(z+i)} \\ &= \frac{i+3}{(7+i6)(i2)} = \frac{-11-i27}{170},\end{aligned}$$

we get

$$\begin{aligned}&\int_{|z|=3}\frac{z+3}{(z^2+6z+8)(z^2+1)}\,dz \\ &\quad= i2\pi\left[\frac{-1}{10}+\frac{-11+i27}{170}+\frac{-11-i27}{170}\right] \\ &\quad= i2\pi\left[\frac{-39}{170}\right] \\ &\quad= \frac{-39\pi i}{85}.\end{aligned}$$

4. Let $f(z) = \dfrac{z^2+2}{(z-2)^2(z+4)}$, $\forall z \in \mathbb{C}\backslash\{2,-4\}$. Obviously, 2 is a double pole and -4 is a simple pole for f. 2 is enclosed by $|z-1| = 3$, and -4 is not enclosed by $|z-1| = 3$. Since

$$(\operatorname{Res} f)(2) = \lim_{z\to 2}\frac{d}{dz}\left(\frac{z^2+2}{z+4}\right) = \frac{(z+4)2z-(z^2+2)}{(z+4)^2} = \frac{1}{2},$$

by Cauchy's residue theorem, we get

$$\int_{|z-1|=3}\frac{z^2+2}{(z-2)^2(z+4)}\,dz = i2\pi\left[(\operatorname{Res} f)(2)\right] = i2\pi\frac{1}{2} = i\pi.$$

5. Let $f(z) = \dfrac{z}{(z^2-2z-15)(z^2+1)}$, $\forall z \in \mathbb{C}\backslash\{5,-3,i,-i\}$. Clearly, $5,-3,i$, and $-i$ are poles of f, and $|z+1-i| = 6$ encloses $-3,i$, and $-i$ but not 5. By Cauchy's residue theorem, we have

$$\begin{aligned}&\int_{|z+1-i|=6}\frac{z}{(z^2-2z-15)(z^2+1)}\,dz \\ &\quad= i2\pi\left[(\operatorname{Res} f)(-3)+(\operatorname{Res} f)(i)+(\operatorname{Res} f)(-i)\right].\end{aligned}$$

Since

$$\begin{aligned}(\operatorname{Res} f)(-3) &= \lim_{z\to -3} \frac{z}{(z-5)(z^2+1)} = \frac{3}{80},\\ (\operatorname{Res} f)(i) &= \lim_{z\to i} \frac{z}{(z^2-2z-15)(z+i)}\\ &= \frac{i}{(-16-i2)(i2)} = \frac{-8+i}{260},\end{aligned}$$

and

$$\begin{aligned}(\operatorname{Res} f)(-i) &= \lim_{z\to -i} \frac{z}{(z^2-2z-15)(z-i)}\\ &= \frac{1}{2(-16+i2)} = \frac{-8-i}{260},\end{aligned}$$

we have

$$\begin{aligned}&\int_{|z+1-i|=6} \frac{z}{(z^2-2z-15)(z^2+1)}\,dz\\ &\quad = i2\pi\left[\frac{3}{80} + \frac{-8+i}{260} + \frac{-8-i}{260}\right]\\ &\quad = i2\pi\left[\frac{3}{80} + \frac{-8+i}{260} + \frac{-8-i}{260}\right]\\ &\quad = \frac{i\pi}{52}.\end{aligned}$$

6. Let $f(z) = \dfrac{\sin(z)}{(z^2+1)^2}$, $\forall z \in \mathbb{C}\backslash\{i,-i\}$. Clearly, i and $-i$ are double poles of f and both are enclosed by γ. Therefore, by Cauchy's residue theorem,

$$\int_\gamma \frac{\sin(z)}{(z^2+1)^2}\,dz = i2\pi\left[(\operatorname{Res} f)(i) + (\operatorname{Res} f)(-i)\right].$$

Since

$$\begin{aligned}(\operatorname{Res} f)(i) &= \lim_{z\to i} \frac{d}{dz}\left(\frac{\sin(z)}{(z+i)^2}\right)\\ &= \lim_{z\to i} \frac{(z+i)\cos(z) - 2\sin(z)}{(z+i)^3}\\ &= \frac{\exp(i)}{-4}\end{aligned}$$

and

$$\begin{aligned}(\operatorname{Res} f)(-i) &= \lim_{z\to -i}\frac{d}{dz}\left(\frac{\sin(z)}{(z-i)^2}\right)\\ &= \lim_{z\to -i}\frac{(z-i)\cos(z)-2\sin(z)}{(z-i)^3}\\ &= \frac{\exp(i)}{-4},\end{aligned}$$

we have

$$\int_{\gamma}\frac{\sin(z)}{(z^2+1)^2}\,dz = i2\pi\frac{2\exp(i)}{-4} = -i\pi\exp(i).$$

7. Let $\dfrac{4z+1}{(z^2+1)(z^2+2)}$, $\forall z\in\mathbb{C}\backslash\{i,-i,-i\sqrt{2},i\sqrt{2}\}$. Clearly, $i,-i,i\sqrt{2}$, and $-i\sqrt{2}$ are simple poles of f, and i and $-i$ are enclosed by $|z|=1$. By Cauchy's residue theorem, we have

$$\int_{|z|=1}\frac{4z+1}{(z^2+1)(z^2+2)}\,dz = i2\pi\left[(\operatorname{Res} f)(i)+(\operatorname{Res} f)(-i)\right].$$

Since

$$\begin{aligned}(\operatorname{Res} f)(i) &= \lim_{z\to i}\frac{4z+1}{(z+i)(z^2+2)}\\ &= \frac{1+i4}{i2}\\ &= \frac{4-i}{2}\end{aligned}$$

and

$$\begin{aligned}(\operatorname{Res} f)(-i) &= \lim_{z\to -i}\frac{4z+1}{(z-i)(z^2+2)}\\ &= \frac{1-i4}{-i2}\\ &= \frac{4+i}{2},\end{aligned}$$

we get

$$\int_{|z|=1}\frac{4z+1}{(z^2+1)(z^2+2)}\,dz = i2\pi\left[\frac{4-i}{2}+\frac{4+i}{2}\right] = i8\pi.$$

8. Let $f(z) = \dfrac{z^3+1}{z(z+1)(z+2)}$, $\forall z \in \mathbb{C}\backslash\{0,-1,-2\}$. Clearly, $0, -1$, and -2 are simple poles of f and all are enclosed by $|z| = 4$. Therefore, by Cauchy's Residue theorem,

$$\int\limits_{|z|=4} \frac{z^3+1}{z(z+1)(z+2)}\,dz = i2\pi\left[(\text{Res}\,f)(0) + (\text{Res}\,f)(-1) + (\text{Res}\,f)(-2)\right].$$

Since

$$\begin{aligned}(\text{Res}\,f)(0) &= \lim_{z\to 0}\frac{z^3+1}{(z+1)(z+2)} = \frac{1}{2},\\ (\text{Res}\,f)(-1) &= \lim_{z\to -1}\frac{z^3+1}{z(z+2)} = 0,\end{aligned}$$

and

$$(\text{Res}\,f)(-2) = \lim_{z\to -2}\frac{z^3+1}{z(z+1)} = \frac{-7}{2},$$

we get

$$\int\limits_{|z|=4} \frac{z^3+1}{z(z+1)(z+2)} = i2\pi\left[\frac{1}{2} + 0 + \frac{-7}{2}\right] = -3i\pi.$$

9. Let $f(z) = \dfrac{\exp(z)-1}{\sin(z)}$, $\forall z \in \mathbb{C}\backslash\{n\pi : n \in \mathbb{Z}\}$. Clearly, 0 is a removable singularity for f as

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\exp(z)-1}{\sin(z)} = \lim_{z\to 0}\frac{\sum\limits_{k=1}^{\infty}\frac{z^k}{k!}}{\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k-1}}{(2k-1)!}} = 0,$$

and $n\pi$ is a pole of order 1, for every $0 \neq n \in \mathbb{Z}$, as

$$\lim_{z\to n\pi} f(z) = \infty \text{ and } \left.\frac{d}{dz}\sin(z)\right|_{n\pi} = \cos(n\pi) = (-1)^n \neq 0.$$

Among the singularities of f, 0 and π are enclosed by $|z-1|=4$. Therefore, by Cauchy's residue theorem, we have

$$\int_{|z-1|=4} \frac{\exp(z)-1}{\sin(z)}\,dz = i2\pi\left[(\text{Res}\,f)(0)+(\text{Res}\,f)(\pi)\right].$$

$$\begin{aligned}(\text{Res}\,f)(0) &= 0.\\ (\text{Res}\,f)(\pi) &= \lim_{z\to\pi}\frac{(z-\pi)(\exp(z)-1)}{\sin(z)}\\ &= (-1)(\exp(\pi)-1).\end{aligned}$$

Therefore,

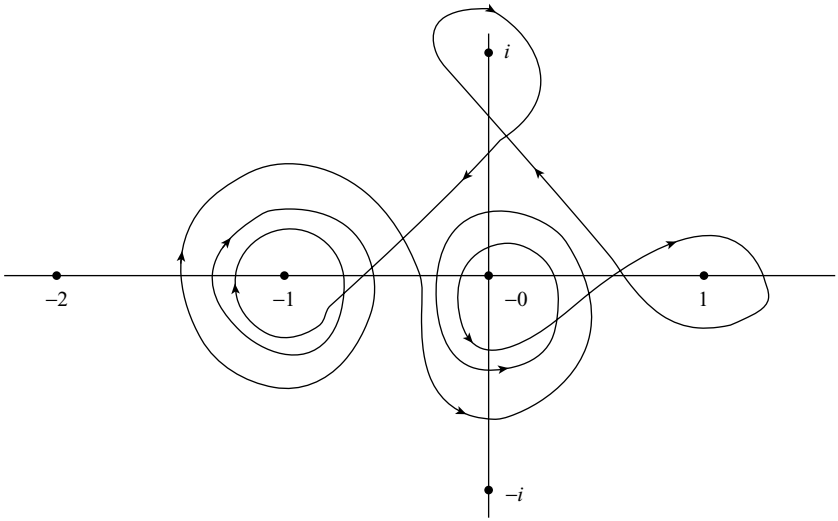
$$\int_{|z-1|=4} \frac{\exp(z)-1}{\sin(z)}\,dz = i2\pi(1-\exp(\pi)).$$

10. Let $f(z)=\dfrac{\exp(z)}{\sin(z)}, \forall z\in\mathbb{C}\backslash\{0\}$. Clearly, 0 is the simple pole of f and is enclosed by $|z|=1$. Therefore, by Cauchy's residue theorem, we have

$$\begin{aligned}\int_{|z|=1}\frac{\exp(z)}{\sin(z)}\,dz &= i2\pi(\text{Res}\,f)(0)\\ &= i2\pi\lim_{z\to 0}\frac{z}{\sin(z)}\\ &= i2\pi.\end{aligned}$$

Note that so far we have evaluated integrals over simple curves. Now, we evaluate an integral over a closed curve, which is not simple.

Example 6.2.3 Evaluate $\int_\gamma \dfrac{2z+3}{(z^4-1)(z^2-5z+6)}\,dz$, where γ is given in the following diagram.



Let $f(z) = \dfrac{2z+3}{(z^4-1)(z^3-5z^2+6z)}$, $\forall z \in \mathbb{C}\backslash\{-1, 0, i, -i, 1, 2, 3\}$. Clearly, $0, 1, -1, i, -i, 2$, and 3 are poles of f. From the diagram, we find[1] that $\text{WN}(\gamma, -1) = -3$, $\text{WN}(\gamma, 0) = 2$, $\text{WN}(\gamma, i) = -1$, $\text{WN}(\gamma, -i) = 0$, $\text{WN}(\gamma, 1) = -1$, $\text{WN}(\gamma, 2) = 0$, and $\text{WN}(\gamma, 3) = 0$.
Therefore, by Cauchy's residue theorem, we get

$$\begin{aligned}
&\int_\gamma \frac{2z+3}{(z^4-1)(z^3-5z^2+6z)}\,dz \\
&\quad = i2\pi\,[\text{WN}(\gamma,-1)\,(\text{Res}\,f)(-1) \\
&\qquad + \text{WN}(\gamma,0)\,(\text{Res}\,f)(0) + \text{WN}(\gamma,i)\,(\text{Res}\,f)(i) \\
&\qquad + \text{WN}(\gamma,-i)\,(\text{Res}\,f)(-i) + \text{WN}(\gamma,1)\,(\text{Res}\,f)(1) \\
&\qquad + \text{WN}(\gamma,2)\,(\text{Res}\,f)(2) + \text{WN}(\gamma,3)\,(\text{Res}\,f)(3)] \\
&\quad = i2\pi\,[-3\,(\text{Res}\,f)(-1) + 2\,(\text{Res}\,f)(0) - (\text{Res}\,f)(i) - (\text{Res}\,f)(1)]\,.
\end{aligned}$$

Since

$$(\text{Res}\,f)(-1) = \lim_{z\to -1} \frac{2z+3}{(z-1)(z^2+1)(z^3-5z^2+6z)} = \frac{1}{48},$$

[1] $\text{WN}(\gamma, a)$ is found by counting the number of times the curve γ winds around a, and the sign is given as $+$ $(-)$ if it winds in anti-clockwise sense (clockwise sense).

$$\begin{aligned}(\text{Res}\, f)(0) &= \lim_{z\to 0} \frac{2z+3}{(z^4-1)(z^2-5z+6)} = \frac{-1}{2},\\ (\text{Res}\, f)(i) &= \lim_{z\to i} \frac{2z+3}{(z^2-1)(z+i)(z^3-5z^2+6z)} = \frac{1+i5}{40},\end{aligned}$$

and

$$(\text{Res}\, f)(1) = \lim_{z\to 1} \frac{2z+3}{(z+1)(z^2+1)(z^3-5z^2+6z)} = \frac{5}{8},$$

we get

$$\begin{aligned}\int_\gamma \frac{2z+3}{(z^4-1)(z^2-5z+6)}\,dz &= i2\pi\left[-3\times\frac{1}{48}+2\times\frac{-1}{2}-\frac{1+i5}{40}-\frac{5}{8}\right]\\ &= i\frac{1}{4}\pi\left[\frac{-1}{2}-8-\frac{1+i5}{5}-5\right]\\ &= \frac{\pi}{4}\left(1-\frac{i137}{10}\right).\end{aligned}$$

Example 6.2.4 Evaluate $\int_{|z-1|=2} \tan(z)\,dz$ using Cauchy's residue theorem.

Solution:
We note that the only pole of tan inside $|z-1|=2$ is $\frac{\pi}{2}$, which is a simple pole. Hence, by Cauchy's residue theorem, $\int_{|z|=2} \tan(z)\,dz = i2\pi(\text{Res}\ \tan)\left(\frac{\pi}{2}\right)$. As $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$, $\sin\left(\frac{\pi}{2}\right) = 1 \neq 0$, and $\cos\left(\frac{\pi}{2}\right) = 0$, then by applying Result 6.1.9, we get $(\text{Res}\ \tan)\left(\frac{\pi}{2}\right) = \dfrac{\sin\left(\frac{\pi}{2}\right)}{\cos'\left(\frac{\pi}{2}\right)} = \dfrac{1}{-1} = -1$. Therefore,

$$\int_{|z|=2} \tan(z)\,dz = -i2\pi.$$

Exercise 6.2.5 Evaluate the following integrals using Cauchy's residue theorem:

1. $\int_{|z|=2} \dfrac{\exp(z)}{z^2-2z-3}\,dz.$

Answer: $\dfrac{-i\pi}{2\exp(1)}$

2. $\displaystyle\int_{|z-1|=2} \frac{z^2-1}{z^2-6z+8}\,dz.$

Answer: $-i3\pi$

3. $\displaystyle\int_{|z-1|=3} \frac{z^2+3z}{(z-1)^2(z-2)}\,dz.$

Answer: $i2\pi$

4. $\displaystyle\int_{|z|=2} \frac{z-1}{(z+1)^2(z-3)}\,dz.$

Answer: $\dfrac{-i\pi}{4}$

5. $\displaystyle\int_{|z+1|=5} \frac{z-1}{(z+2)^2(z+3)^2}\,dz.$

Answer: 0

6. $\displaystyle\int_{|z|=2} \frac{\exp(z)}{(z+1)^2(z-3)}\,dz.$

Answer: $\dfrac{-i5\pi}{8\exp(1)}$

7. $\displaystyle\int_{|z-1-i|=3} \frac{z+4}{(z-7)(z-2)}\,dz.$

Answer: $\dfrac{-i12\pi}{5}$

8. $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^2}\,dz.$

Answer: $i2\pi$

9. $\displaystyle\int_{|z|=4} \frac{z^3}{z^2-2z+2}\,dz.$

Answer: $i4\pi$

10. $\displaystyle\int_{|z|=3} \frac{\exp(z)}{z(z+1+i)^2}\,dz.$

Answer: $\pi i\left(e^{-(1+i)}(-1+2i)-i\right)$

6.3 ARGUMENT PRINCIPLE AND ROUCHE'S THEOREM

THEOREM 6.3.1 (Argument principle)
Let f be a meromorphic function on a region Ω with zeroes $z_j,\ 1 \le j \le n$ and poles $w_k,\ 1 \le k \le m$ including multiplicities. Then, for every closed curve $\gamma \sim 0$ in Ω and not passing through any z_j or w_k,

$$\frac{1}{i2\pi}\int_{\gamma}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{j=1}^{n} \mathrm{WN}(\gamma, z_j) - \sum_{k=1}^{m} \mathrm{WN}(\gamma, w_k).$$

Proof: We know that f' is also analytic wherever f is analytic. Hence, $\dfrac{f'}{f}$ is analytic on Ω except at z_j's and w_k's. By Cauchy's residue theorem, we have

$$\begin{aligned}\frac{1}{i2\pi}\int_{\gamma}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \sum_{z}\mathrm{WN}(\gamma,z)\left(\mathrm{Res}\,\frac{f'}{f}\right)(z)\\ &\quad+\sum_{w}\mathrm{WN}(\gamma,w)\left(\mathrm{Res}\,\frac{f'}{f}\right)(w), \qquad (6.1)\end{aligned}$$

where z varies through the distinct zeroes of f and w varies through the distinct poles of f.

Claim. If f has a zero of order p_z at z, then $\left(\mathrm{Res}\,\dfrac{f'}{f}\right)(z) = p_z$.
Since z is a zero for f of order p_z, by Result 5.2.3, there exists an analytic function g on a neighbourhood $B(z,r)$ of z such that

$$f(\zeta) = (\zeta - z)^{p_z} g(\zeta) \text{ and } g(\zeta) \neq 0,\ \forall \zeta \in B(z,r).$$

Hence, by applying Theorem 4.5.12, we get that $\log(g)$ is analytic in $B(z,r)$, and hence,

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{p_z(\zeta - z)^{p_z-1}g(\zeta) + (\zeta - z)^{p_z}g'(\zeta)}{(\zeta - z)^{p_z}g(\zeta)}$$

$$\begin{aligned} &= \frac{p_z}{g(\zeta)} + \frac{g'(\zeta)}{g(\zeta)} \\ &= \frac{p_z}{g(\zeta)} + \frac{d}{d\zeta}\left[\log(g(\zeta))\right]. \end{aligned}$$

This implies that $\dfrac{f'}{f} - \dfrac{p_z}{\zeta - z}$ is the derivative of an analytic function on $B(z,r)$, and hence, $\left(\text{Res}\,\dfrac{f'}{f}\right)(z) = p_z$.

Claim. If f has a pole of order q_w at w, then $\left(\text{Res}\,\dfrac{f'}{f}\right)(w) = q_w$.

Since w is a pole for f of order q_w, by Result 5.2.11, there exists an analytic function g on a neighbourhood $B(w,r)$ of w such that

$$f(\zeta) = (\zeta - w)^{-q_w} g(\zeta) \text{ and } h(\zeta) \neq 0,\ \forall \zeta \in B(w,r) \setminus \{w\}.$$

Hence, $\log(h)$ is analytic in $B(z,r)$ (by Theorem 4.5.12) and hence,

$$\begin{aligned} \frac{f'(\zeta)}{f(\zeta)} &= \frac{(-q_w)(\zeta - w)^{-q_w - 1} g(\zeta) + (\zeta - w)^{-q_w} h'(\zeta)}{(\zeta - w)^{-q_w} h(\zeta)} \\ &= \frac{-q_w}{h(\zeta)} + \frac{h'(\zeta)}{h(\zeta)} \\ &= \frac{-q_w}{h(\zeta)} + \frac{d}{d\zeta}\left[(\log(h(\zeta)))\right]. \end{aligned}$$

This implies that $\left(\text{Res}\,\dfrac{f'}{f}\right)(w) = -q_w$. Using these observations in equation (6.1), we get

$$\begin{aligned} \frac{1}{i2\pi}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \sum_z \text{WN}(\gamma, z)\left(\text{Res}\,\frac{f'}{f}\right)(z) + \sum_w \text{WN}(\gamma, w)\left(\text{Res}\,\frac{f'}{f}\right)(w) \\ &\qquad \text{(where } z \text{ varies through distinct zeroes of } f \text{ and } w \\ &\qquad \text{varies through distinct poles of } f.) \\ &= \sum_z \text{WN}(\gamma, z) p_z + \sum_w \text{WN}(\gamma, w) q_w \\ &= \sum_{j=1}^{n} \text{WN}(\gamma, z_j) - \sum_{k=1}^{m} \text{WN}(\gamma, w_k). \end{aligned}$$

Thus, the theorem follows. □

THEOREM 6.3.2 Let f be a meromorphic function on a simply connected region Ω. If every zero of f is of even order and if every pole of f is of even order, then there exists a meromorphic function g on Ω such that $f = g^2$.

Proof: Let $z_1, z_2, \ldots, z_n$ be the distinct zeroes of f such that the order of the zero of f at z_j is $2p_j$ for some $p_j \in \mathbb{N}$, $\forall j = 1, 2, \ldots, , n$, and let $w_1, w_2, \ldots, w_m$ be the distinct poles of f such that the order of the pole of f at w_k is $2q_k$ for some $q_k \in \mathbb{N}$, $\forall k = 1, 2, \ldots, m$. By argument principle, for every closed curve γ in $\Omega^* := \Omega \setminus \{z_j, w_k : 1 \le j \le n;\ 1 \le k \le m\}$, we have

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{j=1}^{n} 2p_j \mathrm{WN}(\gamma, z_j) - \sum_{k=1}^{m} 2q_j \mathrm{WN}(\gamma, w_k).$$

Therefore, $\frac{1}{2}\int\limits_{\gamma} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$ is an integral multiple of $2\pi i$. For an arbitrarily fixed $z_0 \in \Omega^*$, we define

$$g(z) = \exp\left(\sqrt{f(z_0)} + \frac{1}{2}\int_{\sigma_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right),\ \forall z \in \Omega^*, \tag{6.2}$$

where σ_z is a curve in Ω^* joining z_0 and z, and $\sqrt{f(z_0)}$ is any one value of square root of $f(z_0) \neq 0$. The definition of g is well defined. Indeed, if z_0 and z are connected by two curves σ_z and τ_z, respectively, in Ω^*, then $\sigma_z \cup (-\tau_z)$ is a closed curve in Ω^*. Therefore, $\frac{1}{2}\int\limits_{\sigma_z \cup (-\tau_z)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$ is an integral multiple of $2\pi i$, and hence, $\exp\left(\frac{1}{2}\int\limits_{\sigma_z \cup (-\tau_z)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right) = 1$, which implies that

$$\exp\left(\frac{1}{2}\int_{\sigma_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right) = \exp\left(\frac{1}{2}\int_{\tau_z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right).$$

Thus, the definition of g is independent of the curve σ_z and hence well defined. Next, we claim that g is analytic on Ω^* and $g' = \frac{gf'}{2f}$ on Ω^*. Let $z \in \Omega^*$ be arbitrary. If $r > 0$ is such that $B(z, r) \subset \Omega^*$ and if $w \in B(z, \frac{r}{3})$, then $z \in B(w, \frac{2r}{3}) \subset B(z, r) \subset \Omega^*$. Hence, we can write

$$g(z) \quad = \quad \exp\left(\sqrt{f(z_0)} + \frac{1}{2}\int_{\sigma_w} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + \frac{1}{2}\int_{L_{w,z}} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\right)$$

$$\begin{aligned} &\text{(where } L_{w,z} \text{ is the line segment joining } w \text{ and } z\text{)} \\ = \quad & g(w)\exp\left(\frac{1}{2}\int\limits_{L_{w,z}}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta\right). \end{aligned}$$

Since $\frac{f'(z)}{f(z)}$ is analytic on $B(w, \frac{2r}{3})$, from the proof of Cauchy's theorem for disc, $\frac{f'(z)}{f(z)}$ is the derivative of the analytic function $\int\limits_{L_{w,z}}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta$ in $B(w, \frac{2r}{3})$. Therefore, g is analytic on Ω^* and

$$g'(z) = g(w)\exp\left(\frac{1}{2}\int\limits_{L_{w,z}}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta\right)\frac{1}{2}\frac{f'(z)}{f(z)} = \frac{g(z)f'(z)}{2f(z)},\ \forall z \in \Omega^*.$$

Since $\frac{(g(z_0))^2}{f(z_0)} = 1$ and

$$\begin{aligned} \frac{d}{dz}\left[\frac{(g(z))^2}{f(z)}\right] &= \frac{f(z)2g(z)g'(z) - (g(z))^2 f'(z)}{(f(z))^2} \\ &= \frac{f(z)2g(z)\frac{g(z)f'(z)}{2f(z)} - (g(z))^2 f'(z)}{(f(z))^2} \\ &= 0, \end{aligned}$$

$\frac{g^2}{f}$ is a constant on Ω^* and the constant value is 1. Thus, $f = g^2$ on Ω^*. It is also easy to see that $\lim\limits_{z\to z_j} g^2(z) = \lim\limits_{z\to z_j} f(z) = f(z_j) = 0$ and $\lim\limits_{z\to w_k} g^2(z) = \lim\limits_{z\to w_k} f(z) = \infty$, where g is the required meromorphic function on Ω such that $g^2 = f$. □

Definition 6.3.3 If f is a meromorphic function on a simply connected region Ω, then we define the 'square root' of f, which is defined by the meromorphic function g on Ω such that $g^2 = f$. We denote this g by $\sqrt{f}$.

RESULT 6.3.4 Let f be a meromorphic function on a region Ω such that $-\Omega = \Omega$. If f^2 is an even function and $\lim\limits_{z\to 0} f(z) = c$ for some $c \in \mathbb{C} \setminus \{0\}$, then f must be an even function.

Proof: First, we note that f^2 is analytic at 0 (since $\lim_{z\to 0} f^2(z) = c^2$), and hence, we can write

$$f^2(z) = \sum_{n=0}^{\infty} a_n z^n, \ \forall z \in B(0,r) \quad \text{for some } r > 0. \tag{6.3}$$

Since f is even, we have

$$0 = f^2(z) - f^2(-z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-z)^n = 2\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1},$$

which implies that $a_{2n+1} = 0, \ \forall n \in \{0, 1, 2, 3, \ldots\}$, by uniqueness theorem for power series. Therefore, we have $f^2(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}, \ \forall z \in B(0,r)$. Then, we claim that f must be even. As f is analytic at 0, then $f(z) = f^2(z) = \sum_{n=0}^{\infty} b_n z^n, \ \forall z \in B(0,r)$ with $b_0 = c \neq 0$. Since $\left(\sum_{n=0}^{\infty} b_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = f^2(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$, equating the coefficient of z on both sides, we get $2b_0 b_1 = 0 \Rightarrow b_1 = 0$. Next by equating the coefficients of z^3 on both sides, we get $2(b_0 b_3 + b_0 b_1) = 0 \Rightarrow 2b_0 b_3 = 0 \Rightarrow b_3 = 0$. Furthermore, we can prove that $b_{2n+1} = 0, \forall n \in \{0, 1, 2, \ldots\}$. Thus, we have $f(z) = \sum_{n=0}^{\infty} b_{2n} z^{2n}, \ \forall z \in B(0,r)$. Therefore, f is even in $B(0,r)$. Then, by analytic continuation, f is an even function on Ω. □

THEOREM 6.3.5 (Rouche's theorem)
Let f and g be analytic functions on a region Ω and γ be a simple closed curve in Ω such that $\gamma \sim 0$ in Ω. If

$$|f(z) - g(z)| < |f(z)|, \ \forall z \in \gamma, \tag{6.4}$$

then f and g have same number of zeroes inside γ including multiplicities.

Proof: Let $a_j, \ 1 \leq j \leq n$ be the zeores of f and $b_k, \ 1 \leq k \leq m$ be the zeores of g including multiplicities.

Case 1: Assume that f and g have no common zeroes.
Let $F = \dfrac{g}{f}$, then F is a meromorphic function on Ω such that F has zeores at the zeores of g, and it has poles at the zeroes of f. We claim

that f and g do not vanish on γ. If $f(z) = 0$ for some $z \in \gamma$, then from equation (6.4), we get $|g(z)| < 0$. If $g(z) = 0$ for some $z \in \gamma$, then again from equation (6.4), we get $|f(z)| < |f(z)|$. As both are not possible, our claim holds. Now, we can apply argument principle to F and we get

$$\frac{1}{i2\pi}\int_{\gamma}\frac{F'(z)}{F(z)}\,dz = \sum_{j=1}^{n} \text{WN}(\gamma, a_j) - \sum_{k=1}^{m} \text{WN}(\gamma, b_k). \tag{6.5}$$

As γ is a simple closed curve, $\text{WN}(\gamma, z) = 1$ or 0, whenever γ encloses z or γ does not enclose z. Hence, the right-hand side of equation (6.5) is simply the number of zeroes of F enclosed by γ including multiplicities $-$ the number of poles of F enclosed by γ including multiplicities which is equal to the number of zeroes of f enclosed by γ including multiplicities $-$ the number of zeroes of g enclosed by γ including multiplicities.

On the other hand, if $F(\gamma) = \Gamma$, then using equation (6.4), we have $|f(z) - 1| < 1, \forall z \in \gamma$. Thus, $\Gamma \subset B(1, 1)$. By Cauchy's theorem for simply connected region, since $\Phi(w) = \dfrac{1}{w}$, $\forall w \in B(1, 1)$ is analytic and Γ is a closed curve in $B(1, 1)$, we get

$$0 = \frac{1}{i2\pi}\int_{\Gamma}\frac{dw}{w} = \int_{\gamma}\frac{F'(z)}{F(z)}\,dz,$$

by applying the change of variable $w = F(z)$. Thus, we get that left-hand side of equation (6.5) is zero. Hence, the theorem follows.

Case 2: Let f and g have the common zeroes $\alpha_1, \alpha_2, \ldots, \alpha_s$. Then, by Result 5.2.3, we can write

$$f(z) = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_s)\phi(z),$$

and

$$g(z) = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_s)\psi(z),$$

where ϕ and ψ are analytic functions on Ω with the following properties:

1. the number of zeroes of ϕ including multiplicities $= n - s$.
2. the number of zeroes of ψ including multiplicities $= m - s$.

3. ϕ and ψ have no common zeroes.
4. $\dfrac{g(z)}{f(z)} = \dfrac{\psi(z)}{\phi(z)}$, $\forall z \in \Omega \setminus \{a_j : j = 1, 2, \ldots, n\}$.

Then, we have

$$\begin{aligned} n - m &= (n - s) - (m - s) \\ &= \int_{\gamma} \frac{\left(\frac{\psi}{\phi}\right)'(z)}{\left(\frac{\psi}{\phi}\right)(z)}\, dz \\ &= 0 \qquad \text{(by applying Case(1)).} \end{aligned}$$

Thus, the theorem follows. □

Example 6.3.6 Find the number of zeroes of $z^7 - 4z^3 + z - 1$ enclosed by $|z| = 1$.
Let $f(z) = -4z^3$ and $g(z) = z^7 - 4z^3 + z - 1$, $\forall z \in \mathbb{C}$. On $|z| = 1$, then we have

$$|f(z) - g(z)| = |z^7 + z - 1| \leq |z|^7 + |z| + 1 \leq 2 < 4 = |-4z^3| = |f(z)|,$$

whence we can apply Rouche's theorem and we get that f and g have same number of zeroes enclosed by $|z| = 1$. As three zeroes of f are enclosed by $|z| = 1$, including multiplicites, g also has three zeroes enclosed by $|z| = 1$.

Example 6.3.7 Prove that one zero of $z^5 + 15z + 1$ belongs to $B\left(0, \dfrac{3}{2}\right)$ and four zeroes of $z^5 + 15z + 1$ belong to $\mathrm{AN}^2_{3/2}(0)$.

Solution: Let $f(z) = z^5$ and $g(z) = z^5 + 15z + 1$, $\forall z \in \mathbb{C}$. Then we note that

$$|f(z) - g(z)| = |15z + 1| \leq 15|z| + 1 = 31 < 32 = 2^5 = |z^5| = |f(z)|,$$

on $|z| = 2$. Hence, we can apply the Rouche's theorem (Theorem 6.5) and we get that $f(z)$ and $g(z)$ have same number of zeroes in $B(0, 2)$ including multiplicities. We know that f has five zeroes (which are obviously $0, 0, 0, 0$, and 0) in $B(0, 2)$ including multiplicities, whence g has five zeroes in $B(0, 2)$. As g has exactly five zeroes, every zero of g belongs to $B(0, 2)$. On the other hand, if we put $f_1(z) = 15z$, then on the circle $|z| = \dfrac{3}{2}$, we have

$$|f_1(z) - g(z)| = |z^5 + 1| \leq |z|^5 + 1 = \frac{243}{32} + 1 < \frac{45}{2} = |15z| = |f_1(z)|,$$

and hence, again by using Rouche's theorem, we conclude that f_1 and g have same number of zeroes in $B\left(0, \frac{3}{2}\right)$. Since f_1 has exactly one zero in $B\left(0, \frac{3}{2}\right)$, g also has exactly one zero in $B\left(0, \frac{3}{2}\right)$. From the proof of Rouche's theorem, we have $g \neq 0$ on $|z| = \frac{3}{2}$. Hence, the remaining four zeroes of g must lie in $\mathrm{AN}^2_{3/2}(0)$.

Example 6.3.8 Prove that $z^6 + 5z^4 + 3$ has exactly 2 zeroes in $\mathrm{AN}^3_1(0)$.

Solution: Let $f(z) = z^6$ and $g(z) = z^6 + 5z^4 + 3$, $\forall z \in \mathbb{C}$. As for $|z| = 3$,

$$|f(z) - g(z)| = |5z^4 + 3| \leq 5|z|^4 + 3 = 408 < 729 = 3^6 = |z^6| = |f(z)|,$$

we can apply Rouche's theorem, and we conclude that f and g have same number of zeroes enclosed by $|z| = 3$, including multiplicities. As f has six zeroes inside $B(0, 3)$, g also has six zeroes inside $B(0, 3)$. If $f_1(z) = 5z^4$, $\forall z \in \mathbb{C}$, then

$$|f_1(z) - g(z)| = |z^6 + 3| \leq |z|^6 + 3 = 4 < 5 = 5|z^4| = |f_1(z)| \text{ on } |z| = 1.$$

Therefore, again using Rouche's theorem, we get that the number of zeroes of f_1 enclosed by $|z| = 1$ (including multiplicities) is same as that of g. Since f_1 has four zeroes enclosed by $|z| = 1$, g has two zeroes enclosed by $|z| = 1$. Thus, g has two zeroes inside the annulus $\mathrm{AN}^3_1(0)$.

Exercise 6.3.9

1. Prove that $z^3 - 9z + 27$ has all zeroes in $B(0, 4)$.

2. Prove that one zero of $z^4 - 7z - 1$ lies in $B(0, 1)$ and three zeroes are in $\mathrm{AN}^2_1(0)$.

3. Find the number of zeroes of $z^8 - 3z^5 + 7z^3 - 2$ in the annulus $\mathrm{AN}^2_1(0)$.
   Answer: 5.

4. Prove that all zeroes of $z^7 - 5z^3 + 12$ lie in the annulus $\mathrm{AN}^2_1(0)$.

5. Prove that exactly 3 zeroes of $z^5 - 10z^2 - 3z + 1$ lie in $\mathrm{AN}^3_2(0)$.

6.4 EVALUATION OF REAL INTEGRALS

Now, we present the formulae to evaluate certain definite real integrals using Cauchy's residue theorem.

THEOREM 6.4.1 (Type 1): If S is a quotient of polynomials in two variables, then $\int_0^{2\pi} S(\cos(\theta), \sin(\theta))\, d\theta = i2\pi \times$ sum of the residues of G at the poles enclosed by $|z| = 1$, where $G(z) = -\frac{i}{z} S\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{i2}\left(z - \frac{1}{z}\right)\right)$.

Proof: If $z = \exp(i\theta)$, then

$$\begin{aligned} \cos(\theta) &= \operatorname{Re} z = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{1}{z}\right), \\ \sin(\theta) &= \operatorname{Im} z = \frac{1}{i2}(z - \bar{z}) = \frac{1}{i2}\left(z - \frac{1}{z}\right), \\ d\theta &= -\frac{i}{z}\, dz \quad (\text{as } dz = i\exp(i\theta)\, d\theta = iz\, d\theta), \end{aligned}$$

and when θ varies along the interval $[0, 2\pi]$, z varies along the circle $|z| = 1$. Therefore, by Cauchy's residue theorem,

$$\begin{aligned} \int_0^{2\pi} S(\cos(\theta), \sin(\theta))\, d\theta &= \int_{|z|=1} S\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{i2}\left(z - \frac{1}{z}\right)\right)\left(-\frac{i}{z}\right) dz \\ &= \int_{|z|=1} G(z)\, dz \\ &= i2\pi \times \text{ sum of the residues of } G \\ &\quad \text{at the poles enclosed by } |z| = 1. \end{aligned}$$

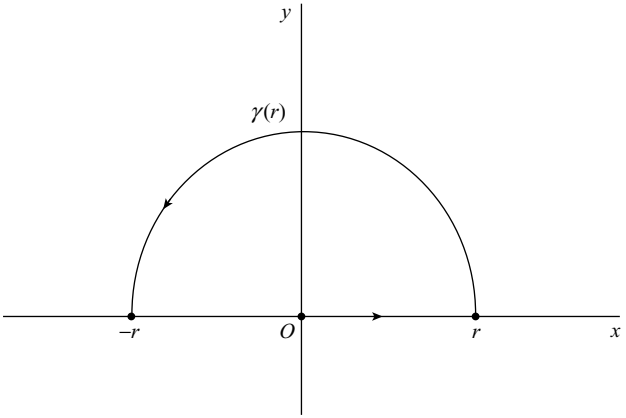
Hence, the theorem follows. □

THEOREM 6.4.2 (Type 2): If R is a rational function with zero at ∞ of order at least 2 and R has no poles on real axis, then $\int_{-\infty}^{\infty} R(x)\, dx = i2\pi \times$ sum of the residues of R at the poles in the upper half-plane.

Proof: First, we construct a closed semicircle $C(r) = \gamma(r) \cup L_{(-r,0),(r,0)}$ with centre 0 and radius r such that $C(r)$ encloses all poles of R in the upper half-plane.

Then, by Cauchy's residue theorem, we get

$$\begin{aligned}\int_{C(r)} R(z)\,dz &= i2\pi \times \text{ sum of the residues of } R \\ &\quad \text{at the poles enclosed by } C(r). \\ &= i2\pi \times \text{ sum of the residues of } R \\ &\quad \text{at the poles in the upper half-plane.}\end{aligned}$$



From the diagram, we have

$$\int_{C(r)} R(z)\,dz = \int_{\gamma(r)} R(z)\,dz + \int_{-r}^{r} R(x)\,dx. \tag{6.6}$$

We claim that $\int_{\gamma(r)} R(z)\,dz \to 0$ as $r \to \infty$.

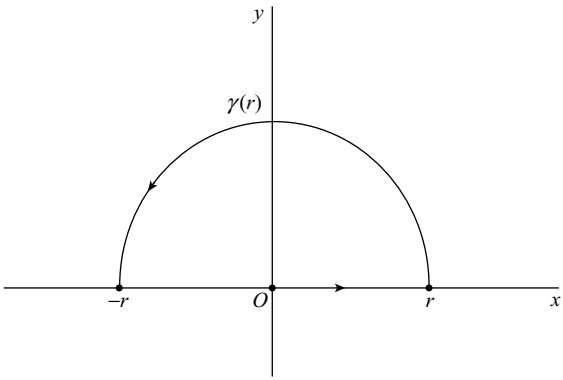
Since R has a zero at ∞ of order at least 2, $\lim_{z\to\infty} z^2 R(z)$ is finite and hence there exist $M > 0$ and $s > 0$ such that $|z^2 R(z)| \leq M$, $\forall |z| > s$. Therefore, using Theorem 4.1.27, for $r > s$,

$$\left|\int_{\gamma(r)} R(z)\,dz\right| \leq \int_{\gamma(r)} |R(z)|\,|dz| \leq \frac{M}{r^2}\int_{\gamma(r)} |dz| = \frac{M}{r^2}\pi r = \frac{M\pi}{r} \to 0 \text{ as } r \to \infty.$$

Hence, allowing $r \to \infty$ in equation (6.6), we get $\lim_{r\to\infty} \int_{C(r)} R(z)\,dz = \int_{-\infty}^{\infty} R(x)\,dx$. Thus, $\int_{-\infty}^{\infty} R(x)\,dx = \lim_{r\to\infty} \int_{C(r)} R(z)\,dz = i2\pi \times$ sum of the residues of R at the poles in the upper half-plane. □

THEOREM 6.4.3 (Type 3(a)): If R is a rational function with zero at ∞ of order at least 2 and R has no poles on real axis, then $\int_{-\infty}^{\infty} R(x)\exp(ix)\,dx = i2\pi \times$ sum of the residues of H at the poles in the upper half-plane, where $H(z) = R(z)\exp(iz), \forall z \in \mathbb{C}$.

Proof: First, we construct a closed semicircle $C(r) = \gamma(r) \cup L_{(-r,0),(r,0)}$ with centre 0 and radius r such that $C(r)$ encloses all poles of R in the upper half-plane.



Then by Cauchy's residue theorem, we get

$$\begin{aligned}\int_{C(r)} R(z)\exp(iz)\,dz &= i2\pi \times \text{ sum of the residues of } H \\ &\quad \text{at the poles enclosed by } C(r). \\ &= i2\pi \times \text{ sum of the residues of } H \\ &\quad \text{at the poles in the upper half-plane.}\end{aligned}$$

Now, we have

$$\int_{C(r)} R(z)\exp(iz)\,dz = \int_{\gamma(r)} R(z)\exp(iz)\,dz + \int_{-r}^{r} R(x)\exp(ix)\,dx. \tag{6.7}$$

We claim that $\int\limits_{\gamma(r)} R(z)\exp(iz)\,dz \to 0$ as $r \to \infty$.

Since R has a zero at ∞ of order at least 2, there exist $M > 0$ and $s > 0$ such that $|z^2 R(z)| \leq M$, $\forall |z| > s$. Using the fact that

$$\text{Im } z > 0 \text{ on } \gamma(r) \Rightarrow \exp(-\text{Im } z) < 1 \text{ on } \gamma(r),$$

for $r > s$, we get

$$\begin{aligned}\left|\int\limits_{\gamma(r)} R(z)\exp(iz)\,dz\right| &\leq \int\limits_{\gamma(r)} |R(z)|\exp(-\text{Im } z)\,|dz| \leq \frac{M}{r^2}\int\limits_{\gamma(r)} |dz| \\ &\quad \text{(using Theorem 4.1.27)} \\ &= \frac{M}{r^2}\pi r = \frac{M\pi}{r} \to 0 \text{ as } r \to \infty.\end{aligned}$$

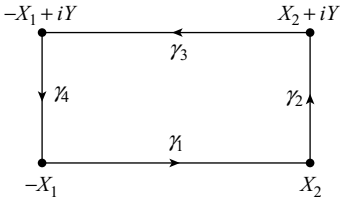
Hence, allowing $r \to \infty$ in equation (6.6), we get

$$\lim_{r\to\infty} \int\limits_{C(r)} R(z)\,dz = \int\limits_{-\infty}^{\infty} R(x)\exp(ix)\,dx.$$

Thus, $\int\limits_{-\infty}^{\infty} R(x)\exp(ix)\,dx = \lim\limits_{r\to\infty} \int\limits_{C(r)} R(z)\exp(iz)\,dz = i2\pi \times$ sum of the residues of H at the poles in the upper half-plane. □

THEOREM 6.4.4 (Type 3(b)): If R is a rational function with zero at ∞ of order 1 and R has no poles on real axis, then $\int\limits_{-\infty}^{\infty} R(x)\exp(ix)\,dx = i2\pi \times$ sum of the residues of H at the poles in the upper half-plane, where $H(z) = R(z)\exp(iz)$, $\forall z \in \mathbb{C}$.

Proof: First, we construct a rectangle $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ with vertices X_2, $X_2 + iY$, $X_2 - iY$, and $-X_2$, where X_1, X_2, and Y are positive real numbers such that the rectangle encloses all poles of R in the upper half-plane.



Then by Cauchy's residue theorem, we get

$$\begin{aligned}\int_{\bigcup_{j=1}^{4}\gamma_j} R(z)\exp(iz)\,dz &= i2\pi \times \text{ sum of the residues of } H \\ &\quad \text{at the poles enclosed by } \bigcup_{j=1}^{4}\gamma_j. \\ &= i2\pi \times \text{ sum of the residues of } H \text{ at the} \\ &\quad \text{poles in the upper half equation plane.} \quad (6.8)\end{aligned}$$

We claim the following:

1. $\int_{\gamma_2} R(z)\exp(iz)\,dz \to 0$ as $X_2 \to \infty$, $\forall Y > 0$, uniformly,
2. $\int_{\gamma_4} R(z)\exp(iz)\,dz \to 0$ as $X_1 \to \infty$, $\forall Y > 0$, uniformly,
3. $\int_{\gamma_3} R(z)\exp(iz)\,dz \to 0$ as $Y \to \infty$, for each fixed $X_1 > 0$ and fixed $X_2 > 0$.

Since R has a simple zero at ∞, there exist $M > 0$ and $K > 0$ such that $|zR(z)| \le M$, $\forall |z| > K$. Therefore, for $X_2 > K$ we get

$$\begin{aligned}\left|\int_{\gamma_2} R(z)\exp(iz)\,dz\right| &= \left|\int_0^Y R(z)\exp(iz)\,dy\right| \quad \text{where } z = X_2 + iy \\ &\le \int_0^Y |R(z)|\exp(-y)\,dy \\ &\le M\int_0^Y \exp(-y)\frac{1}{|z|}\,dy \\ &\le \frac{M}{X_2}\int_0^Y \exp(-y)\,dy \quad (\text{since } |z| \ge X_2) \\ &= \frac{M}{X_2}(-\exp(-Y)+1) \\ &< \frac{M}{X_2} \to 0 \text{ as } X_2 \to \infty.\end{aligned}$$

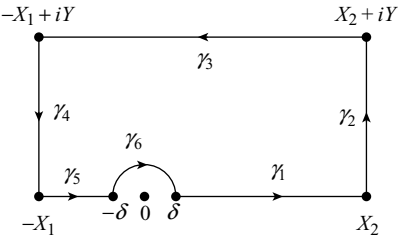
The second claim can also be proved by a similar argument.
For $Y > K$,

$$\begin{aligned}
\left| \int_{\gamma_3} R(z)\exp(iz)\,dz \right| &= \left| \int_{X_2}^{-X_1} R(z)\exp(iz)\,dx \right| \quad \text{where } z = x + iY \\
&\leq \int_{-X_1}^{X_2} |R(z)|\exp(-Y)\,dx \\
&\leq M\exp(-Y)\int_{-X_1}^{X_2} \frac{1}{|z|}\,dx \\
&\leq \frac{M}{Y}\exp(-Y)\int_{-X_1}^{X_2} dx \quad (\text{as } |z| \geq Y) \\
&= \frac{M}{Y}(X_2 + X_1) \to 0 \text{ as } Y \to \infty \\
&\quad \text{for each fixed } X_1 > 0 \text{ and } X_2 > 0.
\end{aligned}$$

Hence, applying $\lim_{X_1\to\infty}$, $\lim_{X_2\to\infty}$, and $\lim_{Y\to\infty}$ in equation (6.8) and using claims (1), (2), and (3), the theorem follows. □

THEOREM 6.4.5 (Type 3(c)): If R is a rational function with simple zero at ∞, a simple pole at 0, and no other real poles, then P.V. $\int_{-\infty}^{\infty} R(x)\exp(ix)\,dx = i2\pi \times$ sum of the residues of H at the poles in the upper half-plane $+\frac{1}{2}(\text{Res } H)(0)$, where $H(z) = R(z)\exp(iz)$, $\forall z \in \mathbb{C}$, where P.V. $\int_{-\infty}^{\infty} R(x)\exp(ix)\,dx$ is called the principal value of the integral and is defined by $\lim_{\delta\to 0}\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) R(x)\exp(ix)\,dx$.

Proof: First, we construct a closed curve $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5 \cup \gamma_6$ as in the following diagram such that the closed curve encloses all poles of R in the upper half-plane, where γ_6 is the upper semicircle with centre 0 and radius δ, for some $\delta \geq 0$.



Then by Cauchy's residue theorem, we get

$$\begin{aligned}\int\limits_{\bigcup_{j=1}^{6}\gamma_j} R(z)\exp(iz)\,dz &= i2\pi \times \text{ sum of the residues of } H \\ &\quad \text{at the poles enclosed by } \bigcup_{j=1}^{6}\gamma_j. \\ &= i2\pi \times \text{ sum of the residues of } H \\ &\quad \text{at the poles in the upper half-plane.} \qquad (6.9)\end{aligned}$$

We claim the following:

1. $\int\limits_{\gamma_2} R(z)\exp(iz)\,dz \to 0$ as $X_2 \to \infty$, $\forall Y > 0$, uniformly,
2. $\int\limits_{\gamma_4} R(z)\exp(iz)\,dz \to 0$ as $X_1 \to \infty$, $\forall Y > 0$, uniformly,
3. $\int\limits_{\gamma_3} R(z)\exp(iz)\,dz \to 0$ as $Y \to \infty$, for each fixed $X_1 > 0$ and fixed $X_2 > 0$,
4. $\int\limits_{\gamma_6} R(z)\exp(iz)\,dz \to -i\pi(\text{Res } H)(0)$ as $\delta \to 0$.

As the first three claims are same as in the previous theorem, we prove the last claim only. Next, let $\beta = (\text{Res } H)(0)$. Then, there exists $\delta_0 > 0$ such that

$$R(z)\exp(iz) = \frac{\beta}{z} + S(z),\ 0 < |z| < \delta_0,$$

where S is analytic on $|z| < \delta_0$. Since $R(z)\exp(iz)$ has a simple pole at '0', the singular part of $R(z)\exp(iz)$ around 0 is $\dfrac{\beta}{z}$ and hence $R(z)\exp(iz) - \dfrac{\beta}{z} = S(z)$ has 0 as a removable singularity. Now,

$$\int\limits_{\gamma_6} R(z)\exp(iz)\,dz = \int\limits_{\gamma_6}\left(\frac{\beta}{z} + S(z)\right)dz$$

$$\begin{aligned} &= \int_{\gamma_6} \frac{\beta}{z}\, dz + \int_{\gamma_6} S(z)\, dz \\ &= \int_{\pi}^{0} \frac{\beta}{\delta \exp(it)} \delta i \exp(it)\, dt + \int_{\gamma_6} S(z)\, dz \\ &= -i\beta\pi + \int_{\gamma_6} S(z)\, dz. \end{aligned}$$

If $0 < \delta < \delta_0$, then $|S(z)| \leq M$ for all z with $|z| \leq \delta$, for some $M > 0$. Therefore, using Theorem 4.1.27, we get

$$\left| \int_{\gamma_6} S(z)\, dz \right| \leq \int_{\gamma_6} |S(z)|\, |dz| \leq M \int_{\gamma_6} |dz| = M\pi\delta \to 0 \text{ as } \delta \to 0.$$

Hence, applying $\lim_{\delta \to 0}$, $\lim_{X_1 \to \infty}$, $\lim_{X_2 \to \infty}$, and $\lim_{Y \to \infty}$ in equation (6.9) and using claims (1), (2), (3), and (4), we get $\lim_{\delta \to 0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) R(x) \exp(ix)\, dx - i\beta\pi =$ $i2\pi \times$ sum of the residues of H at poles in the upper half-plane. Since

$$\lim_{\delta \to 0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) R(x) \exp(ix)\, dx = \text{P.V.} \int_{-\infty}^{\infty} R(x) \exp(ix)\, dx,$$

the theorem follows. □

Example 6.4.6 Find $\int_0^{\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta$.

Solution: Let us first calculate $\int_{-\pi}^{\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta$.

Using the change of variable $z = \exp(i\theta)$, we get $\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$. Hence,

$$\int_{-\pi}^{\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta = \int_{|z|=1} \frac{1}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= -i2 \int_{|z|=1} \frac{1}{3z^2 + 10z + 3}\, dz.$$

The simple poles of $G(z) = \dfrac{1}{3z^2 + 10z + 3}$ are -3 and $-\dfrac{1}{3}$. Since $-\dfrac{1}{3}$ is enclosed by $|z| = 1$, by Cauchy's residue theorem, we get

$$\begin{aligned}
-i2 \int_{|z|=1} \frac{1}{3z^2 + 10z + 3}\, dz &= -\frac{i2}{3} \int_{|z|=1} \frac{1}{(z+3)\left(z + \frac{1}{3}\right)}\, dz \\
&= -i2i2\pi (\text{Res } G)\left(-\frac{1}{3}\right) \\
&= 4\pi \lim_{z \to -\frac{1}{3}} \left(z + \frac{1}{3}\right) G(z) \\
&= \frac{4\pi}{3} \lim_{z \to -\frac{1}{3}} \frac{1}{z+3} \\
&= \frac{4\pi}{3} \times \frac{3}{8} = \frac{\pi}{2}.
\end{aligned}$$

Since $\dfrac{1}{5 + 3\cos(\theta)}$ is an even function,

$$\int_0^{\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{5 + 3\cos(\theta)}\, d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}.$$

Example 6.4.7 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta)}$.

Solution: Let $S(\cos(\theta), \sin(\theta)) = \dfrac{1}{2 + \cos(\theta)}$. This integral is of Type 1. Hence,

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta)} = i2\pi \times \text{ sum of the residues of } G$$

at the poles enclosed by $|z| = 1$,

where

$$\begin{aligned} G(z) &= -\frac{i}{z} S\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{i2}\left(z-\frac{1}{z}\right)\right) \\ &= -\frac{i}{z} \times \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \\ &= \frac{-i2}{z^2+4z+1}. \end{aligned}$$

The poles of G are $-2 \pm \sqrt{3}$, which are the roots of $z^2+4z+1=0$. We note that $-2+\sqrt{3}$ lies inside $|z|=1$ and the other root lies out side the circle. Hence,

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)} &= i2\pi \times (\text{Res } G)(-2+\sqrt{3}) \\ &= i2\pi \lim_{z \to -2+\sqrt{3}} (z+2-\sqrt{3}) G(z) \\ &= i2\pi \lim_{z \to -2+\sqrt{3}} \frac{-i2}{z+2+\sqrt{3}} \\ &= \frac{2\pi}{\sqrt{3}}. \end{aligned}$$

Example 6.4.8 Evaluate $\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\, dx$.

Solution: First, we find $\int_{-\infty}^{\infty} \frac{\exp(i3x)}{(x^2+1)^2}\, dx$. As this integral is of Type 3(a), we have

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\, dx = i2\pi (\text{Res } H)(i),$$

where $H(z) = \frac{\exp(i3z)}{(z^2+1)^2}$, $\forall z \in \mathbb{C}$. Now,

$$\begin{aligned} (\text{Res } H)(i) &= \lim_{z \to i} \frac{d}{dz}\left((z-i)^2 \frac{\exp(i3z)}{(z^2+1)^2}\right) \\ &= \lim_{z \to i} \frac{d}{dz}\left(\frac{\exp(i3z)}{(z+i)^2}\right) \end{aligned}$$

$$= \lim_{z\to i} \frac{d}{dz}\left(\frac{\exp(i3z)}{(z+i)^2}\right)$$
$$= \frac{\exp(-3)(-8)}{(-i8)} = -i\exp(-3),$$

as $\frac{d}{dz}\left(\frac{\exp(i3z)}{(z+i)^2}\right) = \frac{i3\exp(i3z)(z+i)^2 - 2(z+i)\exp(i3z)}{(z+i)^4} = \frac{\exp(i3z)(i3z-5)}{(z+i)^4}$.

Therefore, $\int_{-\infty}^{\infty} \frac{\exp(i3x)}{(x^2+1)^2}\,dx = i2\pi \times (-i)\exp(-3) = 2\pi\exp(-3)$. Equating the real parts on both sides, we get $\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = 2\pi\exp(-3)$.

Example 6.4.9 Evaluate $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$.

Solution: First, we evaluate $\int_{-\infty}^{\infty} \frac{\exp(ix)}{x}\,dx$. This integral comes under Type 3(c). Hence,

$$\int_{-\infty}^{\infty} \frac{\exp(ix)}{x}\,dx = i2\pi \times \text{ sum of the residues of } H \text{ at the poles in the upper half-plane } + \frac{1}{2}(\text{Res } H)(0),$$

where $H(z) = \frac{\exp(iz)}{z}$, $\forall z \in \mathbb{C}$. We note that H has only one simple pole at 0. Hence, $(\text{Res } H)(0) = \lim_{z\to 0} z\frac{\exp(iz)}{z} = \lim_{z\to 0}\exp(iz) = 1$. Therefore, $\int_{-\infty}^{\infty} \frac{\exp(ix)}{x}\,dx = i\pi \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \text{Im}\int_{-\infty}^{\infty} \frac{\exp(ix)}{x}\,dx = \pi$.

Exercise 6.4.10 Evaluate the following integrals:

1. $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta) + \sin(\theta)}$.

   Answer: π

2. $\int_0^{2\pi} \dfrac{\cos(2\theta)\,d\theta}{5+4\sin(\theta)}$.

   Answer: $-\frac{\pi}{6}$

3. $\int_0^{\pi} \dfrac{d\theta}{4+\cos^2(\theta)}$.

   Answer: $\dfrac{\pi}{\sqrt{5}}$

4. $\int_0^{2\pi} \dfrac{d\theta}{(2+\cos(\theta))^2}$.

   Answer: $\dfrac{4\pi}{3\sqrt{3}}$

5. $\int_0^{2\pi} \dfrac{\cos(3\theta)d\theta}{5-\cos(2\theta)}$.

   Answer: 0

6. $\int_{-\infty}^{\infty} \dfrac{dx}{x^2+2x+6}$.

   Answer: $\dfrac{\pi}{2\sqrt{5}}$

7. $\int_{-\infty}^{\infty} \dfrac{x^2}{(x^2+4)^3}\,dx$.

   Answer: $\dfrac{\pi}{32}$

8. $\int_{-\infty}^{\infty} \dfrac{dx}{x^4+1}$.

   Answer: $\dfrac{\pi}{\sqrt{2}}$

9. $\int_0^{\infty} \dfrac{x\sin(x)}{x^2+9}\,dx$.

   Answer: $\dfrac{\pi}{2}\exp(-3)$

10. $\int_{-\infty}^{\infty} \frac{x \exp(i2x)}{1+x^2}\, dx.$

Answer: $i\pi \exp(-2)$

11. $\int_{0}^{\infty} \frac{\cos(x)}{(1+x^2)^3}\, dx.$

Answer: $\frac{7\pi}{16\exp(1)}$

12. $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{(x^2+4)}\, dx.$

Answer: $\frac{\pi}{4}(1-\exp(-4))$

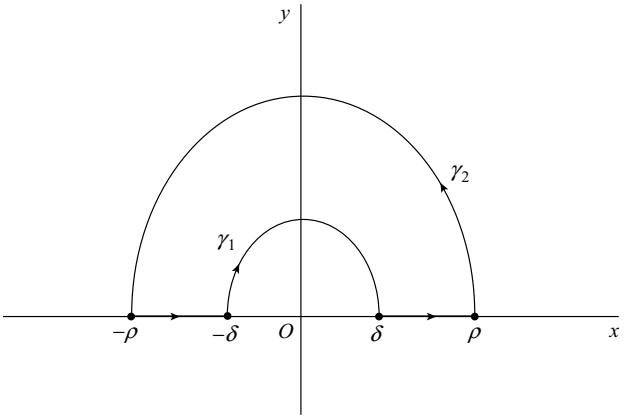
6.5 INTEGRALS OF MULTIVALUED FUNCTIONS

THEOREM 6.5.1 (Type (4)): If $0 < \alpha < 1$, R is a rational function with a zero at infinity of order at least 2 and has no real poles, then $\int_{-\infty}^{\infty} x^{\alpha} R(x)\, dx = \frac{i4\pi}{1-\exp(i2\pi\alpha)} \times$ sum of the residues of Φ at poles in the upper half-plane, where $\Phi(z) = z^{2\alpha+1} R(z^2)$, with a suitable branch in $\mathbb{C}$.

Proof: First, we evaluate $\int_{-\infty}^{\infty} x^{2\alpha+1} R(x^2)\, dx$. Using the branch of log defined on $\Omega_2 = \mathbb{C}\backslash\{(0,y) : y \leq 0\}$ by the unique value of log whose imaginary part lies in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, if $z^{2\alpha}$ is defined on Ω_2 by $\exp(2\alpha \log(z))$, then $z^{2\alpha}$ is analytic on Ω_2. Consider the closed curve

$$\gamma = L_{(-\rho,0),(-\delta,0)} \cup \gamma_1 \cup L_{(\delta,0),(\rho,0)} \cup \gamma_2$$

as shown in the following diagram.



We note that if $Q(z) = R(z^2),\ \forall z \in \mathbb{C}$, then Q is a rational function with zero of order at least 4 at ∞ and Q has no real poles. Then, Q is bounded on $B(0, s)$ for some $s > 0$. By Cauchy's residue theorem, if $0 < \delta < \rho$, then we have

$$\left(\int_{-\rho}^{-\delta} + \int_{\gamma_1} + \int_{\delta}^{\rho} + \int_{\gamma_2} z^{2\alpha+1} R(z)\, dz\right) = i2\pi \times \text{ sum of the residues of } \Phi \text{ at poles in the upper half-plane.} \quad (6.10)$$

where $\Phi(z) = z^{2\alpha+1} R(z^2),\ \forall z \in \mathbb{C}\backslash\{(0, y) : y \leq 0\}$. If we choose $\rho > 0$ and $\delta > 0$ such that γ encloses all poles of Φ in the upper half-plane. We claim the following:

1. $\int_{\gamma_1} z^{2\alpha+1} R(z^2)\, dz \to 0$ as $\delta \to 0$,
2. $\int_{\gamma_2} z^{2\alpha+1} R(z^2)\, dz \to 0$ as $\rho \to \infty$.

If $h(z) = z^{2\alpha+1} Q(z),\ \forall z \in \mathbb{C}\backslash\{(0, y) : y \leq 0\}$, then h is bounded on $A = \{r \exp(i\theta) : 0 < r < s,\ 0 < \theta < \pi\}$ because Q is bounded on $B(0, s)$ and $|z^{2\alpha+1}| \leq s^{2\alpha+1}$, for all $z \in A$. Therefore, there exists $M_1 > 0$ such that

$$|z^{2\alpha+1} R(z^2)| \leq M_1,\ \forall z \in A.$$

If $0 < \delta < s$, then

$$\left|\int_{\gamma_1} z^{2\alpha+1} R(z)\, dz\right| \leq \int_{\gamma_1} |z^{2\alpha+1} R(z^2)|\, |dz|$$

$$\begin{aligned} &\leq \delta M_1 \int_{\gamma_1} |dz| \\ &\leq M_1 \pi \delta^2 \to 0 \text{ as } \delta \to 0. \end{aligned}$$

As Q has zero at ∞ of order at least 4 at ∞, there exist $M_2 > 0$ and $K > 0$ such that $|z^4 R(z^2)| \leq M_2$ whenever $|z| \geq K$. Therefore, for $\rho > K$, we have

$$\begin{aligned} \left| \int_{\gamma_2} z^{2\alpha+1} R(z)\, dz \right| &\leq \int_{\gamma_1} |z^{2\alpha+1}||R(z^2)|\, |dz| \\ &\leq \rho^{2\alpha+1} \int_{\gamma_2} |z^4 R(z^2)| \frac{|dz|}{|z^4|} \\ &\leq M_2 \frac{\rho^{2\alpha+1}}{\rho^4} \int_{\gamma_2} |dz| \\ &= M_2 \frac{\rho^{2\alpha+1}}{\rho^4} \pi\rho \\ &= M_2 \pi \rho^{2\alpha-2} \to 0 \text{ as } \rho \to \infty, \end{aligned}$$

as $2\alpha - 2 < 0$.
Allowing $\rho \to \infty$ and $\delta \to 0$ on both sides of equation (6.10), we get $\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz = i2\pi \times$ sum of residues of Φ at poles in the upper half-plane. Now,

$$\begin{aligned} \int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz &= \int_{-\infty}^{0} z^{2\alpha+1} R(z^2)\, dz + \int_{0}^{\infty} z^{2\alpha+1} R(z^2)\, dz \\ &= \int_{\infty}^{0} (-t)^{2\alpha+1} R(t^2)\, (-dt) + \int_{0}^{\infty} z^{2\alpha+1} R(z^2)\, dz \\ &= \left(1 + (-1)^{2\alpha+1}\right) \int_{0}^{\infty} t^{2\alpha+1} R(t^2)\, dt. \end{aligned}$$

Since $(-1)^{2\alpha} = -\exp(i2\alpha\pi)$, according to this branch of $z^{2\alpha}$, we have

$$\begin{aligned}
&\int_0^\infty x^\alpha R(x)\,dx \\
&\quad = 2\int_0^\infty t^{2\alpha+1}R(t^2)\,dt \quad \text{(using the change of variable } x = t^2\text{)} \\
&\quad = 2 \times \frac{1}{1-\exp(i2\alpha\pi)} \int_{-\infty}^{\infty} z^{2\alpha+1}R(z^2)\,dz \\
&\quad = \frac{i4\pi}{1-\exp(i2\alpha\pi)} \text{ sum of residues of } \Phi \\
&\qquad \text{at poles in the upper half-plane.}
\end{aligned}$$

Thus, the theorem follows. □

THEOREM 6.5.2 (Type (5)): $\int_0^\pi \log(\sin(x))\,dx = -\pi\log(2)$.

Proof: First, we show that $\int_0^\pi \log(1-\exp(i2z))\,dz = 0$. We know that principal branch of log which is defined on $\Omega_1 = \mathbb{C}\backslash\{(x,0) : x \le 0\}$ by the unique value of log whose imaginary part lies in $(-\pi,\pi)$, so that log is analytic on Ω_1. If $z = x + iy$, then

$1 - \exp(i2z)$ is real and ≤ 0

$\Leftrightarrow$ Re $(1 - \exp(i2z)) \le 0$ and Im $(1 - \exp(i2z)) = 0$.

$\Leftrightarrow$ $1 - \exp(-2y)\cos(2x) \le 0$ and $\exp(-2y)\sin(2x) = 0$.

$\Leftrightarrow$ $\exp(-2y)\cos(2x) \ge 1$ and $\sin(2x) = 0$.

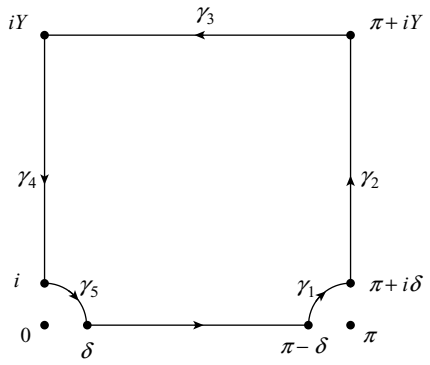
$\Leftrightarrow$ $y \le 0,\ 2x = 2k\pi,\ k \in \mathbb{Z}$ and $2x = n\pi,\ n \in \mathbb{Z}$.

$\Leftrightarrow$ $y \le 0,\ x = k\pi,\ k \in \mathbb{Z}$.

Therefore, $\log(\sin(z))$ is defined for all $z \in \mathbb{C}\backslash\{(k\pi, y) : k \in \mathbb{Z}, y \le 0\}$. If

$$\gamma = L_{(\delta,0),(\pi-\delta,0)} \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$$

is as shown in the following diagram,



then by Cauchy's theorem, we have

$$\left(\int_{\delta}^{\pi-\delta} + \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5}\right) \log(1-\exp(i2z))\,dz = 0. \tag{6.11}$$

We claim the following:

1. $\int_{\gamma_5} \log(1-\exp(i2z))\,dz \to 0$ as $\delta \to 0$,
2. $\int_{\gamma_1} \log(1-\exp(i2z))\,dz \to 0$ as $\delta \to 0$,
3. $\int_{\gamma_3} \log(1-\exp(i2z))\,dz \to 0$ as $Y \to \infty$,
4. $\int_{\gamma_2} \log(1-\exp(i2z))\,dz = -\int_{\gamma_4} \log(1-\exp(i2z))\,dz$.

1. We know that $\dfrac{1-\exp(i2z)}{z} = -\dfrac{\sum_{k=1}^{\infty} \dfrac{(i2z)^n}{n!}}{z} = -\sum_{k=1}^{\infty} \dfrac{(i2z)z^{n-1}}{n!} \to -i2$ as $z \to 0$. Since modulus function is continuous on $\mathbb{C}$ and log is continuous at 2, we get $\log\left|\dfrac{1-\exp(i2z)}{z}\right| \to \log 2$ as $z \to 0$. Therefore, there exists $M > 0$ such that

$$\log\left|\frac{1-\exp(i2z)}{z}\right| \leq M, \ \forall z \in B(0,r) \text{ for some } r > 0.$$

For $0 < \delta < r$, we have

$$\begin{aligned}
&\left| \int_{\gamma_5} \log\left(1 - \exp(i2z)\right)\, dz \right| \\
&\quad \leq \int_{\gamma_5} \left|\log\left(1 - \exp(i2z)\right)\right|\, |dz| \\
&\quad \leq \int_{\gamma_5} \left| \log\left(\frac{1 - \exp(i2z)}{z}\right) + \log(z) \right|\, |dz| \\
&\quad \leq \int_{\gamma_5} \left[\log\left(\left| \frac{1 - \exp(i2z)}{z} \right| \right) + |\log(z)| + \left|\arg\left(1 - \exp(i2z)\right)\right| \right] |dz| \\
&\quad \leq \int_{\gamma_5} \left(M + \log(\delta) + \pi\right) |dz| \\
&\quad \leq (M + \log(\delta) + \pi)\pi \frac{\delta}{2} \to 0 \text{ as } \delta \to 0.
\end{aligned}$$

Since $\lim_{\delta \to 0} \delta \log(\delta) = \lim_{\delta \to 0} \frac{\log(\delta)}{\frac{1}{\delta}} = \lim_{\delta \to 0} \frac{\frac{1}{\delta}}{\frac{-1}{\delta^2}} = 0$, by L'Hospital's rule.

2. Arguing as before, we can prove that $\int_{\gamma_1} \log(1 - \exp(i2z))\, dz \to 0$ as $\delta \to 0$.

3. We know that $|\exp(i2(x + iY))| = \exp(-Y) \to 0$ as $Y \to \infty$ uniformly on $x \in \mathbb{R}$. Therefore, $\log(1 - \exp(x + iY))\, dx \to 0$ as $Y \to \infty$. Therefore, using Result 4.1.5, we get

$$\int_{\gamma_3} \log(1 - \exp(i2z))\, dz = \int_{\pi}^{0} \log(1 - \exp(i2z))\, dz \to 0 \text{ as } Y \to \infty.$$

4. By a direct computation, we get

$$\begin{aligned}
\int_{\gamma_4} \log(1 - \exp(i2z))\, dz &= \int_{Y}^{\delta} \log(1 - \exp(-2y))\, dz \\
&= -\int_{\delta}^{Y} \log(1 - \exp(-2y))\, dz
\end{aligned}$$

$$\begin{aligned} &= -\int_{\delta}^{Y} \log(1-\exp(i2(\pi+iy)))\,dz \\ &= -\int_{\gamma_2} \log(1-\exp(i2z))\,dz. \end{aligned}$$

Allowing $Y \to \infty$ and $\delta \to 0$ on both sides of equation (6.11), we get

$$\int_0^{\pi} \log(1-\exp(i2z))\,dz = 0.$$

Since

$$1-\exp(i2z) = \exp(iz)\,(\exp(-iz)-\exp(iz)) = -i2\exp(iz)\sin(z),$$

we have

$$\begin{aligned} \int_0^{\pi} \log(\sin(x))\,dx &= \int_0^{\pi} \log(1-\exp(i2x))\,dx \\ &\quad -\int_0^{\pi} [\log(2)+\log(-i)+\log(\exp(ix))]\,dx \\ &= 0-\left[\pi\log(2)-i\frac{\pi}{2}\pi+i\frac{\pi^2}{2}\right] \\ &= -\pi\log(2). \end{aligned}$$

Hence, the theorem follows. □

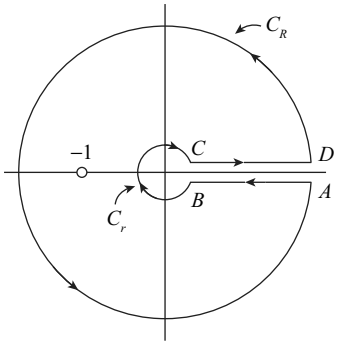
Example 6.5.3 Evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

Solution: Let $f(z) = \frac{1}{\sqrt{z}(z+1)}$. Then, $z=-1$ is a pole for f. Note that if $g(z) = \sqrt{z}$, then g is not a single-valued function. Hence, first we define a branch of g as follows so that it is analytic at -1.

We take $\Omega = \mathbb{C} \setminus \{(x,0) : x \geq 0\}$ and define $\sqrt{z} = r^{1/2}\exp\left(i\frac{\theta}{2}\right)$, where $\theta \in (0, 2\pi)$. As there is a unique argument of z in this range $(0, 2\pi)$,

it is a well-defined function. Using the continuity of this branch of argument function (See Lemma 3.3.8), we get that this branch of $\sqrt{z}$ is analytic.

To evaluate the integral, first we consider $\int_\gamma f(z)\,dz$, where γ is the closed curve as in the following diagram.



Although it seems that there is a gap between the line segments AB and CD, actually they are lying on the positive real axis. However, they are given different notation because their parametric equations are given in different ways as $AB = \{t\exp(i2\pi) : R \le t \le r\}$ and $CD = \{t\exp(i0) : r \le t \le R\}$.
(The difference between AB and CD is that the image of AB under $\sqrt{z}$ is on the negative real axis, whereas the same for CD is on the positive real axis.)

Since -1 is the only pole of f inside γ, by Cauchy's residue theorem, we have

$$\int_\gamma \frac{dz}{\sqrt{z}(z+1)} = i2\pi(\operatorname{Res} f)(-1) = i2\pi(-i) = 2\pi,$$

as

$$(\operatorname{Res} f)(-1) = \lim_{z\to -1}\frac{dz}{\sqrt{z}} = (-1)^{1/2} = \exp\left(-i\frac{\pi}{2}\right) = -i.$$

Since

$$2\pi = \int_\gamma \frac{1}{\sqrt{z}(z+1)} = \int_{C_R} + \int_{AB} + \int_{C_r} + \int_{CD} \frac{dz}{\sqrt{z}(z+1)}, \tag{6.12}$$

we find

$$\int_{AB} \frac{dz}{\sqrt{z}(z+1)} = \int_R^r \frac{\exp(i2\pi)\,dt}{t^{1/2}\exp(i\pi)(t\exp(i2\pi)+1)} = -\int_R^r \frac{dt}{t^{1/2}(t+1)}$$

and

$$\int_{CD} \frac{dz}{\sqrt{z}(z+1)} = \int_r^R \frac{\exp(i0)\,dt}{t^{1/2}\exp(i0)(t\exp(i0)+1)} = \int_r^R \frac{dt}{t^{1/2}(t+1)}.$$

We also note that

$$\begin{aligned}
\left|\int_{C_R} \frac{dz}{\sqrt{z}(z+1)}\right| &= \left|\int_0^{2\pi} \frac{Ri\exp(i\theta)\,d\theta}{R^{1/2}\exp\left(i\frac{\theta}{2}\right)(R\exp(i\theta)+1)}\right| \\
&\leq \int_0^{2\pi} \frac{R\,d\theta}{R^{1/2}|R\exp(i\theta)+1|} \\
&\leq R^{-1/2}\int_0^{2\pi} \frac{d\theta}{|\exp(i\theta)+\frac{1}{R}|} \\
&\leq R^{-1/2}2\pi\frac{1}{1-\frac{1}{R}} \to 0 \text{ as } R\to\infty
\end{aligned}$$

(In the above inequality, we have used $|\exp(i\theta)+\frac{1}{R}| \geq ||\exp(i\theta)| - |\frac{1}{R}|| = 1-\frac{1}{R}$ because $1 > \frac{1}{R}$ for sufficiently large R.)
and

$$\begin{aligned}
\left|\int_{C_r} \frac{dz}{\sqrt{z}(z+1)}\right| &= \left|\int_{2\pi}^0 \frac{ri\exp(i\theta)\,d\theta}{r^{1/2}\exp\left(i\frac{\theta}{2}\right)(r\exp(i\theta)+1)}\right| \\
&\leq r^{1/2}\int_0^{2\pi} \frac{d\theta}{1-r} \\
&= r^{1/2}2\pi\frac{1}{1-r} \to 0 \text{ as } r\to\infty.
\end{aligned}$$

Using these observations in equation (6.12), we get

$$2\int_r^R \frac{dt}{t^{1/2}(t+1)} = 2\int_r^R \frac{dt}{t^{1/2}(t+1)} + \lim_{r\to 0}\int_{C_r} \frac{dz}{\sqrt{z}(z+1)}$$

$$+ \lim_{R\to\infty} \int_{C_R} \frac{dz}{\sqrt{z}(z+1)} = 2\pi.$$

Hence, $\int_r^R \frac{dt}{t^{1/2}(t+1)} = \pi.$

LEMMA 6.5.4 For any $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, $\left| \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{1+\exp(-2ct)}\, dt \right| < \infty.$

Proof: Let $\phi(t) = 1 + \exp(-2ct)$, $t \in \mathbb{R}$. As ϕ is continuous on $\mathbb{R}$ and $\phi(0) = 2$ for any $\epsilon \in (0,2)$, there exists $\delta > 0$ such that

$$|t| < \delta \Rightarrow |1 + \exp(-2ct) - 2| < \epsilon \Rightarrow |1 + \exp(-2ct)| > 2 - \epsilon.$$

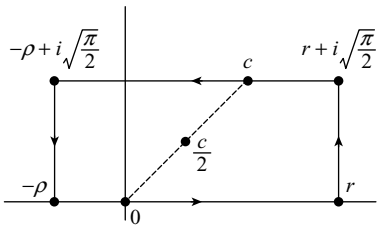
If $t > \delta$, then $|1 + \exp(-2ct)| \geq 1 - |\exp(-2ct)| = 1 - \exp(-2t \operatorname{Re} c) > 1 - \exp(-2\delta \operatorname{Re} c)$, and if $t < -\delta$, then $|1 + \exp(-2ct)| \geq \exp(-2t \operatorname{Re} c) - 1 > \exp(2\delta \operatorname{Re} c) - 1$.
So, if $\mu = \min\{2 - \epsilon, 1 - \exp(-2\delta \operatorname{Re} c), \exp(2\delta \operatorname{Re} c)\} - 1$, then $|1 + \exp(-2ct)| \geq \mu$, $\forall t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{1+\exp(-2ct)}\, dt \right| \\
&\leq \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{|1+\exp(-2ct)|}\, dt \\
&\leq \frac{1}{\mu} \int_{-\infty}^{\infty} \exp(-t^2)\, dt \\
&= \frac{2}{\mu} \int_{0}^{\infty} \exp(-t^2)\, dt \\
&= \frac{2}{\mu} \left(\int_0^1 \exp(-t^2)\, dt + \int_1^{\infty} \exp(-t^2)\, dt \right) \\
&< \frac{2}{\mu} \left(1 + \int_1^{\infty} \exp(-t)\, dt \right) \\
&\qquad (\text{as } t < 1 \Rightarrow \exp(-t^2) < 1 \text{ and } t > 1 \Rightarrow \exp(-t^2) < \exp(-t))
\end{aligned}$$

$$= \frac{2}{\mu}\left(1+\frac{1}{e}\right) < \infty. \qquad \square$$

PROBLEM 6.5.5 Prove that $\int_{-\infty}^{\infty} \exp(-t^2)\,dt = \sqrt{\pi}$.

Solution: Let $c = \sqrt{i\pi} = \sqrt{\frac{\pi}{2}}(1+i)$. If $f(z) = \dfrac{\exp(-z^2)}{1+\exp(-2cz)}$, then f has poles at every $w \in \{w \in \mathbb{C} : \exp(-2cw)+1 = 0\}$. Then, $2cw = i(2k-1)\pi \Rightarrow w = \dfrac{i(2k-1)}{2c}$ for some $k \in \mathbb{Z}$. Moreover, each $\dfrac{i(2k-1)}{2c}$ is a simple pole of f because it is a simple zero of $1+\exp(-2cz)$ as $\dfrac{d}{dz}(1+\exp(-2cz)) \neq 0$. Now, we consider the following rectangle R with vertices $-\rho, r, r+i\sqrt{\frac{\pi}{2}}$, and $-\rho+i\sqrt{\frac{\pi}{2}}$, which passes through the point $c = \sqrt{\frac{\pi}{2}}(1+i)$, where $r > 1+\sqrt{\frac{\pi}{2}}$ and $\rho > 0$ (later, both of these positive real numbers r and ρ will be tending to $+\infty$).



Since $\operatorname{Im} z \in \left(0, \sqrt{\dfrac{\pi}{2}}\right)$, $\forall z \in \operatorname{Int} R$, the only pole of f in $\operatorname{Int} R$ is $\frac{c}{2}$. In other words, $\frac{c}{2}$ is the only pole of f enclosed by the boundary ∂R of R. Therefore, by Cauchy's residue theorem, we have

$$\int_{\partial R} \frac{\exp(-z^2)}{1+\exp(-2cz)}\,dz = i2\pi(\operatorname{Res} f)\left(\frac{c}{2}\right) = \sqrt{\pi}, \tag{6.13}$$

and by applying Result 6.1.9, we have

$$(\operatorname{Res} f)\left(\frac{c}{2}\right) = \frac{\exp\left(-\dfrac{c^2}{4}\right)}{(-2c)\exp\left(-c^2\right)} = \frac{\exp\left(-i\dfrac{\pi}{4}\right)}{(-\sqrt{2\pi}(1+i))\exp\left(-i\pi\right)}$$

$$= \frac{\frac{1}{\sqrt{2}}(1-i)}{\sqrt{2\pi}(1+i)} = \frac{-i}{2\sqrt{\pi}}.$$

By direct calculation, we also get

$$\begin{aligned}\int_{\partial R} f(z)\,dz &= \int_{-\rho}^{r} f(t)\,dt + i\int_{0}^{\sqrt{\frac{\pi}{2}}} f(r+it)\,dt \\ &\quad + \int_{r}^{-\rho} f\left(t + i\sqrt{\frac{\pi}{2}}\right)dt + i\int_{\sqrt{\frac{\pi}{2}}}^{0} f(-\rho+it)\,dt. \quad (6.14)\end{aligned}$$

We note that

$$\begin{aligned}\left|\int_{0}^{\sqrt{\frac{\pi}{2}}} f(r+it)\,dt\right| &\le \int_{0}^{\sqrt{\frac{\pi}{2}}} \frac{|\exp(-(r^2-t^2+i2rt))|}{|1+\exp(-2a(r+it))|}\,dt \\ &\le \int_{0}^{\sqrt{\frac{\pi}{2}}} \frac{\exp(-(r^2-t^2))}{1-\exp(-\sqrt{2\pi}(r-t))}\,dt \\ &\quad [\text{ as } |1-\exp(-2a(r+it))| \ge 1-|\exp(-2a(r+it))| \\ &\quad = 1-\text{Re }\exp(-2a(r+it)) = 1-\exp(-\sqrt{2\pi}(r-t))\,] \\ &\le \exp(-r^2)\int_{0}^{\sqrt{\frac{\pi}{2}}} \frac{\exp(t^2)}{1-\exp(-\sqrt{2\pi})}\,dt \\ &\quad [\text{ as } r > 1+\sqrt{\frac{\pi}{2}} \text{ and } t \le \sqrt{\frac{\pi}{2}} \Rightarrow r-t > 1 \\ &\quad \Rightarrow \frac{1}{1-\exp(-\sqrt{2\pi}(r-t))} \le \frac{1}{1-\exp(-\sqrt{2\pi})}\,] \\ &\le \frac{\exp\left(-r^2+\frac{\pi}{2}\right)}{1-\exp(-\sqrt{2\pi})}\int_{0}^{\sqrt{\frac{\pi}{2}}} dt \\ &\le \sqrt{\frac{\pi}{2}}\frac{\exp\left(-r^2+\frac{\pi}{2}\right)}{1-\exp(-\sqrt{2\pi})} \to 0 \text{ as } r \to +\infty.\end{aligned}$$

Similarly, we can show that

$$\int_{\sqrt{\frac{\pi}{2}}}^{0} f(-\rho + it)\, dt \to 0 \text{ as } \rho \to +\infty.$$

Therefore, applying $\lim_{r\to+\infty}$ and $\lim_{\rho\to+\infty}$ on both sides of equation (6.14) and using equation (6.13), we get

$$\sqrt{\pi} = \lim_{r\to+\infty} \lim_{\rho\to+\infty} \int_{\partial R} f(z)\, dz = \int_{-\infty}^{\infty} \left[f(t) - f\left(t + i\sqrt{\frac{\pi}{2}} \right) \right] dt. \quad (6.15)$$

Applying the following observations in equation (6.15)

$$\begin{aligned} \int_{-\infty}^{\infty} f\left(t + i\sqrt{\frac{\pi}{2}} \right) dt &= \int_{-\infty}^{\infty} f\left(s + \sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}} \right) ds \left(\text{using } s = t + \sqrt{\frac{\pi}{2}} \right) \\ &= \int_{-\infty}^{\infty} f\,(s + c)\; ds \end{aligned}$$

and

$$\begin{aligned} f\,(t + c) &= \frac{\exp(-(t + c)^2)}{1 + \exp(-2c(t + c))} \\ &= \frac{\exp(-(t^2 + c^2 + 2ct)}{1 + \exp(-2ct - 2c^2)} \\ &= \frac{-\exp(-t^2)\exp(-2ct)}{1 + \exp(-2ct)} \\ &\quad (\text{as } c^2 = \pi i,\ \exp(-i\pi) = -1, \exp(-i2\pi) = 1.) \\ &= \frac{-\exp(-t^2)(1 + \exp(-2ct)}{1 + \exp(-2ct)} + \frac{\exp(-t^2)}{1 + \exp(-2ct)} \\ &= -\exp(-t^2) + f(t), \end{aligned}$$

we obtain

$$\sqrt{\pi} = \int_{-\infty}^{\infty} f(t)\, dt - \int_{-\infty}^{\infty} (-\exp(-t^2) + f(t))\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\,dt + \int_{-\infty}^{\infty} \exp(-t^2)\,dt - \int_{-\infty}^{\infty} f(t)\,dt \text{ (using Lemma 6.5.4)}$$

$$= \int_{-\infty}^{\infty} \exp(-t^2)\,dt.$$

The integral in the previous problem is called the error integral.

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7

Some Interesting Theorems

7.1 MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS

In Chapter 2, we have already introduced the definition of harmonic functions, and in this section, we shall see some properties of harmonic functions.

RESULT 7.1.1 (Laplace equation in polar form)

If u is a harmonic function given in polar form, then $r\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{\partial^2 u}{\partial \theta^2}=0$.

Proof: We know that the Cartesian coordinates and polar coordinates are related by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. As

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos(\theta)+\frac{\partial u}{\partial y}\sin(\theta),$$

we get

$$r\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}r\cos(\theta)+\frac{\partial u}{\partial y}r\sin(\theta)=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y},$$

which implies

$$\begin{aligned} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) &= r\frac{\partial}{\partial r}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \\ &= x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)+y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \end{aligned}$$

$$\begin{aligned} &= x\left(\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}\right)+y\left(x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}\right)\\ &= x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}. \end{aligned}$$

Similarly, using

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=-\frac{\partial u}{\partial x}r\sin(\theta)+\frac{\partial u}{\partial y}r\cos(\theta)=x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x},$$

we get

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}\right)\\ &= x\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}\right)-y\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}\right)\\ &= x\left(x\frac{\partial^2 u}{\partial y^2}-y\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial u}{\partial x}\right)-y\left(x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}-y\frac{\partial^2 u}{\partial x^2}\right)\\ &= x^2\frac{\partial^2 u}{\partial y^2}+y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}. \end{aligned}$$

Using the Laplace equation in Cartesian form $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, we get

$$r\frac{\partial u}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial \theta^2}=x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+y^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0.$$

Hence, the result follows. □

LEMMA 7.1.2 Let v be a function on a region Ω such that it has continuous partial derivatives. Then, $\int\limits_{\gamma} dv = 0$ for every closed curve γ in Ω.

Proof: By direct computation, if $z(t),\ t \in [a,b]$ is a parametric equation of γ, then $\int\limits_{\gamma} dv = \int\limits_{\gamma}\left(\dfrac{\partial v}{\partial x}\,dx+\dfrac{\partial v}{\partial y}\,dy\right)=\int\limits_{a}^{b} z'(t)\,dt = z(x(b),y(b))-z(x(a),y(a))=0$, as γ is a closed curve. □

RESULT 7.1.3 Let u_1 and u_2 be harmonic functions on a region Ω. If v_1 and v_2 are harmonic conjugates of u_1 and u_2, respectively, then $\int\limits_{\gamma}(u_1\,dv_2-u_2\,dv_1)=0$ for every cycle $\gamma \sim 0$ in Ω.

Proof: If $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$, then f_1 and f_2 are analytic functions on Ω. Therefore, f_2' is analytic, and hence, $f_1 f_2'$ is also analytic on Ω. Therefore, by general version of Cauchy's theorem (Theorem 4.4.3), we get

$$\int_\gamma f_1 f_2' \, dz = 0 \Rightarrow \text{Im} \int_\gamma f_1 f_2' \, dz = 0.$$

Using

$$\begin{aligned} \text{Im}\,(f_1 f_2' \, dz) = \text{Im}\,(f_1 \, d(f_2)) &= \text{Im}\,((u_1 + iv_1)(du_2 + i dv_2)) \\ &= u_1 dv_2 + v_1 du_2 \\ &= u_1 dv_2 + d(u_2 v_1) - u_2 dv_1 \end{aligned}$$

and using Lemma 7.1.2, we obtain $\int_\gamma d(u_2 v_1) = 0$, and hence,

$$\int_\gamma (u_1 \, dv_2 - u_2 \, dv_1) = 0.$$

Thus, the result follows. □

THEOREM 7.1.4 (Mean value property for harmonic function)

If u is a harmonic function on Cl $B(z, r)$, then $u(z) = \dfrac{1}{2\pi} \int_0^{2\pi} u(z + r\exp(i\theta))\, d\theta$.

Proof: Let v be a harmonic conjugate of u. Then, $f = u + iv$ is an analytic function on Cl $B(z, r)$. Then, by Cauchy's integral formula, we have

$$f(z) = \frac{1}{i2\pi} \int_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Since the parametric equation of $|\zeta - z| = r$ is $\zeta(\theta) = z + r\exp(i\theta)$, $\forall \theta \in [0, 2\pi]$, we have

$$f(z) = \frac{1}{i2\pi} \int_0^{2\pi} \frac{f(z + r\exp(i\theta))}{r\exp(i\theta)} \, ir\exp(i\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z + r\exp(i\theta))\, d\theta.$$

Hence, we get $u(z) = \dfrac{1}{2\pi} \int_0^{2\pi} u(z + r\exp(i\theta))\, d\theta$ by equating the real parts on both sides. □

PROBLEM 7.1.5 If u is a harmonic function on $B(0,s)\backslash\{0\}$, then $\frac{1}{2\pi}\int\limits_{|z|=r} u\,d\theta = a\log(r)+b, \forall 0<r<s$ for some $A, B \in \mathbb{R}$.

Solution:

Let $u_1 = \log(r)$ and $u_2 = u$. Applying Theorem 7.1.1, u_1 is a harmonic function, as $r\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) + \frac{\partial^2 u_1}{\partial \theta^2} = r\frac{\partial}{\partial r}\left(r\,\frac{1}{r}\right) + 0 = 0$. Therefore, by Theorem 7.1.3, we have $\int\limits_{\gamma}(u_1\,dv_2 - u_2\,dv_1) = 0$ for every cycle $\gamma \sim 0$ in Ω, where v_1 and v_2 are harmonic conjugates of u_1 and u_2, respectively. For a harmonic function u and its harmonic conjugate v, we first convert $u\,dv$ into the polar form (on $|z| = r$) as follows. Using C–R equations, we have

$$dv = \frac{\partial v}{dx}dx + \frac{\partial v}{dy}dy = -\frac{\partial u}{dy}dx + \frac{\partial u}{dx}dy.$$

As in the proof of Theorem 7.1.1, from $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$r\frac{\partial u}{\partial r}\,d\theta = \left(\frac{\partial u}{\partial x}r\cos(\theta) + \frac{\partial u}{\partial y}r\sin(\theta)\right)d\theta = \frac{\partial u}{\partial x}\,x\,d\theta + \frac{\partial u}{\partial y}\,y\,d\theta,$$

and using the fact that r is constant on $|z| = r$, we also have

$$dx = -r\sin(\theta)\,d\theta = -y\,d\theta \text{ and } dy = r\cos(\theta)\,d\theta = x\,d\theta.$$

From the above two equations, we get

$$r\frac{\partial u}{\partial r}\,d\theta = \frac{\partial u}{dx}dy - \frac{\partial u}{dy}dx = dv \text{ on } |z| = r.$$

If C_1 and C_2 are concentric circles with centre 0 and radii r_1 and r_2, respectively, with $0 < r_1 < r_2 < s$, then we claim that $C_1 \cup (-C_2) \sim 0$ in $B(0,s)\backslash\{0\}$. If $w \notin B(0,s)\backslash\{0\}$, then either $w = 0$ or $|w| > s$. As $\text{WN}(C_1 \cup (-C_2), z) = \text{WN}(C_1, z) - \text{WN}(C_2, z) = \begin{cases} 1-1=0, & \text{if } w = 0 \\ 0-0=0, & \text{if } |w| > s \end{cases}$, our claim holds.

Therefore, we have

$$0 = \int\limits_{C_1 \cup (-C_2)} (u_1\,dv_2 - u_2\,dv_1)$$

$$
\begin{aligned}
&= \int_{C_1 \cup (-C_2)} \left(u_1 \, r \frac{\partial u_2}{\partial r} \, d\theta - u_2 \, r \frac{\partial u_1}{\partial r} \, d\theta \right) \\
&= \int_{C_1 \cup (-C_2)} \log(r) \left(r \frac{\partial u}{\partial r} \, d\theta \right) - \int_{C_1 \cup (-C_2)} u \, d\theta \\
&= \log(r_1) \int_{C_1} \left(r_1 \frac{\partial u}{\partial r} \, d\theta \right) - \log(r_2) \int_{C_2} \left(r_2 \frac{\partial u}{\partial r} \, d\theta \right) - \int_{C_1} u \, d\theta + \int_{C_2} u \, d\theta,
\end{aligned}
$$

which implies that

$$
\int_{C_1} u \, d\theta - \log(r_1) \int_{C_1} r_1 \frac{\partial u}{\partial r} \, d\theta = \int_{C_2} u \, d\theta - \log(r_2) \int_{C_2} r_2 \frac{\partial u}{\partial r} \, d\theta.
$$

Therefore, $\int_{|z|=r} u \, d\theta - \log(r) \int_{|z|=r} r \frac{\partial u}{\partial r} \, d\theta$ is a constant, say b. *(Reason: After integrating these functions with respect to θ, we get only a function of r and by previous equation, it is also independent of r.)*

By a same argument using $\int_{\gamma} dv = 0$, whenever $\gamma \sim 0$ in $B(0,r)\backslash\{0\}$ (see Lemma 7.1.2.), we get $\int_{|z|=r} r \frac{\partial u}{\partial r} \, d\theta$, which is a constant, say a.

Thus, we get $\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = A \log(r) + B$, where $A = \frac{a}{2\pi}$ and $B = \frac{b}{2\pi}$.

Exercise 7.1.6 Deduce mean value property for a harmonic function on $B(0,r)$ from the previous problem.

THEOREM 7.1.7 (Maximum principle for harmonic function)
If u is a non-constant harmonic function on a region Ω, then u cannot have maximum in Ω.

Proof: Suppose u attains maximum at $z \in \Omega$, then we have $u(w) \leq u(z), \forall w \in \Omega$. Now, choose $r > 0$ such that $\mathrm{Cl}\, B(z,r) \subseteq \Omega$. By mean value property for harmonic function, we have

$$
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r \exp(i\theta)) \, d\theta.
$$

We claim that $u(w) = u(z)$, $\forall w$ with $|w - z| = r$. If there exists w_0 with $|w_0 - z| = r$ and $u(w_0) < u(z)$, then let $w_0 = z + r\exp(i\theta_0)$ for some $\theta_0 \in [0, 2\pi]$. Using the continuity of u and the exponential function, we choose an interval $[a, b]$ such that $\theta_0 \in [a, b] \subseteq [0, 2\pi]$, where $a < b$, and for every $\theta \in [a, b]$,

$$|u(z + r\exp(i\theta)) - u(z + r\exp(i\theta_0))| < \frac{u(z) - u(z + r\exp(i\theta_0))}{2},$$

which implies

$$\begin{aligned} u(z + r\exp(i\theta)) &< u(z + r\exp(i\theta_0)) + \frac{u(z) - u(z + r\exp(i\theta_0))}{2} \\ &= \frac{u(z) + u(z + r\exp(i\theta_0))}{2} \\ &= \frac{u(z) + u(w_0)}{2}. \end{aligned}$$

Hence, we get

$$\begin{aligned} u(z) &= \frac{1}{2\pi}\int_0^{2\pi} u(z + r\exp(i\theta))\, d\theta \\ &= \frac{1}{2\pi}\left(\int_0^a u(z + r\exp(i\theta))\, d\theta + \int_a^b u(z + r\exp(i\theta))\, d\theta \right. \\ &\qquad \left. + \int_b^{2\pi} u(z + r\exp(i\theta))\, d\theta \right) \\ &\leq \frac{1}{2\pi}\left(\int_0^a u(z)\, d\theta + \int_a^b \left(\frac{u(z) + u(w_0)}{2}\right) d\theta + \int_b^{2\pi} u(z)\, d\theta \right) \\ &< \frac{u(z)}{2\pi}\,(a + b - a + 2\pi - b) \,(\text{since } u(z) + u(w_0) < 2u(z)) \\ &= u(z), \end{aligned}$$

which is a contradiction. Hence, u is constant on $\{w : |w - z| = r\}$. Therefore, $u(w) = u(z)$ on $\{w : |w - z| = s\}$, $\forall s \in (0, r)$, and hence, $u(w) = u(z)$ on $B(z, r)$. If $A = \{w \epsilon \Omega : u(w) = u(z)\}$, then A is an open set, by the previous argument. As u is continuous and $\{z\}$ is a closed set, we see that $A = u^{-1}(\{u(z)\})$ is a closed set, and hence, A^c is an open set. Using that Ω

is connected and $\Omega = A \cup A^c$ with both A and A^c are open, either $A = \varnothing$ or $A^c = \varnothing$. Since $z \in A$, we get $A^c = \varnothing$, which implies that $\Omega = A$. Therefore, u is a constant on Ω, which is a contradiction. Hence, there is no maximum for u in Ω. □

THEOREM 7.1.8 (Minimum principle for harmonic function)
If u is a non-constant harmonic function on a region Ω, then u cannot have minimum in Ω.
Proof of this theorem is an immediate consequence of the previous theorem.

If we examine the proof of the above two theorems, we can notice that the mean value property and the continuity of the function u are sufficient to obtain these theorems. Hence, we can have the following theorem.

THEOREM 7.1.9 If u is a real-valued continuous function on Ω such that u is continuous and satisfies

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r\exp(i\theta))\, d\theta$$

whenever Cl $B(z, r) \subseteq \Omega$, then u has neither maximum nor minumum in Ω.

THEOREM 7.1.10 (Poisson's formula)
If u is harmonic on $B(0, r)$ and continuous on Cl $B(0, r)$, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|r\exp(i\theta) - z|^2} u(r\exp(i\theta))\, d\theta,\ \forall z \in B(0, r).$$

Proof: Poisson's formula at $z = 0$ is same as the mean value property of the harmonic function u. So, we prove this result for $z \in B(0, r)\backslash\{0\}$. First, we note that

$$\text{Re}\ \frac{w+z}{w-z} = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}}\right) = \frac{r^2 - |z|^2}{|w-z|^2},\ \forall w \text{ with } |w| = r.$$

Case 1. u is harmonic on Cl $B(0, r)$.

We know that $\eta = \Lambda(w) = \dfrac{r(w-z)}{r^2 - \overline{z}w}$ maps Cl $B(0, r)$ onto Cl $B(0, 1)$ (see Example 3.2.26)). As (by Result 3.2.5)

$$w = \Lambda^{-1}(\eta) = \frac{r(r\eta + z)}{r + \overline{z}\eta},$$

We also know that Λ^{-1} analytically maps Cl $B(0,1)$ onto Cl $B(0,r)$. If f is an analytic function on Cl $B(0,r)$ such that Re $f = u$, then by Chain rule (Theorem 2.1.10), $f \circ \Lambda^{-1}$ is analytic on Cl $B(0,1)$, and hence, Re $(f \circ \Lambda^{-1})$ is a harmonic function on Cl $B(0,1)$.

Using $w = r\exp(i\theta) \Rightarrow dw = ir\exp(i\theta)d\theta = iwd\theta$ and $\eta = \exp(i\phi)$, we get

$$\begin{aligned}
d\phi &= \frac{d(\exp(i\phi))}{i\exp(i\phi)} \\
&= \frac{d\eta}{i\eta} \\
&= -id\log\left(\frac{r(w-z)}{r^2-\bar{z}w}\right) \\
&= -i\left(\frac{r}{r(w-z)} + \frac{\bar{z}}{r^2-\bar{z}w}\right)dw \\
&= -i\left(\frac{1}{w-z} + \frac{\bar{z}}{w\overline{w}-\bar{z}w}\right)iw\,d\theta \\
&= \left(\frac{w}{w-z} + \frac{\bar{z}}{\overline{w}-\bar{z}}\right)d\theta \\
&= \left(\frac{r^2-|z|^2}{|w-z|^2}\right)d\theta \\
&= \left(\frac{r^2-|z|^2}{|r\exp(i\theta)-z|^2}\right)d\theta.
\end{aligned}$$

Then, by mean value property for harmonic function, we get

$$\begin{aligned}
u(\Lambda^{-1}(0)) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\Lambda^{-1}(\exp(i\phi))\right)d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} u\left(r\exp(i\theta)\right)\left(\frac{r^2-|z|^2}{|r\exp(i\theta)-z|^2}\right)d\theta.
\end{aligned}$$

Case 2. u is harmonic on $B(0,r)$ and continuous on Cl $B(0,r)$.
Then, for every $s \in (0,1)$, if $u_s(w) = u(sw)$, $\forall w \in$ Cl $B(0,r)$, then u_s is harmonic on Cl $B(0,r)$, and hence, by Case 1, we get

$$u(sz) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2r^2-|z|^2}{|sr\exp(i\theta)-z|^2}u\left(sr\exp(i\theta)\right)\,d\theta.$$

By using the uniform continuity of u on Cl $B(0,r)$, we get

$$u(z) = \lim_{s\to 1} u(sz) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-|z|^2}{|r\exp(i\theta)-z|^2} u\left(r\exp(i\theta)\right)\, d\theta.$$

Thus, the theorem follows. □

Remark 7.1.11: The Poisson's formula can also be written in the following ways.

$$u(z) = \frac{1}{2\pi}\int_a^{a+2\pi} \frac{r^2-|z|^2}{|r\exp(i\theta)-z|^2} u\left(r\exp(i\theta)\right)\, d\theta,\ (a\in\mathbb{R}). \quad (7.1)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\left(\frac{r\exp(i\theta)+z}{r\exp(i\theta)-z}\right) u(r\exp(i\theta))\, d\theta. \quad (7.2)$$

$$= \frac{1}{2\pi}\text{Re}\left(\int_{|w|=r} \frac{w+z}{w-z}\,\frac{u(w)}{iw}\, dw\right),\ \forall z\in B(0,r). \quad (7.3)$$

THEOREM 7.1.12 (Jensen's formulae)
Let f be an analytic function on Cl $B(0,r)$ such that f is not identically zero.

Case 1. If $f\neq 0$ on $B(0,r)$, then

$$\log(|f(0)|) = \int_0^{2\pi} \log(|f(r\exp(i\theta))|)\, d\theta.$$

Case 2. If the zeroes of f in $B(0,r)$ are $a_j\in B(0,r)\backslash\{0\}$, $j=1,2,\ldots,n$, including multiplicities, then

$$\log(|f(0)|) = -\sum_{j=1}^{n}\log\left(\frac{r}{|a_j|}\right) + \int_0^{2\pi}\log(|f(r\exp(i\theta))|)\, d\theta.$$

Case 3. If f has a zero of order m at 0 and the zeroes of f in $B(0,r)$ are $a_j\in B(0,r)\backslash\{0\}, j=1,2,\ldots,n$, including multiplicities, then

$$\log(|c_m|) + m\log(r) = -\sum_{j=1}^{n} \log\left(\frac{r}{|a_j|}\right) + \int_0^{2\pi} \log(|f(r\exp(i\theta))|)\, d\theta,$$

where c_m is the coefficient of z^m in the Taylor's series expansion of f in $B(0,r)$.

Proof:

Case 1. **Subcase (i)** If f is nowhere zero on Cl $B(0,r)$, then $\log(f)$ is an analytic function on Cl $B(0,r)$ (by Theorem 4.5.12). Therefore, by Result 2.2.6, Re $\log(f) = \log(|f|)$ is harmonic on Cl $B(0,r)$. Therefore, by mean value property of harmonic function, we have

$$\log(|f(0)|) = \frac{1}{2\pi} \int_0^{2\pi} \log(f(r\exp(i\theta)))\, d\theta.$$

Subcase (ii) Suppose f is nowhere zero on $B(0,r)$ and the zeroes of f on $\{z : |z| = r\}$ are $r\exp(i\theta_1), r\exp(i\theta_2), \ldots, r\exp(i\theta_\nu)$, including the multiplicities, for some $\theta_k \in [0, 2\pi]$, $\forall k = 1, 2, \ldots, \nu$. Then, let $f_1(z) = \frac{f(z)}{\prod_{k=1}^{\nu}(z - r\exp(i\theta_k))}$. Then, f_1 has no zeroes on Cl $B(0,r)$ after removing the removable singularities at $r\exp(i\theta_k)$, $k = 1, 2, \ldots, \nu$. Therefore, by replacing f by f_1 in the Subcase (i), we get

$$\begin{aligned}
&\log(|f(0)|) - \nu\log r \\
&= \log\left(\frac{|f(0)|}{\prod_{k=1}^{\nu} |0 - r\exp(i\theta_k)|}\right) \\
&= \log(|f_1(0)|) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log(f_1(r\exp(i\theta)))\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log\left(\frac{|f(r\exp(i\theta))|}{\prod_{k=1}^{\nu} |r\exp(i\theta) - r\exp(i\theta_k)|}\right) d\theta
\end{aligned}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \log(|f(r\exp(i\theta))|) - \sum_{k=1}^{\nu} [\log(r) + \log(|\exp(i\theta) - \exp(i\theta_k)|)] \right\} d\theta.$$

Thus, we have

$$\log(|f(0)|) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \log(|f(r\exp(i\theta))|) - \sum_{k=1}^{\nu} \log(|\exp(i\theta) - \exp(i\theta_k)|) \right\} d\theta. \quad (7.4)$$

We claim that $\int_0^{2\pi} \log(|\exp(i\theta) - \exp(i\theta_k)|) = 0$, $\forall k \in \{1, 2, \ldots, \nu\}$. Using the fact that $\exp(it)$ is a periodic function on $\mathbb{R}$ of period 2π (Result 2.4.20) and using Lemma 2.4.23, we obtian

$$\begin{aligned} &\int_0^{2\pi} \log(|\exp(i\theta) - \exp(i\theta_k)|) \\ &= \int_0^{2\pi} \log(|1 - \exp(i(\theta - \theta_k)|)) \\ &= \int_{\theta_k}^{\theta_k + 2\pi} \log(|1 - \exp(it)|) \\ &\quad \text{(using the change of variable } t = \theta - \theta_k) \\ &= \int_0^{2\pi} \log(|1 - \exp(it)|)\, dt. \end{aligned} \quad (7.5)$$

Since

$$\begin{aligned} \log(|1 - \exp(it)|) &= \frac{1}{2} \log((1 - \cos(t))^2 + \sin^2(t)) \\ &= \frac{1}{2} \log(1 + \cos^2(t) - 2\cos(t) + \sin^2(t)) \\ &= \frac{1}{2} \log(2(1 - \cos(t))) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}[\log(2) + \log(1 - \cos(t))] \\ &= \frac{1}{2}[\log(2) + \log(2\sin^2(t/2))] \\ &= \log(2) + \log(\sin(t/2)), \end{aligned}$$

the right-hand side of equation (7.5) is equal to

$$\begin{aligned} \int_0^{2\pi} [\log(2) + \log(\sin(t/2))]\, dt &= 2\pi \log(2) + 2\int_0^{\pi} \log(\sin(u))\, du \\ &\quad \text{(by the change of variable } u = t/2) \\ &= 2\pi \log(2) - 2\pi \log(2) \\ &\quad \text{(by Theorem 6.5)} \\ &= 0. \end{aligned}$$

Using these observations in equation (7.4), we get

$$\log(|f(0)|) = \frac{1}{2\pi}\int_0^{2\pi} \log(|f(r\exp(i\theta))|)\, d\theta.$$

Case 2. Let $f_2(z) = f(z)\prod_{j=1}^{n} \dfrac{r^2 - \overline{a_j}z}{r(z - a_j)}$, $\forall z \in \text{Cl}\, B(0, r)$. Clearly, f_2 becomes an analytic function on Cl $B(0, r)$, and it has no zeroes in $B(0, r)$. If $|z| = r$, then let $z = r\exp(i\theta)$ for some $\theta \in [0, 2\pi]$, and hence,

$$\left|\frac{r^2 - \overline{a_j}z}{r(z - a_j)}\right| = \left|\frac{r\exp(i\theta)(r\exp(-i\theta) - \overline{a_j})}{r(r\exp(i\theta) - a_j)}\right| = \left|\frac{r\exp(-i\theta) - \overline{a_j}}{r\exp(i\theta) - a_j}\right| = 1.$$

Hence it follws that $|f_2| = |f|$ on $|z| = r$ Therefore, replacing $f(z)$ by $f_2(z)$ in Case 1, we obtain

$$\begin{aligned} \log(|f(0)|) + \sum_{j=1}^{n} \log\left(\frac{r}{|a_j|}\right) &= \log\left(|f(0)|\prod_{j=1}^{n} \frac{r}{|a_j|}\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log(|f_2(r\exp(i\theta))|)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log(|f(r\exp(i\theta))|)\, d\theta. \end{aligned}$$

Case 3. Now, let $f_3(z) = f(z)\frac{r^m}{z^m}$, $\forall z \in \text{Cl}\, B(0,r)$ so that f_3 is analytic and $f_3(0) \neq 0$. Hence, applying Case 2, we get

$$\begin{aligned}\log(|f_3(0)|) &= -\sum_{j=1}^{n} \log\left(\frac{r}{|a_j|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f_3(r\exp(i\theta))|\, d\theta.\\ &= -\sum_{j=1}^{n} \log\left(\frac{r}{|a_j|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(r\exp(i\theta))|\, d\theta,\end{aligned}$$

since $|f_3| = |f|$ on $|z| = r$. If the coefficient of z^m in the Taylor's series expansion of f is c_m, then we have $f_3(0) = c_m r^m$, and hence, the right-hand side of the above equation becomes $\log(|c_m|) + m\log(r)$. □

Remark 7.1.13: In the Jensen's formulae, if f has zeroes on $\{z \in \mathbb{C} : |z| = r\}$, then the integral on the right-hand side has no meaning as in Definition 4.1.6, and hence, the value of the integral should be understood by the above formulae.

THEOREM 7.1.14 (Poisson–Jensen's formula)
Let f be an analytic function on $\text{Cl}\, B(0,r)$ such that f is not identically zero. If the zeroes of f in $\text{Cl}\, B(0,r)$ are $a_j \in B(0,r)\backslash\{0\}, j = 1,2,\ldots,n$, including multiplicities, then

$$\log(|f(z)|) = -\sum_{j=1}^{n} \log\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\,\frac{r\exp(i\theta)+z}{r\exp(i\theta)-z} \log(|f(r\exp(i\theta))|)\, d\theta$$

provided $f(z) \neq 0$.

Proof: Let $F(z) = f(z)\prod_{j=1}^{n} \frac{r^2 - \overline{a_j}z}{r(z-a_j)} \forall z \in \text{Cl}\, B(0,r)$. Then, clearly F is nowhere zero analytic on $\text{Cl}\, B(0,r)$. (For details, see the proof of the previous theorem.) Using the fact that $\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right| = 1$ on $|z| = r$, we obtain $|F(z)| = |f(z)|$ on $|z| = r$. Hence, $\log(F)$ is analytic and by applying the Poisson's formula (Theorem 7.1.10) for $\log(|f|)$, we get

$$\begin{aligned}&\log(|f(z)|) + \sum_{j=1}^{n} \log\left(\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right|\right)\\ &\quad = \log(|F(z)|)\end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\,\frac{r\exp(i\theta)+z}{r\exp(i\theta)-z}\log|F(r\exp(i\theta))|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\,\frac{r\exp(i\theta)+z}{r\exp(i\theta)-z}\log|f(r\exp(i\theta))|\,d\theta.$$

Thus, the theorem follows. □

7.2 POISSON'S INTEGRAL

Definition 7.2.1 Let u be a real-valued bounded piecewise continuous function on the unit circle C. Then, the Poisson integral of u is defined by

$$P_u(z) = \frac{1}{2\pi}\text{Re}\left(\int_C \frac{\zeta+z}{\zeta-z}\,\frac{u(\zeta)}{i\zeta}\,d\zeta\right),\ \forall z \in B(0,1).$$

LEMMA 7.2.2 Let u be a real-valued bounded piecewise continuous function on the unit circle C, and let γ be a circular arc given by $\gamma(t) = \exp(it),\ t \in [a,b]$, where $a, b \in \mathbb{R}$ with $0 \le b-a \le 2\pi$.

1. If $P_{u,\gamma}(z) = \frac{1}{2\pi}\text{Re}\left(\int_\gamma \frac{\zeta+z}{\zeta-z}\,\frac{u(\zeta)}{i\zeta}\,d\zeta\right),\ \forall z \in \gamma^c$, then $P_{u,\gamma}$ is harmonic on γ^c.
2. If u is continuous at $\zeta_0 \in C$, then $\lim_{z\to\zeta_0} P_u(z) = u(\zeta_0)$.

Proof:

1. If

$$f_1(z) = \frac{1}{2\pi i}\int_\gamma \frac{u(\zeta)}{\zeta-z}\,d\zeta,\ f_2(z) = \frac{1}{2\pi i}\int_\gamma \frac{u(\zeta)}{\zeta(\zeta-z)}\,d\zeta,\ \forall z \in \gamma^c,$$

then as in the proof of Theorem 4.3.2, we get that f_1 and f_2 are analytic on γ^c.
(*Note that the only difference between the current situation and Theorem 4.3.2 is that here, we have a piecewise continuous function u on γ, whereas ϕ is a continuous function on γ in Theorem 4.3.2. However, to follow the proof of Theorem 4.3.2, we need continuity of ϕ to get*

$\int_\gamma |\phi(\zeta)||d\zeta| < +\infty$. *As* u *is bounded piecewise continuous on* γ, *we also have* $\int_\gamma |u(\zeta)||d\zeta| < +\infty$. *Thus, the same proof will work in the current situation also.*)

Therefore, if $f(z) = f_1(z) + zf_2(z)$, $\forall z \in \gamma^c$, then f is analytic on γ^c and

$$\begin{aligned} \operatorname{Re} f(z) &= \operatorname{Re}\left(\frac{1}{2\pi}\int_\gamma \left[\frac{u(\zeta)}{\zeta - z} + \frac{zu(\zeta)}{\zeta(\zeta - z)}\right]\frac{d\zeta}{i}\right) \\ &= \operatorname{Re}\left(\frac{1}{2\pi}\int_\gamma \left[\frac{(\zeta + z)u(\zeta)}{i\zeta(\zeta - z)}\right] d\zeta\right) \\ &= P_{u,\gamma}. \end{aligned}$$

Thus, $P_{u,\gamma}$ is a harmonic function on γ^c.

2. Let $\zeta_0 = \exp(it_0)$ for some $t_0 \in \mathbb{R}$. First, we assume that $u(\zeta_0) = 0$. If $v : \mathbb{R} \to \mathbb{R}$ is defined by $v(t) = u(\exp(it))$, $\forall t \in \mathbb{R}$, then v is continuous at t_0. Hence, for a given $\epsilon > 0$, there exists $r > 0$ such that

$$t \in (t_0 - r, t_0 + r) \Rightarrow |v(t) - v(t_0)| < \frac{\epsilon}{2} \Rightarrow |u(\exp(it))| < \frac{\epsilon}{2}.$$

Let $a = t_0 - r$ and $b = t_0 + r$, so that $C = C_1 \cup C_2$, where

$$C_1 = \{\exp(it) : b - 2\pi \le t \le a\} \text{ and } C_2 = \{\exp(it) : a < t < b\}.$$

Since u is a piecewise continuous function on C_1 and on Cl C_2 P_{u,C_1} is harmonic on C_1^c and P_{u,C_2} is harmonic on $(\text{Cl } C_2)^c$. Now for each $\xi \in C_2$, using $\operatorname{Re}\left(\frac{\exp(it) + \xi}{\exp(it) - \xi}\right) = \frac{1 - |\xi|^2}{|\exp(it) - \xi|^2}$ (Cf. Proof of Theorem 7.1.10), we have

$$\begin{aligned} P_{u,C_1}(\xi) &= \frac{1}{2\pi}\operatorname{Re}\left(\int_{C_1} \frac{\zeta + \xi}{\zeta - \xi}\,\frac{u(\zeta)}{i\zeta}\,d\zeta\right) \\ &= \frac{1}{2\pi}\operatorname{Re}\left(\int_{b-2\pi}^{a} \frac{\exp(it) + \xi}{\exp(it) - \xi}\,u(\exp(it))\,dt\right) \\ &= \frac{1}{2\pi}\int_{b-2\pi}^{a} \frac{1 - |\xi|^2}{|\exp(it) - \xi|^2}\,u(\exp(it))\,dt = 0, \end{aligned}$$

as u is real valued and $|\xi| = 1$. In particular, $P_{u,C_1}(\zeta_0) = 0$. Being P_{u,C_1} is harmonic on C_1^c, it is continuous on C_1^c. Therefore, P_{u,C_1} is continuous at ζ_0. Hence, for a given $\epsilon > 0$, there exists $\delta > 0$ such that $|z - \zeta_0| < \delta \Rightarrow |P_{u,C_1}(z)| < \dfrac{\epsilon}{2}$. By the choice of C_2, we have $|u(\zeta)| < \dfrac{\epsilon}{2}, \forall \zeta \in C_2$. Therefore, for each $z \in B(0,1)$, we have

$$\begin{aligned}
|P_{u,C_2}(z)| &= \frac{1}{2\pi}\left|\mathrm{Re}\left(\int\limits_{\mathrm{Cl}\,C_2} \frac{\zeta+\xi}{\zeta-\xi}\,\frac{u(\zeta)}{i\zeta}\,d\zeta\right)\right| \\
&= \frac{1}{2\pi}\left|\int\limits_a^b \mathrm{Re}\left(\frac{\exp(it)+z}{\exp(it)-z}\right)\,u(\exp(it))\,dt\right| \\
&\quad \text{(as } u \text{ is real valued)} \\
&\leq \frac{1}{2\pi}\int\limits_a^b \left|\mathrm{Re}\left(\frac{\exp(it)+z}{\exp(it)-z}\right)\right|\,|u(\exp(it))|\,dt \\
&= \frac{1}{2\pi}\int\limits_a^b \frac{1-|z|^2}{|\exp(it)-z|^2}\,|u(\exp(it))|\,dt \\
&\leq \frac{\epsilon}{2}\frac{1}{2\pi}\int\limits_a^b \frac{1-|z|^2}{|\exp(it)-z|^2}\,dt \\
&\leq \frac{\epsilon}{2}\frac{1}{2\pi}\int\limits_{b-2\pi}^b \frac{1-|z|^2}{|\exp(it)-z|^2}\,dt \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

The last equality is obtained by using the Poisson's formula in equation (7.1) for the constant function 1. Thus, for $z \in B(0,1)$ with $|z-\zeta_0| < \delta$, we have

$$|P_u(z)| = |P_{u,C_1}(z) + P_{u,C_2}(z)| \leq |P_{u,C_1}(z)| + |P_{u,C_2}(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, we have proved that $\lim\limits_{z\to\zeta_0} u(z) = 0$.

Suppose $u(\zeta_0) \neq 0$, if $v = u - u(\zeta_0)$, then v is also a real-valued bounded piecewise continuous function on C with $v(\zeta_0) = 0$. So, applying the

previous case for v, we get that P_v is a harmonic function on $B(0,1)$ and $\lim_{z\to\zeta_0} P_v(z) = 0$. However,

$$\begin{aligned}
P_v(z) &= \frac{1}{2\pi}\mathrm{Re}\left(\int_C \frac{\zeta+z}{\zeta-z}\,\frac{v(\zeta)}{i\zeta}\,d\zeta\right)\\
&= \frac{1}{2\pi}\mathrm{Re}\left(\int_C \frac{\zeta+z}{\zeta-z}\frac{u(\zeta)}{i\zeta}d\zeta\right) - \frac{1}{2\pi}\mathrm{Re}\left(\int_C \frac{\zeta+z}{\zeta-z}\frac{u(\zeta_0)}{i\zeta}d\zeta\right)\\
&= P_u(z) - u(\zeta_0)\frac{1}{2\pi}\mathrm{Re}\left(\int_C \frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{i\zeta}\right), \quad (\text{as } u(\zeta_0)\in\mathbb{R})\\
&= P_u(z) - u(\zeta_0)\frac{1}{2\pi}\int_{b-2\pi}^{b} \frac{1-|z|^2}{|\exp(it)-z|^2}\,dt\\
&= P_u(z) - u(\zeta_0) \qquad (\text{applying Poisson's formula for } u).
\end{aligned}$$

Thus, we have $\lim_{z\to\zeta_0} P_u(z) = u(z_0)$. □

From Poisson's formula and from the proof of previous theorem, we can see that if u is a harmonic function on $B(0,r)$ and continuous on Cl $B(0,r)$, then the analytic function f on $B(0,r)$, whose real part is the given u, can be obtained by

$$f(z) = \frac{1}{i2\pi}\int_{|\zeta|=r} \frac{\zeta+z}{\zeta-z}\,\frac{u(\zeta)}{\zeta}\,d\zeta,\ \forall z\in B(0,r),$$

which is called the Schwarz's formula. The following theorem is also due to Schwarz.

THEOREM 7.2.3 (Schwarz's theorem)

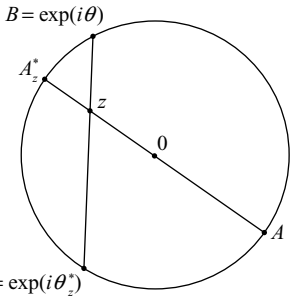
For a fixed $z\in B(0,1)$, let $u_z : [0,2\pi) \to [0,2\pi)$ be the function defined by $u_z(\theta) = \theta_z^*$, where θ_z^* is such that $\exp(i\theta)$, $\exp(i\theta_z^*)$ and z are collinear. If

$$U_z : C \to [0,2\pi) \text{ is defined by } U_z(\zeta) = u_z(\theta) = \theta_z^*,$$

where $\theta\in[0,2\pi)$ is unique (however depending on ζ) such that $\zeta = \exp(i\theta)$, $\forall\zeta\in C$, where $C = \{\zeta\in\mathbb{C} : |\zeta| = 1\}$. Then,

$$P_{U_z}(z) = \int_0^{2\pi} U_z(\exp(i\theta))\,d\theta_z^* = \int_0^{2\pi} U_z(\exp(i\theta_z^*))\,d\theta, \forall z\in B(0,1).$$

Proof: Note that $u_z : [0, 2\pi) \to [0, 2\pi)$ is a bijection, with $u_z = u_z^{-1}$.



By the intersecting chord theorem[1], we obtain that the area of the rectangle with sides $|z - A|$ and $|z - A_z^*|$ is equal to the area of the rectangle with sides $|z - B|$ and $|z - B_z^*|$. From the diagram, $|z - A|$ is the length of the line segment joining z and A, which is equal to $|z| + |A| = |z| + 1$ and $|z - A^*| = |A^*| - |z| = 1 - |z|$, and hence, we have

$$1 - |z|^2 = (1 - |z|)(1 + |z|) = |z - \exp(i\theta)|\ |z - \exp(i\theta_z^*)|. \tag{7.6}$$

As z lies between the points $\exp(i\theta)$ and $\exp(i\theta_z^*)$, we get that $z - \exp(i\theta)$ and $z - \exp(i\theta_z^*)$ lie on the same line but in opposite directions from z. Therefore, it follows that $\dfrac{z - \exp(i\theta)}{z - \exp(i\theta_z^*)} < 0$. If $\arg(z - \exp(i\theta)) = \varphi$ and $\arg(z - \exp(i\theta_z^*)) = \psi$, then we have $\varphi - \psi = \pi$ (if the arguments are written in $[0, 2\pi)$). Moreover, we have

$$\begin{aligned}(z - \exp(i\theta))(\bar{z} - \exp(-i\theta_z^*)) &= |z - \exp(i\theta)||\bar{z}\exp(-i\theta_z^*)|\exp(i(\varphi - \psi)) \\ &= -|z - \exp(i\theta)|\ |z - \exp(i\theta_z^*)| \\ &= |z|^2 - 1 \quad \text{(using equation 7.6).}\end{aligned}$$

Hence, by differentiating on both sides of the above equation, we get

$$\begin{aligned}& i\exp(i\theta)\,d\theta(\bar{z} - \exp(-i\theta_z^*)) - i\exp(-i\theta_z^*)d\theta_z^*(z - \exp(i\theta)) = 0 \\ \Rightarrow\quad & \frac{d\theta}{d\theta_z^*} = \exp(-i(\theta + \theta_z^*))\frac{z - \exp(i\theta)}{\bar{z} - \exp(-i\theta_z^*)}.\end{aligned}$$

[1] It is a theorem in 2D analytic geometry, proved by Euclid, which says that *in a circle, two straight lines cut one another, and the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

Using the fact that $d\theta$ and $d\theta_z^*$ are the usual Lebesgue measures on $[0, 2\pi]$ (positive measures) and using equation (7.6), we have

$$\frac{d\theta}{d\theta_z^*} = \left|\frac{d\theta}{d\theta_z^*}\right| = \left|\frac{z - \exp(i\theta)}{\bar{z} - \exp(-i\theta_z^*)}\right| = \frac{1 - |z|^2}{|\bar{z} - \exp(-i\theta_z^*)|^2}.$$

Therefore, by definition of P_{U_z}, we have

$$\begin{aligned} P_{U_z}(z) &= \frac{1}{2\pi}\mathrm{Re}\int_C \left(\frac{\zeta + z}{\zeta - z}\right) U_z(\zeta)\frac{d\zeta}{i\zeta} \\ &= \frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi} \left(\frac{\exp(i\theta) + z}{\exp(i\theta) - z}\right) U_z(\exp(i\theta))\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z|^2}{|\exp(i\theta) - z|^2} U_z(\exp(i\theta))\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} U_z(\exp(i\theta))\, d\theta_z^*. \end{aligned}$$

Using the fact that $u_z^{-1} = u_z$ and interchanging θ and θ_z^* in the above expression, we also get

$$P_{U_z}(z) = \frac{1}{2\pi}\int_0^{2\pi} U_z(\exp(i\theta_z^*))\, d\theta.$$

Hence, the theorem follows. □

It is interesting to note that the converse of the mean value property for harmonic functions is also true.

THEOREM 7.2.4 If u is a real-valued continuous function on Ω, satisfying the mean value property, $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + r\exp(it))\, dt$, $\forall a \in \Omega$, for some $r > 0$ such that Cl $B(a, r) \subseteq \Omega$, then u is harmonic on Ω.

Proof: Let $a \in \Omega$ be arbitrary and $r > 0$ be such that Cl $B(a, r) \subseteq \Omega$. If v : Cl $B(0, 1) \to \mathbb{R}$ is defined by

$$v(z) = u(a + rz), \forall z \in \text{Cl } B(0, 1),$$

then v is a continuous function on Cl $B(0,1)$. Therefore, by Lemma 7.2.2, if

$$P_v(z) = \begin{cases} \frac{1}{2\pi}\text{Re}\left(\int\limits_{|\zeta|=1} \frac{\zeta+z}{\zeta-z}\,\frac{v(\zeta)}{i\zeta}\,d\zeta\right) & \text{if } |z|<1 \\ v(z) & \text{if } |z|=1, \end{cases}$$

then P_v is a harmonic function on $B(0,1)$ and continuous on Cl $B(0,1)$. We define $\phi : \text{Cl}\, B(a,r) \to \mathbb{R}$ by

$$\phi(w) = P_v\left(\frac{w-a}{r}\right),\ \forall z \in \text{Cl}\, B(a,r).$$

If f is an analytic function on $B(0,1)$ such that $P_v = \text{Re}\, f$, then we have $\phi = \text{Re}\,(f \circ T)$, where $T : \text{Cl}\, B(a,r) \to \text{Cl}\, B(0,1)$ is defined by $T(w) = \frac{w-a}{r}$, $\forall w \in \text{Cl}\, B(a,r)$. Since T is analytic, we get that ϕ is harmonic on $B(a,r)$ and is continuous on Cl $B(a,r)$. In particular, if $|w-a| = r$, then

$$\phi(w) = P_v\left(\frac{w-a}{r}\right) = v\left(\frac{w-a}{r}\right) = u\left(a + r\left(\frac{w-a}{r}\right)\right) = u(w).$$

By hypothesis and by Theorem **??**, both u and ϕ satisfy mean value property, and hence, $u-\phi$ is a real function on Cl $B(a,r)$, satisfying the following properties:

1. $u-\phi$ is a continuous function on Cl $B(a,r)$,
2. $u-\phi = 0$ on $|w-a| = r$,
3. $u-\phi$ satisfies the mean value property.

As $u-\phi$ is a continuous function on the compact set Cl $B(a,r)$, it attains its maximum and minimum on Cl $B(a,r)$ (Theorem 1.6.28). However, applying Theorem 7.1.9, it follows that $u-\phi$ has neither maximum nor minimum on $B(a,r)$. Hence, $u-\phi$ attains its maximum and minimum on $|w-a| = r$ (on which this function vanishes). Thus, $u-\phi = 0$ on Cl $B(a,r)$. Therefore, $u = \phi$ is a harmonic function on $B(a,r)$. As $a \in \Omega$ is arbitrary, u is harmonic on Ω. □

THEOREM 7.2.5 (Harnack's inequality)
Let u be a non-negative real-valued continuous function on Cl $B(a,\rho)$ and harmonic on $B(a,\rho)$. If $|z-a| \le r < \rho$, then

$$\frac{\rho-r}{\rho+r}\,u(a) \le u(z) \le \frac{\rho+r}{\rho-r}\,u(a).$$

Proof: Let $z_0 \in \mathbb{C}$ be arbitrary such that $|z_0 - a| = r$. As

$$\rho - r = |\rho \exp(it)| - |z_0 - a| \leq |a + \rho \exp(it) - z_0| \leq |\rho \exp(it)| + |z_0 - a| \leq \rho + r,$$

we have $(\rho - r)^2 \leq |a + \rho \exp(it) - z_0|^2 \leq (\rho + r)^2$. Therefore, we have

$$\frac{\rho - r}{\rho + r} = \frac{\rho^2 - r^2}{(\rho + r)^2} \leq \frac{\rho^2 - r^2}{|a + \rho \exp(it) - z_0|^2} \leq \frac{\rho^2 - r^2}{(\rho - r)^2} = \frac{\rho + r}{\rho - r}.$$

As $u(\zeta) \geq 0$ for all $\zeta \in \mathbb{C}$ with $|\zeta - a| = \rho$, we have

$$\begin{aligned} \frac{\rho - r}{\rho + r}\, u\left(a + \rho \exp(it)\right) &\leq \frac{\rho^2 - r^2}{|a + \rho \exp(it) - z_0|^2}\, u\left(a + \rho \exp(it)\right) \\ &\leq \frac{\rho + r}{\rho - r}\, u\left(a + \rho \exp(it)\right), \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{\rho - r}{\rho + r}\, \frac{1}{2\pi} \int_0^{2\pi} u\left(a + \rho \exp(it)\right)\, dt \\ &\quad \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{|a + \rho \exp(it) - z_0|^2}\, u\left(a + \rho \exp(it)\right)\, dt \\ &\quad \leq \frac{\rho + r}{\rho - r}\, \frac{1}{2\pi} \int_0^{2\pi} u\left(a + \rho \exp(it)\right)\, dt, \forall t \in [0, 2\pi). \end{aligned}$$

Now, using mean value property for harmonic functions, we have

$$\frac{\rho - r}{\rho + r}\, u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{|a + \rho \exp(it) - z_0|^2}\, u\left(a + \rho \exp(it)\right)\, dt \leq \frac{\rho + r}{\rho - r}\, u(a).$$

If $v(w) = u(a + w),\ \forall w \in B(0, \rho)$, then using the Poisson's formula for v as in equation (7.1), we obtain that the integral in the above inequality is equal to $v(z_0) = u(a + z_0)$, and hence, we have

$$\frac{\rho - r}{\rho + r}\, u(a) \leq u(a + z_0) \leq \frac{\rho + r}{\rho - r}\, u(a), \forall z_0 \in \mathbb{C} \text{ with } |z_0 - a| = r.$$

Applying the maximum and minimum principles for harmonic functions, we get

$$\frac{\rho - r}{\rho + r}\, u(a) \le u(z) \le \frac{\rho + r}{\rho - r}\, u(a),\ \forall z \in \text{Cl}\, B(a, r).$$

Thus, the inequality is proved. □

THEOREM 7.2.6 (Harnack's principle)
Let u_n be harmonic on Ω_n, $\forall n \in \mathbb{N}$. Let Ω be a region such that for each $a \in \Omega$, there exists $r > 0$ and $N \in \mathbb{N}$ such that $\text{Cl}\, B(a, \rho) \subseteq \Omega_n$ and $u_n \le u_{n+1}$ on $\text{Cl}\, B(a, \rho)$, $\forall n \ge N$. Then, either $u_n \to \infty$ uniformly on every compact subset of Ω as $n \to \infty$ or $u_n \to u$ uniformly on every compact subset of Ω as $n \to \infty$ for some harmonic function u on Ω.

Proof: For an arbitrary $a \in \Omega$, there exist $\rho > 0$ and $N \in \mathbb{N}$ such that $\text{Cl}\, B(a, \rho) \subseteq \Omega_n$ and $u_n \le u_{n+1}$ on $\text{Cl}\, B(a, \rho)$, $\forall n \ge N$. If $n \ge m \ge N$, then $u_n - u_m$ is a non-negative harmonic function on $\text{Cl}\, B(a, \rho)$, and hence, by Harnack's inequality for $u_n - u_m$,

$$R_1\, (u_n - u_m)(a) \le (u_n - u_m)(z) \le R_2\, (u_n - u_m)(a),\ \forall z \in \text{Cl}\, B(a, r) \quad (7.7)$$

holds, where $R_1 = \frac{\rho - r}{\rho + r}$, $R_2 = \frac{\rho + r}{\rho - r}$, and $0 < r < \rho$ be arbitrary but fixed.

[2]Since $(u_{n+N}(a))$ is an increasing sequence of real numbers, either $u_n(a) \to +\infty$ as $n \to \infty$ or $u_n(a) \to b$ as $n \to \infty$, for some $b \in \mathbb{R}$.
We first assume that $u_n(a) \to +\infty$ as $n \to \infty$.

If $\mu = \inf\{u_m(z) : z \in \text{Cl}\, B(a, r)\}$ (which exists because u_m is a continuous function and $\text{Cl}\, B(a, r)$ is a compact set), then using the first inequality in equation (7.7), for every $z \in \text{Cl}\, B(a, r)$, we get

$$\begin{aligned} u_n(z) &\ge R_1\, (u_n - u_m)(a) + u_m(z) \\ &\ge R_1\, u_n(a) + [\mu - R_1\, u_m(a)] \to +\infty \text{ as } n \to \infty, \end{aligned}$$

as $R_1 > 0$. Therefore, $u_n \to +\infty$ uniformly on $\text{Cl}\, B(a, r)$ as $n \to \infty$. As every compact subset of Ω is contained in a finite union of such closed discs, we conclude that $u_n \to +\infty$ uniformly on every compact subset of Ω, as $n \to \infty$.

Suppose $u_n(a) \to b$ as $n \to \infty$ for some $b \in \mathbb{C}$. Then, arguing as before, using the second inequality in equation (7.7), for every $z \in \text{Cl}\, B(a, r)$, we get

$$0 \le u_n(z) - u_m(z) \le R_2\, (u_n - u_m)(a) \to 0 \text{ as } m, n \to \infty$$

[2]It is a simple exercise to the reader.

Therefore, (u_n) is a uniformly Cauchy sequence on Cl $B(a,r)$, and hence, it is uniformly convergent on Cl $B(a,r)$ and also on every compact subset of Ω. If $u = \lim_{n\to\infty} u_n$, then u is a uniform limit of a sequence of continuous functions on every compact subset of Ω and so it is a continuous function on Ω. Furthermore, for every $z \in \Omega$, if $r > 0$ and $n_0 \in \mathbb{N}$ are such that Cl $B(z,r) \subseteq \Omega_n$, for all $n \geq n_0$, then we have

$$u_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u_n(z + r\exp(it))\, dt, \ \forall n \geq n_0.$$

Allowing $n \to \infty$ on both sides, the uniform convergence of u_n on compact subsets of Ω implies that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r\exp(it))\, dt.$$

Hence, applying Theorem 7.2.4, we get that u is a harmonic function on Ω. □

7.3 SCHWARZ REFLECTION PRINCIPLE

In the following sequel, we shall use $\Omega^+ = \Omega \cap \{z \in \mathbb{C} : \text{Im}\, z > 0\}$, $\Omega^- = \Omega \cap \{z \in \mathbb{C} : \text{Im}\, z < 0\}$ and $\sigma = \Omega \cap \mathbb{R}$, where Ω is a region.

THEOREM 7.3.1 (Schwarz reflection principle)
Let Ω be a region such that $\Omega = \{\bar{z} : z \in \Omega\}$.

1. If $v : \Omega^+ \cup \sigma \to \mathbb{R}$ is continuous on $\Omega^+ \cup \sigma$ and harmonic on Ω^+ and if $v(z) = 0$, $\forall z \in \sigma$, then there exists a harmonic function $V : \Omega \to \mathbb{R}$ such that $V(z) = v(z)$, $\forall z \in \Omega^+ \cup \sigma$ and $V(\bar{z}) = -V(z)$, $\forall z \in \Omega$.

2. Furthermore, if $f = u + iv$ is analytic on Ω^+, where v is as in (1), then there exists an analytic function F on Ω such that $F(z) = f(z)$, $\forall z \in \Omega^+$ and $\overline{F(\bar{z})} = F(z)$, $\forall z \in \Omega$.

Proof: Define $V : \Omega \to \mathbb{R}$ by $V(z) = \begin{cases} v(z) & \text{if } z \in \Omega^+ \cup \sigma \\ -v(\bar{z}) & \text{if } z \in \Omega^- \end{cases}$ $\forall z \in \Omega$.
Immediately, it follows that $V(\bar{z}) = -V(z)$, $\forall z \in \Omega$; indeed,

$$\begin{aligned} V(\bar{z}) &= \begin{cases} v(\bar{z}) & \text{if } \bar{z} \in \Omega^+ \\ 0 & \text{if } \bar{z} \in \sigma \\ -v(z) & \text{if } \bar{z} \in \Omega^- \end{cases} \\ &= \begin{cases} v(\bar{z}) & \text{if } z \in \Omega^- \\ 0 & \text{if } z \in \sigma \\ -v(z) & \text{if } z \in \Omega^+ \end{cases} \\ &= -V(z). \end{aligned}$$

To prove that V is harmonic on Ω, we shall show that V is harmonic on $\sigma \cup \Omega^-$. For $(x,y) \in \Omega^-$, we have $\dfrac{\partial V}{\partial x}(x,y) = -v_x(x,-y)$, $\dfrac{\partial^2 V}{\partial x^2}(x,y) = -v_{xx}(x,-y)$, $\dfrac{\partial V}{\partial x}(x,y) = v_y(x,-y)$, and $\dfrac{\partial^2 V}{\partial x^2}(x,y) = -v_{yy}(x,-y)$. Therefore,

$$\frac{\partial^2 V}{\partial x^2}(x,y) + \frac{\partial^2 V}{\partial x^2}(x,y) = -v_{xx}(x,-y) - v_{yy}(x,-y) = 0,$$

as v is harmonic on Ω^+ and $(x,-y) \in \Omega^+$ whenever $(x,y) \in \Omega^-$. Therefore, V is harmonic on Ω^-. Let $x_0 \in \sigma \subseteq \Omega$ be arbitrary. Since Ω is open, choose $r > 0$ such that $B(x_0,r) \subseteq \Omega$, and hence, $B(x_0,r) \cap \mathbb{R} = (x_0 - r, x_0 + r)$. If $T(z) = \dfrac{z - x_0}{r}$, $\forall z \in \mathbb{C}$, then T maps $B(x_0,r)$ onto $B(0,1)$. Define $\varphi : B(0,r) \to \mathbb{R}$ by

$$\varphi(w) = (V \circ T^{-1})(w) = V(rw + x_0),\ \forall w \in B(0,1).$$

Clearly, φ is a continuous function on $\{\exp(i\theta) : 0 < \theta < 2\pi\}$; in other words, it is a piecewise continuous function on the unit circle. Therefore, by Lemma 7.2.2, P_φ is a harmonic function on $B(0,1)$, where

$$\begin{aligned} P_\varphi(w) &= \frac{1}{2\pi} \mathrm{Re} \left(\int_{|\xi|=1} \frac{w+\xi}{w-\xi} \frac{\varphi(\xi)}{i\xi}\, d\xi \right) \\ &= \frac{1}{2\pi} \int_{|\xi|=1} \frac{1-|w|^2}{|\xi - w|^2} \frac{\varphi(\xi)}{\xi}\, d\xi,\ \forall w \in B(0,1). \end{aligned}$$

Clearly, $P_\varphi \circ T$ is a harmonic function on $B(x_0,r)$, and for $z \in B(x_0,r)$,

$$(P_\varphi \circ T)(z) = P_\varphi \left(\frac{z - x_0}{r} \right)$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \int\limits_{|\xi|=1} \frac{r^2(1-|\frac{z-x_0}{r}|^2)}{|r\xi-(z-x_0)|^2} \frac{\varphi(\xi)}{\xi}\, d\xi \\
&= \frac{1}{2\pi} \int\limits_{|\zeta-x_0|=r} \frac{r^2-|z-x_0|^2}{|\zeta-z|^2} \frac{V(\zeta)}{(\zeta-x_0)}\, d\zeta \\
&\quad \text{(by the change of variable } \xi = T(\zeta) = \frac{\zeta-x_0}{r} \\
&\quad \text{and using the fact that } \varphi \circ T = V).
\end{aligned}
$$

We also see that

$$
\begin{aligned}
(P_\varphi \circ T)(\overline{z}) &= \frac{1}{2\pi} \int\limits_{|\zeta-x_0|=r} \frac{r^2-|\overline{z}-x_0|^2}{|\zeta-\overline{z}|^2} \frac{V(\zeta)}{(\zeta-x_0)}\, d\zeta \\
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{r^2-|\overline{z}-x_0|^2}{|x_0+r\exp(i\theta)-\overline{z}|^2}\, V(x_0+r\exp(i\theta))\, id\theta \\
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{r^2-|\overline{z}-x_0|^2}{|x_0+r\exp(-i\varphi)-\overline{z}|^2}\, V(x_0+r\exp(-i\varphi))\, id\varphi \\
&\quad \text{(using the change of variable } \varphi = -\theta) \\
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{r^2-|\overline{z-x_0}|^2}{|\overline{x_0+r\exp(i\varphi)-z}|^2}\, V(\overline{x_0+r\exp(i\varphi)})\, id\varphi \\
&\quad \text{(using } x_0 \in \mathbb{R}) \\
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{r^2-|z-x_0|^2}{|x_0+r\exp(i\varphi)-z|^2}\, [-V(x_0+r\exp(i\varphi))]\, id\varphi \\
&= -\frac{1}{2\pi} \int\limits_{|\omega-x_0|=r} \frac{r^2-|z-x_0|^2}{|\omega-z|^2} \frac{V(\omega)}{(\omega-x_0)}\, d\omega \\
&= -(P_\varphi \circ T)(z).
\end{aligned}
$$

Now, $V - P_\varphi \circ T$ is a harmonic function on $B(x_0, r) \cap \Omega^+$ with the following properties:

1. $(V - P_\varphi \circ T)(z) = 0, \forall z \in \{x_0 + r\exp(i\theta) : 0 < \theta < \pi\}$.

   Justification. Using Lemma 7.2.2, for $0 < s < r$, $(P_\varphi \circ T)(x_0 + s\exp(i\theta)) = P_\varphi(\frac{s}{r}\exp(i\theta)) \to \varphi(\exp(i\theta)) = V(x_0 + r\exp(i\theta))$ as $s \to r$. Hence, $(P_\varphi \circ T)(x_0 + r\exp(i\theta)) = \lim_{s\to r}(P_\varphi \circ T)(x_0 + s\exp(i\theta)) = V(x_0 + r\exp(i\theta))$.

2. $(V - P_\varphi \circ T)(z) = 0, \forall x \in B(x_0, r) \cap \sigma$.

   Justification. By definition, $V(x) = 0, \forall x \in \sigma$. From the fact that $P_\varphi \circ T$ is continuous on $B(x_0, r) \cap \sigma$ and $(P_\varphi \circ T)(\bar{z}) = (P_\varphi \circ T)(\bar{z})$, we have $(P_\varphi \circ T)(x) = \lim_{y\to 0}(P_\varphi \circ T)(x + iy) = -\lim_{y\to 0}(P_\varphi \circ T)(x - iy) = -(P_\varphi \circ T)(x) \Rightarrow (P_\varphi \circ T)(x) = 0, \forall x \in B(x_0, r) \cap \sigma$.

3. $(V - P_\varphi \circ T)(z) = 0, \forall z \in B(x_0, r) \cap \Omega^+$.

   Justification. From the above two properties, we have $V - P_\varphi \circ T$, which is a harmonic function on $B(x_0, r) \cap \Omega^+$ that vanishes on $\partial(B(x_0, r) \cap \Omega^+)$. Hence, by maximum and minimum principles for harmonic functions, $V - P_\varphi \circ T = 0$ on $B(x_0, r) \cap \Omega^+$.

Hence, $V = P_\varphi \circ T$ on $B(x_0, r) \cap \Omega^+$. Applying the same technique, we can prove that $V = P_\varphi \circ T$ on $B(x_0, r) \cap \Omega^-$. Thus, $V = P_\varphi \circ T$ is harmonic on σ, and hence, it is harmonic on Ω.

Next, for the given analytic function $f = u + iv$ on Ω^+, we define

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \\ U(z) & \text{if } z \in \sigma \\ \overline{f(\bar{z})} & \text{if } z \in \Omega^-, \ \forall z \in \Omega, \end{cases}$$

where $-U$ is a harmonic conjugate of V such that $f = U + iV$ on Ω^+. Using Result 2.2.29, we immediately get that F is analytic on Ω^-. As $-U$ is a harmonic conjugate of V, $U + iV$ is an analytic function, wherever they are defined. In particular, U is analytic on σ, as $V = 0$ on σ. Therefore, F is analytic on Ω, and by definition, $F(z) = f(z), \forall z \in \Omega^+$ and $F(\bar{z}) = F(z)$, $\forall z \in \Omega \setminus \sigma$. Let $z_0 \in \sigma$ be arbitrary. Find $r > 0$ such that $B(z_0, r) \subseteq \Omega$. If $\psi : B(z_0, r) \to \mathbb{R}$ is defined by $\psi(z) = U(z) - U(\bar{z})$ and $\Phi(z) = \dfrac{\partial \psi}{\partial x}(z) - i\dfrac{\partial \psi}{\partial y}(z)$, $\forall z \in B(z_0, r)$, then we claim that ψ is an harmonic function and Φ is an analytic function on $B(z_0, r)$. For $(x, y) \in B(x_0, r)$,

$$\psi_{xx}(x, y) + \psi_{yy}(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\psi(x, y)]$$

$$\begin{aligned} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[U(x,y) - U(x,-y)] \\ &= U_{xx}(x,y) + U_{yy}(x,y) - [U_{xx}(x,-y) + U_{yy}(x,-y)] \\ &= 0 \text{ (as } U \text{ is harmonic).} \end{aligned}$$

Therefore, ψ is a harmonic function on $B(x_0, r)$. Clearly, Φ has continuous partial derivatives with respect to x and y. Therefore, to prove Φ is analytic, we verify C–R equations only.

$$\begin{aligned} \frac{\partial}{\partial x}(\text{Re } \Phi) &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2} \\ &= -\frac{\partial^2 \psi}{\partial y^2} \qquad \text{(as } \psi \text{ is harmonic)} \\ &= \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}(\text{Im } \Phi) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial y}(\text{Re } \Phi) &= \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial y \partial x} \\ &= \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) \\ &= -\frac{\partial}{\partial x}(\text{Im } \Phi). \end{aligned}$$

Therefore, our claim holds.

Since $\psi(x) = 0$, $\forall x \in B(x_0, r) \cap \sigma \subseteq \mathbb{R}$, we have $\dfrac{\partial \psi}{\partial x} = 0$, $\forall x \in B(x_0, r)$. Furthermore, $\dfrac{\partial \psi}{\partial y} = \dfrac{\partial}{\partial y}[U(x,y) - U(x,-y)] = U_y(x,y) + U_y(x,-y)$ which implies that $\dfrac{\partial \psi}{\partial y}(x,0) = 2U_y(x,0) = -2V_x(x,0) = 0$, $\forall (x,0) \in B(x_0, r) \cap \sigma$, as V is a harmonic conjugate of U and $V = 0$ on σ. Thus, Φ vanishes on $B(x_0, r) \cap \sigma$, and hence, it vanishes on $B(x_0, r)$, by analytic continuation. Hence, $\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \psi}{\partial y} = 0$ on $B(x_0, r)$. Therefore, ψ is constant on $B(x_0, r)$. As $x = \bar{x}$, $\forall x \in \sigma$, we have $\psi(x) = U(x) - U(\bar{x}) = 0$, for all $x \in B(x_0, r) \cap \sigma$, and hence, we obtain $\psi(z) = 0$, $\forall z \in B(x_0, r)$. Thus, $U(z) = U(\bar{z})$, $\forall z \in B(x_0, r)$.

Note that for each $x_0 \in \sigma$, we have found a neighbourhood $B(x_0, r)$ in which U is a harmonic conjugate with the following properties:

(1) $U(z) + iV(z)$ is analytic on $B(x_0, r)$,

(2) $f(z) = U(z) + iV(z)$, on $B(x_0, r) \cap \Omega^+$,

(3) $U(z) = U(\bar{z})$, $\forall z \in B(x_0, r)$.

From the properties (2) and (3) of U, we can also have

$$\overline{f(\overline{z})} = \overline{U(\overline{z}) + iV(\overline{z})} = U(z) + iV(z) = f(z),\ \forall z \in B(x_0, r) \cap \Omega^-.$$

Therefore, if $x_j \in \sigma$ and U_j harmonic conjugate of V_j in $B(x_j, r_j)$ for some $r_j > 0, j = 1, 2$ with $B(x_1, r_1) \cap B(x_2, r_2) \neq \varnothing$, then $U_1(z) + iV(z) = f(z) = U_2(z) + iV(z), \forall z \in B(x_1, r_1) \cap B(x_2, r_2) \cap \Omega^+$. Therefore, by analytic continuation,

$$U_1(z) + iV(z) = f(z) = U_2(z) + iV(z),\ \forall z \in B(x_1, r_1) \cap B(x_2, r_2).$$

Therefore, F is a well-defined analytic function on Ω, satisfying the desired properties. □

7.4 RIEMANN MAPPING THEOREM

Definition 7.4.1 Let (f_n) be a sequence of analytic functions on a region Ω and let $A \subseteq \Omega$. Then, we say that (f_n) is uniformly bounded on A if there exists $M > 0$ such that $|f(z)| \leq M$, $\forall z \in \Omega$ and $\forall n \in \mathbb{N}$.

THEOREM 7.4.2 (Montel's theorem)
Let (f_n) be a sequence of analytic functions on a region Ω. If (f_n) is uniformly bounded on every compact subset of Ω, then there exists a subsequence of (f_n), which converges uniformly on every compact subset of Ω.

Proof: Fix $w \in \Omega$. Choose $\delta(w) > 0$ such that $\mathrm{Cl}\, B(w, \delta(w)) \subseteq \Omega$. First, we show that we can find a subsequence that converges uniformly on $B\left(w, \dfrac{\delta(w)}{2}\right)$. Now applying Theorem 5.1.5, we can expand each f_m as

$$f_m(z) = \sum_{n=0}^{\infty} \frac{f_m^{(n)}(w)}{n!}(z - w)^n,\ \forall z \in \mathrm{Cl}\, B\left(w, \frac{\delta(w)}{2}\right).$$

If $M = \sup\left\{|f_m(z)| : z \in \mathrm{Cl}\, B\left(w, \dfrac{\delta(w)}{2}\right), m \in \mathbb{N}\right\}$, then $M < +\infty$ and by Cauchy's estimate, we have

$$\left|\frac{f_m^{(n)}(w)}{n!}\right| \leq \frac{M2^n}{(\delta(w))^n},\ \forall m \in \mathbb{N}.$$

As $(f_m(w))$ is a bounded sequence in $\mathbb{C}$, there exists a subsequence $(f_{m,0})$ of (f_m) such that

$$(f_{m,0}(w)) := f_{1,0}(w), f_{2,0}(w), f_{3,0}(w), \ldots$$

converges. Using the fact that $\left(\frac{f'_{m,0}(w)}{1!}\right)$ is bounded, then there exists a subsequence $(f_{m,1})$ of $(f_{m,0})$ such that

$$\left(\frac{f'_{m,1}(w)}{1!}\right) := \frac{f'_{1,1}(w)}{1!}, \frac{f'_{2,1}(w)}{1!}, \frac{f'_{3,1}(w)}{1!}, \ldots$$

converges. Proceeding further, we get that for every $n \in \mathbb{N}$, there exists a subsequence $(f_{m,n})$ of $(f_{m,n-1})$ such that

$$\left(\frac{f^{(n)}_{m,n}(w)}{n!}\right) := \frac{f^{(n)}_{1,n}(w)}{n!}, \frac{f^{(n)}_{2,n}(w)}{n!}, \frac{f^{(n)}_{3,n}(w)}{n!}, \ldots$$

converges. Then, the diagonal subsequence $(f_{m,m})$ of (f_m),

$$\left(\frac{f^{(k)}_{m,m}(w)}{k!}\right) := \frac{f^{(k)}_{1,1}(w)}{k!}, \frac{f^{(k)}_{2,2}(w)}{k!}, \frac{f^{(k)}_{3,3}(w)}{k!}, \ldots$$

converges, for every $k = 0, 1, 2, 3, \ldots$. Now, we claim that if $(g_k) = (f_{m,m})$, then (g_k) converges uniformly on $B\left(w, \frac{\delta(w)}{2}\right)$. For a given $\epsilon > 0$, we choose $N_1, N_2 \in \mathbb{N}$ such that

$$\frac{1}{2^{N_1}} < \frac{\epsilon}{4M} \text{ and } \left|(g_j^{(n)} - g_k^{(n)})(w)\right| < \frac{\epsilon n! 2^{n-1}}{(N_1+1)(\delta(w))^n}, \ \forall k, j \geq N_2, \ \forall 1 \leq n \leq N_1.$$

For $j, k \geq N_2$ and $z \in B\left(w, \frac{\delta(w)}{2}\right)$, we have

$$\begin{aligned}
|g_j(z) - g_k(z)| &= \left|\sum_{n=0}^{\infty} \frac{(g_j - g_k)^{(n)}}{n!}(z-w)^n\right| \\
&\leq \sum_{n=0}^{N_1} \left|\frac{(g_j - g_k)^{(n)}}{n!}(z-w)^n\right| + \sum_{n=N_1+1}^{\infty} \left|\frac{(g_j - g_k)^{(n)}}{n!}(z-w)^n\right| \\
&\leq \sum_{n=0}^{N_1} \frac{\epsilon 2^{n-1}}{(N_1+1)(\delta(w))^n}|z-w|^n + \sum_{n=N_1+1}^{\infty} \frac{2M}{(\delta(w))^n}|z-w|^n \\
&< \frac{\epsilon}{2} + \frac{2M}{2^{N_1+1}} \sum_{n=0}^{\infty} \frac{1}{2^n}
\end{aligned}$$

$$\begin{aligned} &= \frac{\epsilon}{2} + \frac{4M}{2^{N_1+1}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\mathbb{C}$ is second countable (Theorem 1.4.48), we can choose a sequence of points $w_j \in \Omega$ such that $\Omega = \bigcup_{j=1}^{\infty} B\left(w_j, \frac{\delta(w_j)}{2}\right)$. Given a sequence of functions (f_m), which is uniformly bounded on compact sets, we find a subsequence $(\phi_{n,1})$ of (f_m), which converges uniformly on $B\left(w_1, \frac{\delta(w_1)}{2}\right)$. Next, we find a subsequence $(\phi_{n,2})$ of $(\phi_{n,1})$, which converges uniformly on $B\left(w_2, \frac{\delta(w_2)}{2}\right)$, and so on. The diagonal subsequence $(\phi_{n,n})$ converges uniformly on $B\left(w_j, \frac{\delta(w_j)}{2}\right)$, $\forall j \in \mathbb{N}$. Thus, $(\phi_{n,n})$ converges uniformly on each compact subset of Ω because every compact subset of Ω is contained in a finite union of the $B\left(w_j, \frac{\delta(w_j)}{2}\right)$. □

LEMMA 7.4.3 (Koebe's lemma)
Let Ω be a simply connected region such that $0 \in \Omega \subseteq B(0,1)$. Then, there exists a one-to-one analytic function g from Ω into $B(0,1)$ such that $g(0) = 0$ and $|g(z)| > |z|$, $\forall z \in \Omega \backslash \{0\}$.

Proof: For a given $\zeta \in B(0,1)\backslash\Omega$, we choose $\xi \in \mathbb{C}$ such that $\xi^2 = -\zeta$. Define $g_1(z) = T_{-\zeta}\left([T_{-\xi}(z)]^2\right)$, $\forall z \in \mathbb{C}$, where $T_w(z) = \dfrac{z-w}{1-\overline{w}z}$, $\forall z \in \mathbb{C}$. Now, $g_1(0) = T_{-\zeta}(\xi^2) = T_{-\zeta}(-\zeta) = 0$. We note that g_1 is not one-to-one because using the bijectiveness of $T_{-\xi}$ on $\mathbb{C}_\infty$, we can find $a, b_1, b_2 \in \mathbb{C}$ such that $a \neq 0$, $b_1 \neq b_2$, $T_{-\xi}(b_1) = a$, and $T_{-\xi}(b_2) = -a$, so that $g_1(b_1) = g_1(b_2)$. Therefore, there is no constant $c \in \mathbb{C}$ such that $g_1(z) = cz$, $\forall z \in \mathbb{C}$. Applying Schwarz's lemma, we have $|g_1(z)| < |z|$, $\forall z \in \Omega$. As T_ζ is nowhere zero on Ω and it is analytic, using Corollary 4.5.13, there exists an analytic function g_2 on Ω such that $g_2^2 = T_\zeta$ and $g_2(0) = \xi$. (By choosing a suitable value of $\log(-\zeta)$. This is possible because $[g_2(0)]^2 = T_\zeta(0) = -\zeta$ and $\xi^2 = -\zeta$). Define $g(z) = T_\xi\left(g_2(z)\right)$, $\forall z \in \Omega$. Then, g is analytic on Ω and

$$g(0) = T_\xi\left(g_2(0)\right) = T_\xi(\xi) = 0.$$

As $T_w^{-1} = T_{-w}$, $\forall w \in \mathbb{C}$, for $z \in \Omega\backslash\{0\}$, we have $g_1(z) \neq 0$ and

$$\begin{aligned} |g(z)| &> |g_1(g(z))| \\ &= \left|T_{-\zeta}\left([T_{-\xi} \circ T_\xi \circ g_2](z)\right)^2\right| \end{aligned}$$

$$\begin{aligned} &= \left|T_{-\zeta}\left(g_2(z)\right)^2\right| \\ &= \left|T_{-\zeta}\left(T_\zeta(z)\right)\right| \\ &= |z|. \end{aligned}$$

Thus, the lemma follows. □

THEOREM 7.4.4 (Riemann mapping theorem)
If $\Omega \subsetneq \mathbb{C}$ is a simply connected region, then there exists an analytic bijection from Ω onto $B(0,1)$.

Proof: Let $\zeta \in \mathbb{C}\backslash\Omega$ be fixed. If

$$f_1(z) = \sqrt{z},\ z \in \mathbb{C}\backslash\{(x,0) : x \leq 0\}$$

is the principal branch of $\sqrt{\cdot}$ and

$$f_2(z) = z - a,\ \forall z \in \Omega,$$

then $f_1 \circ f_2$ is analytic and one-to-one on Ω. Furthermore, if $\xi \in (f_1 \circ f_2)(\Omega)$, then the principal argument of ξ should be in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and hence, $-\xi \notin (f_1 \circ f_2)(\Omega)$. By open mapping theorem (Theorem 4.5.4), $(f_1 \circ f_2)(\Omega)$ turns to be an open set, and hence, for a given $\xi_0 \in (f_1 \circ f_2)(\Omega)$, there exists $\epsilon > 0$ such that $B(\xi_0, \epsilon) \subseteq (f_1 \circ f_2)(\Omega)$. Since $B(-\xi_0, \epsilon) = -B(\xi_0, \epsilon)$, no point of $B(-\xi_0, \epsilon)$ lies in $(f_1 \circ f_2)(\Omega)$. Therefore, for every $z \in \Omega$, $(f_1 \circ f_2)(z) \notin B(-\xi_0, \epsilon)$, and hence, $|(f_1 \circ f_2)(z) + \xi_0| \geq \epsilon$. Hence, it follows that if

$$f_3(z) = \frac{1}{(f_1 \circ f_2)(z) + \xi_0},\ \forall z \in \Omega,$$

then $f_3 : \Omega \to \operatorname{Cl} B\left(0, \epsilon^{-1}\right)$ is analytic and injective on Ω. For a fixed $z_0 \in \Omega$, if

$$f(z) = \frac{\epsilon(f_3(z) - f_3(z_0))}{3},\ \forall z \in \Omega,$$

then f is analytic on Ω, and $f(\Omega)$ is a region containing 0, by open mapping theorem (Theorem 4.5.4) and by Theorem 1.6.29. Since f_3 is injective on Ω, f is also injective on Ω. As $f_3 : \Omega \to \operatorname{Cl} B\left(0, \epsilon^{-1}\right)$, we have $f : \Omega \to \operatorname{Cl} B\left(0, \frac{2}{3}r\right)$. Thus, $f : \Omega \to G$ is an analytic bijective map, where $G = f(\Omega) \subsetneq B(0,1)$ is a region containing 0.

Let $\mathscr{H} = \{h : G \to B(0,1) : h$ be analytic injective with $h(0) = 0\}$. Let $a \in G\backslash\{0\}$ be arbitrarily fixed. Using $f \in \mathscr{H}$, we have $\mathscr{H} \neq \varnothing$. If

$M = \sup_{h \in \mathscr{H}} |h(a)|$, then $0 < M$ (since $0 \neq f \in \mathscr{H}$) and $M \leq 1$ (as $|h(a)| < 1$, $\forall z \in G$, $\forall h \in \mathscr{H}$). From the definition of M, we can find a sequence $\{h_n\}$ from $\mathscr{H}$ such that

$$|h_n(a)| \to M \text{ as } n \to \infty.$$

As $\{h_n\}$ is uniformly bounded on every compact subset of G, by Montel's theorem (Theorem 7.4.2), there exists an analytic function h on G such that $h_{n_j} \to h$ as $j \to \infty$ for some subsequence $\{h_{n_j}\}$ of $\{h_n\}$. Using the fact that each h_{n_j} is analytic, injective and $h_{n_j}(0) = 0$, we observe that h is also analytic (by Theorem 4.3.4), injective and $h(0) = 0$. For every $z \in G$, $h(z) = \lim_{j \to \infty} h_{n_j}(z) \in \text{Cl}\, B(0,1)$. Furthermore,

$$|h(a)| = \lim_{j \to \infty} |h_{n_j}(a)| = M > 0$$

implies that h is non-constant. Therefore, by open mapping theorem, $h(G)$ should be an open subset, and hence, no point of G should be in $\{w \in \mathbb{C} : |w| = 1\}$. Therefore, $h : G \to B(0,1)$, and hence, $h \in \mathscr{H}$. Now, we claim that $h : G \to B(0,1)$ is onto. Suppose $h(G) \subsetneqq B(0,1)$, then by Koebe's lemma, there exists an analytic injective function $g : h(G) \to B(0,1)$ such that

$$g(0) = 0 \text{ and } |g(z)| > |z|, \ \forall \in f(G).$$

Then, $(g \circ h) : G \to B(0,1)$ belongs to $\mathscr{H}$ and

$$|(g \circ h)(z)| > |h(z)| = M,$$

which contradicts the choice of M. Hence, h is onto. Therefore, $h \circ f : \Omega \to B(0,1)$ is the required analytic bijection. Thus, the proof is completed. □

The map constructed in the previous theorem is usually called a Riemann mapping from Ω onto $B(0,1)$.

RESULT 7.4.5 (Uniqueness of Riemann mapping)
If $F_j : \Omega \to B(0,1), j = 1,2$ are two Riemann mappings such that $F_j(z_0) = 0$, $j = 1,2$, and $\arg(F_1'(z_0)) = \arg(F_2'(z_0))$, for some $z_0 \in \Omega$, then $F_1 = F_2$ on Ω.

Proof: By Corollary 4.5.5, $F_2^{-1} : B(0,1) \to \Omega$ is an analytic bijection. Then, clearly $F_1 \circ F_2^{-1} : B(0,1) \to B(0,1)$ is an analytic bijection, satisfying

$$(F_1 \circ F_2^{-1})(0) = F_1(F_2^{-1})(0) = F_1(z_0) = 0.$$

Hence, by Corollary 4.5.10, we get $(F_1 \circ F_2^{-1})(z) = cz, \forall z \in B(0,1)$ for some $c \in \mathbb{C}$ with $|c| = 1$. For a given $z \in \Omega$, let $w = F_2(z)$ so that $F_1(z) = (F_1 \circ F_2^{-1})(w) = cw = cF_2(z)$. Using $\arg(F_1'(z_0)) = \arg(F_2'(z_0))$ and $F_1'(z_0) = cF_2'(z_0)$, we get $\arg(c) = 0$. Thus, $c = 1$. Hence the result follows. □

7.5 SCHWARZ–CHRISTOFFEL FORMULA

Note that the Riemann mapping theorem states the existence of an analytic bijection between a simply connected region and the open unit ball; for a practical purpose, this theorem does not give any procedure for constructing such a function. Such a formula is obtained by Schwarz and Christoffel separately. In this section, we shall see how this formula is obtained.

Definition 7.5.1 Let Ω be a region. We say that a sequence $\{z_n\}$ in Ω tends to the boundary $\partial\Omega$ of Ω; if for a given $z \in \Omega$, then there exist $\epsilon_z > 0$ and $N_z \in \mathbb{N}$ such that

$$|z_n - z| \geq \epsilon_z, \ \forall n \geq N_z.$$

We write this by $z_n \to \partial\Omega$ as $n \to \infty$.

Definition 7.5.2 Let Ω be a region and γ be a curve in Ω given by $\gamma(t)$, $t \in [a,b]$. We say that a curve γ in Ω tends to the boundary $\partial\Omega$ of Ω; if for a given $z \in \Omega$, there exist $\epsilon_z > 0$ and $t_z \in [a,b)$ such that

$$|\gamma(t) - z| \geq \epsilon_z, \ \forall t \geq t_z.$$

In this case, we write $\gamma(t) \to \partial\Omega$ as $t \to b$.

LEMMA 7.5.3 Let $\{z_n\}$ be a sequence in a region Ω. Then, $z_n \to \partial\Omega$ as $n \to \infty$ iff for each compact set $K \subseteq \Omega$, there exists $N_K \in \mathbb{N}$ such that $z_n \notin K, \forall n \geq N_K$.

Proof: Assume that $z_n \to \partial\Omega$ as $n \to \infty$ and $K \subseteq \Omega$ is the given compact set. For each $z \in K$, there exists $\epsilon_z > 0$ and $N_z \in \mathbb{N}$ such that $|z_n - z| \geq \epsilon_z, \ \forall n \geq N_z$. Since $\{B\left(x, \frac{\epsilon_x}{2}\right) : x \in K\}$ is a collection of open sets satisfying $K \subseteq \bigcup_{x \in K} B\left(x, \frac{\epsilon_x}{2}\right)$ and K is compact, there exist $x_1, x_2, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} B\left(x_j, \frac{\epsilon_{x_j}}{2}\right)$. If $N_K = \max_{1 \leq j \leq n} N_{x_j}$, then for each $x \in K$ and for $n \geq N_K$, there exists $j \in \{1, 2, \ldots, n\}$ such that $x \in B(x_j, \epsilon_{x_j})$ so that

$$|x - z_n| \geq |z_n - x_j| - |x_j - x| > \epsilon_{x_j} - \frac{\epsilon_{x_j}}{2} > 0.$$

Thus, $z_n \notin K, \forall n \geq N_K$.

Conversely, if for each compact set $K \subseteq \Omega$, there exists $N_K \in \mathbb{N}$ such that $z_n \notin K$, $\forall n \geq N_K$, then for a given $z \in \Omega$, we choose $r > 0$ such that $\text{Cl}\, B(z,r) \subseteq \Omega$. Clearly, $\text{Cl}\, B(z,r)$ is a compact subset of Ω, then by assumption, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $z_n \notin \text{Cl}\, B(z,r)$, $\forall n \geq N$, and hence, $|z - z_n| > \epsilon$. Thus, $z_n \to \partial\Omega$ as $n \to \infty$. □

THEOREM 7.5.4 Let f be a homeomorphism[3] of a region Ω onto another region G. Then, $z_n \to \partial\Omega$ as $n \to \infty$ iff $f(z_n) \to \partial G$ as $n \to \infty$.

Proof: Assume that $z_n \to \partial\Omega$ as $n \to \infty$. For a given compact set K of G, using the continuity of f^{-1}, we conclude that $f^{-1}(K)$ is compact (by Theorem 1.6.27). Then, by hypothesis, there exists $N \in \mathbb{N}$ such that $z_n \notin f^{-1}(K)$ for all $n \geq N$. Since f is a bijection, we have $f(z_n) \notin K$, $\forall n \geq N$. Thus, $f(z_n) \to \partial G$ as $n \to \infty$.

Conversely, assume that $f(z_n) \to \partial G$ as $n \to \infty$. For a given compact set T of Ω, we have $f(T)$, which is compact in G. Therefore, there exists $N \in \mathbb{N}$ such that $f(z_n) \notin f(T)$ for all $n \geq N$. Thus, $z_n \notin T$, $\forall n \geq N$.

Similarly, we can have the following theorem. □

THEOREM 7.5.5 If $f : \Omega \to G$ is a homeomorphism and $\gamma(t)$, $t \in [a,b]$ is a curve in Ω, then $\gamma(t) \to \partial\Omega$ as $t \to b$ iff $f(\gamma(t)) \to \partial G$ as $t \to b$.

Definition 7.5.6 Let Ω be a region such that $\partial\Omega$ contains a line segment γ with a parametric equation $\gamma(t) = (1-t)\zeta + t\xi$, $t \in (0,1)$. *(Note that we use open interval* $(0,1)$ *and not* $[0,1]$ *in the parametric equation.)* We say that γ is a free boundary arc if for every $\zeta \in \gamma$ and for a sufficiently small $r > 0$, then $B(\zeta, r) \cap \partial\Omega$ is an actual diameter[4] of $B(\zeta, r)$.

If γ is a free boundary arc, for $\zeta \in \gamma$, then there exists $r_0 > 0$ such that $B(\zeta, r) \cap \gamma$ is a diameter of $B(\zeta, r)$ for every $r \in (0, r_0)$. Hence, the two open half-discs (say H_1 and H_2) determined by this diameter does not intersect $\partial\Omega$. Then, we have two possible cases.

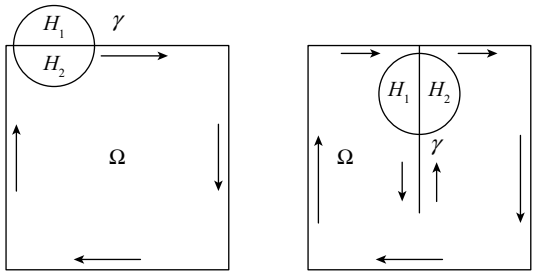
Case 1. $H_1 \subseteq \Omega$ and $H_2 \subseteq \Omega^c$.

Case 2. $H_1 \subseteq \Omega$ and $H_2 \subseteq \Omega$.

See the following diagrams, to understand the above possibilities.

[3] Homeomorphism is a continuous bijection with continuous inverse.

[4] Here, we mean the diameter by any chord passing through the centre of the disc.



Definition 7.5.7 Let $\zeta \in \gamma$ and $r > 0$ be such that $B(\zeta, r) \cap \partial\Omega$ is a diameter of $B(\zeta, r)$.

1. If one of the two half-open discs determined by the diameter of $B(\zeta, r)$ lies in Ω and the other half-disc lie in $(\text{Cl}\ \Omega)^c$, then ζ is called a one-sided free boundary point of Ω.

2. If both of the half-open discs determined by the diameter of $B(\zeta, r)$ lie in Ω, then ζ is called a two-sided free boundary point of Ω.

Definition 7.5.8 Let γ be a free boundary arc in Ω. Then,

1. ζ is a one-sided free boundary point of Ω, $\forall \zeta \in \gamma$ iff γ is called a one-sided free boundary arc.

2. ζ is a two-sided free boundary point of Ω, $\forall \zeta \in \gamma$ iff γ is called a two-sided free boundary arc.

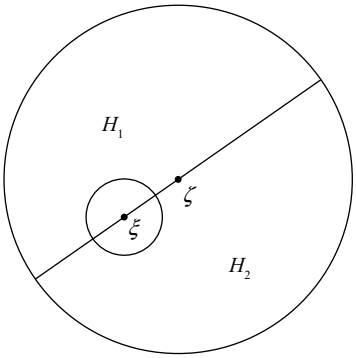
LEMMA 7.5.9 Let γ be a free boundary arc in Ω. Then,

1. γ is a one-sided free boundary arc of Ω iff γ has a one-sided free boundary point.

2. γ is a two-sided free boundary arc Ω iff γ has a two-sided free boundary point.

Proof: As the first statement of this theorem is equivalent to the second statement, we prove only the first statement. Obviously, it follows that if γ is a one-sided free boundary arc, then γ has a one-sided free boundary point. Conversely, assume that γ has a one-sided free boundary point ζ. By definition of free boundary arc, there exists $r > 0$ such that $B(\zeta, r) \cap \partial\Omega$ is a diameter of $B(\zeta, r)$, and by the definition of one-sided free boundary point of Ω, if H_1 and H_2 are the two half-open discs determined by the diameter,

then let $H_1 \subseteq \Omega$, $H_2 \subseteq (\text{Cl}\ \Omega)^c$. Then, for every $\xi \in B(\zeta, r) \cap \partial\Omega$ and for every $s \in (0, r - d(\zeta, \xi))$, we have $B(\xi, s) \cap \partial\Omega$, which is a diameter of $B(\xi, s)$. Furthermore, one of the two half-open discs determined by this diameter lies in $H_1 \subseteq \Omega$ and the other half-open disc lies in $H_2 \subseteq (\text{Cl}\ \Omega)^c$.



Thus, we have proved that if $A = \{\zeta \in \gamma : \zeta$ is a one-sided free boundary point$\}$, then A is an open subset of γ. By the same argument, we can show that if $B = \{\zeta \in \gamma : \zeta$ is a two-sided free boundary point$\}$, then B is an open subset of γ. However, γ is connected (begin a continuous image of $(0, 1)$), $\gamma = A \cup B$ and $A \cap B = \varnothing$. Given that $A \neq \varnothing$. Therefore, $B = \varnothing$ (otherwise, Ω is disconnected), and hence, $A = \gamma$. □

THEOREM 7.5.10 Let Ω be a simply connected region such that $\partial\Omega$ contains a one-sided free boundary arc γ. If f is a Riemann mapping of Ω onto $B(0, 1)$, then there exist a simply connected region Ω_0 containing $\Omega \cup \gamma$ and an analytic function F_0 on Ω_0 such that $F_0(z) = f(z)$, $\forall z \in \Omega$, and F_0 is one-to-one on γ and $F_0(\gamma) \subseteq \partial B(0, 1) = \{w \in \mathbb{C} : |w| = 1\}$.

Proof:

Case 1. $\gamma \subseteq \mathbb{R}$ and Ω is a subset of the upper half-plane.
Using that $f : \Omega \to B(0, 1)$ is a bijection and $0 \in B(0, 1)$, we find a unique point $z_0 \in \Omega$ such that $f(z_0) = 0$. As γ is a one-sided free boundary arc, for each $x_0 \in \gamma$, find $r > 0$ such that $z_0 \notin B(x_0, r)$ and $B(x_0, r) \cap \sigma$ is a diameter of $B(x_0, r)$, and let H be the semi-open disc $B(x_0, r) \cap \Omega$. Since H is a simply connected region and $f \neq 0$ on H, $\log(f)$ can be suitably defined as an analytic function on H (see Theorem 4.5.12). Let (z_n) be a sequence in H such that $z_n \to \gamma$ as

$n \to \infty$ be arbitrary. Using Theorem 7.5.4, we get $|f(z_n)| \to 1$ as $n \to \infty$. Therefore,

$$\text{Re } \log(f(z_n)) \to 0 \text{ as } n \to \infty.$$

If $g = i\log(f)$, then $\text{Im } g(z_n) = \log(|f|)(z_n) \to 0$ as $n \to \infty$, whenever $z_n \to \gamma$ as $n \to \infty$. Therefore, using Schwarz reflection principle (Theorem 7.3.1), there exists an analytic function G on $B(x_0, r)$ such that $G(z) = g(z)$, $\forall z \in H$ and $\overline{G(\overline{z})} = G(z)$, $\forall z \in B(x_0, r)$. If $F = \exp(-iG)$, then F is analytic on $B(x_0, r)$ such that $F(z) = f(z)$, $\forall z \in H$. As $x_0 \in \gamma$ is arbitrary, for each $x \in \gamma$, there exists $r_x > 0$, and analytic function F_x is analytic on $B(x, r_x)$ and $F_x(z) = f(z)$, $\forall z \in B(x, r_x) \cap \Omega$. We take $\Omega_0 = \Omega \cup \left(\underset{x \in \gamma}{\cup} B(x, r_x) \right)$. Clearly, Ω_0 is a simply connected region containing $\Omega \cup \gamma$. If $x_1, x_2 \in \gamma$ such that $B(x_1, r_{x_1}) \cap B(x_2, r_{x_2}) \neq \varnothing$, then $F_{x_1}(z) = F_{x_2}(z) = f(z)$, $\forall z \in B(x_1, r_{x_1}) \cap B(x_2, r_{x_2}) \cap \Omega$. Therefore, by analytic continuation, we have $F_{x_1} = F_{x_2}$ on $B(x_1, r_{x_1}) \cap B(x_2, r_{x_2})$. Therefore, if we define $F_0 : \Omega_0 \to \mathbb{C}$ by

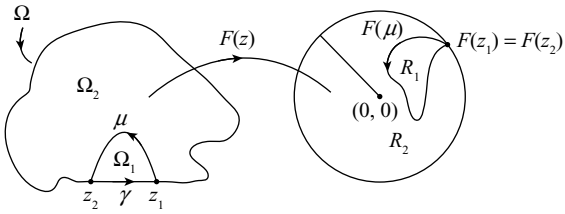
$$F_0(z) = \begin{cases} f(z) & \text{if } z \in \Omega \\ F_x(z) & \text{if } z \in B(x, r_x), \end{cases}$$

then F_0 is a well-defined analytic function on Ω_0 with $F_0(z) = f(z)$, $\forall z \in \Omega$. Since $F_0(z) = f(z)$ on Ω and

$$|f(z_n)| \to 1 \text{ as } n \to \infty, \text{ whenever } z_n \to \gamma \text{ as } n \to \infty, \tag{7.8}$$

we have $F_0(\gamma) \subseteq \partial B(0, 1)$.

To conclude the proof of this theorem, it remains to show that F_0 is one-to-one on γ. Suppose there exist $z_1, z_2 \in \gamma$ such that $F(z_1) = F(z_2)$. Then, choose a simple curve μ (with a parametric equation $\mu(t)$, $t \in [a, b]$) in Ω_0 such that $\mu(t) \in \Omega$, $\forall t \in (a, b)$, $\mu(a) = z_1$ and $\mu(b) = z_2$.



Clearly, $\mu \cup L_{z_2,z_1}$ is a simple closed curve in $\Omega \cup \gamma$ and $F(\mu) \cup \{F(z_1)\}$ is also a simple closed curve in $\{w \in \mathbb{C} : |w| \leq 1\}$; because $F(\mu \setminus \{z_1, z_2\}) = f(\mu \setminus \{z_1, z_2\}) \subseteq B(0,1)$, f is one-to-one and $F(z_1) = F(z_2) \in \partial B(0,1)$. The simple curve μ in Ω determines two regions Ω_1 and Ω_2 such that $\Omega \setminus \mu = \Omega_1 \cup \Omega_2$ (see the diagram). Similarly, the closed curve $F(\mu) \cup \{F(z_1)\}$ determines two regions R_1 and R_2 in $B(0,1)$ such that $B(0,1) \setminus f(\mu) = R_1 \cup R_2$. As $f : \Omega \to B(0,1)$ is a continuous bijection, we have $f(\Omega_1) \cup f(\Omega_2) = R_1 \cup R_2$, and $f(\Omega_1)$ and $f(\Omega_2)$ are connected subsets of $B(0,1) \setminus f(\mu)$. Therefore, either $f(\Omega_1) = R_1$ or $f(\Omega_1) = R_2$. In both cases, $(\mathrm{Cl}\, f(\Omega_1)) \cap \partial B(0,1) = \{F(z_1)\}$ (see the above diagram).

If $z \in \gamma$ is such that z lies between z_1 and z_2, then choose a sequence (z_n) in Ω_1 such that $z_n \to z$ as $n \to \infty$. This is possible because $z \in \mathrm{Cl}\, \Omega_1$ (see the diagram). By Theorem 7.5.4, we have $|f(z_n)| \to 1$ as $n \to \infty$, and by the continuity of f, we also have $|f(z_n)| \to |f(z)|$ as $n \to \infty$. Therefore, $|f(z)| = 1$. As $f(z) \in (\mathrm{Cl}\, f(\Omega_1)) \cap \partial B(0,1) = \{F(z_1)\}$, we must have $f(z) = F(z) = F(z_1)$. Hence, F is a constant function on the line segment L_{z_2,z_1} joining z_2 and z_1. Thus, by analytic continuation, F becomes constant on $\Omega_0 \Rightarrow F = f$, which is constant on Ω. This is a contradiction to the bijectiveness of f. Therefore, F is one-to-one on γ.

Case 2. For an arbitrary γ and an arbitrary Ω as in the statement of the theorem, first we find a linear fractional transform S such that

1. S maps the straight line passing along γ onto the real axis,
2. the entire region Ω is mapped into the upper half-plane,
3. its unique finite pole lies in the complement of $\mathrm{Cl}\, \Omega$.

If $\Omega_S = S(\Omega)$, then Ω_S is a region such that $S(\gamma)$ is a part of its boundary on the real axis. If $g = f \circ S^{-1}$ on Ω_S, then g is an analytic function on Ω_S, and g and Ω_S are suitable for applying Case 1 of proof of this theorem. Therefore, we get an analytic function G on a region $\Omega_{S,0}$ containing $\Omega_S \cup S(\gamma)$ such that $G(z) = g(z)$, $\forall z \in \Omega_S$. Thus, we take $\Omega_0 = S^{-1}(\Omega_{S,0})$ and the required analytic function as $F = G \circ S$ on Ω_0. □

Remark 7.5.11: We point out that according to the notations in the proof of previous theorem, $|F_0(z)| > 1$, $\forall z \in \Omega_0 \setminus \mathrm{Cl}\, \Omega$.

We justify the above statement for the case $\gamma \subseteq \mathbb{R}$, and the general case follows by applying a similar technique used in Case 2 of proof of previous

theorem. If $z \in B(x, r_x) \cap (\text{Cl } \Omega)^c$ for some $x \in \gamma$, then $F_0(z) = F_x(z)$, and by the construction of F_x (from the previous proof), $F_x(z) = \exp(-iG(z))$ for some analytic function G on $B(x, r_x)$, thereby satisfying

$$G(z) = \exp(i \log(f)),\ \forall z \in B(x, r_x) \cap \Omega \text{ and } \overline{G(\overline{z})} = G(z),\ \forall z \in B(x, r_x).$$

Therefore, for $z \in B(x, r_x) \cap (\text{Cl } \Omega)^c$, find $z_1 \in B(x, r_x) \cap \Omega$ such that $\overline{z_1} = z$. We have

$$\begin{aligned}
|F(z)| &= \exp(-iG(z)) \\
&= |\exp(-iG(\overline{z_1}))| \\
&= |\exp(-i\overline{G(z_1)})| \\
&= |\exp(-i\overline{[i \log(f(z_1))]})| \\
&= |\exp(-\log(\overline{f(z_1)}))| \\
&= \exp(\text{Re } [-\log(\overline{f(z_1)})]) \\
&= \exp(\log(|\overline{f(z_1)}|)^{-1}) \\
&= |f(z_1)|^{-1} > 1, \text{ as } f : \Omega \to B(0, 1) \text{ and } z_1 \in \Omega.
\end{aligned}$$

COROLLARY 7.5.12 If f is a Riemann mapping of $B(a, r)$ onto $B(0, 1)$ and μ is an circular arc on the boundary of $B(a, r)$ with a parametric equation $a + r\exp(i\theta)$, $\theta_1 < \theta < \theta_2$ for some $\theta_1, \theta_2 \in \mathbb{R}$, then there exist a simply connected region U_0 containing $B(a, r) \cup \mu$ and an analytic function F_0 on U_0 such that $F_0(z) = f(z),\ \forall z \in B(a, r)$, where F_0 is one-to-one on μ and $F_0(\mu) \subseteq \partial B(0, 1)$.

Proof of this corollary is similar to the proof of Theorem 7.5.10 with a small change. Here, first we choose a linear fractional transform, which maps the unit circle to the real line (as in Case 2 of proof of Theorem 7.5.10), and then, we follow the proof of Case 1 as in the proof of the same theorem.

THEOREM 7.5.13 Let Ω be a simply connected region enclosed by the given closed polygon $\bigcup_{j=1}^{n-1} L_{z_j, z_{j+1}} \cup L_{z_n, z_1}$. If the interior angle at z_j is $\alpha_j \pi$, $\forall j = 1, 2, \ldots, n$, then, the unique Riemann mapping $f : \Omega \to B(0, 1)$ can be extended as a homeomorphism $F : \text{Cl } \Omega \to \text{Cl } B(0, 1)$ such that $F(z_j) = w_j$ for some w_j with $|w_j| = 1$, and $F(L_{z_j, z_{\rho(j)}})$ (including the end points) is the circular arc joining w_j and $w_{\rho(j)}$ (including the end points), where $\rho : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the bijection defined by $\rho(j) = \begin{cases} j+1 & \text{if } j < n \\ 1 & \text{if } j = n \end{cases}$, $\forall j = 1, 2, \ldots, n$ so that $F(\partial\Omega) = \partial B(0, 1)$.

Proof: If $\gamma_j = \{(1-t)z_j + tz_{\rho(j)} : (0,1)\}$ and $\rho(j)$ is as in the statement of the theorem, $\forall j = 1, 2, \ldots, n$, then γ_j is a one-sided free boundary arc. (Note that γ_j does not include the end points z_j and $z_{\rho(j)}$, whereas $L_{z_j, z_{\rho(j)}}$ includes the end points z_j and $z_{\rho(j)}$.) Then, applying the previous theorem repeatedly, f can be extended as an analytic function G on a region containing Ω and γ_j, $\forall j = 1, 2, \ldots, n$ and $G(\gamma_j) \subseteq \partial B(0,1)$. Note that G is not defined at z_j, $\forall j = 1, 2, \ldots, n$. Fix $j \in \{1, 2, \ldots, n\}$ arbitrarily. Now, find $r_j > 0$ such that $z_k \notin B(z_j, r_j)$ if $j \neq k \in \{1, 2, \ldots, n\}$. Denote $B(z_j, r_j) \cap \Omega$ by S_j. As $z - z_j \neq 0$, $\forall z \in S_j$, $\log(z - z_j)$ is defined as an analytic function on S_j, and hence, if

$$\zeta = \phi_j(z) = (z - z_j)^{1/\alpha_j} = \exp((1/\alpha_j)\log(z - z_j)),\ \forall z \in S_j,$$

then $\phi_j : S_j \to H_j$ is an analytic bijection, where

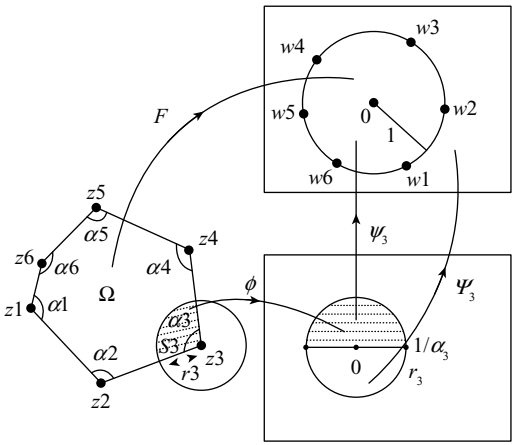
$$H_j = \left\{ r_j^{1/\alpha_j} \exp(i\theta) \in \mathbb{C} : \frac{\theta_j}{\alpha_j} < \theta < \frac{\theta_j}{\alpha_j} + \pi \right\},\ \text{where } \theta_j = \arg(z_{\rho(j)} - z_j),$$

an open semi-disc with centre 0 and radius r_j^{1/α_j}. (Because

$$\begin{aligned} z \in S_j &\Leftrightarrow \theta_j < \arg(z - z_j) < \theta_j + \alpha_j\pi \text{ and } |z - z_j| < r_j. \\ &\Leftrightarrow \frac{\theta_j}{\alpha_j}\pi < \arg(z - z_j)^{1/\alpha_j} < \frac{\theta_j}{\alpha_j} + \pi \text{ and } |z - z_j|^{1/\alpha_j} < r_j^{1/\alpha_j}. \\ &\Leftrightarrow \phi_j(z) \in H_j). \end{aligned}$$

Then, the inverse of ϕ_j is also an analytic function, which is given by

$$z = \phi_j^{-1}(\zeta) = z_j + \zeta^{\alpha_j} = z_j + \exp(\alpha_j \log(\zeta)),\ \forall \zeta \in H_j.$$



Define $\psi_j : H_j \to \mathbb{C}$ by $\psi_j = f \circ \phi_j^{-1}$. Then, ψ_j is analytic on H_j. If ζ tends to the diameter of H_j, then $z \to [L_{z_{\rho^{-1}(j)}z_j} \cup L_{z_j z_{\rho(j)}}] \cap B(z_j, r_j)$ (that is, z tends to the union of two line segments in the boundary of S_j).
$\Rightarrow f(z) \to \partial U$ (by Theorem 7.5.4) $\Rightarrow |\psi_j(\zeta)| = |f(\phi_j^{-1}(\zeta))| = |f(z)| \to 1$. Therefore, as in the proof of previous theorem, by reflection principle, there exists an analytic function $\Psi_j : B(0, r_j^{1/\alpha_j}) \to \mathbb{C}$ such that $\Psi_j(\zeta) = \psi_j(\zeta)$, $\forall \zeta \in H_j$. Now, we define $F : \text{Cl}\ \Omega \to \mathbb{C}$ by

$$F(z) = \begin{cases} G(z) & \text{if } z \in (\text{Cl}\ \Omega) \setminus \{z_j : j = 1, 2, \ldots, n\} \\ \Psi_j(0) & \text{if } z = z_j,\ j = 1, 2, \ldots, n. \end{cases}$$

To prove F is continuous on Cl Ω, it is sufficient to check its continuity at z_j, $\forall j = 1, 2, \ldots, n$. Let $j \in \{1, 2, \ldots, n\}$ be arbitrary and (x_m) be a sequence from Cl Ω such that $x_m \to z_j$ as $m \to \infty$. We first choose $M \in \mathbb{N}$ such that $x_{M+m} \in B(z_j, r_j)$, $\forall m \in \mathbb{N}$.

Case 1. $x_{M+m} \in \Omega$, $\forall m \in \mathbb{N}$.
Using $\lim\limits_{z \to z_j} \phi_j(z) = 0$, we get $\phi_j(x_{M+m}) \to 0$ as $m \to \infty$. As Ψ_j is continuous at 0, we have $\Psi_j(\phi_j(x_{M+m})) \to \Psi_j(0)$ as $m \to \infty$. However,

$$\Psi_j(\phi_j(x_{M+m})) = \psi_j(\phi_j(x_{M+m})) = f(x_{M+m}) = G(x_{M+m}) = F(x_{M+m}),$$

and hence, $F(x_{M+m}) \to \Psi_j(0) = F(z_j)$ as $m \to \infty$.

Case 2. $x_{M+m} \in \gamma_j \cap B(z_j, r_j)$, $\forall m \in \mathbb{N}$.
Since G is continuous on γ_j for each $m \in \mathbb{N}$, there exists $0 < \delta_m < \frac{1}{m}$ such that

$$|z - x_{M+m}| < \delta_m \Rightarrow |G(z) - G(x_{M+m})| < \frac{1}{m}.$$

As $\gamma_j \subseteq \partial S_j$, we choose $u_{M+m} \in B(x_{M+m}, \delta_m) \cap S_j \subseteq \Omega$, $\forall m \in \mathbb{N}$. Using

$$\begin{aligned} |u_{M+m} - z_j| &\leq |u_{M+m} - x_{M+m}| + |x_{M+m} - z_j| \\ &\leq \frac{1}{m} + |x_{M+m} - z_j| \to 0 \text{ as } m \to \infty, \end{aligned}$$

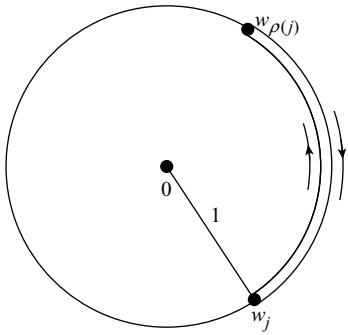
it follows that $u_{M+m} \to z_j$ as $m \to \infty$. Therefore, by Case 1, we have $F(u_{M+m}) \to F(z_j)$ as $m \to \infty$. Therefore,

$$\begin{aligned} |F(x_{M+m}) - F(z_j)| &\leq |F(x_{M+m}) - F(u_{M+m})| + |F(u_{M+m}) - F(z_j)| \\ &\leq \frac{1}{m} + |F(u_{M+m}) - F(z_j)| \to 0 \text{ as } m \to \infty. \end{aligned}$$

Case 3. $x_{M+m} \in \gamma_{\rho^{-1}(j)} \cap B(z_j, r_j)$, $\forall m \in \mathbb{N}$.
This case is similar to the previous case.

Thus, F is continuous on Cl Ω. If we denote $\Psi_j(0) = w_j$, then clearly we have $F(z_j) = w_j$ and $|w_j| = 1$, as $F(z_j) = f(z_j) = \lim_{m\to\infty} z_{j,m}$ for some $z_{j,m} \in \Omega$ such that $z_{j,m} \to z_j$ as $m \to \infty$, and by Theorem 7.5.4, $|f(z_{j,m})| \to 1$ as $m \to \infty$, $\forall j = 1, 2, \ldots, n$. Therefore, $F(\partial\Omega) \subseteq \partial B(0, 1)$. As $\partial\Omega$ is a closed curve in Cl Ω whose image under the continuous function F should be a closed curve in $\partial B(0, 1)$, there are only two possibilities:

1. $F(\partial\Omega) = \partial B(0, 1)$.
2. $F(\partial\Omega)$ is a circular arc contained in $\partial B(0, 1)$, which is traversed in both directions as in the following diagram.



(In the above diagram, actually there is a single circular arc from w_j and $w_{\rho(j)}$, which is traversed in both directions. To make it clear, we have drawn the arc using double lines.)

We claim that the second case is not possible because in the second case, we have

$$\begin{aligned} 0 &= \text{WN}(F(\partial\Omega), 0) \\ &= \frac{1}{i2\pi} \int_{F(\partial\Omega)} \frac{dw}{w} \\ &= \frac{1}{i2\pi} \int_{\partial\Omega} \frac{f'(z)}{f(z)}\, dz \end{aligned}$$

(by the change of variable $w = f(z)$)

$$
\begin{aligned}
&= \text{WN}(\partial\Omega, z_0), \text{ where } z_0 \in \Omega \text{ such that } F(z_0) = 0 \\
&\quad \text{(by Theorem 4.5.2)} \\
&= 1,
\end{aligned}
$$

which is a contradiction. Therefore, our claim holds. Therefore, the images of γ_j and γ_k under F are pairwise disjoint and $w_j \neq w_k$ if $j \neq k$. Hence, $F : \text{Cl}\,\Omega \to \text{Cl}\,B(0,1)$ is a bijection. As Cl Ω is a closed and bounded subset of $\mathbb{C}$, it is a compact set. Therefore, for every open set E of Cl Ω, (Cl Ω) $\setminus E$ is closed, and hence, it is compact, whose image under the continuous function F is also a compact subset of Cl $B(0,1)$, by Theorem 1.6.27. Therefore, $F((\text{Cl}\,\Omega) \setminus E) = (\text{Cl}\,B(0,1)) \setminus F(E)$ is closed in Cl $B(0,1)$. Therefore, $(F^{-1})^{-1}(E) = F(E)$ is open in Cl $B(0,1)$ whenever E is open in Cl Ω. Thus, we get that $F^{-1} : \text{Cl}\,B(0,1) \to \text{Cl}\,\Omega$ is a continuous map, by Theorem 1.6.24, and hence, $F : \text{Cl}\,\Omega \to \text{Cl}\,B(0,1)$ is a homeomorphism. □

THEOREM 7.5.14 (Schwarz–Christoffel formula for open unit disc)
Let Ω be a bounded simply connected region enclosed by a polygon with vertices z_j and interior angles $\alpha_j\pi$ at z_j, $j = 1, 2, \ldots, n$. If f is a Riemann mapping from Ω onto $B(0,1)$, then there exist constants c_1 and c_2 such that

$$f^{-1}(w) = c_1 \int_0^w \prod_{j=1}^{n} (\zeta - w_j)^{-\beta_j}\, d\zeta + c_2,$$

where the integral is defined over any curve joining 0 and w in $B(0,1)$, $\beta_j = 1 - \alpha_j$, $w_j = \lim_{z \to z_j} f(z)$, $\forall j = 1, 2, \ldots, n$.

Proof: From geometry, we first note that sum of the exterior angles of a closed polygon is 2π. That is, $\sum_{j=1}^{n} \beta_j = 2$. We shall use the notations as in the proof of previous theorem. By applying Corollary 7.5.12, repeatedly, for the the circular arc μ_j joining w_j and $w_{\rho(j)}$ (which excludes the end points), $j = 1, 2, \ldots, n$, there exists a simply connected region U_0 containing Cl $B(0,1) \setminus \{w_j : 1 \leq j \leq n\}$ and an analytic function $\mathscr{F}$ on U_0 such that $\mathscr{F} = F^{-1}$ on $B(0,1)$.

Next, we prove that the function $\Psi_j'(0) \neq 0$. Suppose that $\Psi_j'(0) = 0$. Then, by local correspondence theorem, there exist $0 < \delta < r_j^{1/\alpha_j}$ and $0 < \epsilon < 1$ such that for each $w \in B(w, \epsilon)$, there exist at least two distinct points $a, b \in B(0, \delta)$ such that $\Psi_j(a) = w = \Psi_j(b)$. By Theorem 7.5.10 and by Remark 7.5.11, we see that

$$|\Psi_j| = 1 \text{ on the diameter of the disc } B(0, r_j^{1/\alpha_j})$$

and

$$|\Psi_j| > 1 \text{ on } \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

Therefore, $a, b \in H_j$. Observing, $\psi_j : H_j \to \mathbb{C}$ is defined by $\psi_j = f \circ \phi_j^{-1}$ and f is one-to-one on $\phi^{-1}\left(H_j\right)$, ψ_j is one-to-one on H_j. Since $\Psi_j = \psi_j$, $\forall z \in H_j$, Ψ_j is one-to-one on H_j. Therefore, $\Psi_j(a) = \Psi_j(b)$ implies that $a = b$, which is a contradiction. Hence, our claim holds. Then, again using local corresponding theorem, there exist $0 < \delta_j < r_j^{1/\alpha_j}$ and $0 < \epsilon_j < 1$ such that $\Psi_j : B(0, \delta_j) \to B(w_j, \epsilon_j)$ is an analytic bijection. Then, by Corollary 4.5.5, we have Ψ_j^{-1}, which is also differentiable at w_j and $(\Psi_j^{-1})'(w_j) = \dfrac{1}{\Psi_j'(0)} \neq 0$. Observing $\Psi_j^{-1}(w_j) = 0$, we conclude that Ψ_j^{-1} has simple zero at w_j, and hence, there exists an analytic function P_j on $B(w_j, t_j)$ for some $0 < t_j < \epsilon$ such that $P_j(w) \neq 0$ and $\zeta = \Psi_j^{-1}(w) = (w - w_j)P_j(w)$, $\forall w \in B(w_j, t_j)$. Then, we can write,

$$\zeta^{\alpha_j} = (\Psi_j^{-1}(w))^{\alpha_j} = (w - w_j)^{\alpha_j} P_j^{\alpha_j}(w), \ \forall w \in B(w_j, t_j) \setminus L_{w_j, w_j - t_j},$$

where $(\Psi_j^{-1}(w))^{\alpha_j}$, and $(w - w_j)^{\alpha_j}$ are analytic functions on $B(w_j, t_j) \setminus L_{w_j, w_j - t_j}$, and $P_j^{\alpha_j}$ is an analytic function on $B(w_j, t_j)$, each of which is defined by using exp and principal branch of log.

By the previous theorem, $F : \operatorname{Cl} \Omega \to \operatorname{Cl} B(0, 1)$ is a homeomorphism such that $F(z) = f(z)$, $\forall z \in \Omega$. Therefore, for each $w \in B(w_j, t_j)$, first we choose $\zeta \in B(0, \delta_j)$ such that $\Psi_j(\zeta) = w$, and hence,

$$F^{-1}(w) = F^{-1}(\Psi_j(\zeta)) = (F^{-1} \circ F \circ \phi^{-1})(\zeta) = \phi^{-1}(\zeta) = z_j + \zeta^{\alpha_j}.$$

Therefore, we have

$$F^{-1}(w) - z_j = \zeta^{\alpha_j} = (w - w_j)^{\alpha_j} Q_j(w), \ \forall w \in B(w_j, t_j) \setminus L_{w_j, w_j - t_j}, \quad (7.9)$$

where $Q_j = P_j^{\alpha_j}$. By differentiating on both sides of equation (7.9), we get

$$\begin{aligned} (F^{-1})'(w) &= \alpha_j (w - w_j)^{\alpha_j - 1} Q_j(w) + (w - w_j)^{\alpha_j} Q_j'(w) \\ \Rightarrow (w - w_j)^{1 - \alpha_j} (F^{-1})'(w) &= \alpha_j Q_j(w) + (w - w_j) Q_j'(w). \\ \Rightarrow (w - w_j)^{\beta_j} (F^{-1})'(w) &= \alpha_j Q_j(w) + (w - w_j) Q_j'(w). \end{aligned} \quad (7.10)$$

We note that the function on the right-hand side of equation (7.10) is an analytic function on $B(w_j, t_j)$, whereas the function on the left-hand side is an analytic function on $B(w_j, t_j) \setminus L_{w_j, w_j - t_j}$. Define $\mathscr{I}_j : B(w_j, t_j) \to \mathbb{C}$ by

$$\mathscr{I}_j(w) = \alpha_j Q_j(w) + (w - w_j) Q_j'(w), \ \forall w \in B(w_j, t_j). \tag{7.11}$$

Using equations (7.10) and (7.11) and by analytic continuation, we have

$$\mathscr{F}'(w)(w - w_j) = \mathscr{I}_j(w)(w - w_j), \ \forall w \in B(w_j, t_j). \tag{7.12}$$

If $V_0 = U_0 \cup \bigcup_{j=1}^{n} B(w_j, t_j)$, then $H : V_0 \to \mathbb{C}$ is defined by

$$H(w) = \mathscr{F}'(w) \prod_{j=1}^{n} (w - w_j)^{\beta_j}, \ \forall w \in V_0. \tag{7.13}$$

We claim that H is analytic on Cl $B(0, 1)$. As $|w_j| = 1$, $\forall j = 1, 2, \ldots, n$, clearly, each $(w - w_j)^{\beta_j}$ is analytic on $B(0, 1)$, and hence, H is analytic on $B(0, 1)$. If $j \neq k$, then $(w - w_j)^{\beta_j}$ is analytic on $B(w_k, t_k)$ and by equation (7.12),

$$\mathscr{F}'(w)(w - w_k)^{\beta_k} = \mathscr{I}_\rho(w),$$

which is analytic on $B(w_k, t_k)$. Therefore, H is analytic on $B(w_k, t_k)$, $\forall k = 1, 2, \ldots, n$. If $w_0 \in (\text{Cl}\, B(0, 1)) \setminus \left(\bigcup_{j=1}^{n} B(w_j, t_j) \cup B(0, 1) \right)$, then $|w_0| = 1$ but $w_0 \notin B(w_j, t_j)$ for any j. Therefore, $w - w_j \neq 0$ in a neighbourhood of w_0 in which $(w - w_j)^{\beta_j}$ is an analytic function, $\forall j = 1, 2, \ldots, n$ and $\mathscr{F}'$ is also analytic in a neighbourhood of w_0. Thus, our claim follows.

We claim that H is nowhere zero on Cl $B(0, 1)$. Using $\mathscr{F} = F^{-1}$ on $B(0, 1)$ and F^{-1} is a one-to-one function, we get that $\mathscr{F}'$ is nowhere zero on Cl $B(0, 1)$. (The proof of this statement is similar to the proof of $\Psi_j'(0) \neq 0$ at the beginning of this proof.) Furthermore, $w - w_j \neq 0$, $\forall w \in B(0, 1) \Rightarrow (w - w_j)^{\beta_j} \neq 0$, $\forall w \in B(0, 1) \Rightarrow H(w) \neq 0$, $\forall w \in B(0, 1)$. Hence, the only possible zeroes of H are $w_j, j = 1, 2, \ldots, n$. However, the value of $\mathscr{F}'(w)(w - w_k)^{\beta_k}$ at w_k is $\mathscr{I}_\rho(w_k) = \alpha_k Q_\rho(w_k)$, which is non-zero and $(w_k - w_j)^{\beta_j} \neq 0$, if $j \neq k$. Thus, H is nowhere zero on Cl $B(0, 1)$.

Next, we find $s > 0$ such that H is analytic and nowhere zero on $B(0, 1 + s)$. For each $u \in \partial B(0, 1)$, find $s_u > 0$ such that $\mathscr{F}$ is analytic and nowhere zero on $B(u, s_u)$. As $\partial B(0, 1)$ is a closed and bounded subset of $\mathbb{C}$, it is a compact set, and hence, there exist $u_1, u_2, \ldots, u_\nu$ such that $\partial B(0, 1) \subseteq \bigcup_{p=1}^{\nu} B\left(u_p, \frac{s_p}{2}\right)$. If $s = \min\left\{\frac{s_p}{2} : 1 \leq p \leq \nu\right\}$, then $s > 0$, and for

each $w \in B(0, 1+s) \setminus \mathrm{Cl}\, B(0,1)$, let $u = \exp(i\theta)$, where $\theta = \arg w$. Then, $u \in \partial B(0,1) \Rightarrow u \in B\left(u_p, \frac{s_p}{2}\right)$ for some $p \in \{1, 2, \ldots, v\}$. As

$$|w-u_p| \leq |w-u| + |u-u_p| < |w| - 1 + \frac{s_p}{2} < 1 + r - 1 + \frac{s_p}{2} \leq \frac{s_p}{2} + \frac{s_p}{2} = s_p$$

$w \in B\left(u_p, s_p\right)$, and hence, $\mathscr{F}$ is analytic and non-zero at w. Therefore, $\mathscr{F}$ is analytic and nowhere zero on $B(0, 1+s)$, which is a simply connected region. Therefore, by Theorem 4.5.12, we can find a suitable branch for $\log(H)$, so that it is analytic on $B(0, 1+s)$. In particular, with respect to this branch, $\arg(H(w)) = \mathrm{Im}\, H(w)$ is continuous on $B(0, 1+s)$. If $j \in \{1, 2, \ldots, m\}$ and $w \in \{\exp(i\theta) : \theta_j < \theta < \theta_{\rho(j)}\}$, where $\arg(w_j) = \theta_j$ and $\arg(w_{\rho(j)}) = \theta_{\rho(j)}$, then from equation (7.13), we have

$$\arg H(\exp(i\theta)) = \arg(\mathcal{F}'(\exp(i\theta))) + \sum_{j=1}^{n} \arg(\exp(i\theta) - \exp(i\theta_j)). \quad (7.14)$$

Since

1. $\arg F'(\exp(i\theta))$ is the angle between the tangent of the line segment γ_j joining z_j and $z_{\rho(j)}$ at $F(\exp(i\theta))$ and the tangent of the circular arc μ_j joining $w_j = \exp(i\theta_j)$, $w_{\rho(j)} = \exp(i\theta_{\rho(j)})$ at $w = \exp(i\theta)$,

2. the slope of the the tangent of line segment γ_j is constant (say φ_j) at each point on γ_j,

3. the slope of the tangent of the circular arc μ_j at $\exp(i\theta)$ is $\theta + \frac{\pi}{2}$,

we have

$$\arg(F'(\exp(i\theta))) = \varphi_j - \left(\theta + \frac{\pi}{2}\right).$$

As
$\exp(i\theta) - \exp(i\theta_j)$

$$\begin{aligned}
&= \exp\left(i\left(\frac{\theta+\theta_j}{2}\right)\right)\left[\exp\left(i\left(\frac{\theta-\theta_j}{2}\right)\right) - \exp\left(-i\left(\frac{\theta-\theta_j}{2}\right)\right)\right] \\
&= \exp\left(i\left(\frac{\theta+\theta_j}{2}\right)\right) 2i \sin\left(\frac{\theta-\theta_j}{2}\right) \\
&= 2\sin\left(\frac{\theta-\theta_j}{2}\right) \exp\left(i\left(\frac{\theta+\theta_j+\pi}{2}\right)\right)
\end{aligned}$$

and $2\sin\left(\frac{\theta-\theta_j}{2}\right) > 0$, we have $\arg(\exp(i\theta) - \exp(i\theta_j)) = \frac{\theta+\theta_j+\pi}{2}$. Consolidating the above observations, we get

$$\begin{aligned}
\arg H(\exp(i\theta)) &= \varphi_j - \left(\theta + \frac{\pi}{2}\right) + \sum_{j=1}^{n} \beta_j \frac{\theta + \theta_j + \pi}{2} \\
&= \varphi_j - \left(\theta + \frac{\pi}{2}\right) + \frac{\theta + \pi}{2} \sum_{j=1}^{n} \beta_j + \sum_{j=1}^{n} \beta_j \frac{\theta_j}{2} \\
&= \varphi_j - \left(\theta + \frac{\pi}{2}\right) + \theta + \pi + \sum_{j=1}^{n} \beta_j \frac{\theta_j}{2} \left(\text{using } \sum_{j=1}^{n} \beta_j = 2 \right) \\
&= \varphi_j + \frac{\pi}{2} + \sum_{j=1}^{n} \beta_j \frac{\theta_j}{2},
\end{aligned}$$

which is independent of θ. Hence, $\arg H$ is a constant function on μ_j, $\forall j \in \{1, 2, \ldots, n\}$. As $\arg H$ is continuous, its value on each circular arc μ_j must be the same, and hence, it is constant on $\partial B(0, 1)$. Since $\arg H$ is the imaginary part of the analytic function $\log(H)$, it is a harmonic function and by maximum and minimum principles of harmonic functions, $\arg H$ is constant on Cl $B(0, 1)$. Hence, by Result 2.2.20, $\log(H)$ is a constant function on Cl $B(0, 1)$. Thus, H is a constant function on $B(0, 1)$, and hence,

$$F'(w) = (f^{-1})'(w) = c_1 \prod_{j=1}^{n} (w - w_j)^{-\beta_j}, \ \forall w \in B(0, 1),$$

for some $c_1 \in \mathbb{C}$. Integrating on both sides of the above equation over a curve joining 0 and w in $B(0, 1)$, we get

$$f^{-1}(w) = c_1 \int_0^w \prod_{j=1}^{n} (w - w_j)^{-\beta_j} \, dw + c_2, \ \forall w \in B(0, 1),$$

for some $c_1, c_2 \in \mathbb{C}$. □

THEOREM 7.5.15 (Schwarz–Christoffel formula for upper half-plane)
Let Ω be a bounded simply connected region enclosed by a polygon with vertices z_j and interior angles $\alpha_j\pi$ at z_j, $j = 1, 2, \ldots, n$. If f is a Riemann

mapping from Ω onto the upper half-plane H^+, then there exist constants C_1 and C_2 such that

$$f^{-1}(\zeta) = C_1 \int_i^{\zeta} \prod_{j=1}^{n-1} (\zeta - \zeta_j)^{-\beta_j}\, d\zeta + C_2,$$

where the integral is defined over any curve joining i and ζ in H^+ and $\beta_j = 1 - \alpha_j$, $\zeta_j = \lim_{z \to z_j} f(z)$, $\forall j = 1, 2, \ldots, n$, with $\zeta_n = \infty$.

Proof: To find a linear fractional transform, which maps the unit disc onto the the upper half-plane, we find a linear fractional transform T, which maps $1, -i, i$ to $1, 0, \infty$, respectively, by using the definition of cross ratio as given below.

$$T(w) = (w, 1, -i, i) = \frac{w+i}{w-i}\frac{1-i}{1+i} = -i\frac{w+i}{w-i}.$$

As T maps the unit circle onto the real axis and $T(0) = -i\dfrac{i}{-i} = i$, it maps $B(0,1)$ onto H^+. We find $w_j \in \partial B(0,1)$ such that $T(w_j) = \zeta_j, j = 1, \ldots, n-1$, and we know that if $w_n = i$, then $T(w_n) = \infty$, and $T^{-1} \circ f : \Omega \to B(0,1)$ is a bijective analytic map such that $\lim_{z \to z_j} (T^{-1} \circ f)(z) = w_j, j = 1, 2, \ldots, n$. Hence, by applying the proof of previous theorem, we obtain

$$c_1 \prod_{j=1}^{n} (w - w_j)^{-\beta_j} = (f^{-1} \circ T)'(w) = (f^{-1})'(T(w))T'(w),$$

which implies that

$$(f^{-1})'(T(w)) = \frac{c_1}{T'(w)} \prod_{j=1}^{n} (w - w_j)^{-\beta_j}.$$

Now, using the change of variable $\zeta = T(w)$ in the last equation, we also get

$$(f^{-1})'(\zeta) = c_1 (T^{-1})'(\zeta)(T^{-1}(\zeta) - T^{-1}(\zeta_n))^{-\beta_n} \prod_{j=1}^{n-1} (T^{-1}(\zeta) - T^{-1}(\zeta_j))^{-\beta_j}. \tag{7.15}$$

Using the following

$$\begin{aligned} w = T^{-1}(\zeta) &= \frac{-i\zeta - 1}{-\zeta - i} = \frac{i\zeta + 1}{\zeta + i} = i\frac{\zeta - i}{\zeta + i}, \\ (T^{-1})'(\zeta) &= i\frac{(\zeta - i) - (\zeta + i)}{(\zeta + i)^2} = \frac{-2}{(\zeta + i)^2}, \end{aligned}$$

and

$$\begin{aligned} T^{-1}(\zeta) - T^{-1}(\zeta_j) &= i\left[\frac{\zeta - i}{\zeta + i} - \frac{\zeta_j - i}{\zeta_j + i}\right] \\ &= i\left[\frac{\zeta\zeta + i\zeta - i\zeta_j + 1 - (\zeta\zeta_j - i\zeta + i\zeta_j + 1)}{(\zeta + i)(\zeta_j + i)}\right] \\ &= \frac{-2(\zeta - \zeta_j)}{(\zeta + i)(\zeta_j + i)} \end{aligned}$$

in equation (7.15), we get

$$\begin{aligned} (f^{-1})'(\zeta) &= \frac{-2c_1}{(\zeta + i)^2}\left(\frac{2}{\zeta + i}\right)^{-\beta_n}\prod_{j=1}^{n-1}\left[\frac{-2(\zeta - \zeta_j)}{(\zeta + i)(\zeta_j + i)}\right]^{-\beta_j} \\ &= \frac{-2c_1}{(\zeta + i)^2}\left(\frac{2}{\zeta + i}\right)^{-\beta_n}\left(\frac{-2}{\zeta + i}\right)^{-\sum_{j=1}^{n-1}\beta_j}\prod_{j=1}^{n-1}\left[\frac{\zeta - \zeta_j}{\zeta_j + i}\right]^{-\beta_j} \\ &= \frac{-2c_1}{(\zeta + i)^2}\left(\frac{2}{\zeta + i}\right)^{-\beta_n}\left(\frac{-2}{\zeta + i}\right)^{-2+\beta_n}\frac{\prod_{j=1}^{n-1}(\zeta - \zeta_j)^{-\beta_j}}{\prod_{j=1}^{n-1}(\zeta_j + i)^{-\beta_j}} \\ &= C_1\prod_{j=1}^{n-1}(\zeta - \zeta_j)^{-\beta_j}, \text{ where } C_1 = -\frac{c_1}{2}\prod_{j=1}^{n-1}(\zeta_j + i)^{\beta_j}. \end{aligned}$$

Therefore, applying integral over a curve joining i and ζ in H^+, we get

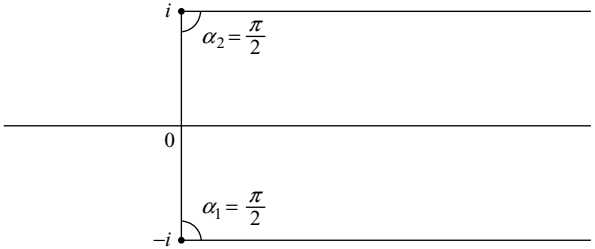
$$f^{-1}(\zeta) = C_1\int_i^{\zeta}\prod_{j=1}^{n}(\zeta - \zeta_j)^{-\beta_j}\,d\zeta + C_2,\ \forall\zeta \in H^+,$$

for some $C_1, C_2 \in \mathbb{C}$. □

Remark 7.5.16: The Schwarz–Christoffel formula is valid for semi-infinite regions determined by a polygon with vertices $z_j, j = 1, 2, \cdots, n-1$. In this case, z_n should be considered as a point at infinity.

Example 7.5.17 Find a Reimann mapping that maps the upper half-plane to the infinite strip $\{(x,y) : x \geq 0,\ -1 \leq y \leq 1\}$.

Solution: The polygon considered in this example is as follows.



We take $z_1 = -i$, $z_2 = i$; $\alpha_1 = \alpha_2 = \dfrac{1}{2}$ and $\zeta_1 = -1$, $\zeta_2 = 1$. (The choice of ζ_1 and ζ_2 are arbitrary but distinct.) Then, by Schwarz–Christoffel formula (Theorem 7.5.15), the required conformal mapping g is given by

$$g(\zeta) = C_1 \int (\zeta+1)^{-1/2}(\zeta-1)^{-1/2}\, d\zeta + C_2 = C_1(\zeta^2-1)^{-1/2} + C_2, \forall \zeta \in H^+,$$

for some $C_1, C_2 \in \mathbb{C}$, where H^+ denotes the upper half-plane. Since $\zeta \in H^+$, we find the value of $(\zeta^2 - 1)^{1/2}$ by $i\sqrt{1-\zeta^2}$, where the latter function is defined according to the branch of $\sqrt{\cdot}$, as defined in Result 3.3.11. Therefore,

$$g'(\zeta) = -iC_1 \int \frac{1}{\sqrt{1-\zeta^2}}\, d\zeta + C_2 \Rightarrow g(z) = -iC_1 \arcsin(\zeta) + C_2,\ \forall \zeta \in H^+.$$

Now, we find the values of C_1 and C_2 by using $g(-1) = -i$ and $g(1) = i$ as follows:

$$\begin{aligned} -iC_1 \arcsin(-1) + C_2 &= iC_1\frac{\pi}{2} + C_2 = -i, \\ -iC_1 \arcsin(1) + C_2 &= -iC_1\frac{\pi}{2} + C_2 = i. \end{aligned}$$

Adding these two equations, we get $2C_2 = 0 \Rightarrow C_2 = 0$. Substituting this value in the first equation, we get $C_1 = -\dfrac{2}{\pi}$. Hence, $g(\zeta) = \dfrac{2i}{\pi}\arcsin(\zeta),\ \forall \zeta \in H$.

8

Elliptic Functions

8.1 BASIC CONCEPTS

We would first like to make clear to the readers the common features among ellipse, elliptic curve, elliptic integrals, and elliptic functions. We know that an ellipse is a closed curve satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for some $a, b \in \mathbb{R} \notin \{0\}$, whose parametric equations are given by

$$x(t) = a\cos(t),\ y(t) = b\sin(t),\ \forall t \in [0, 2\pi].$$

(As there is no loss of generality, we assume that $a \leq b$).
Hence, the arc length of this ellipse at a given point $t_0 \in [0, 2\pi]$, from the point on the ellipse corresponding to $t = 0$, is given by the integral

$$\begin{aligned}
S(t_0) &= \int_0^{t_0} |(x'(t), y'(t))|\, dt \\
&= \int_0^{t_0} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt \\
&= \int_0^{t_0} \sqrt{a^2 \sin^2(u) + b^2(1 - \sin^2(u))}\, du \\
&= \int_0^{t_0} \sqrt{b^2 - (b^2 - a^2)\sin^2(u)}\, du \\
&= b\int_0^{t_0} \sqrt{1 - k^2 \sin^2(u)}\, du, \text{ where } k = \frac{(b^2 - a^2)}{b}.
\end{aligned}$$

Hence, the integrals of the form $\int_0^{\phi} \sqrt{1 - k^2 \sin^2(u)}\, du$ are called elliptic integrals. More specifically, they are called the elliptic integrals of second kind. There are two more kinds of elliptic integrals, namely, elliptic integrals of first kind and elliptic integrals of third kind.

Elliptic integrals of first kind : $\int_0^{\phi} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}}$.

Elliptic integrals of third kind: $\int_0^{\phi} \frac{du}{\sqrt{1 + n \sin^2(u)}\sqrt{1 - k^2 \sin^2(u)}}$.

By making the change of variable $x = \sin u$, we get

Elliptic integrals of first kind : $\int_0^{x} \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}}$,

Elliptic integrals of second kind : $\int_0^{x} \frac{\sqrt{1 - k^2x^2}\, dx}{\sqrt{1 - x^2}}$, and

Elliptic integrals of third kind: $\int_0^{x} \frac{dx}{\sqrt{1 + nx^2}\sqrt{1 - x^2}\sqrt{1 - k^2x^2}}$.

A general elliptic integral is of the form $\int R(x, \sqrt{P(x)})\, dx$, where R is a rational function in two variables, and P is a cubic or quadratic polynomial. It is interesting to see from the literature on elliptic integrals that every elliptic integral can be expressed by using those standard forms, together with elementary integrals.

An elliptic curve is a curve in $\mathbb{C}$, which is given by the following equation;

$$y^2 = x^3 + ax + b \text{ where } a, b \in \mathbb{Q} \text{ with the discriminant } 4a^3 + 27b^2 \neq 0.$$

The condition $4a^3 + 27b^2 \neq 0$ is equivalent to say that $x^3 + ax + b = 0$ has no multiple roots in $\mathbb{C}$. We are going to see that the Weierstrass function $\wp_{a,b}$ is an elliptic function satisfying the differential equation

$$(\wp'_{a,b}(z))^2 = 4(\wp(z))^3 - 60G_4\wp(z) - 140G_6$$

and hence, $\wp_{a,b}$ can be viewed as the inverse of an elliptic integral. We shall show that every elliptic function is a rational function of $\wp_{a,b}$ and its derivative, for some suitable choices of a's and b's. From the literature of elliptic curves, we see that every elliptic curve can be parameterized by using $\wp_{a,b}$. This is the history behind the similarity in the nomenclature. It should be noted that ellipse is not an elliptic curve. Now, we are going to discuss the elliptic functions in this chapter.

Definition 8.1.1 Let $f : \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$. We say that 'a' is a period of f if $f(z + a) = f(z)$, $\forall z \in \mathbb{C}$.

The set of all periods of f is denoted by P_f.

Definition 8.1.2 A non-zero period 'a' of f is said to be a fundamental period of f if there is no $b \in P_f \setminus \{0\}$ such that $\frac{a}{b} \in \mathbb{Z}$ with $|b| < |a|$.

By the above definition, it is clear that 'a' is a fundamental period of f if and only if '$-a$' is a fundamental period of f.

LEMMA 8.1.3 For a given $f : \mathbb{C} \to \mathbb{C}$,

1. $ma + nb \in P_f$, whenever $m, n \in \mathbb{Z}$ and $a, b \in P_f$.

   And if f is a non-constant meromorphic function on $\mathbb{C}$, then

2. P_f has no limit point in $\mathbb{C}$.

3. P_f is exactly equal to one of the following three sets:

   (a) $\{0\}$,

   (b) $\{ma : m \in \mathbb{Z}\}$ for some $a \in \mathbb{C}$,

   (c) $\{ma + nb : m, n \in \mathbb{Z}\}$ for some $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$.

Proof:

1. First, we prove that if $a \in P_f$ and $m \in \mathbb{Z}$, then $ma \in P_f$. For $m = 0$, clearly $ma = 0 \in P_f$. For $m > 0$, using the condition $a \in P_f$ repeatedly, we get

$$\begin{aligned} f(z + ma) &= f(z + (m-1)a + a) \\ &= f(z + (m-1)a) \\ &= f(z + (m-2)a) \\ &\vdots \\ &= f(z + a) \\ &= f(z), \ \forall z \in \mathbb{C} \end{aligned}$$

   and hence, $ma \in P_f$. For $m < 0$,

$$\begin{aligned} f(z + ma) &= f(z + ma + a) \\ &= f(z + (m+1)a) \end{aligned}$$

$$\begin{aligned} &= f(z+(m+2)a) \\ &\vdots \\ &= f(z-a) \\ &= f(z-a+a) \\ &= f(z),\ \forall z \in \mathbb{C}. \end{aligned}$$

Hence, $ma \in P_f, \forall m \in \mathbb{Z}$. Clearly, $p, q \in P_f \Rightarrow f(z+p+q) = f(z+q) = f(z),\ \forall z \in \mathbb{C} \Rightarrow p+q \in P_f$. Thus, (1) follows.

2. If S_f is the set of all poles of f and if

$$\Omega_f = \{z \in \mathbb{C} : f \text{ is analytic at } z\} = \mathbb{C} \setminus S_f,$$

then either $P_f \subseteq \Omega_f$ or $P_f \subseteq S_f$. If $P_f \subseteq \Omega_f$ and if P_f has a limit point, then Ω_f has a limit point, and hence, f becomes a constant function on Ω_f, by analytic continuation. If w is a pole of f, then it is isolated, and hence, there exists $r > 0$ such that $B(w,r) \setminus \{w\} \subset \Omega_f$. As f is constant on Ω_f, f is constant on $B(w,r)\setminus\{w\}$. Then, $\lim_{z \to w} f(z)$ must be the constant (and is not infinity), which is a contradiction. This implies that f cannot have any pole. Thus, f is constant on $\mathbb{C}$. This contradiction leads to conclude that P_f has no limit point in $\mathbb{C}$. If $P_f \subseteq S_f$, then obviously P_f has no limit point since every pole of f is isolated.

3. If $P_f \neq \{0\}$, then we choose $a \in P_f \setminus \{0\}$ such that $|a| = \inf\{|p| : p \in P_f \setminus \{0\}\}$. Such an a exists by using (2). If there are more than one such a, then we choose any one of them. Using (1), we have $\{ma : m \in \mathbb{Z}\} \subseteq P_f$. If $\{ma : m \in \mathbb{Z}\} = P_f$, then the theorem is over. Otherwise, we choose $b \in P_f \setminus \{ma : m \in \mathbb{Z}\}$ such thatI $|b| = \inf\{|p| : p \in P_f \setminus \{ma : m \in \mathbb{Z}\}\}$. Then by using (1), $\{ma + nb : m, n \in \mathbb{Z}\} \subseteq P_f$. Now, we first claim that $\frac{a}{b} \notin \mathbb{R}$. Suppose that $\frac{a}{b} \in \mathbb{R}$, then either $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Z}$ or $\frac{b}{a} \in \mathbb{R} \setminus \mathbb{Z}$. (Note that $\frac{a}{b} \neq 0$.) If $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Z}$, then we can find $m \in \mathbb{Z}$ such that

$$m < \frac{a}{b} < m+1 \Rightarrow |mb-a| < |b| \text{ and } mb - a \in P_f,$$

which is a contradiction to the choice of b. Similarly, we get a contradiction by assuming that $\frac{b}{a} \in \mathbb{R} \setminus \mathbb{Z}$.

Next, we claim that for each $c \in P_f$, there exist $r, s \in \mathbb{R}$ such that $c = ra + sb$. Using $\frac{a}{b} \neq \frac{\overline{a}}{\overline{b}}$, it follows that $\begin{vmatrix} a & b \\ \overline{a} & \overline{b} \end{vmatrix} = a\overline{b} - \overline{a}b \neq 0$, and hence, the system of linear equations $ra + sb = c;\ \ r\overline{a} + s\overline{b} = \overline{c}$ in the variables r and s has a unique solution $(r, s) \in \mathbb{C}^2$. We note that (r, s) is a solution of the system of linear equations if and only if $(\overline{r}, \overline{s})$ is also a solution of the same system of linear equations. Hence, by the uniqueness of the solution, we get

$$(r, s) = (\overline{r}, \overline{s}) \Rightarrow r = \overline{r} \text{ and } s = \overline{s} \Rightarrow r, s \in \mathbb{R}.$$

Thus, our claim holds.

Now, we choose $m, n \in \mathbb{Z}$ such that

$$|r + m| \leq \frac{1}{2} \text{ and } |s + n| \leq \frac{1}{2}.$$

If $d = c + ma + nb$, then $d \in P_f$, and using the necessary condition for equality in triangle inequality, as $\frac{b}{a} \notin \mathbb{R}$, we get

$$|d| = |(r+m)a+(s+n)b| \underset{\neq}{\leq} |(r+m)a|+|(s+n)b| \leq \frac{1}{2}(|a|+|b|) \leq \frac{1}{2}(2|b|) = |b|.$$

If d is not an integral multiple of a, then this inequality contradicts the minimality of $|b|$. Hence, it follows that $d = la$ for some $l \in \mathbb{Z}$. Therefore, from the definition of d, we have

$$c = d - ma - nb = (l - m)a - nb.$$

Thus, the proof is complete. □

From the above proof, we see the following observations:

1. If $P_f = \{ma : m \in \mathbb{Z}\}$, then a is a fundamental period of f.
2. If $P_f = \{ma + nb : m, n \in \mathbb{Z}\}$ with $\frac{a}{b} \notin \mathbb{R}$, then a and b are fundamental periods of f.

Definition 8.1.4 A meromorphic function f on $\mathbb{C}$ is called an elliptic function if $P_f = \{ma + nb : m, n \in \mathbb{Z}\}$ for some $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$, where P_f is the set of all periods of f.

If f is a meromorphic function on $\mathbb{C}$, then there exists $S_f \subset \mathbb{C}$ such that $f : \mathbb{C} \setminus S_f \to \mathbb{C}$ and $\lim_{z \to w} f(z) = \infty$, $\forall w \in S_f$. In this case, by *"a is a period of f"*, we mean that $f(z + a) = f(z)$, $\forall z \in \mathbb{C} \setminus S_f$ and $w \in S_f$ if and only if $a + w \in S_f$.

In the language of *linear algebra*, the condition $\frac{a}{b} \notin \mathbb{R}$ can be stated as '*a* and *b are linearly independent in* $\mathbb{C}$ *over* $\mathbb{R}$' or briefly '*a* and *b are independent*'. Therefore, in the above definition, as there exist two independent fundamental periods of f, an elliptic function is also called a doubly periodic function.

Definition 8.1.5 For a given pair of complex numbers $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$, a period module generated by a and b is defined by $Q_{a,b} = \{ma + nb : m, n \in \mathbb{Z}\}$.

Remark 8.1.6: Hereafter, we write 'f is an elliptic function with periodic module $Q_{a,b}$' to mean that f is an elliptic function with $P_f = Q_{a,b}$.

THEOREM 8.1.7 If f and g are elliptic functions with $P_f = P_g$ and $c \in \mathbb{C}$, then

1. $f + g$ is an elliptic function.
2. cf is an elliptic function.
3. fg is an elliptic function.
4. $\frac{1}{g}$ is an elliptic function, and hence, $\frac{f}{g}$ is an elliptic function.

Proof:

1. As f and g are meromorphic functions, for each $w \in \mathbb{C}$, the Laurent series expansions of f and g around w have at most finite number of non-zero terms in their singular parts. Hence, the Laurent series expansion of $f + g$ also has at most finite number of terms in its singular part. If w is a pole of exactly one of f and g, then w is a pole of $f + g$. If w is a pole of both f and g, then either w is a removable singularity of $f + g$ (and hence, it may be assumed as a regular point after removing its singularity) or w is a pole of $f + g$. Hence, $S_{f+g} \subseteq S_f \cup S_g$. Thus, $f + g$ is a meromorphic function on $\mathbb{C}$. As f and g have same periods, for each $a \in P_f = P_g$,

$$(f + g)(z + a) = f(z + a) + g(z + a) = f(z) + g(z), \ \forall z \in \mathbb{C} \setminus S_{f+g}.$$

We know that for each $w \in \mathbb{C}$, there exists $r > 0$ such that f is analytic on $B(w, r) \setminus \{w\}$. Therefore

$$\lim_{z \to w} (f + g)(z) = \lim_{z \to w} (f + g)(z + a) = \lim_{\zeta \to w+a} (f + g)(\zeta), \ \forall w \in \mathbb{C}.$$

Thus, w is a pole of $f+g$ if and only if $w+a$ is a pole of $f+g$. Hence, $f+g$ is an elliptic function.

2. Clearly, w is a pole of f if and only if w is a pole of cf. Hence, cf is a meromorphic function and $S_f = S_{cf}$. Furthermore, if $a \in P_f$, then

$$(cf)(z+a) = cf(z+a) = cf(z),\ \forall z \in \mathbb{C} \setminus S_{cf} = S_f.$$

3. First, we show that if f and g are meromorphic functions on $\mathbb{C}$, then fg is also a meromorphic function. If w is a pole of order m for f, then

   (a) w is a pole of order m for fg if $w \in \Omega_g$ and $g(w) \neq 0$.

   (b) w is a pole of order $m-k$ for fg if w is a zero of order k for g with $k < m$.

   (c) w is a removable singularity for fg with $(fg)(w) \neq 0$ if w is a zero of order m for g.

   (d) w is a removable singularity for fg and after removing the singularity, w is a zero of order $k-m$ for fg if w is a zero of order k for g with $k > m$.

   Thus, every pole of f is either a pole of fg or a removable singularity of fg. Similarly, every pole of g is either a pole or removable singularity of fg. Moreover, if $z \notin S_f \cup S_g$, then f and g are analytic at z, and hence, fg is also analytic at fg. Thus, fg is a meromorphic function on $\mathbb{C}$. Using the periodicity of f and g as in (1), one can prove that fg is doubly periodic with same fundamental periods of f and g.

4. If g is a meromorphic function, then obviously, every zero of g is a pole of $\frac{1}{g}$. Every pole w of g is a removable singularity of $\frac{1}{g}$; and it becomes a zero of $\frac{1}{g}$, after removing the singularity at w. Hence, $\frac{1}{g}$ is a meromorphic function and for every $a \in P_f$,

$$\frac{1}{g}(z+a) = \frac{1}{g(z+a)} = \frac{1}{g(z)} = \frac{1}{g}(z).$$

   Thus, $\frac{1}{g}$ is an elliptic function. Finally, using this with (3), we conclude that $\frac{f}{g}$ is also an elliptic function. □

Exercise 8.1.8 If f and g are elliptic functions such that $P_f \subsetneq P_g$, then prove that $P_{f+g} = P_{fg} = P_{\frac{f}{g}} = P_g$.

LEMMA 8.1.9 If f is an elliptic function, then f' is also an elliptic function.

Proof: Let $z_0 \in \mathbb{C}$ and $a \in P_f$ be fixed. If f is differentiable at z_0, then we get

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{f(z+a) - f(z_0+a)}{(z+a) - (z_0+a)} \qquad (\text{using } a \in P_f) \\ &= \lim_{w \to z_0 + a} \frac{f(w) - f(z_0+a)}{w - (z_0+a)} \qquad (\text{by substituting } w = z + a) \\ &= f'(z_0 + a). \end{aligned}$$

Therefore, $f'(z_0) = f'(z_0 + a)$. If f has a pole at w of order m, then we can write $f(z) = (z - w)^{-m} g(z)$, $\forall z \in B(w, r) \setminus \{w\}$ for some analytic function g and for some $r > 0$. Therefore,

$$\begin{aligned} f'(z) &= (z - w)^{-m} g'(z) - m(z - w)^{-m-1} g(z) \\ &= (z - w)^{-m-1} [(z - w) g'(z) - m g(z)], \ \forall z \in B(w, r) \setminus \{w\}. \end{aligned}$$

As $(z - w)g'(z) - mg(z)$ is analytic, we obtain f' has a pole of order $m + 1$ at w. Thus, f' is also an elliptic function. □

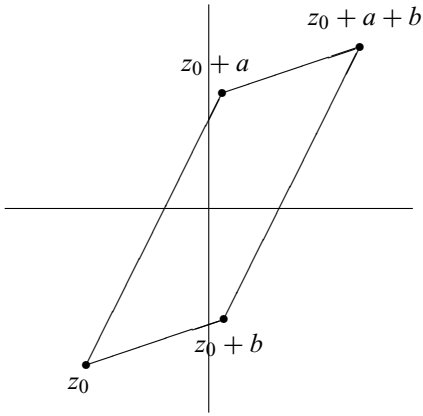
8.2 FUNDAMENTAL PARALLELOGRAM

Definition 8.2.1 Let a and b be complex numbers such that $\frac{a}{b} \notin \mathbb{R}$. For each $z_0 \in \mathbb{C}$, the *semi-open* parallelogram

$$D_{a,b}(z_0) = \{z_0 + \lambda a + \mu b : 0 \leq \lambda < 1,\ 0 \leq \mu < 1\}$$

with vertices z_0, $z_0 + a$, $z_0 + b$, and $z_0 + a + b$ is called a fundamental parallelogram generated by a and b.

If f is an elliptic function with period module $Q_{a,b}$, then we also call $D_{a,b}(z_0)$ a fundamental parallelogram of f. If f is a constant (elliptic) function, then there is no fundamental period for f. In this case, for every pair $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$ and for every $z_0 \in \mathbb{C}$, we call $D_{a,b}(z_0)$ a fundamental parallelogram of f.



A fundamental parallelogram

LEMMA 8.2.2 Let a and b be such that $\frac{a}{b} \notin \mathbb{R}$ and let $z_0 \in \mathbb{C}$.

1. For each $z \in \mathbb{C}$, there exists unique $w \in D_{a,b}(0)$ such that $z - w \in Q_{a,b}$.
2. There exists a bijection $\phi : D_{a,b}(z_0) \to D_{a,b}(0)$ such that $\phi(z) - z \in Q_{a,b}$, $\forall z \in D_{a,b}(z_0)$.
3. $f(D_{a,b}(z_0)) = f(D_{a,b}(0)) = f(\mathbb{C})$, $\forall f$ with $P_f = Q_{a,b}$.

Proof:

1. As in the proof of Lemma 8.1.3(3), for the given $z \in \mathbb{C}$, we get $z = ra + sb$ for some $r, s \in \mathbb{R}$. Next, we choose $m, n \in \mathbb{Z}$ such that $0 \leq r+m < 1$ and $0 \leq s + n < 1$ so that

$$z + na + mb = (r + m)a + (s + n)b \in D_{a,b}(0).$$

   If we take $w = z + na + mb$, then obviously, $z - w \in Q_{a,b}$. Suppose there exists $\zeta \in D_{a,b}(0)$ with $z - \zeta \in Q_{a,b}$, then let $\zeta = z + la + kb$ for some $l, k \in \mathbb{Z}$. Then, $0 \leq r + l < 1$ and $0 \leq s + k < 1$, and hence, $1 > |(r + m) - (r + l)| = |m - l|$. Being $m - l \in \mathbb{Z}$, the only option is $m - l = 0$. Similarly, we can prove that $n = k$. Thus, $w = \zeta$, and hence, the uniqueness follows.

2. Using (1), for the given $z_0 \in \mathbb{C}$, there exist unique $w_0 \in D_{a,b}(0)$ and $p \in Q_{a,b}$ such that $w_0 = z_0 + p$. If $\zeta_0 = a + b - p$, then we claim that

$\zeta_0 \in \text{Cl}\, D_{a,b}(z_0)$. As $w_0 \in D_{a,b}(0)$, there exist $r_0, s_0 \in [0,1)$ such that $w_0 = r_0 a + s_0 b$. Therefore, if $u_0 = 1 - r_0$ and $v_0 = 1 - s_0$, then

$$\begin{aligned}\zeta_0 &= a + b - p \\ &= a + b - w_0 + z_0 \\ &= z_0 + (1 - r_0)a + (1 - s_0)b \\ &= z_0 + u_0 a + v_0 b \in \text{Cl}\, D_{a,b}(z_0)\end{aligned}$$

as $u_0, v_0 \in (0,1]$.

Case 1: $w_0 = 0$.

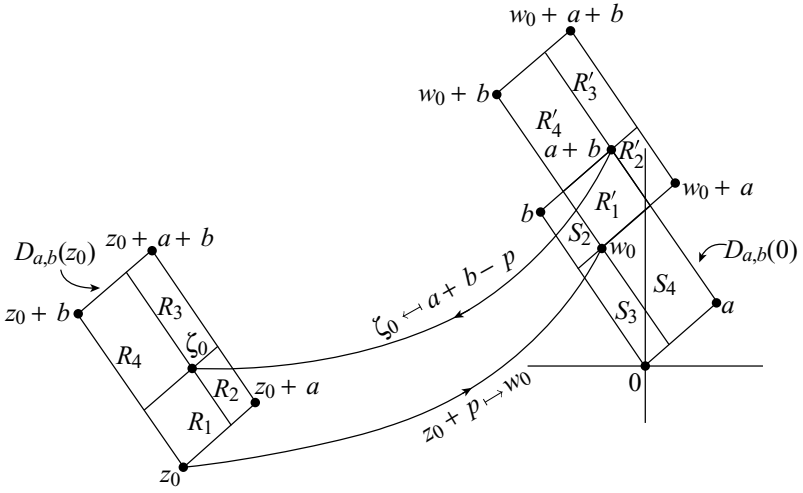
In this case, $\zeta_0 = a + b + z_0$, $p = z_0$, and hence, the required map $\phi : D_{a,b}(z_0) \to D_{a,b}(0)$ is simply the translation map given by $\phi(z) = z + p$, $\forall z \in D_{a,b}(z_0)$.

Case 2: $w_0 \neq 0$.

Then, we assume that $\zeta_0 \notin \partial D_{a,b}(z_0)$, and hence, we can partition $D_{a,b}(z_0)$ as $\bigcup_{j=1}^{4} R_j$, where

$$\begin{aligned}R_1 &= \{z_0 + \lambda a + \mu b : \lambda \in [0, u_0),\ \mu \in [0, v_0)\} \\ R_2 &= \{z_0 + \lambda a + \mu b : \lambda \in [u_0, 1),\ \mu \in [0, v_0)\} \\ R_3 &= \{z_0 + \lambda a + \mu b : \lambda \in [u_0, 1),\ \mu \in [v_0, 1)\} \\ R_4 &= \{z_0 + \lambda a + \mu b : \lambda \in [0, u_0),\ \mu \in [v_0, 1)\}\end{aligned}$$

The following diagram explains this proof geometrically.



$$\left.\begin{aligned} R_1' &= R_1 + p \\ S_2 + a = R_2' &= R_2 + p \\ S_3 + a + b = R_3' &= R_3 + p \\ S_4 + b = R_4' &= R_4 + p \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} R_1' &= R_1 + p \\ S_2 &= R_2 + p - a \\ S_3 &= R_3 + p - a - b \\ S_4 &= R_4 + p - b \end{aligned}\right.$$

Now, if we define $\phi : D_{a,b}(z_0) \to D_{a,b}(0)$ by

$$\phi(z) = \begin{cases} z + p & \text{if } z \in R_1 \\ z + p - a & \text{if } z \in R_2 \\ z + p - a - b & \text{if } z \in R_3 \\ z + p - b & \text{if } z \in R_4 \end{cases}$$

then ϕ is the required bijection between $D_{a,b}(z_0)$ and $D_{a,b}(0)$.
If $\zeta_0 \neq a + b + z_0$ and $\zeta_0 \in \partial D_{a,b}(0)$, then we can prove this part of the lemma by splitting the rectangle $D_{a,b}(z_0)$ into two suitable subrectangles. This part is a simple exercise to the reader.

3. Obviously, we have $f(D_{a,b}(0)) \subseteq f(\mathbb{C})$. If $\zeta \in f(\mathbb{C})$, then there exists $z \in \mathbb{C}$ such that $f(z) = \zeta$. Using part (1) of this lemma, there exists $w \in D(a, b)(0)$ such that $z - w \in Q_{a,b} = P_f$. Thus, $\zeta = f(z) = f(w) \in f(D_{a,b}(0))$. Hence, we get $f(D_{a,b}(0)) = f(\mathbb{C})$. Therefore, it immediately follows that $f(D_{a,b}(z_0)) \subseteq f(D_{a,b}(0))$. If $w \in D_{a,b}(0)$, then $f(w) = f(\phi^{-1}(w)) \in f(D_{a,b}(z_0))$, as $\phi^{-1}(w) - w \in Q_{a,b}$, from the proof of part (2) of the lemma. Therefore, $f(D_{a,b}(z_0)) = f(D_{a,b}(0))$. □

Using the previous lemma, if f is an elliptic function, then the number of poles (including multiplicities) of f in a fundamental parallelogram of f is independent of z_0. Therefore, we can define the following.

Definition 8.2.3 Let f be an elliptic function. Then, the number of poles (including multiplicities) of f in a fundamental parallelogram of f is called the order of the elliptic function f.

LEMMA 8.2.4 Let f be an elliptic function. The number of poles of f in a fundamental parallelogram of f is finite, and the sum of the residues of f at all poles in $D_{a,b}(z_0)$ is 0.

Proof: We first note that if f is a meromorphic function on a region Ω, then for each $z \in \Omega$, there exists $r_z > 0$ such that f is analytic on $B(z, r_z) \setminus \{z\}$. (If z is a pole, then this statement follows from Lemma 5.2.11, and if it is not a pole, then this statement is straightforward.) Let $D_{a,b}(z_0)$ be a fundamental parallelogram of f. If there exist infinite number of poles of f in Cl $D_{a,b}(z_0)$ (which is compact), then they should have a limit point w in Cl $D_{a,b}(z_0)$ by

Theorem 1.4.39. Therefore, f becomes analytic on $B(w,r) \setminus \{w\}$, for some $r > 0$. By the choice of w, for this $r > 0$, we can find a pole p of f such that $p \in \mathrm{Cl}\, D_{a,b}(z_0)B(w,r) \setminus \{w\}$, which is a contradiction. Thus, there exist at most finite number of poles of f in $D_{a,b}(z_0)$.

Since every pole p of f in $D_{a,b}(z_0)$ corresponds to a pole q of f in a translate $D_{a,b}(z+z_0)$ of $D_{a,b}(z_0)$ such that $p - q \in Q_{a,b}$, by Lemma 8.2.2, instead of finding the sum of residues of f in $D_{a,b}(z_0)$, we can find the same in $D_{a,b}(z+z_0)$ for any $z \in \mathbb{C}$. As there are only finite number of zeroes and poles in $D_{a,b}(z_0)$, by translating $D_{a,b}(z_0)$ with a suitable $z \in \mathbb{C}$, we can obtain that there is no zero or pole on the boundary of $D_{a,b}(z+z_0)$. Therefore, we now assume that there is no pole on $\partial D_{a,b}(z_0)$.

If γ is the boundary of $D_{a,b}(z_0)$, then by Cauchy's residue theorem,

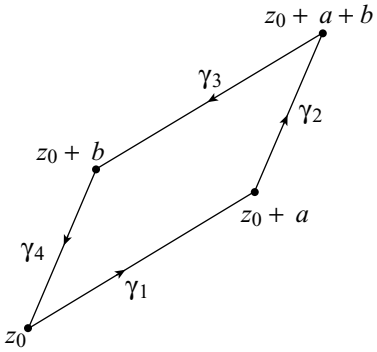
$$\begin{aligned} \frac{1}{i2\pi}\int_{\gamma} f(z)\,dz &= \text{Sum of the residues of } f \text{ at poles enclosed by } \gamma. \\ &= \text{Sum of the residues of } f \text{ at poles in } D_{a,b}(z_0). \end{aligned}$$

On the other hand, as $D_{a,b}(z_0) = \{z_0 + \lambda a + \mu b : \lambda, \mu \in [0,1)\}$, where $a, b \in P_f$ with $\frac{a}{b} \notin \mathbb{R}$ and $z_0 \in \mathbb{C}$ is fixed, then

$$\frac{1}{i2\pi}\int_{\gamma} f(z)\,dz = \frac{1}{i2\pi}\sum_{j=1}^{4}\int_{\gamma_j} f(z)\,dz.$$

where

$$\begin{aligned} \gamma_1(t) &= z_0 + ta,\ t \in [0,1] \\ \gamma_2(t) &= z_0 + a + tb,\ t \in [0,1] \\ \gamma_3(t) &= z_0 + (1-t)a + b,\ t \in [0,1] \\ \gamma_4(t) &= z_0 + (1-t)b,\ t \in [0,1] \end{aligned}$$



Therefore, using $a, b \in P_f$,

$$\begin{aligned}\int_{\gamma_1} f(z)\,dz &= \int_0^1 f(z_0 + ta)\,a\,dt \\ &= \int_1^0 f(z_0 + (1-s)a)\,a(-ds) \\ &= -\int_0^1 f(z_0 + (1-s)a)\,a(-ds) \\ &= -\int_{\gamma_3} f(z)\,dz.\end{aligned}$$

Similarly, we can prove that $\int_{\gamma_2} f(z)\,dz = -\int_{\gamma_4} f(z)\,dz$, and hence, $\int_{\gamma} f(z)\,dz = 0$. Thus, we get the sum of the residues of f at poles in $D_{a,b}(z_0) = 0$. □

LEMMA 8.2.5 If $D_{a,b}(z_0)$ is a fundamental parallelogram of a non-zero elliptic function, then the number of zeroes of f in $D_{a,b}(z_0)$ including multiplicities is same as the number of zeroes of f in $D_{a,b}(z_0)$ including multiplicities.

Proof: Using Theorem 8.1.7 and Lemma 8.1.9, we get that $\dfrac{f'}{f}$ is also an elliptic function. Therefore, as in the proof of the previous lemma, we can prove that $\int_{\partial D_{a,b}(z_0)} \dfrac{f'(z)}{f(z)}\,dz = 0$. However, using argument principle, we get $\frac{1}{i2\pi} \int_{\partial D_{a,b}(z_0)} \dfrac{f'(z)}{f(z)}\,dz$ = number of zeroes of f (including multiplicities) enclosed by $\partial D_{a,b}(z_0)-$ number of poles of f (including multiplicities) enclosed by $\partial D_{a,b}(z_0)$. This completes the proof of the lemma. □

LEMMA 8.2.6 If $\{z_1, z_2, \ldots, z_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ are the zeroes and poles of an elliptic function f including multiplicities in a fundamental parallelogram $D_{a,b}$ of f, respectively, then $z_1 + z_2 + \cdots + z_n - w_1 - w_2 - \cdots - w_n \in P_f$.

Proof: Without loss of generality, we assume that no pole or zero of f is on $\partial D_{a,b}$ because, if necessary, as the number of poles and zeroes are finite, we can translate this fundamental parallelogram to another fundamental parallelogram with this property. Furthermore, by Lemma 8.2.2, we conclude that proving this result for one fundamental parallelogram implies this result for all fundamental parallelogram. As $\{z_1, z_2, \ldots, z_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ are the zeroes and poles of f including multiplicities in $D_{a,b}$, applying Results 5.2.3 and 5.2.11 repeatedly, we observe that if $\phi(z) = \dfrac{\prod_{j=1}^{n}(z - w_j)}{\prod_{j=1}^{n}(z - z_j)} f(z)$, then ϕ has removable singularity at each $\zeta \in \{z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n\}$, and it has neither a pole nor a zero at any point. Hence,

$$f(z) = \frac{\prod_{j=1}^{n}(z - z_j)}{\prod_{j=1}^{n}(z - w_j)} \phi(z), \ \forall z \in D_{a,b} \setminus \{w_1, w_2, \ldots, w_n\}.$$

Therefore, finding the logarithmic derivative of f, we get

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \left(\frac{1}{(z - z_j)} - \frac{1}{(z - w_j)} \right) + \frac{\phi'(z)}{\phi(z)}.$$

As $\frac{\phi'}{\phi}$ is analytic, by applying Cauchy's theorem and Cauchy's integral formula, we get

$$\begin{aligned} \frac{1}{i2\pi} \int_{\partial D_{a,b}} \frac{zf'(z)}{f(z)}\, dz &= \sum_{j=1}^{n} \left(\frac{1}{i2\pi} \int_{\partial D_{a,b}} \frac{z}{(z - z_j)}\, dz - \frac{1}{i2\pi} \int_{\partial D_{a,b}} \frac{z}{(z - w_j)}\, dz \right) \\ &\quad + \frac{1}{i2\pi} \int_{\partial D_{a,b}} \frac{\phi'(z)}{\phi(z)}\, dz \\ &= z_1 + z_2 + \cdots + z_n - w_1 - w_2 - \cdots - w_n. \end{aligned}$$

Next, using the parametric equation of $\partial D_{a,b}$ (from Lemma 8.2.4), we get

$$\frac{1}{i2\pi} \int_{\partial D_{a,b}} \frac{zf'(z)}{f(z)}\, dz = \frac{1}{i2\pi} \sum_{j=1}^{4} \int_{\gamma_j} \frac{zf'(z)}{f(z)}\, dz,$$

where $\gamma_1(t) = \zeta + ta,\ \gamma_2(t) = \zeta + a + tb,\ \gamma_3(t) = \zeta + (1-t)a + b,\ \gamma_4(t) = \zeta + (1-t)b,\ t \in [0,1]$. Therefore, using $a, b \in P_f = P_{f'}$, we get

$$\begin{aligned}
\int_{\gamma_1} \frac{zf'(z)}{f(z)}\,dz &= \int_0^1 \frac{(\zeta+ta)f'(\zeta+ta)}{f(\zeta+ta)}\,adt \\
&= \int_0^1 \frac{(\zeta+b+ta)f'(\zeta+b+ta)}{f(\zeta+b+ta)}\,adt \\
&\quad - \int_0^1 \frac{bf'(\zeta+b+ta)}{f(\zeta+b+ta)}\,adt \\
&= \int_1^0 \frac{(\zeta+b+(1-s)a)f'(\zeta+b+(1-s)a)}{f(\zeta+b+(1-s)a)}\,a(-ds) \\
&\quad - \int_0^1 \frac{bf'(\zeta+ta)}{f(\zeta+ta)}\,adt \\
&= -\int_0^1 \frac{(\zeta+b+(1-s)a)f'(\zeta+b+(1-s)a)}{f(\zeta+b+(1-s)a)}\,a(-ds) \\
&\quad - b\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz \\
&= -\int_{\gamma_3} \frac{zf'(z)}{f(z)}\,dz - b\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz.
\end{aligned}$$

Now, using a is a period of f and f has no zero or pole on $\partial D_{a,b}$, we get that $f(\gamma_1)$ is a closed curve not passing through zero. Then, by using the change of variable $w = f(z)$, we get

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}\,dz = \int_{f(\gamma_1)} \frac{dw}{w-0}\,dw = \mathrm{WN}(f(\gamma_1), 0) \in \mathbb{Z}.$$

Hence, $\int\limits_{\gamma_1+\gamma_3} \dfrac{zf'(z)}{f(z)}\,dz = nb$ for some $n \in \mathbb{Z}$. Similarly, we can prove that $\int\limits_{\gamma_2+\gamma_4} \dfrac{zf'(z)}{f(z)}\,dz = ma$, for some $m \in \mathbb{Z}$. Thus,

$$z_1 + z_2 + \cdots + z_n - w_1 - w_2 - \cdots - w_n = \frac{1}{i2\pi} \int\limits_{\partial D_{a,b}} \frac{zf'(z)}{f(z)}\,dz = ma + nb$$

which belongs to P_f. □

LEMMA 8.2.7 Every entire elliptic function is a constant function.

Proof: Let f be an entire elliptic function. First, we show that f is a bounded function on $\mathbb{C}$. If a and b are independent periods of f, then consider the closed parallelogram

$$K = \{\lambda a + \mu b : 0 \le \lambda \le 1,\ 0 \le \mu \le 1\}$$

with vertices 0, a, b, and $a + b$. As f is continuous on the compact set K, we get that f is bounded on K. Next, we show that for every $z \in \mathbb{C}$, there exist $m, n \in \mathbb{Z}$ such that $z + na + mb \in K$.

As in the proof of Lemma 8.2.2, we get $z = ra + sb$ for some $r, s \in \mathbb{R}$. Next, we choose $m, n \in \mathbb{C}$ such that

$$0 \le r + m \le 1 \text{ and } 0 \le s + n \le 1$$

so that

$$z + na + mb = (r + m)a + (s + n)b \in K.$$

Clearly, $f(z + na + mb) = f(z)$, using $a, b \in P_f$. Thus, $f(\mathbb{C}) = f(K)$, which is bounded. Therefore, f is a bounded entire function, and hence, it is constant, by Liouville's theorem. □

Exercise 8.2.8 Prove that the order of a non-constant elliptic function is at least 2.
Hint: Apply Lemma 8.2.4.

8.3 WEIERSTRASS $\wp_{a,b}$ FUNCTION

There are no obvious examples of non-constant elliptic functions. However, the Weierstrass function is an example of elliptic function with desirable fundamental periods, which is defined as follows.

Definition 8.3.1 For given $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$, we define

$$\wp_{a,b}(z) = \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z},\\ m\neq 0 \text{ or } n\neq 0}} \left(\frac{1}{(z-(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right)$$

$\forall z \in \mathbb{Z} \setminus \{ma + nb : m, n \in \mathbb{Z}\}$.

The above infinite series means $\lim_{J\to\infty} S_J$, whenever the limit exists, where $S_J = \sum_{N=1}^{J} \sum_{\substack{m,n\in\mathbb{Z}\\ |m|+|n|=N}} \left(\frac{1}{(z-(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right)$, $\forall J \in \mathbb{N}$. For notational convenience, we write $\wp_{a,b}$ as follows.

$$\wp_{a,b}(z) = \frac{1}{z^2} + \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2} \right), \forall z \in \mathbb{Z} \setminus Q_{a,b},$$

where $Q_{a,b} = \{ma + nb : m, n \in \mathbb{Z}\}$.

LEMMA 8.3.2 If $a, b \in \mathbb{C}$ with $\frac{a}{b} \notin \mathbb{R}$, then there exists $\mu > 0$ such that

$$|ra + sb| \geq \mu(|r| + |s|), \ \forall r, s \in \mathbb{R}.$$

Proof: If $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\phi(r, s) = |ra + sb|, \ \forall (r, s) \in \mathbb{R} \times \mathbb{R}$$

then ϕ is continuous on $\mathbb{R} \times \mathbb{R}$. Indeed for a given $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{|a| + |b|}$, so that whenever $|(r, s) - (u, v)| < \delta$, we have $|r - u| \leq |(r, s) - (u, v)| < \delta$, $|s - v| \leq |(r, s) - (u, v)| < \delta$, and hence,

$$\begin{aligned} |\phi(r, s) - \phi(u, v)| &= \big|\, |ra + sb| - |ua + vb| \,\big| \\ &\leq \big|\, (ra + sb) - (ua + vb) \,\big| \\ &= |(r - u)a + (s - v)b| \\ &\leq |r - u||a| + |s - v||b| \\ &< \delta(|a| + |b|) \\ &= \epsilon. \end{aligned}$$

Next, we define $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\psi(r,s) = |r| + |s|$, $\forall (r,s) \in \mathbb{R} \times \mathbb{R}$. We also see that ψ is continuous because for a given $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{2}$ so that $|(r,s) - (u,v)| < \delta$ implies that

$$\begin{aligned} |\psi(r,s) - \psi(u,v)| &= |\,|r| + |s| - |u| - |v|\,| \\ &\leq |\,|r| - |u|\,| + |\,|s| - |v|\,| \\ &\leq |r-u| + |s-v| \\ &\leq 2|(r,s) - (u,v)| \\ &< 2\delta \\ &= \epsilon. \end{aligned}$$

If $K = \{(r,s) \in \mathbb{R} : |r| + |s| = 1\}$, then K is a closed set because ψ is continuous, $\{1\}$ is a closed set, and $K = \psi^{-1}(\{1\})$. Clearly, K is bounded, as $|(r,s)| = \sqrt{|r|^2 + |s|^2} \leq \sqrt{2}$, $\forall (r,s) \in K$. Therefore, K is a compact subset of $\mathbb{C}$. Since ϕ is a continuous real-valued function on K, if $\mu = \inf \phi(K)$, then $\mu = \phi(r_0, s_0)$ for some $(r_0, s_0) \in K$.

We claim that $\mu > 0$. Suppose that $\mu = 0$. As $(r_0, s_0) \in K$, we have $|r_0| + |s_0| = 1$, and hence, at least one of them is non-zero.

Case 1: $r_0 \neq 0$.
Then, $|r_0 a + s_0 b| = 0 \Rightarrow r_0 a + s_0 b = 0 \Rightarrow \frac{a}{b} = \frac{-s_0}{r_0} \in \mathbb{R}$, which is a contradiction.

Case 2: $s_0 \neq 0$.
Here, $|r_0 a + s_0 b| = 0 \Rightarrow r_0 a + s_0 b = 0 \Rightarrow \frac{b}{a} = \frac{-r_0}{s_0} \in \mathbb{R}$, which is a contradiction to $\frac{b}{a} \notin \mathbb{R}$. (Note that $\frac{b}{a} \notin \mathbb{R} \Leftrightarrow \frac{a}{b} \notin \mathbb{R}$.)

Thus, $\mu > 0$. Now, for $(r,s) = (0,0)$, the inequality $|ra + sb| \geq \mu(|r| + |s|)$ holds and for $(r,s) \in \mathbb{R} \times \mathbb{R} \setminus (0,0)$, $\left|\frac{r}{|r|+|s|}\right| + \left|\frac{s}{|r|+|s|}\right| = 1$, and hence,

$$\left|\frac{r}{|r| + |s|}a + \frac{s}{|r| + |s|}b\right| \geq \mu \Rightarrow |ra + sb| \geq \mu(|r| + |s|).$$

LEMMA 8.3.3 If a and b are independent complex numbers, then the series

$$\sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2}\right)$$

converges uniformly on every compact subset of $\mathbb{C} \setminus Q_{a,b}$.

Proof: Let K be a compact subset of $\mathbb{C} \setminus Q_{a,b}$. Since K is bounded, there exists $C > 0$ such that $|z| \leq C,\ \forall z \in K$. We choose $N_1 \in \mathbb{N}$, such that

$$|p| > 2C,\ \forall p = ma + nb \text{ such that } m, n \in \mathbb{Z} \text{ with } |m| + |n| \geq N_1. \quad (8.1)$$

For a given $z \in K$ and p as in equation (8.1), we have $|p| > 2C \geq 2|z|$, and hence, $|z - p| \geq |p| - |z| > |p| - \dfrac{|p|}{2} = \dfrac{|p|}{2}$. Therefore,

$$\begin{aligned}
\left|\frac{1}{(z-p)^2} - \frac{1}{p^2}\right| &= \left|\frac{2pz - z^2}{p^2(z-p)^2}\right| \\
&\leq \left|\frac{pz}{p^2(z-p)^2}\right| + \left|\frac{z(p-z)}{p^2(z-p)^2}\right| \\
&< \frac{|z|}{|p|\dfrac{|p|^2}{4}} + \frac{|z|}{|p|^2\dfrac{|p|}{2}} \\
&\leq \frac{6C}{|p|^3}. \qquad (8.2)
\end{aligned}$$

Now, using Lemma 8.3.2, we choose $\mu > 0$ such that

$$|ma + nb| > \mu(|m| + |n|),\ \forall m, n \in \mathbb{Z}. \quad (8.3)$$

Next, we note that for every $N \in \mathbb{N}$, there exist exactly $4N$ ordered pairs (m, n) such that $|m| + |n| = N$. Indeed, the required ordered pairs are given as follows.

If N is even, then

$$\begin{aligned}
&\text{(4 pairs)} && (\pm N, 0), (0, \pm N) \\
&\text{(8 pairs)} && (\pm(N-1), \pm 1), (\pm 1, \pm(N-1)) \\
&\text{(8 pairs)} && (\pm(N-2), \pm 2), (\pm 2, \pm(N-2)) \\
& && \vdots \\
&\text{(8 pairs)} && \left(\pm\left(\frac{N}{2}+1\right), \pm\left(\frac{N}{2}-1\right)\right), \left(\pm\left(\frac{N}{2}-1\right), \pm\left(\frac{N}{2}+1\right)\right) \\
&\text{(4 pairs)} && \left(\pm\frac{N}{2}, \pm\frac{N}{2}\right) \\
&\text{(Total } 4N \text{ pairs)} && \left(= 4 + \left(\frac{N}{2}-1\right) \times 8 + 4\right)
\end{aligned}$$

and if N is odd, then

$$\begin{aligned}
&(4 \text{ pairs}) && (\pm N, 0), (0, \pm N)\\
&(8 \text{ pairs}) && (\pm(N-1), \pm 1), (\pm 1, \pm(N-1))\\
&(8 \text{ pairs}) && (\pm(N-2), \pm 2), (\pm 2, \pm(N-2))\\
&&& \vdots\\
&(8 \text{ pairs}) && \left(\pm\frac{N+1}{2}, \pm\frac{N-1}{2}\right), \left(\pm\frac{N-1}{2}, \pm\frac{N+1}{2}\right)\\
&(\text{Total } 4N \text{ pairs}) && \left(= 4 + \frac{N-1}{2} \times 8\right).
\end{aligned}$$

Now, for a given $\epsilon > 0$, we can find $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and $\sum_{N=N_2}^{\infty} \frac{1}{N^2} < \epsilon$. For this N_1, using the above observation and the equations (8.2) and (8.3), we get

$$\begin{aligned}
&\left| \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2} \right) - \sum_{N=1}^{N_2} \sum_{\substack{m,n \in \mathbb{Z} \\ |m|+|n|=N}} \left(\frac{1}{(z-(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right) \right| \\
&= \left| \sum_{N=N_2}^{\infty} \sum_{\substack{m,n \in \mathbb{Z} \\ |m|+|n|=N}} \left(\frac{1}{(z-(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right) \right| \\
&\leq \sum_{N=N_2}^{\infty} \sum_{\substack{m,n \in \mathbb{Z} \\ |m|+|n|=N}} \frac{6C}{|ma+nb|^3} \\
&\leq 6C \sum_{N=N_2}^{\infty} \sum_{\substack{m,n \in \mathbb{Z} \\ |m|+|n|=N}} \frac{1}{\mu^3(|m|+|n|)^3} \\
&\leq 6C \sum_{N=N_2}^{\infty} \frac{4N}{\mu^3 N^3} \\
&\leq \frac{24C}{\mu^3} \sum_{N=N_2}^{\infty} \frac{1}{N^2} < \frac{24C}{\mu^3}\epsilon.
\end{aligned}$$

Thus, the series converges uniformly on every compact subset of $\mathbb{C} \setminus Q_{a,b}$. □

RESULT 8.3.4 The Weierstrass $\wp_{a,b}$ function is an

1. even function;
2. elliptic function with $P_{\wp_{a,b}} = Q_{a,b}$.

Proof: For $z \in \mathbb{C} \setminus Q_{a,b}$, replacing $-(m,n)$ by (u,v) and using

$$\{(m,n) : m,n \in \mathbb{Z},\ (m,n) \neq (0,0)\} = \{-(m,n) : m,n \in \mathbb{Z},\ (m,n) \neq (0,0)\},$$

we get

$$\begin{aligned}
\wp_{a,b}(-z) &= \frac{1}{(-z)^2} + \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z} \\ |m|+|n|=N}} \left(\frac{1}{(-z-(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right) \\
&= \frac{1}{z^2} + \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z} \\ |m|+|n|=N}} \left(\frac{1}{(z+(ma+nb))^2} - \frac{1}{(ma+nb)^2} \right) \\
&= \frac{1}{z^2} + \sum_{N=1}^{\infty} \sum_{\substack{u,v\in\mathbb{Z} \\ |u|+|v|=N}} \left(\frac{1}{(z-(ua+vb))^2} - \frac{1}{(ua+vb)^2} \right) \\
&= \wp_{a,b}(z).
\end{aligned}$$

From the previous lemma, $\wp_{a,b}$ is an analytic function on $\mathbb{C} \setminus Q_{a,b}$ and

$$\wp'_{a,b}(z) = -\frac{2}{z^3} - \sum_{p\in Q_{a,b}\setminus\{0\}} \frac{2}{(z-p)^3} = \sum_{p\in Q_{a,b}} \frac{-2}{(z-p)^3} = \sum_{m,n\in\mathbb{Z}} \frac{-2}{(z-(ma+nb))^3}.$$

Hence, for any $ja+kb \in Q_{a,b}$, using $\{(m+k,n+j) : m,n \in \mathbb{Z}\} = \{(m,n) : m,n \in \mathbb{Z}\}$, we get

$$\begin{aligned}
\wp'_{a,b}(z+ja+kb) &= \sum_{m,n\in\mathbb{Z}} \frac{-2}{(z+ja+kb-(ma+nb))^3} \\
&= \sum_{m,n\in\mathbb{Z}} \frac{-2}{(z+(j-m)a+(k-n)b)^3} \\
&= \sum_{r,s\in\mathbb{Z}} \frac{-2}{(z+ra+sb)^3} \\
&= \wp'_{a,b}(z).
\end{aligned}$$

Therefore, $\wp'_{a,b}$ is a doubly periodic with $P_{\wp'_{a,b}} = Q_{a,b}$. If

$$F(z) = \wp_{a,b}(z+a) - \wp_{a,b}(z),\ \forall z \in \mathbb{C} \setminus Q_{a,b},$$

then

$$F'(z) = \wp'_{a,b}(z+a) - \wp'_{a,b}(z) = 0,\ \forall z \in \mathbb{C} \setminus Q_{a,b}.$$

Thus, F is constant on $\mathbb{C}\backslash Q_{a,b}$, and hence, on $\mathbb{C}$. Therefore, by taking $z = -\frac{a}{2}$, we get $F(-\frac{a}{2}) = \wp_{a,b}(\frac{a}{2}) - \wp_{a,b}(-\frac{a}{2}) = 0$, as $\wp_{a,b}$ is an even function. Thus, F is identically zero on $\mathbb{C}$, and hence, $\wp_{a,b}(z+a) = \wp_{a,b}(z)$, $\forall z \in \mathbb{C} \setminus Q_{a,b}$. Similarly, we can show that $\wp_{a,b}(z+b) = \wp_{a,b}(z)$, $\forall z \in \mathbb{C} \setminus Q_{a,b}$. Therefore, $\wp_{a,b}$ is an elliptic function with $P_{\wp_{a,b}} = Q_{a,b}$. □

LEMMA 8.3.5 If f and g are elliptic functions such that both have same zeroes and poles of the same orders, then prove that $f = cg$ for some constant $c \in \mathbb{C}$.

Proof: As f and g are elliptic functions with same zeroes and poles, it is obvious that they have same fundamental periods. Then, $\frac{f}{g}$ is an entire function after removing the singularities at the zeroes and poles of f. Thus, by Lemma 8.2.7, $\frac{f}{g}$ is a constant, and hence, the lemma follows. □

THEOREM 8.3.6 Every even elliptic function with fundamental periods a and b is a rational function of $\wp_{a,b}$.

Proof: Let f be an even elliptic function with fundamental periods a and b.

Case 1: f has neither a pole nor a zero at 0.

Then, using the doubly periodicity of f, we immediately conclude that f has neither a pole nor a zero at w, $\forall w \in P_{\wp_{a,b}}$. If $w \in \{\frac{a}{2}, \frac{b}{2}, \frac{a+b}{2}\}$, then $w-(-w) = 2w \in \{a, b, a+b\}$, and hence, $\wp'_{a,b}(w) = \wp'_{a,b}(-w)$. On the other hand, as $\wp'_{a,b}$ is an odd function (equivalently, $\wp_{a,b}$ is an even function), we get $\wp'_{a,b}(w) = -\wp_{a,b}(-w)$. Therefore, $\wp'_{a,b}$ vanishes at each $w \in \{\frac{a}{2}, \frac{b}{2}, \frac{a+b}{2}\}$. As 0 is the only pole for $\wp'_{a,b}$ in the fundamental parallelogram $D_{a,b}(0)$ and its order is 3, by Lemma 8.2.5, the number of zeroes of $\wp'_{a,b}$ in $D_{a,b}(0)$ must be 3. Therefore, $\frac{a}{2}, \frac{b}{2}, \frac{a+b}{2}$ must be simple zeroes of $\wp'_{a,b}$, and there is no other zero for $\wp'_{a,b}$ in $D_{a,b}(0)$. Hence, obviously, if $w \in \{\frac{a}{2}, \frac{b}{2}, \frac{a+b}{2}\}$, then w must be a zero of order 2 for $\wp_{a,b} - \wp_{a,b}(w)$. Using the doubly periodicity of $\wp_{a,b}$, we obtain that $\wp_{a,b} - \wp_{a,b}(w)$ has a zero of order 2 at each $w \in \frac{1}{2}P_{\wp_{a,b}} \left(= \{\frac{p}{2} : p \in P_{\wp_{a,b}}\}\right)$. Next, if $w \notin \frac{1}{2}P_{\wp_{a,b}}$, then $\wp_{a,b} - \wp_{a,b}(w)$ has two distinct simple zeroes at w and $-w$.

Since f is an even function, for a given zero z_0 of f in $D_{a,b}(0)$, $-z_0$ is also a zero of f, and hence, there exists a unique point w_0 in $D_{a,b}(0)$ such that $w_0 + z_0 \in Q_{a,b}$ and hence, $f(-z_0) = f(w_0)$. If $w_0 \neq z_0$, then we get two distinct zeroes (z_0 and w_0) of f in $D_{a,b}(0)$ with $z_0 + w_0 \in Q_{a,b}$. If $w_0 = z_0$ (equivalently $z_0 \in \frac{1}{2}P_{\wp_{a,b}}$), then f has

a zero at z_0 of even order. (Because, being f is even, $f^{(2k-1)}$ is an odd function for every $k \in \mathbb{N}$, and hence, $f^{(2k-1)}(z_0) = -f^{(2k-1)}(-z_0)$. From $z_0 \in \frac{1}{2}P_{\wp_{a,b}}$ and by the fact that $f^{(2k-1)}$ is a doubly periodic function of periods a and b, we also have $f^{(2k-1)}(z_0) = f^{(2k-1)}(-z_0)$. Therefore, $f^{(2k-1)}(z_0) = 0$. Therefore, if the positive integer m at which $f^{(m)}(z_0) \neq 0$, then m must be even.) Thus, if the distinct zeroes of f in $D_{a,b}(0)$ are $z_1, z_2, \ldots, z_n$, then we can rename the zeroes such that

$$z_1, z_2, \ldots, z_{2k} \notin \frac{1}{2}P_{\wp_{a,b}},\ z_{2k+1}, z_{2k+2}, \ldots, z_n \in \frac{1}{2}P_{\wp_{a,b}}$$

and

$$z_{k+j} + z_j \in Q_{a,b},\ \forall j = 1, 2, \ldots, k$$

for some $k \in \mathbb{N}$ with $2k \leq n$.

If the order of the zero of f at z_j is m_j, then clearly $m_j \in 2\mathbb{N}$ for $2k + 1 \leq j \leq n$ and

$$\prod_{j=1}^{k}(\wp_{a,b} - \wp_{a,b}(z_j))^{m_j} \cdot \prod_{j=2k+1}^{n}(\wp_{a,b} - \wp_{a,b}(z_j))^{\frac{m_j}{2}} \tag{8.4}$$

has exactly the same zeroes $z_1, z_2, \ldots, z_n$ with the same orders m_j.

Similarly, if $\zeta_1, \zeta_2, \ldots, \zeta_\nu$ are the distinct poles of order μ_j, respectively, for f in $D_{a,b}(0)$, then by the previous argument, we can say that

$$\prod_{j=1}^{\lambda}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\mu_j} \cdot \prod_{j=2\lambda+1}^{\nu}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\frac{\mu_j}{2}} \tag{8.5}$$

has exactly the same zeroes $\zeta_1, \zeta_2, \ldots, \zeta_\nu$ with the orders μ_j, for a suitable $\lambda \in \mathbb{N}$.

Then by Lemma 8.2.5, we have $\sum_{j=1}^{n} m_j = \sum_{j=1}^{\nu} \mu_j$, and hence, both of the functions in equations (8.4) and (8.5) are having same poles with same orders. Thus, after removing the removable singularities (the poles of the functions in equations (8.4) and (8.5)),

$$\frac{\prod_{j=1}^{k}(\wp_{a,b} - \wp_{a,b}(z_j))^{m_j} \cdot \prod_{j=2k+1}^{n}(\wp_{a,b} - \wp_{a,b}(z_j))^{\frac{m_j}{2}}}{\prod_{j=1}^{\lambda}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\mu_j} \cdot \prod_{j=2\lambda+1}^{\nu}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\frac{\mu_j}{2}}}$$

has the zeroes z_j with orders m_j, $1 \le j \le n$, and poles ζ_j, $1 \le j \le \mu$, with orders μ_j in $D_{a,b}(0)$. Thus, by applying Lemma 8.3.5, we conclude that

$$f = c\frac{\prod_{j=1}^{k}(\wp_{a,b} - \wp_{a,b}(z_j))^{m_j} \cdot \prod_{j=2k+1}^{n}(\wp_{a,b} - \wp_{a,b}(z_j))^{\frac{m_j}{2}}}{\prod_{j=1}^{\lambda}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\mu_j} \cdot \prod_{j=2\lambda+1}^{\nu}(\wp_{a,b} - \wp_{a,b}(\zeta_j))^{\frac{\mu_j}{2}}}.$$

for some $c \in \mathbb{C}$.

Case 2: f can have a zero or a pole at 0.

If f has a zero of order n at 0, then $n \in 2\mathbb{N}$, as f is even. We note that $\wp_{a,b}$ has a double pole at 0, and hence, $\wp_{a,b}^{\frac{n}{2}}$ has a pole at 0 of order n. Therefore, $f\wp_{a,b}^{\frac{n}{2}}$ has neither a pole nor a zero at 0 after removing the singularity at 0.

If f has a pole of order m, then $m \in 2\mathbb{N}$, as f is even. Therefore, $f\wp_{a,b}^{\frac{-m}{2}}$ has neither a pole or a zero at 0 after removing the singularity at 0.

Hence, the theorem follows. □

THEOREM 8.3.7 Every elliptic function can be written as $R(\wp_{a,b}, \wp'_{a,b})$, where R is a rational function in two variables.

Proof: Let f be a given elliptic function. If we define $f_1 = \frac{f(z)-f(-z)}{2}$ and $f_2(z) = \frac{f(z)+f(-z)}{2}$, $\forall z \in \Omega_f$, then f_1 is an odd function, f_2 is an even function, and $f = f_1 + f_2$ on Ω_f. (Recall that Ω_f is the set of all points at which f is analytic.) As $\dfrac{f_1}{\wp'_{a,b}}$ and f_2 are elliptic even functions, applying the previous theorem, we get $\dfrac{f_1}{\wp'_{a,b}} = R_1(\wp_{a,b})$ and $f_2 = R_2(\wp_{a,b})$, where R_1 and R_2 are rational functions in single variable. Thus, $f = \wp'_{a,b}R_1(\wp_{a,b}) + R_2(\wp_{a,b})$, which is a rational function in $\wp_{a,b}$ and $\wp'_{a,b}$. □

In the following sequel, we use the notation $G_n = \sum_{p \in Q_{a,b} \setminus \{0\}} \frac{1}{p^n}$, $\forall n \ge 3$.

RESULT 8.3.8 The Weierstrass $\wp_{a,b}$ function satisfies the following differential equation.

$$(\wp'_{a,b}(z))^2 = 4(\wp_{a,b}(z))^3 - 60G_4\wp_{a,b}(z) - 140G_6.$$

Proof: We first find the Laurent series expansion of $\wp_{a,b}$ in $B(0,r) \setminus \{0\}$, for some $r > 0$.

$$\begin{aligned}
\wp_{a,b}(z) &= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2} \right) \\
&= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{d}{dz} \left[\frac{1}{(p-z)} \right] - \frac{1}{p^2} \right) \\
&= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{p} \frac{d}{dz} \left[\sum_{k=0}^{\infty} \left(\frac{z}{p} \right)^k \right] - \frac{1}{p^2} \right) \\
&= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{p^2} \sum_{k=1}^{\infty} k \left(\frac{z}{p} \right)^{k-1} - \frac{1}{p^2} \right) \\
&= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left(\frac{1}{p^2} \left[1 + \sum_{k=2}^{\infty} k \left(\frac{z}{p} \right)^{k-1} \right] - \frac{1}{p^2} \right) \\
&= \frac{1}{z^2} + \sum_{p \in Q_{a,b} \setminus \{0\}} \frac{1}{p^2} \sum_{k=2}^{\infty} k \left(\frac{z}{p} \right)^{k-1} \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} (k+1) \sum_{p \in Q_{a,b} \setminus \{0\}} \frac{1}{p^{k+2}} z^k \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}
\end{aligned}$$

as $G_{2n+1} = 0$, $\forall n \in \mathbb{N}$. Differentiating on both sides, we get

$$\wp'_{a,b}(z) = \frac{-2}{z^3} + \sum_{n=1}^{\infty} 2n(2n+1) G_{2n+2} z^{2n-1}.$$

Now, clearly $(\wp'_{a,b}(z))^2 - 4(\wp_{a,b}(z))^3 + 60G_4 \wp_{a,b}(z) + 140G_6$ is an elliptic function, which can have a possible pole only at 0 in $D_{a,b}(0)$. The Laurent series expansion of this function around 0 is obtained as follows.

If $\sum_{n=-\infty}^{\infty} A_n z^n$ is the Laurent series expansion of $(\wp'_{a,b}(z))^2$, then we first find $\sum_{n=-\infty}^{0} A_n z^n$ (the singular part along with A_0) as

$$\begin{aligned}(\wp'_{a,b}(z))^2 &= \left(-\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots\right)^2 \\ &= \frac{4}{z^6} - 24G_4\frac{1}{z^2} - 80G_6 + A_2 z^2 + A_3 z^3 + \cdots\end{aligned}$$

(In the first line of the above expression, we have written the series expansion of $\wp'_{a,b}(z)$ up to the term corresponding to z^3 explicitly. The reason is that only these three terms which contribute to get the terms corresponding to z^n, $n \leq 0$ in the square of this expansion.

Similarly, we write only the terms corresponding to z^n, $n \leq 0$, in the expansion of $\wp_{a,b}(z))^3$ as follows.

$$\begin{aligned}(\wp_{a,b}(z))^3 &= \left(\frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^6 + \cdots\right)^3 \\ &= \left(\frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \cdots\right).\end{aligned}$$

Therefore, the Laurent series expansion of $(\wp'_{a,b}(z))^2 - 4(\wp_{a,b}(z))^3 + 60G_4\wp_{a,b}(z) + 140G_6$ is

$$\left(\frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots\right) - 4\left(\frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \cdots\right) + 60G_4\frac{1}{z^2} + 140G_6 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

which is equal to $B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ for some $B_1, B_2, B_3, \cdots \in \mathbb{C}$. Therefore, $\lim_{z\to 0}(\wp'_{a,b}(z))^2 - 4(\wp_{a,b}(z))^3 + 60G_4\wp_{a,b}(z) + 140G_6 = 0$. Thus, $(\wp'_{a,b}(z))^2 - 4(\wp_{a,b}(z))^3 + 60G_4\wp_{a,b}(z) + 140G_6$ has no poles in $D_{a,b}(0)$. Hence, it is an entire elliptic function, and hence, it is a constant function, by Lemma 8.2.7. As the value of the constant is same as the limit of the function at 0, we conclude that $(\wp'_{a,b}(z))^2 - 4(\wp_{a,b}(z))^3 + 60G_4\wp_{a,b}(z) + 140G_6 = 0$. Hence, the result follows. □

Exercise 8.3.9 Prove that $\wp''_{a,b} = 6\wp_{a,b} - 30G_4$.

RESULT 8.3.10 The Weierstrass function $\wp_{a,b}$ satisfies the following differential equation

$$\left(\wp'_{a,b}\right)^2 = 4\left(\wp_{a,b} - e_1\right)\left(\wp_{a,b} - e_2\right)\left(\wp_{a,b} - e_3\right),$$

where $e_1 = \wp_{a,b}\left(\frac{a}{2}\right)$, $e_2 = \wp_{a,b}\left(\frac{b}{2}\right)$ and $e_3 = \wp_{a,b}\left(\frac{a+b}{2}\right)$.

Proof: From Case 1 of proof Theorem 8.3.6, we have the following observations:

1. The only zeroes of $\wp'_{a,b}$ in the fundamental parallelogram $D_{a,b}(0)$ are $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{a+b}{2}$, and each is a simple zero. Thus, the only zeroes of $(\wp'_{a,b})^2$ are double zeroes at $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{a+b}{2}$.
2. The only zeroes of $\left(\wp_{a,b} - e_1\right)\left(\wp_{a,b} - e_2\right)\left(\wp_{a,b} - e_3\right)$ in $D_{a,b}(0)$ are $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{a+b}{2}$, and each zero has multiplicity 2.

Since 0 is the only pole of $\wp_{a,b}$ in $D_{a,b}(0)$, which is of order 2,

$$\left(\wp_{a,b} - e_1\right)\left(\wp_{a,b} - e_2\right)\left(\wp_{a,b} - e_3\right)$$

has a pole at 0 in $D_{a,b}(0)$ of order 6. Similarly, $\wp'_{a,b}$ has the only pole at 0 in $D_{a,b}(0)$ of order 3, and hence $(\wp'_{a,b})^2$ has a pole at 0 in $D_{a,b}(0)$ which is of order 6. Therefore, $(\wp'_{a,b})^2$ and $\left(\wp_{a,b} - e_1\right)\left(\wp_{a,b} - e_2\right)\left(\wp_{a,b} - e_3\right)$ have same number of zeroes and poles of same orders in $\mathbb{C}$. Therefore, $\frac{(\wp'_{a,b})^2}{(\wp_{a,b}-e_1)(\wp_{a,b}-e_2)(\wp_{a,b}-e_3)}$ is a constant, say c. Thus, we have $\left(\wp'_{a,b}\right)^2 = c\left(\wp_{a,b} - e_1\right)\left(\wp_{a,b} - e_2\right)\left(\wp_{a,b} - e_3\right)$. Next, we find the value of c as follows. As

$$\lim_{z\to 0} z^6(\wp'_{a,b}(z))^2 = \lim_{z\to 0} z^6 \left(\sum_{m,n\in\mathbb{Z}} \frac{2}{(z-(ma+nb))^3}\right)^2 = 2^2 = 4$$

and

$$\lim_{z\to 0} z^2\left(\wp_{a,b}(z) - e_1\right) = \lim_{z\to 0} z^2 \left(\frac{1}{z^2} + \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2}\right)\right) = 1,$$

we get

$$\begin{aligned} c &= \frac{(\wp'_{a,b})^2}{(\wp_{a,b}-e_1)(\wp_{a,b}-e_2)(\wp_{a,b}-e_3)} \\ &= \lim_{z\to 0}\frac{(\wp'_{a,b}(z))^2}{(\wp_{a,b}(z)-e_1)(\wp_{a,b}(z)-e_2)(\wp_{a,b}(z)-e_3)} \\ &= \frac{\lim_{z\to 0} z^6(\wp'_{a,b}(z))^2}{\lim_{z\to 0} z^6(\wp_{a,b}(z)-e_1)(\wp_{a,b}(z)-e_2)(\wp_{a,b}(z)-e_3)} \\ &= 4. \end{aligned}$$

□

THEOREM 8.3.11 Prove that the unique elliptic function of order 2 with the fundamental periods a and b satisfying $\lim_{z\to 0}\left(f(z)-\frac{1}{z^2}\right)=0$ is $\wp_{a,b}$.

Proof: From the definition of $\wp_{a,b}$, clearly we have that $\wp_{a,b}$ is an elliptic function of order 2 (as the only pole of $\wp_{a,b}$ in $D_{a,b}(0)$ is at 0 and it is a double pole) and

$$\lim_{z\to 0}\left(\wp_{a,b}(z)-\frac{1}{z^2}\right)=\lim_{z\to 0}\sum_{p\in Q_{a,b}\setminus\{0\}}\left(\frac{1}{(z-p)^2}-\frac{1}{p^2}\right)=0.$$

Conversely, let f be an elliptic function of order 2 with the fundamental periods a and b satisfying $\lim_{z\to 0}\left(f(z)-\frac{1}{z^2}\right)=0$. First, we note that f should have a pole of order 2 at 0. As a and b are the fundamental periods of f, f should have double poles at every point of $Q_{a,b}$. Since f is an elliptic function of order 2, it should not have non-zero poles in $D_{a,b}(0)$. Thus, the set of all poles of f are same as $Q_{a,b}$. Now, we have

$$\lim_{z\to 0}\left(f(z)-\wp_{a,b}(z)\right)=\lim_{z\to 0}\left(f(z)-\frac{1}{z_2}\right)-\left(\wp_{a,b}(z)-\frac{1}{z_2}\right)=0$$

and hence, $f-\wp_{a,b}$ has a removable singularity at 0. As $f-\wp_{a,b}$ is elliptic with fundamental periods a and b, it has removable singularity at every point of $Q_{a,b}$. By defining $f(z)-\wp_{a,b}(z)=0,\ \forall z\in Q_{a,b}$, the singularity at every point of $Q_{a,b}$ is removed and $f-\wp_{a,b}$ becomes an entire elliptic function. Thus, $f-\wp_{a,b}$ becomes a constant function. By seeing its value at a point of $Q_{a,b}$, we conclude that $f=\wp_{a,b}$. □

THEOREM 8.3.12 (First addition theorem for $\wp_{a,b}$)
For any $u, v \in \mathbb{C} \setminus Q_{a,b}$, we have

$$\begin{vmatrix} \wp'_{a,b}(u) & \wp_{a,b}(u) & 1 \\ \wp'_{a,b}(v) & \wp_{a,b}(v) & 1 \\ -\wp'_{a,b}(u+v) & \wp_{a,b}(u+v) & 1 \end{vmatrix} = 0.$$

Proof: Since $\wp_{a,b}$ is doubly periodic and even if $u \pm v \in Q_{a,b}$, then $\wp_{a,b}(u) = \wp_{a,b}(v)$ and $\wp'_{a,b}(u) = \wp'_{a,b}(v)$, and hence, the determinant is 0. Therefore, we assume that $u \pm v \notin Q_{a,b}$. Now, we consider the following system of linear equations in two variables α and β.

$$\wp'_{a,b}(u) = \alpha\wp_{a,b}(u) + \beta \quad \text{and} \quad \wp'_{a,b}(v) = \alpha\wp_{a,b}(v) + \beta.$$

As the determinant of the coefficient matrix

$$\begin{vmatrix} \wp_{a,b}(u) & 1 \\ \wp_{a,b}(v) & 1 \end{vmatrix} = \wp_{a,b}(u) - \wp_{a,b}(v) \neq 0,$$

the above system of linear equations has a unique solution, say (α_0, β_0). If

$$f(z) = \wp'_{a,b}(z) - \alpha_0\wp_{a,b}(z) - \beta_0$$

then clearly f is an elliptic function of order 3 and it has a triple pole at 0, as $\wp'_{a,b}$ is of order 3 and it has a pole of order 3 at 0. Hence, by Lemmas 8.2.5 and 8.2.6, there are only three zeroes including multiplicities (say z_1, z_2 and z_3) in $D_{a,b}(0)$, and $z_1 + z_2 + z_3 \in Q_{a,b}$. By the choice of α_0 and β_0, we know that u and v are the two known zeroes of f, and hence, the other zero must be $-(u+v) + p$ for some $p \in Q_{a,b}$. In other words, we have

$$\wp'_{a,b}(-(u+v)) - \alpha_0\wp_{a,b}(-(u+v)) - \beta_0 = 0.$$

We combine this equation with the known two equations

$$\wp'_{a,b}(u) - \alpha_0\wp_{a,b}(u) - \beta = 0 \quad \text{and} \quad \wp'_{a,b}(v) - \alpha_0\wp_{a,b}(v) - \beta_0 = 0$$

into a matrix equation as follows.

$$\begin{pmatrix} \wp'_{a,b}(u) & \wp_{a,b}(u) & 1 \\ \wp'_{a,b}(v) & \wp_{a,b}(v) & 1 \\ \wp'_{a,b}(-(u+v)) & \wp_{a,b}(-(u+v)) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\alpha_0 \\ -\beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the system of homogeneous linear equation

$$\begin{pmatrix} \wp'_{a,b}(u) & \wp_{a,b}(u) & 1 \\ \wp'_{a,b}(v) & \wp_{a,b}(v) & 1 \\ \wp'_{a,b}(-(u+v)) & \wp_{a,b}(-(u+v)) & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

in three variables A, B and C has a non-trivial solution $A = 1, B = -\alpha_0$ and $C = \alpha_0$. Hence, from the theory of system linear equations, the coefficient matrix should be singular or equivalently

$$\begin{vmatrix} \wp'_{a,b}(u) & \wp_{a,b}(u) & 1 \\ \wp'_{a,b}(v) & \wp_{a,b}(v) & 1 \\ -\wp'_{a,b}(u+v) & \wp_{a,b}(u+v) & 1 \end{vmatrix} = 0.$$

Here, we have used the fact that $\wp_{a,b}$ is an even function and $\wp'_{a,b}$ is an odd function. □

THEOREM 8.3.13 (Second addition theorem for $\wp_{a,b}$)

1. For every $u, v \in \mathbb{C} \setminus Q_{a,b}$ with $u \pm v \notin Q_{a,b}$, we have

$$\wp_{a,b}(u+v) = \frac{1}{4}\left(\frac{\wp'_{a,b}(u) - \wp'_{a,b}(v)}{\wp_{a,b}(u) - \wp_{a,b}(v)}\right)^2 - \wp_{a,b}(u) - \wp_{a,b}(v).$$

2. For every $u \in \mathbb{C} \setminus Q_{a,b}$ with $2u \notin Q_{a,b}$, then

$$\wp_{a,b}(2u) = \frac{1}{4}\left(\frac{\wp''_{a,b}(u)}{\wp'_{a,b}(u)}\right)^2 - 2\wp_{a,b}(u).$$

Proof: Let $u, v \in \mathbb{C} \setminus Q_{a,b}$ with $u \pm v \notin Q_{a,b}$. Retaining the notations used in the previous theorem, we have $\wp_{a,b}(z) = \alpha_0\wp_{a,b}(z) + \beta_0,\ \forall z \in S$, where S consists of all points of $\mathbb{C} \setminus Q_{a,b}$ with $z - u \in Q_{a,b}$ or $z - v \in Q_{a,b}$ or $z + u + v \in Q_{a,b}$. Hence, we have

$$(\wp'_{a,b}(z))^2 = (\alpha_0\wp_{a,b}(z) + \beta_0)^2,\ \forall z \in S.$$

Using Result 8.3.8, we have

$$4(\wp_{a,b}(z))^3 - 60G_4\wp_{a,b}(z) - 140G_6 - (\alpha_0^2(\wp_{a,b}(z))^2 + 2\alpha_0\beta_0\wp_{a,b}(z) + \beta_0^2) = 0$$

and hence,

$$4(\wp_{a,b}(z))^3-\alpha_0^2(\wp_{a,b}(z))^2-(2\alpha_0\beta_0+60G_4)\wp_{a,b}(z)-(140G_6+\beta_0^2)=0,\ \forall z\in S.$$

By the choice of u and v, the three complex numbers $\wp_{a,b}(u)$, $\wp_{a,b}(v)$, and $\wp_{a,b}(u+v)$ are pairwise distinct, and hence, they are the only solutions of the cubic polynomial equation

$$4X^3-\alpha_0^2X^2-(2\alpha_0\beta_0+60G_4)X-(140G_6+\beta_0^2)=0.$$

Therefore, the sum of the roots of the equation $\wp_{a,b}(u)+\wp_{a,b}(v)+\wp_{a,b}(u+v)$ equals $\frac{1}{4}\alpha_0^2$. Now, we find $\alpha_0=\frac{\wp'_{a,b}(u)-\wp'_{a,b}(v)}{\wp_{a,b}(u)-\wp_{a,b}(v)}$ by solving from the system of linear equations

$$\wp'_{a,b}(u)=\alpha_0\wp_{a,b}(u)+\beta_0\quad\text{and}\quad\wp'_{a,b}(v)=\alpha_0\wp_{a,b}(v)+\beta_0.$$

Thus, we get

$$\wp_{a,b}(u+v)=\frac{1}{4}\left(\frac{\wp'_{a,b}(u)-\wp'_{a,b}(v)}{\wp_{a,b}(u)-\wp_{a,b}(v)}\right)^2-\wp_{a,b}(u)-\wp_{a,b}(v).\tag{8.6}$$

If $u\in\mathbb{C}\setminus Q_{a,b}$ such that $2u\notin Q_{a,b}$, then allowing $v\to u$ on both sides of equation (8.6), we obtain

$$\begin{aligned}
\wp_{a,b}(2u) &= \lim_{v\to u}\wp_{a,b}(u+v)\\
&= \lim_{v\to u}\frac{1}{4}\left(\frac{\wp'_{a,b}(u)-\wp'_{a,b}(v)}{\wp_{a,b}(u)-\wp_{a,b}(v)}\right)^2-\wp_{a,b}(u)-\lim_{v\to u}\wp_{a,b}(v)\\
&= \lim_{v\to u}\frac{1}{4}\left(\frac{\frac{\wp'_{a,b}(v)-\wp'_{a,b}(u)}{v-u}}{\frac{\wp_{a,b}(v)-\wp_{a,b}(u)}{v-u}}\right)^2-\wp_{a,b}(u)-\wp_{a,b}(u)\\
&= \frac{1}{4}\left(\frac{\wp''_{a,b}(u)}{\wp'_{a,b}(u)}\right)^2-2\wp_{a,b}(u)(\text{as } 2u\notin Q_{a,b}\Rightarrow\wp'_{a,b}(u)\neq 0).
\end{aligned}$$

Thus, the theorem follows. □

The second assertion of the above theorem is called the duplication formula for $\wp_{a,b}$.

COROLLARY 8.3.14 For $u,v\in\mathbb{C}\setminus Q_{a,b}$, $\wp_{a,b}$ satisfies

$$\wp_{a,b}(u+v)+\wp_{a,b}(u-v)=\frac{(2\wp_{a,b}(u)\wp_{a,b}(v)-30G_4)(\wp_{a,b}(u)+\wp_{a,b}(v))-140G_6}{(\wp_{a,b}(u)-\wp_{a,b}(v))^2}.$$

Proof: Applying Result 8.3.8 in equation (8.6) and using the oddity of $\wp'_{a,b}$ and the evenness of $\wp_{a,b}$, we get
$\wp_{a,b}(u+v)+\wp_{a,b}(u-v)$

$$
\begin{aligned}
&= \frac{1}{4}\left(\frac{\wp'_{a,b}(u)-\wp'_{a,b}(v)}{\wp_{a,b}(u)-\wp_{a,b}(v)}\right)^2-\wp_{a,b}(u)-\wp_{a,b}(v)\\
&\quad+\frac{1}{4}\left(\frac{\wp'_{a,b}(u)+\wp'_{a,b}(v)}{\wp_{a,b}(u)-\wp_{a,b}(v)}\right)^2-\wp_{a,b}(u)-\wp_{a,b}(v)\\
&= \frac{1}{2}\frac{(\wp'_{a,b}(u))^2+(\wp'_{a,b}(v))^2}{\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}-2(\wp_{a,b}(u)+\wp_{a,b}(v))\\
&= \frac{(\wp'_{a,b}(u))^2+(\wp'_{a,b}(v))^2-4(\wp_{a,b}(u)+\wp_{a,b}(v))\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}{2\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}\\
&= \frac{1}{2\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}\Big(\Big[4(\wp_{a,b}(u))^3-60G_4\wp_{a,b}(u)-140G_6\\
&\quad+4(\wp_{a,b}(v))^3-60G_4\wp_{a,b}(v)-140G_6\Big]\\
&\quad-4\Big[(\wp_{a,b}(u))^3+(\wp_{a,b}(v))^3-\wp_{a,b}(u)(\wp_{a,b}(v))^2-(\wp_{a,b}(u))^2\wp_{a,b}(v)\Big]\Big)\\
&= \frac{1}{2\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}\big(-60G_4(\wp_{a,b}(u)+\wp_{a,b}(v))-280G_6\\
&\quad+4\wp_{a,b}(u)\wp_{a,b}(v)(\wp_{a,b}(u)+\wp_{a,b}(v))\big)\\
&= \frac{(2\wp_{a,b}(u)\wp_{a,b}(v)-30G_4)(\wp_{a,b}(u)+\wp_{a,b}(v))-140G_6}{\left(\wp_{a,b}(u)-\wp_{a,b}(v)\right)^2}.
\end{aligned}
$$

Hence, the corollary follows. □

COROLLARY 8.3.15 For $u, v \in \mathbb{C} \setminus Q_{a,b}$, $\wp_{a,b}$ satisfies

$$\wp_{a,b}(u+v)-\wp_{a,b}(u-v)=-\frac{\wp_{a,b}(u)\wp_{a,b}(v)}{(\wp_{a,b}(u)-\wp_{a,b}(v))^2}.$$

The proof of this corollary is similar to that of the previous one.

COROLLARY 8.3.16 For every $u \in \mathbb{C} \setminus Q_{a,b}$,

$$\wp_{a,b}\left(u+\frac{a}{2}\right)=\frac{(e_1-e_2)(e_1-e_3)}{\wp_{a,b}(u)-e_1}+e_1,$$

where $e_1=\wp_{a,b}\left(\frac{a}{2}\right)$, $e_2=\wp_{a,b}\left(\frac{b}{2}\right)$ and $e_3=\wp_{a,b}\left(\frac{a+b}{2}\right)$.

Proof: Using the value of $\left(\wp_{a,b}(u)\right)^2$ from Result 8.3.10 in equation (8.6), we get

$$\begin{aligned}
&\wp_{a,b}\left(u+\tfrac{a}{2}\right)\\
&= \frac{1}{4}\left(\frac{\wp'_{a,b}(u)}{\wp_{a,b}(u)-e_1}\right)^2 - \wp_{a,b}(u) - e_1\\
&= \frac{(\wp_{a,b}(u)-e_1)(\wp_{a,b}(u)-e_2)(\wp_{a,b}(u)-e_3)}{(\wp_{a,b}(u)-e_1)^2} - \wp_{a,b}(u) - e_1\\
&= \frac{(\wp_{a,b}(u)-e_2)(\wp_{a,b}(u)-e_3)}{(\wp_{a,b}(u)-e_1)} - (\wp_{a,b}(u)+e_1)\\
&= \frac{(\wp_{a,b}(u))^2-(e_2+e_3)\wp_{a,b}(u)+e_2e_3-(\wp_{a,b}(u))^2+e_1^2}{(\wp_{a,b}(u)-e_1)}\\
&= \frac{e_1\wp_{a,b}(u)+e_2e_3+e_1^2}{(\wp_{a,b}(u)-e_1)}\\
&\quad (\text{as } e_1+e_2+e_3=0)\\
&= \frac{e_1\wp_{a,b}(u)+(e_1-e_2)(e_1-e_3)-2e_1^2+e_1^2}{(\wp_{a,b}(u)-e_1)}\\
&\quad (\text{as } (e_1-e_2)(e_1-e_3)=e_1^2+e_2e_3-(e_2+e_3)e_1=2e_1^2+e_2e_3)\\
&= \frac{(e_1-e_2)(e_1-e_3)}{(\wp_{a,b}(u)-e_1)}+e_1.
\end{aligned}$$

Thus, the corollary follows. □

COROLLARY 8.3.17 $\wp_{a,b}\left(\frac{a}{4}\right) = e_1 \pm \sqrt{(e_1-e_2)(e_1-e_3)}$.

Proof: In the previous corollary, substituting $u = -\frac{a}{4}$, we get

$$\wp_{a,b}\left(\frac{a}{4}\right) = \wp_{a,b}\left(-\frac{a}{4}+\frac{a}{2}\right) = \frac{(e_1-e_2)(e_1-e_3)}{(\wp_{a,b}\left(\frac{a}{4}\right)-e_1)}+e_1$$

and hence, $\left[\wp_{a,b}\left(\frac{a}{4}\right)-e_1\right]^2 = (e_1-e_2)(e_1-e_3)$. Thus, the corollary follows. □

8.4 THE FUNCTIONS $\zeta_{a,b}$ AND $\sigma_{a,b}$

Definition 8.4.1 Define

$$\zeta_{a,b}(z) = \frac{1}{z} + \sum_{0\neq p\in Q_{a,b}} \left(\frac{1}{z-p}+\frac{1}{p}+\frac{z}{p^2}\right),\ \forall z\in\mathbb{C}\setminus Q_{a,b}.$$

LEMMA 8.4.2 The series

$$\sum_{0\neq p\in Q_{a,b}} \left(\frac{1}{z-p}+\frac{1}{p}+\frac{z}{p^2}\right)$$

converges uniformly in $\mathbb{C}\setminus Q_{a,b}$, and the anti-derivative of $\wp_{a,b}$ is $-\zeta_{a,b}$.

Proof: We shall show that the series converges uniformly on $\mathrm{Cl}\, B(0,r)$ for every $r>0$. Using Lemma 8.3.2, first we find $\mu>0$ such that $|ma+nb|\geq \mu(|m|+|n|)$, $\forall (m,n)\in\mathbb{Z}$. Now, choose $N_0\in\mathbb{N}$ such that $N_0>\dfrac{2r}{\mu}$ so that

$$|ma+nb|\geq \mu(|m|+|n|)\geq \mu N_0>2r,\ \forall (m,n)\in\mathbb{Z}^2 \text{ with } |m|+|n|\geq N_0.$$

(Note that as $\dfrac{a}{b}\notin\mathbb{R}$, we have $a\neq 0\neq b$.)

Now, for $|z|\leq r$ and for $(m,n)\in\mathbb{Z}^2$ with $|m|+|n|\geq N_0$, we have $\left|\frac{z}{ma+nb}\right|<\frac{1}{2}$, and hence,

$$\begin{aligned}
&\left|\frac{1}{z-(ma+nb)}+\frac{1}{ma+nb}+\frac{z}{(ma+nb)^2}\right|\\
&= \left|\frac{-1}{ma+nb}\left[\frac{1}{1-\frac{z}{ma+nb}}\right]+\frac{1}{ma+nb}+\frac{z}{(ma+nb)^2}\right|\\
&= \left|\frac{-1}{ma+nb}\left[1+\frac{z}{ma+nb}+\sum_{k=2}^{\infty}\frac{z^k}{|ma+nb|^k}\right]+\frac{1}{ma+nb}+\frac{z}{(ma+nb)^2}\right|\\
&= \left|\frac{-1}{ma+nb}\sum_{k=2}^{\infty}\frac{z^k}{(ma+nb)^k}\right|\\
&\leq \frac{|z|^2}{|ma+nb|^3}\sum_{k=0}^{\infty}\frac{|z|^k}{|ma+nb|^k}\\
&\leq \frac{|z|^2}{|ma+nb|^3}\frac{1}{1-\frac{|z|}{|ma+nb|}}\\
&\leq \frac{|z|^2}{|ma+nb|^3}\frac{1}{1-\frac{1}{2}}\\
&\leq \frac{2|z|^2}{|ma+nb|^3}\leq 4r^2\frac{1}{|ma+nb|^3},
\end{aligned}$$

which is a typical term of the convergent series $\sum_{\substack{m,n\in\mathbb{Z}\\ |m|+|n|\neq 0}} \frac{1}{|ma+nb|^3}$. (See the proof of Lemma 8.3.3.) Thus, by comparison test, the series

$$\sum_{\substack{m,n\in\mathbb{Z}\\ |m|+|n|\neq 0}} \left(\frac{1}{z-(ma+nb)} + \frac{1}{ma+nb} + \frac{z}{(ma+nb)^2}\right)$$

converges uniformly on Cl $B(0,r)\setminus Q_{a,b}$, and hence, on every compact subset of $\mathbb{C}\setminus Q_{a,b}$. Therefore we have, $\zeta'_{a,b}(z) = \frac{-1}{z^2} + \sum_{0\neq p\in Q_{a,b}} \left(\frac{-1}{(z-a)^2} + \frac{1}{p^2}\right) = -\wp_{a,b}(z)$. □

RESULT 8.4.3 There exist constants $\alpha, \beta \in \mathbb{C}$ such that $\zeta_{a,b}(z+a)-\zeta_{a,b}(z) = \alpha$ and $\zeta_{a,b}(z+b)-\zeta_{a,b}(z) = \beta$, $\forall z \in \mathbb{C}\setminus Q_{a,b}$. Furthermore, they satisfy $\alpha b - \beta a = 2\pi i$.

Proof: From the previous lemma, we have $\zeta'_{a,b} = -\wp_{a,b}$. As a is a period of $\wp_{a,b}$, we get

$$\zeta'_{a,b}(z+a) - \zeta'_{a,b}(z) = 0 \Rightarrow \wp_{a,b}(z+a) - \wp_{a,b}(z) = 0,\ \forall z \in \mathbb{C}\setminus Q_{a,b}.$$

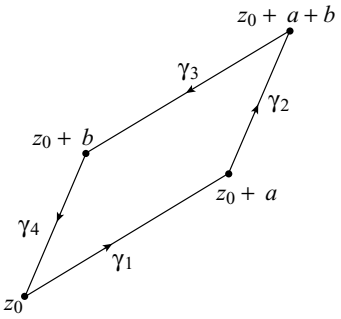
Thus, there exists a constant $\alpha \in \mathbb{C}$ such that

$$\zeta_{a,b}(z+a) - \zeta_{a,b}(z) = \alpha,\ \forall z \in \mathbb{C}\setminus Q_{a,b}.$$

Similarly, we can prove that there exists $\beta \in \mathbb{C}$ such that

$$\zeta_{a,b}(z+b) - \zeta_{a,b}(z) = \beta, \forall z \in \mathbb{C}\setminus Q_{a,b}$$

Let $D_{a,b}(z_0)$ be a fundamental parallelogram such that $\partial D_{a,b}(z_0)$ does not have any pole of $\zeta_{a,b}$. Since the set of all poles of $\zeta_{a,b}$ is same as $Q_{a,b}$, $D_{a,b}(z_0)$ should contain exactly one pole of $\zeta_{a,b}$. We also know that Res $(\zeta_{a,b}, p) = 1$, $\forall p \in Q_{a,b}$, as the coefficient of $\frac{1}{z-p}$ in the Laurent series expansion of $\zeta_{a,b}$ in $B(a,r)\setminus\{a\}$ is 1, for some $r > 0$, $\forall p \in Q_{a,b}$. Note that a parametric equation of $\partial D_{a,b}(z_0)$ is given as follows:



$\gamma_1(t) = z_0 + ta,\ t \in [0,1]$

$\gamma_2(t) = z_0 + a + tb,\ t \in [0,1]$

$\gamma_3(t) = z_0 - ta + b,\ t \in [-1,0]$

$\gamma_4(t) = z_0 - ta,\ t \in [-1,0].$

Therefore, by Cauchy's residue theorem, we get

$$\begin{aligned}
i2\pi &= \int_{\partial D_{a,b}(z_0)} \zeta_{a,b}(z)\,dz \\
&= \int_0^1 \zeta_{a,b}(z_0+ta)\,a\,dt + \int_0^1 \zeta_{a,b}(z_0+a+tb)\,b\,dt \\
&\quad - \int_{-1}^0 \zeta_{a,b}(z_0-ta+b)\,a\,dt - \int_{-1}^0 \zeta_{a,b}(z_0-tb)\,b\,dt \\
&= a\int_0^1 \zeta_{a,b}(z_0+ta)\,dt + b\int_0^1 \left[\zeta_{a,b}(z_0+tb)+\alpha\right] dt \\
&\quad - a\int_0^1 \left[\zeta_{a,b}(z_0+sa)+\beta\right] ds - \int_0^1 \zeta_{a,b}(z_0+sb)\,ds \\
&= -a\int_0^1 \beta\,dt + b\int_0^1 \alpha\,dt \\
&= \alpha b - \beta a.
\end{aligned}$$

Remark 8.4.4: From the previous proposition, we note that $\zeta_{a,b}$ is not a periodic function.

PROBLEM 8.4.5 Prove that $\zeta_{a,b}$ is an odd function.

Solution: Let $z \in \mathbb{C} \setminus Q_{a,b}$ be arbitrary. Now,

$$\begin{aligned}
\zeta_{a,b}(-z) &= -\frac{1}{z} + \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{-z-p} + \frac{1}{p} + \frac{-z}{p^2}\right) \\
&= -\left[\frac{1}{z} + \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{z+p} - \frac{1}{p} + \frac{z}{p^2}\right)\right]
\end{aligned}$$

$$\begin{aligned} &= -\left[\frac{1}{z} + \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{z-p} + \frac{1}{p} + \frac{z}{p^2}\right)\right] \\ &\quad (\text{as } Q_{a,b}\setminus\{0\} = -(Q_{a,b}\setminus\{0\})) \\ &= -\zeta_{a,b}(z). \end{aligned}$$

PROBLEM 8.4.6 Prove that

$$\wp_{a,b}(z+w) + \wp_{a,b}(z) + \wp_{a,b}(w) = \left[\zeta_{a,b}(z+w) - \zeta_{a,b}(z) - \zeta_{a,b}(w)\right]^2, \quad \forall z, w \in \mathbb{C}\setminus Q_{a,b}.$$

Solution: Let $w \in \mathbb{C}$ be fixed arbitrarily in $\mathbb{C}\setminus Q_{a,b}$. If

$$f(z) = \wp_{a,b}(z+w) + \wp_{a,b}(z) + \wp_{a,b}(w) - \left[\zeta_{a,b}(z+w) - \zeta_{a,b}(z) - \zeta_{a,b}(w)\right]^2$$

for every $z \in \mathbb{C}\setminus Q_{a,b}$, then f is analytic on $\mathbb{C}\setminus Q_{a,b}$. Now, we show that f is an entire function after removing the removable singularities. If

$$\begin{aligned} f_1(z) &= \wp_{a,b}(z+w) + \wp_{a,b}(w) \\ f_2(z) &= \zeta_{a,b}(z+w) - \zeta_{a,b}(w) \\ R_{a,b}(z) &= \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{(z-p)^2} - \frac{1}{p^2}\right) \\ \rho_{a,b}(z) &= \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{(z-p)} + \frac{1}{p}\right) + \frac{z}{p^2},\ \forall z \in \mathbb{C} \end{aligned}$$

then we can write

$$\begin{aligned} f(z) &= \frac{1}{z^2} + R_{a,b}(z) + f_1(z) - \left(f_2(z) - \frac{1}{z} - \rho_{a,b}(z)\right)^2 \\ &= \frac{1}{z^2} + R_{a,b}(z) + f_1(z) - \left(f_2(z) - \rho_{a,b}(z)\right)^2 - \frac{1}{z^2} + \frac{2}{z}\left(f_2(z) - \rho_{a,b}(z)\right) \\ &= R_{a,b}(z) + f_1(z) - \left(f_2(z) - \rho_{a,b}(z)\right)^2 + \frac{2}{z}\left(f_2(z) - \rho_{a,b}(z)\right). \end{aligned}$$

Since

$$f_2(0) - \rho_{a,b}(0) = \zeta_{a,b}(w) - \zeta_{a,b}(w) - \sum_{p\in Q_{a,b}\setminus\{0\}} \left(\frac{1}{(-p)} + \frac{1}{p}\right) = 0,$$

$\frac{2}{z}(f_2(z) - \rho_{a,b}(z))$ has a removable singularity at 0. Hence, after removing the singularity at 0, $\frac{2}{z}(f_2(z) - \rho_{a,b}(z))$ becomes analytic in a neighbourhood of 0. Hence, f becomes analytic in a neighbourhood of 0. By a similar argument, we can prove that f is analytic in a neighbourhood of p, $\forall p \in Q_{a,b}$. Thus, f becomes analytic on $\mathbb{C}$.

Next, we claim that f is a periodic function with $P_f = Q_{a,b}$. To prove this claim, first we prove that $\zeta_{a,b}(z+w) - \zeta_{a,b}(z) - \zeta_{a,b}(w)$ is a periodic function of periods a and b (as a function of z only). As in the proof of Result 8.3.4, let

$$F(z) = \zeta_{a,b}(z+w+a) - \zeta_{a,b}(z+a) - \zeta_{a,b}(z+w) + \zeta_{a,b}(z), \forall z \in \mathbb{C} \setminus Q_{a,b}.$$

Then, F is analytic on $\mathbb{C} \setminus Q_{a,b}$ and

$$\begin{aligned} F'(z) &= \zeta'_{a,b}(z+w+a) - \zeta'_{a,b}(z+a) - \zeta'_{a,b}(z+w) + \zeta'_{a,b}(z) \\ &= -\wp_{a,b}(z+w+a) + \wp_{a,b}(z+a) + \wp_{a,b}(z+w) - \wp_{a,b}(z) \\ &\quad (\text{ by Lemma 8.4.2 }) \\ &= 0. \quad (\text{as } a \text{ is a period of } \wp_{a,b}). \end{aligned}$$

Therefore, F is constant on $\mathbb{C} \setminus \{0\}$, and hence, it is constant on $\mathbb{C}$. To find the value of the constant, using the fact that $\zeta_{a,b}$ is an odd function, we find

$$\begin{aligned} F\left(-\frac{w+a}{2}\right) &= \zeta_{a,b}\left(\frac{w+a}{2}\right) - \zeta_{a,b}\left(\frac{-w+a}{2}\right) - \zeta_{a,b}\left(\frac{w-a}{2}\right) \\ &\quad + \zeta_{a,b}\left(\frac{-w-a}{2}\right) \\ &= \zeta_{a,b}\left(\frac{w+a}{2}\right) + \zeta_{a,b}\left(\frac{w-a}{2}\right) - \zeta_{a,b}\left(\frac{w-a}{2}\right) \\ &\quad - \zeta_{a,b}\left(\frac{w+a}{2}\right) \\ &= 0. \end{aligned}$$

Therefore, F is identically zero on $\mathbb{C}$, and hence, $\zeta_{a,b}(z+w) - \zeta_{a,b}(z) - \zeta_{a,b}(w)$ is a periodic function of period a. Similarly, we can prove that b is also a period of $\zeta_{a,b}(z+w) - \zeta_{a,b}(z) - \zeta_{a,b}(w)$. Recalling that $P_{\wp_{a,b}} = Q_{a,b}$, we conclude that $P_f = Q_{a,b}$. □

LEMMA 8.4.7 The infinite product $\prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}}$ converges uniformly on every compact subset of $\mathbb{C}$.

Proof: For an arbitrarily fixed $w \in \mathbb{C}$ with $|w| < 1$, if we define $\varphi_w(t) : [0,1] \to \mathbb{C}$ by

$$\varphi(t) = (1 - tw)e^{tw + \frac{(tw)^2}{2}} - 1, \ t \in [0,1]$$

then φ is a differentiable function on $[0,1]$. Applying complex version of mean value theorem (Theorem 2.1.19), we first observe that if $|w| < 1$, then

$$\begin{aligned} \left|(1-w)e^{w+\frac{w^2}{2}} - 1\right| &= |\varphi(1) - \varphi(0)| \\ &\leq \left|\varphi'(s)\right| \qquad (\text{ for some } s \in (0,1)\ ) \\ &= s^2 e^{|sw| + \frac{|sw|^2}{2}} |w|^3 \\ &< |w|^3 e^{3/2}. \end{aligned}$$

Let K be a compact subset of $\mathbb{C}$. Then, we choose $M >$ such that $|z| \leq M, \ \forall z \in K$. Now, for $z \in K$, we have

$$\sum_{p \in Q_{a,b} \setminus \{0\}} \left| \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}} - 1 \right| \leq \sum_{p \in Q_{a,b} \setminus \{0\}} \left|\frac{z}{p}\right|^3 e^{3/2} \leq M^3 e^{3/2} \sum_{p \in Q_{a,b} \setminus \{0\}} \frac{1}{|p|^3}.$$

The last series is convergent. (Indeed, see the proof of Lemma 8.3.3). Hence, by Theorem 5.4.5, we get that the product $\prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}}$ converges absolutely and uniformly on K (because $\sum_{p \in Q_{a,b} \setminus \{0\}} \left| \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}} - 1 \right|$ converges uniformly on K). □

Definition 8.4.8 Define

$$\sigma_{a,b}(z) = z \prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}}, \ \forall z \in \mathbb{C}$$

Clearly, $\sigma_{a,b}$ is an entire function with simple zeroes at every $p \in Q_{a,b}$.

RESULT 8.4.9 The function $\sigma_{a,b}$ is an odd function and it satisfies $\frac{\sigma'_{a,b}(z)}{\sigma_{a,b}(z)} = \zeta_{a,b}(z), \ \forall z \in \mathbb{C} \setminus Q_{a,b}$.

Proof: For $z \in \mathbb{C}$, using the fact that $Q_{a,b} \setminus \{0\} = -(Q_{a,b} \setminus \{0\})$, we get

$$\begin{aligned}
\sigma_{a,b}(-z) &= (-z) \prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{(-z)}{p}\right) e^{\frac{(-z)}{p} + \frac{1}{2}\frac{(-z)^2}{p^2}} \\
&= -z \prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{z}{(-p)}\right) e^{\frac{z}{(-p)} + \frac{1}{2}\frac{z^2}{p^2}} \\
&= -z \prod_{p \in Q_{a,b} \setminus \{0\}} \left(1 - \frac{z}{p}\right) e^{\frac{z}{p} + \frac{1}{2}\frac{z^2}{p^2}} \\
&= -\sigma_{a,b}(z).
\end{aligned}$$

Thus, $\sigma_{a,b}$ is an odd function.

Finding the logarithmic derivative of $\sigma_{a,b}$, we immediately get

$$\begin{aligned}
\frac{\sigma'_{a,b}(z)}{\sigma_{a,b}(z)} &= \frac{1}{z} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left[\frac{1}{1 - \frac{z}{p}}\left(\frac{-1}{p}\right) + \frac{1}{p} + \frac{z}{p^2}\right] \\
&= \frac{1}{z} + \sum_{p \in Q_{a,b} \setminus \{0\}} \left[\frac{1}{z - p} + \frac{1}{p} + \frac{z}{p^2}\right] \\
&= \zeta_{a,b}(z).
\end{aligned}$$

Hence, the result follows. □

RESULT 8.4.10 Let f be an elliptic function such that

1. $P_f = Q_{a,b}$,
2. the only zeroes of f in $D_{a,b}(0)$ are $z_1, z_2, \ldots, z_n$ including multiplicities,
3. the only poles of f in $D_{a,b}(0)$ are $w_1, w_2, \ldots, w_n$ including multiplicities.

Then, there exists $c \in \mathbb{C}$ such that

$$f(z) = c \prod_{j=1}^{n} \frac{\sigma_{a,b}(z - z_j)}{\sigma_{a,b}(z - w_j)}, \ \forall z \in \mathbb{C} \setminus S_f,$$

where S_f is the set of all singularities of f.

Proof: As $\sigma_{a,b}$ is an entire function with simple zero at 0, we observe that $\sigma_{a,b}(z-s)$ is also an entire function with a simple zero at s, for each fixed $s \in \mathbb{C}$. If $g(z) = \prod_{j=1}^{n} \frac{\sigma_{a,b}(z-z_j)}{\sigma_{a,b}(z-w_j)}$, $\forall z \in \mathbb{C} \setminus S_f$, then g has zeroes at z_j and poles at w_j. We claim that a and b are periods of g. From Result 8.4.3, we have

$$\zeta_{a,b}(z+a) - \zeta_{a,b}(z) = \alpha \text{ and } \zeta_{a,b}(z+b) - \zeta_{a,b}(z) = \beta$$

for some $\alpha, \beta \in \mathbb{C}$. Using the previous result, we obtain

$$\frac{\sigma'_{a,b}(z+a)}{\sigma_{a,b}(z+a)} - \frac{\sigma'_{a,b}(z)}{\sigma_{a,b}(z)} = \alpha \text{and} \frac{\sigma'_{a,b}(z+b)}{\sigma_{a,b}(z+b)} - \frac{\sigma'_{a,b}(z)}{\sigma_{a,b}(z)} = \beta,$$

which imply that

$$\sigma'_{a,b}(z+a)\sigma_{a,b}(z) - \sigma'_{a,b}(z)\sigma_{a,b}(z+a) - \alpha\sigma_{a,b}(z+a)\sigma_{a,b}(z) = 0. \quad (8.7)$$

$$\sigma'_{a,b}(z+b)\sigma_{a,b}(z) - \sigma'_{a,b}(z)\sigma_{a,b}(z+b) - \beta\sigma_{a,b}(z+b)\sigma_{a,b}(z) = 0. \quad (8.8)$$

For $z \in \mathbb{C} \setminus S_f$, consider
$\frac{d}{dz}\left[\frac{\sigma_{a,b}(z+a)}{\sigma_{a,b}(z)} \exp\left(-\alpha\left(z+\frac{a}{2}\right)\right)\right]$

$$\begin{aligned}
&= \exp\left(-\alpha\left(z+\frac{a}{2}\right)\right)\left[\frac{\sigma'_{a,b}(z+a)}{\sigma_{a,b}(z)} - \frac{\sigma_{a,b}(z+a)\sigma'_{a,b}(z)}{\sigma^2_{a,b}(z)}\right] \\
&\quad -\alpha \exp\left(-\alpha\left(z+\frac{a}{2}\right)\right)\frac{\sigma_{a,b}(z+a)}{\sigma_{a,b}(z)} \\
&= \frac{\exp\left(-\alpha\left(z+\frac{a}{2}\right)\right)}{\sigma^2_{a,b}(z)}\left[\sigma'_{a,b}(z+a)\sigma_{a,b}(z) - \sigma_{a,b}(z+a)\sigma'_{a,b}(z)\right. \\
&\quad \left.-\alpha\sigma_{a,b}(z+a)\sigma_{a,b}(z)\right] \\
&= 0 \text{(by applying equation (8.7)).}
\end{aligned}$$

Therefore, $\frac{\sigma_{a,b}(z+a)}{\sigma_{a,b}(z)} \exp\left(-\alpha\left(z+\frac{a}{2}\right)\right)$ is a constant function on $\mathbb{C} \setminus Q_{a,b}$, and hence, it is a constant function (say C) on $\mathbb{C}$. By substituting $z = -\frac{a}{2}$, we get $C = \frac{\sigma_{a,b}\left(\frac{a}{2}\right)}{\sigma_{a,b}\left(-\frac{a}{2}\right)} = -1$ because $\sigma_{a,b}$ is an odd function and $\sigma_{a,b}\left(\frac{a}{2}\right) \neq 0$. Thus, we have

$$\sigma_{a,b}(z+a) = -\sigma_{a,b}(z)\exp\left(\alpha\left(z+\frac{a}{2}\right)\right) \quad (8.9)$$

For $z \in \mathbb{C} \setminus S_f$, using equation (8.9), it follows that

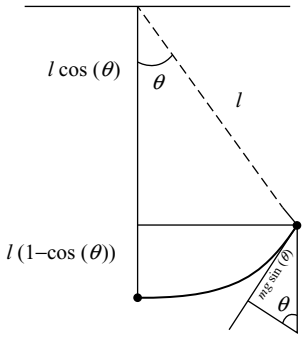
$$\frac{\sigma_{a,b}(z - z_j + a)}{\sigma_{a,b}(z - w_j + a)} = \frac{\sigma_{a,b}(z - z_j)}{\sigma_{a,b}(z - w_j)}$$

and hence, $g(z + a) = g(z)$, $\forall z \in \mathbb{C} \setminus Q_{a,b}$. By a similar argument, we prove that b is also a period of g. Thus, g is an elliptic function with fundamental[1] periods a and b. Therefore, the zeroes of g and the poles of g are precisely $\{z_j + p : p \in Q_{a,b},\ 1 \leq j \leq n\}$ and $\{w_j + p : p \in Q_{a,b},\ 1 \leq j \leq n\}$, respectively, as the the zeroes and poles of g inside the fundamental region $D_{a,b}(0)$ are $\{z_j : 1 \leq j \leq n\}$ and $\{w_j : 1 \leq j \leq n\}$, respectively. Thus, by Lemma 8.3.5, we get $f = cg$ for some $c \in \mathbb{C}$. □

8.5 JACOBI'S ELLIPTIC FUNCTIONS sn_k, cn_k AND dn_k

We start this section with the motivation for introducing elliptic integrals from the problem of simple pendulum.

A simple pendulum consists of a weightless thread of length l, whose one end is attached to a sphere of mass m and the other end is attached to a fixed point on the roof. By drawing the sphere aside from the equilibrium position, we assume that the sphere oscillates in a same vertical plane. The problem is to determine the angle θ between the current position of thread with its equilibrium position, as a function of time t.



[1] This statement can be obtained from the fact that a and b are fundamental periods of $\wp_{a,b}$ and by the relations among $\wp_{a,b}$, $\zeta_{a,b}$ and $\sigma_{a,b}$.

Applying the Newton's second law, the force acting upon the sphere due to this movement is same as the tangential component of force acting upon the sphere due to gravity. Thus, we get

$$m\frac{d^2}{dt^2}(l\theta) = -mg\sin(\theta), \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0,$$

where $l\theta$ is the arc length of the path of the sphere from equilibrium position to the present time. Multiplying on both sides of the above equation by $ml^2\frac{d\theta}{dt}$, we get

$$ml^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} = 0 \Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos(\theta))\right] = 0.$$

Thus, we get the total energy (kinetic energy + potential energy) acting upon the sphere at time t, which is constant. As the energy required to move the sphere from the equilibrium position (corresponding to $\theta = 0$) to the highest position (corresponding to $\theta = \pi$) is $k^2(2mgl)$ for some $0 < k < 1$ (because of the oscillatory motion), we have

$$\frac{1}{2}\left(l\frac{d\theta}{dt}\right)^2 + gl(1-\cos(\theta)) = k^2(2gl). \tag{8.10}$$

If the initial velocity of the sphere is v_0, then by the law of conservation of energy, then we have $\frac{1}{2}m\left(v_0^2 - \left(l\frac{d\theta}{dt}\right)^2\right) = mgl(1-\cos(\theta))$ and so we have

$$\left(l\frac{d\theta}{dt}\right)^2 = v_0^2 - 2gl(1-\cos(\theta)). \tag{8.11}$$

Comparing equations (8.10) and (8.11), we find that $k^2 = \frac{v_0^2}{4gl}$. Thus, using the change of variable $x = \sqrt{\frac{g}{l}}t$ in equation (8.11), we get

$$\left(\frac{d\theta}{dx}\right)^2 = 4\left(k^2 - \sin^2\left(\frac{\theta}{2}\right)\right). \tag{8.12}$$

To obtain the Euler's normal form and Jacobi's normal form of equation (8.12), we consider the change of variable $k\sin(\phi) = \sin\left(\frac{\theta}{2}\right)$, which is introduced by Euler and Jacobi. As $k\cos(\phi)\frac{d\phi}{dx} = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\frac{d\theta}{dx}$, we have

$$\frac{4k^2\cos^2(\phi)}{\cos^2\left(\frac{\theta}{2}\right)}\left(\frac{d\phi}{dx}\right)^2 = \left(\frac{d\theta}{dx}\right)^2 = 4\left(k^2 - \sin^2\left(\frac{\theta}{2}\right)\right) = 4k^2(1-\sin^2(\phi))$$

and so $\left(\frac{d\phi}{dx}\right)^2 = \cos^2\left(\frac{\theta}{2}\right) = 1 - \sin^2\left(\frac{\theta}{2}\right) = 1 - k^2\sin^2(\phi)$. Thus, the Euler's normal form is obtained as

$$\left(\frac{d\phi}{dx}\right)^2 = 1 - k^2\sin^2(\phi). \tag{8.13}$$

In this equation, if we use another change of variable (due to Jacobi) $y = \sin(\phi) = k^{-1}\sin\left(\frac{\theta}{2}\right)$, then we obtain $\frac{dy}{dx} = \frac{k^{-1}}{2}\cos\left(\frac{\theta}{2}\right)\frac{d\theta}{dx}$, and hence, using equation (8.12), we also get

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \frac{k^{-2}}{4}\cos^2\left(\frac{\theta}{2}\right)4\left(k^2 - \sin^2\left(\frac{\theta}{2}\right)\right)\\ &= \left(1 - \sin^2\left(\frac{\theta}{2}\right)\right)\left(1 - k^{-2}\sin^2\left(\frac{\theta}{2}\right)\right)\\ &= (1 - k^2y^2)(1 - y^2),\end{aligned} \tag{8.14}$$

which is called Jacobi's normal form. Now, we find the solution of equation (8.13) as follows.
From equation (8.13), we have

$$\frac{d\phi}{dx} = \sqrt{1 - k^2\sin^2(\phi)}. \tag{8.15}$$

Using the initial conditions $\theta = 0$ and $\dfrac{d\theta}{dt} = 0$ at $t = 0$, we have $\theta = 0$ and $\dfrac{d\theta}{dx} = 2k$, and hence, we get $\phi = 0$ and $\frac{d\phi}{dx} = 1$ at $x = 0$. Thus, $\frac{d\phi}{dx}$ is invertible in a small neighbourhood of $x = 0$, and hence, the derivative of x with respect to ϕ is obtained simply by

$$\frac{dx}{d\phi} = \frac{1}{\sqrt{1 - k^2\sin^2(\phi)}}, \tag{8.16}$$

which has the solution

$$x = F_k(\phi) = \int_0^{\phi} \frac{du}{\sqrt{1 - k^2\sin^2(u)}}. \tag{8.17}$$

The integral equation (8.17) is called the elliptic integral of first kind (See first section of this chapter). The constant k is called the modulus, and x is called the argument. If F_k is the function denoting $\phi \mapsto x$, then F_k is an odd and increasing function. Indeed,

1. using the change of variable $v = -u$ in what follows, we get $F_k(-\phi) = \int_0^{-\phi} \frac{du}{\sqrt{1-k^2\sin^2(u)}} = -\int_0^{\phi} \frac{dv}{\sqrt{1-k^2\sin^2(v)}} = -F_k(\phi)$.

2. the derivative $\frac{dF_k}{d\phi} = \frac{1}{\sqrt{1-k^2\sin^2(u)}} > 0$ (by Theorem 4.1.11), and hence, F_k is an increasing function (by Lemma 2.2.18).

Hence, the function F_k is invertible, and hence, its inverse map is also odd and increasing function. It is customary to denote the inverse of F_k by $\mathbf{am}_k$. We also note that

$$\mathbf{am}_k'(x) = \sqrt{1 - k^2\sin^2(\mathbf{am}_k(x))}. \tag{8.18}$$

If we denote $F_k(\frac{\pi}{2})$ by K_k, then K_k is the time taken (in the variable x) when ϕ varies from 0 to $\frac{\pi}{2}$. Now, we see that the map F_k is a bijection from $\mathbb{R}$ onto itself when $0 \le k < 1$ and is a bijection from $\mathbb{R}$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ when $k = 1$.

Definition 8.5.1 The basic Jacobian elliptic functions are defined as follows:

1. $\mathbf{sn}_k(x) = \sin(\mathbf{am}_k(x)) = \sin(\phi)$,
2. $\mathbf{cn}_k(x) = \cos(\mathbf{am}_k(x)) = \cos(\phi)$,
3. $\mathbf{dn}_k(x) = \sqrt{1 - k^2\sin^2(\phi)} = \sqrt{1 - k^2\mathbf{sn}_k^2(x)}$,

where x and ϕ are related by the one-to-one correspondence $x = F_k(\phi)$.

For the case $k = 0$, as $x = F_0(\phi) = \int_0^{\phi} du = \phi$, we have, by definition,

$$\mathbf{sn}_0(x) = \sin(x),\ \mathbf{cn}_0(x) = \cos(x),\ \mathbf{dn}_0(x) = 1, \text{ and } K_0 = \frac{\pi}{2}.$$

When $k = 1$, as

$$x = F_1(\phi) = \int_0^{\phi} \sec(u)du = \ln(\sec(\phi) + \tan(\phi))$$

we have

$$\exp(x) = \sec(\phi) + \tan(\phi) = \frac{1 + \sin(\phi)}{\cos(\phi)}$$

and hence,

$$\exp(2x) = \frac{(1+\sin(\phi))^2}{\cos^2(\phi)} = \frac{(1+\sin(\phi))^2}{(1-\sin^2(\phi))} = \frac{1+\sin(\phi)}{1-\sin(\phi)}.$$

Solving for $\sin(\phi)$ from the above two equations, we get

$$(1-\sin(\phi))\exp(2x) = 1+\sin(\phi) \Rightarrow \sin(\phi) = \frac{\exp(2x)-1}{\exp(2x)+1} = \tanh(x).$$

Hence,

$$\begin{aligned}
\mathbf{sn}_1(x) &= \sin(\phi) = \tanh(x), \\
\mathbf{cn}_1(x) &= \cos(\phi) = \sqrt{1-\sin^2(\phi)} = \sqrt{1-\tanh^2(x)} = \operatorname{sech}(x), \\
\mathbf{dn}_1(x) &= \sqrt{1-\sin^2(\phi)} = \sqrt{1-\tanh^2(x)} = \operatorname{sech}(x), \\
K_1 &= \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\sin^2(u)}} = \int_0^{\frac{\pi}{2}} \sec^2(u)\,du = \infty.
\end{aligned}$$

RESULT 8.5.2 For any $0 \le k < 1$,

1. $\mathbf{sn}_k(0) = 0$ and $\mathbf{cn}_k(0) = \mathbf{dn}_k(0) = 1$.
2. $\mathbf{sn}_k(K_k) = 1$, $\mathbf{cn}_k(K_k) = 0$ and $\mathbf{dn}_k(K_k) = j$, where $j = \sqrt{1-k^2}$ and is called the complementary modulus.
3. $\mathbf{sn}_k(-x) = -\mathbf{sn}_k(x)$, $\mathbf{cn}_k(-x) = \mathbf{cn}_k(x)$ and $\mathbf{dn}_k(-x) = \mathbf{dn}_k(x)$.
4. $\mathbf{sn}_k^2 + \mathbf{cn}_k^2 = 1$ and $k^2\mathbf{sn}_k^2 + \mathbf{dn}_k^2 = 1$.

Proof: Directly from the definition of $\mathbf{sn}_k$, $\mathbf{cn}_k$, and $\mathbf{dn}_k$, we get the following:

1. From the definition of F_k, we have $F_k(0) = 0$, and hence, $\mathbf{am}_k(0) = 0$. Therefore, $\mathbf{sn}_k(0) = \sin(\mathbf{am}_k(0)) = \sin(0) = 0$, $\mathbf{cn}_k(0) = \cos(\mathbf{am}_k(0)) = \cos(0) = 1$, and $\mathbf{dn}_k(0) = \sqrt{1-k^2\mathbf{sn}_k^2(0)} = 1$.
2. We note that $F_k\left(\frac{\pi}{2}\right) = K_k$, and hence, $\mathbf{am}_k(K_k) = \frac{\pi}{2}$. Therefore,

$$\mathbf{sn}_k(K_k) = \sin(\mathbf{am}_k(K_k)) = \sin\left(\frac{\pi}{2}\right) = 1,$$

$$\mathbf{cn}_k(K_k) = \cos(\mathbf{am}_k(K_k)) = \cos\left(\frac{\pi}{2}\right) = 0,$$

$$\mathbf{dn}_k(K_k) = \sqrt{1-k^2\mathbf{sn}_k^2(K_k)} = \sqrt{1-k^2}.$$

3. We know that $\mathbf{am}_k$ and sin are odd functions, and hence,

$$\mathbf{sn}_k(-x) = \sin(\mathbf{am}_k(-x)) = \sin(-\mathbf{am}_k(x)) = -\sin(\mathbf{am}_k(x)) = -\mathbf{sn}_k(x),$$

$$\mathbf{cn}_k(-x) = \cos(-\mathbf{am}_k(x)) = \cos(\mathbf{am}_k(x)) = \mathbf{cn}_k(x),$$

$$\mathbf{dn}_k(-x) = \sqrt{1 - k^2\mathbf{sn}_k(-x)} = \sqrt{1 - k^2\mathbf{sn}_k^2(x)} = \mathbf{dn}_k(x).$$

4. Finally, we also have

$$\mathbf{sn}_k^2(x) + \mathbf{cn}_k^2(x) = \sin^2(\mathbf{am}_k(x)) + \cos^2(\mathbf{am}_k(x)) = 1$$

and

$$\mathbf{dn}_k^2(x) = 1 - k^2\mathbf{sn}_k^2(x) \Rightarrow \mathbf{dn}_k^2(x) + k^2\mathbf{sn}_k^2(x) = 1.$$

Thus, the result follows. □

THEOREM 8.5.3 For $0 \le k \le 1$,

$$\mathbf{sn}_k(x + 2K_k) = -\mathbf{sn}_k(x),\ \mathbf{cn}_k(x + 2K_k) = -\mathbf{cn}_k(x),\ \mathbf{dn}_k(x + 2K_k) = \mathbf{dn}_k(x).$$

Proof: First, we note that $\dfrac{1}{\sqrt{1 - k^2\sin^2(u)}}$ is a periodic function of period π (as $\sin^2$ is a periodic function of period π). Let $\phi \in \mathbb{R}$ be arbitrary. Applying Lemma 2.4.23, twice (for $a = -\frac{\pi}{2}$ and for $a = \phi$), we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2\sin^2(u)}} = \int_{\phi}^{\phi+\pi} \frac{du}{\sqrt{1 - k^2\sin^2(u)}}. \tag{8.19}$$

For the given $\phi \in \mathbb{R}$, if $x = F_k(\phi)$, then

$$\begin{aligned} x + 2K_k &= \int_0^{\phi} \frac{du}{\sqrt{1 - k^2\sin^2(u)}} + 2\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2\sin^2(u)}} \\ &= \int_0^{\phi} \frac{du}{\sqrt{1 - k^2\sin^2(u)}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2\sin^2(u)}} \\ &\quad \text{(as the integrand of last integral is an even function.)} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\phi} \frac{du}{\sqrt{1-k^2\sin^2(u)}} + \int_{\phi}^{\phi+\pi} \frac{du}{\sqrt{1-k^2\sin^2(u)}} \\
&\qquad \text{(Using equation (8.19))} \\
&= \int_0^{\phi+\pi} \frac{du}{\sqrt{1-k^2\sin^2(u)}}.
\end{aligned}
$$

Therefore, we have proved that

$$F_k(\phi+\pi) = x + 2K_k. \tag{8.20}$$

Hence, $\mathbf{sn}_k(x+2K_k) = \sin(\phi+\pi) = -\sin(\phi) = -\mathbf{sn}_k(x)$. Similarly, we can prove that $\mathbf{cn}_k(x+2K_k) = -\mathbf{cn}_k(x)$ is a periodic function of period $4K_k$. Now,

$$\mathbf{dn}_k(x+2K_k) = \sqrt{1-k^2\mathbf{sn}_k^2(x+2K_k)} = \sqrt{1-k^2\mathbf{sn}_k^2(x)} = \mathbf{dn}_k(x).$$

Hence, the theorem follows. □

As an immediate consequence of the previous theorem, we have

$$\mathbf{sn}_k(x+4K_k) = -\mathbf{sn}_k(x+2K_k) = \mathbf{sn}_k(x).$$

Similarly, we also get $\mathbf{cn}_k(x+4K_k) = \mathbf{cn}_k(x)$, and hence, the the following corollary holds.

COROLLARY 8.5.4 $\mathbf{sn}_k$ and $\mathbf{cn}_k$ are periodic functions of period $4K_k$, and $\mathbf{dn}_k$ is a periodic function of period $2K_k$.

RESULT 8.5.5 $\mathbf{sn}_k' = \mathbf{cn}_k\mathbf{dn}_k$, $\mathbf{cn}_k' = -\mathbf{sn}_k\mathbf{dn}_k$, and $\mathbf{dn}_k' = -k^2\mathbf{sn}_k\mathbf{cn}_k$.

Proof: Let x and ϕ be related by $x = F_k(\phi)$. By a direct calculation, we get

$$
\begin{aligned}
\mathbf{sn}_k'(x) &= \tfrac{d}{dx}\mathbf{sn}_k(x) &&= \tfrac{d}{dx}(\sin(\phi)) \\
&= \left[\tfrac{d}{d\phi}\sin(\phi)\right]\tfrac{d\phi}{dx} &&= \cos(\phi)\tfrac{d\mathbf{am}_k}{dx} \\
&= \cos(\phi)\sqrt{1-k^2\mathbf{sn}_k^2(x)} &&= \cos(\phi)\mathbf{dn}_k(x) \\
&= \mathbf{cn}_k(x)\mathbf{dn}_k(x), \\
\mathbf{cn}_k'(x) &= \left[\tfrac{d}{d\phi}\cos(\phi)\right]\tfrac{d\phi}{dx} = -\sin(\phi)\mathbf{dn}_k(x).
\end{aligned}
$$

$$\begin{aligned}
\mathbf{dn}_k'(x) &= \frac{d}{dx}\sqrt{1-k^2\mathbf{sn}_k^2(x)} \\
&= \frac{1}{2}\frac{1}{\sqrt{1-k^2\mathbf{sn}_k^2(x)}}(-k^2)2\mathbf{sn}_k(x)\mathbf{sn}_k'(x) \\
&= -k^2\frac{1}{\sqrt{1-k^2\mathbf{sn}_k^2(x)}}\mathbf{sn}_k(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x) \\
&= -k^2\mathbf{sn}_k(x)\mathbf{cn}_k(x).
\end{aligned}$$

Hence, the result follows. □

The quotients of the functions from $\{\mathbf{sn}_k,\ \mathbf{cn}_k,\ \mathbf{dn}_k,\ 1\}$ are denoted by Glaisher as follows:

$$\begin{aligned}
\mathbf{ns}_k &= \frac{1}{\mathbf{sn}_k}, & \mathbf{nc}_k &= \frac{1}{\mathbf{cn}_k}, & \mathbf{nd}_k &= \frac{1}{\mathbf{dn}_k}, \\
\mathbf{sc}_k &= \frac{\mathbf{sn}_k}{\mathbf{cn}_k}, & \mathbf{sd}_k &= \frac{\mathbf{sn}_k}{\mathbf{dn}_k}, & \mathbf{cs}_k &= \frac{\mathbf{cn}_k}{\mathbf{sn}_k}, \\
\mathbf{cd}_k &= \frac{\mathbf{cn}_k}{\mathbf{dn}_k}, & \mathbf{ds}_k &= \frac{\mathbf{dn}_k}{\mathbf{sn}_k}, & \mathbf{dc}_k &= \frac{\mathbf{dn}_k}{\mathbf{cn}_k}.
\end{aligned}$$

So far the Jacobi's elliptic functions are defined from $\mathbb{R}$ into $\mathbb{R}$. To extend them as functions from $\mathbb{C}$ into $\mathbb{C}$, we use the Jacobi's imaginary transformation which is given as follows:

$$\sin(\psi) = i\tan(\phi). \tag{8.21}$$

RESULT 8.5.6 If ϕ and ψ are related by equation (8.21), then $F_k(\psi) = iF_j(\phi)$, where $j = \sqrt{1-k^2}$. Moreover, $\psi = \log(\sec(\phi) + \tan(\phi))$.

Proof: Using the change of variable $\sin(v) = i\tan(u)$, we get the following identities;

1. $\cos(v) = \sqrt{1-\sin^2(v)} = \sqrt{1+\tan^2(u)} = \sec(u)$, and hence, $\cos(u) = \sec(v)$.

2. $\sin(u) = \sqrt{1-\cos^2(u)} = \sqrt{1-\sec^2(v)} = i\tan(v)$.

3. Differentiating $\sin(v) = i\tan(u)$ and using the first identity, we obtain $\cos(v)dv = i\sec^2(u)du \Rightarrow dv = i\sec(u)du$.

4.
$$\begin{aligned}\sqrt{1-k^2\sin^2(v)} &= \sqrt{1+k^2\tan^2(u)}\\ &= \sqrt{\sec^2(u)-\tan^2(u)+k^2\tan^2(u)}\\ &= \sqrt{\sec^2(u)-(1-k^2)\tan^2(u)}\\ &= \frac{\sqrt{1-j^2\sin^2(u)}}{\cos(u)}.\end{aligned}$$

Therefore,

$$\begin{aligned}F_k(\psi) &= \int_0^{\psi}\frac{dv}{\sqrt{1-k^2\sin^2(v)}}\\ &= \int_0^{\phi}\frac{i\sec(u)du}{\sqrt{1+k^2\tan^2(u)}}\\ &= \int_0^{\phi}\frac{i\sec(u)\cos(u)du}{\sqrt{1-j^2\sin^2(u)}}\\ &= iF_j(\phi).\end{aligned}$$

Next, from the third identity, replacing u and v, respectively, by ϕ and ψ, we get

$$d\psi = i\sec(\phi)d\phi \Rightarrow \psi = i\log(\sec(\phi)+\tan(\phi)).$$

Thus, the result follows. □

Using the previous result, we can define the function $\mathbf{am}_k$, $\mathbf{sn}_k$, $\mathbf{cn}_k$, and $\mathbf{dn}_k$ at ix, for $x \in \mathbb{R}$, consistently with their definitions at x.

Definition 8.5.7 Let $0 \le k \le 1$ and $j = \sqrt{1-k^2}$. For a given $x \in \mathbb{R}$, if $\phi = \mathbf{am}_j(x)$, then we define the following:

1. $\mathbf{am}_k(ix) = \psi = i\log(\sec(\phi)+\tan(\phi))$.
2. $\mathbf{sn}_k(ix) = \sin(\psi) = i\tan(\phi) = i\dfrac{\mathbf{sn}_j(x)}{\mathbf{cn}_j(x)} = i\mathbf{sc}_j(x)$.
3. $\mathbf{cn}_k(ix) = \cos(\psi) = \dfrac{1}{\cos(\phi)} = \dfrac{1}{\mathbf{cn}_j(x)} = \mathbf{nc}_j(x)$.
4. $\mathbf{dn}_k(ix) = \sqrt{1-k^2\sin^2(\psi)} = \dfrac{\sqrt{1-j^2\sin^2(\phi)}}{\cos(\phi)} = \dfrac{\mathbf{dn}_j(x)}{\mathbf{cn}_j(x)}$.

THEOREM 8.5.8 For every $x \in \mathbb{R}$, we have

$$\mathbf{sn}_k(ix+i4K_j) = \mathbf{sn}_k(ix);\ \mathbf{cn}_k(ix+i4K_j) = \mathbf{cn}_k(ix);\ \mathbf{dn}_k(ix+i4K_j) = \mathbf{dn}_k(ix),$$

where $K_j = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-j^2\sin^2(u)}}$ and $j = \sqrt{1-k^2}$.

Proof: Let $x \in \mathbb{R}$ be arbitrary. Using Definition 8.5.7, we have

$$\begin{aligned}
\mathbf{sn}_k(i(x+4K_j)) &= i\frac{\mathbf{sn}_j(x+4K_j)}{\mathbf{cn}_j(x+4K_j)} \\
&= i\frac{\mathbf{sn}_j(x)}{\mathbf{cn}_j(x)} \text{ (as } 4K_j \text{ is a period for } \mathbf{sn}_j \text{ and } \mathbf{cn}_j), \\
&= \mathbf{sn}_k(ix), \\
\mathbf{cn}_k(i(x+4K_j)) &= \frac{1}{\mathbf{cn}_j(x+4K_j)} \\
&= \frac{1}{\mathbf{cn}_j(x)} \text{ (as } 4K_j \text{ is a period for } \mathbf{cn}_j) \\
&= \mathbf{cn}_k(ix), \\
\mathbf{dn}_k(i(x+4K_j)) &= \frac{\mathbf{dn}_j(x+4K_j)}{\mathbf{cn}_j(x+4K_j)} \\
&= \frac{\mathbf{dn}_j(x)}{\mathbf{cn}_j(x)} \text{ (as } 4K_j \text{ is a period for } \mathbf{cn}_j \text{ and } \mathbf{dn}_j) \\
&= \mathbf{dn}_k(ix).
\end{aligned}$$

Hence, the theorem follows. □

Next we shall prove the addition theorems for the Jacobi's elliptic functions using which these functions can be extended to the complex plane. Although there are different proofs of the following theorem in the literature, the following proof is relatively easier and is not using the properties of any other elliptic functions such as theta function.

THEOREM 8.5.9

$$\begin{aligned}
\mathbf{cn}_k(x+y) &= \frac{\mathbf{cn}_k(x)\mathbf{cn}_k(y) - \mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)}{1-k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)} \\
\mathbf{sn}_k(x+y) &= \frac{\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) + \mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1-k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}
\end{aligned}$$

$$\mathbf{dn}_k(x+y) = \frac{\mathbf{dn}_k(x)\mathbf{dn}_k(y) - k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}.$$

Proof: Let $F_k(\phi) = x$, $F_k(\psi) = y$, and $F_k(\mu) = x + y$. Then, using the fact that F_k is an odd function, we have

$$F_k(\phi) + F_k(\psi) + F_k(-\mu) = 0.$$

We claim that the expression

$$\cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi)\sqrt{1 - k^2\sin^2(\mu)} = \cos(\mu) \tag{8.22}$$

(where μ is a constant) satisfies the following differential equation

$$\frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} + \frac{d\psi}{\sqrt{1 - k^2\sin^2(\psi)}} = 0. \tag{8.23}$$

Dividing on both sides of equation (8.22) by $\sin(\phi)\sin(\psi)$ and then differentiating, we get

$$d\left[\frac{\cos(\phi)\cos(\psi)}{\sin(\phi)\sin(\psi)}\right] = d\left[\frac{\cos(\mu)}{\sin(\phi)\sin(\psi)}\right].$$

As

$$\begin{aligned} d\left[\frac{\cos(\phi)\cos(\psi)}{\sin(\phi)\sin(\psi)}\right] &= \frac{1}{\sin^2(\phi)\sin^2(\psi)}[\sin(\phi)\sin(\psi)(-\sin(\phi)d\phi\cos(\psi) \\ &\quad -\cos(\phi)\sin(\psi)d\psi) - \cos(\phi)\cos(\psi)(\cos(\phi)d\phi \\ &\quad \sin(\psi) + \sin(\phi)\cos(\psi)d\psi)] \\ &= \frac{-1}{\sin^2(\phi)\sin^2(\psi)}[\sin(\psi)\cos(\psi)d\phi + \sin(\phi)\cos(\phi)d\psi] \\ &= -\left[\frac{\cos(\psi)d\phi}{\sin(\psi)\sin^2(\phi)} + \frac{\cos\phi d\psi}{\sin(\phi)\sin^2(\psi)}\right] \end{aligned}$$

and

$$\begin{aligned} d\left[\frac{\cos(\mu)}{\sin(\phi)\sin(\psi)}\right] &= \frac{-\cos(\mu)}{\sin^2(\phi)\sin^2(\psi)}[\cos(\phi)d\phi\sin(\psi) + \sin(\phi)\cos(\psi)d\psi] \\ &= -\left[\frac{\cos(\mu)\cos(\phi)d\phi}{\sin^2(\phi)\sin(\psi)} + \frac{\cos(\mu)\cos(\psi)d\psi}{\sin(\phi)\sin^2(\psi)}\right] \end{aligned}$$

we have

$$\frac{d\phi}{\sin^2(\phi)\sin(\psi)}[\cos(\psi) - \cos(\mu)\cos(\phi)] + \frac{d\psi}{\sin(\phi)\sin^2(\psi)} [\cos(\phi) - \cos(\mu)\cos(\psi)] = 0$$

which implies that

$$\left(\frac{\cos(\psi) - \cos(\mu)\cos(\phi)}{\sin(\phi)}\right) d\phi + \left(\frac{\cos(\phi) - \cos(\mu)\cos(\psi)}{\sin(\psi)}\right) d\psi = 0. \tag{8.24}$$

The expression in equation (8.22) can be rewritten as

$$(\cos(\phi)\cos(\psi) - \cos(\mu))^2 = \sin^2(\phi)\sin^2(\psi)(1 - k^2\sin^2(\mu)) \tag{8.25}$$

$$\begin{aligned}
\Rightarrow \quad & \cos^2(\phi)\cos^2(\psi) + \cos^2(\mu) - 2\cos(\phi)\cos(\psi)\cos(\mu) \\
& = (1 - \cos^2(\phi))(1 - \cos^2(\psi)) - k^2\sin^2(\phi)\sin^2(\psi)\sin^2(\mu) \\
& = 1 - \cos^2(\phi) - \cos^2(\psi) + \cos^2(\phi)\cos^2(\psi) \\
& \quad - k^2\sin^2(\phi)\sin^2(\psi)\sin^2(\mu) \\
\Rightarrow \quad & \cos^2(\phi) + \cos^2(\psi) + \cos^2(\mu) - 2\cos(\phi)\cos(\psi)\cos(\mu) \\
& \quad + k^2\sin^2(\phi)\sin^2(\psi)\sin^2(\mu) = 1.
\end{aligned}$$

The last equation is unaltered by permuting ϕ, ψ, and μ; equivalently, equation (8.22) is also unaltered by replacing μ by $-\mu$ and then by permuting ϕ, ψ, and $-\mu$. Therefore, by interchanging ψ and $-\mu$, and by interchanging ϕ and $-\mu$ in equation (8.22), we get the following two more equations:

1. $$\begin{aligned}
\cos(\phi)\cos(\mu) + \sin(\phi)\sin(\mu)\sqrt{1 - k^2\sin^2(\psi)} &= \cos(\psi) \\
\Rightarrow \frac{\cos(\psi) - \cos(\mu)\cos(\phi)}{\sin(\phi)} &= \frac{\sin(\mu)}{\sqrt{1 - k^2\sin^2(\psi)}}.
\end{aligned} \tag{8.26}$$

2. $$\begin{aligned}
\cos(\mu)\cos(\psi) + \sin(\mu)\sin(\psi)\sqrt{1 - k^2\sin^2(\phi)} &= \cos(\phi) \\
\Rightarrow \frac{\cos(\phi) - \cos(\mu)\cos(\psi)}{\sin(\psi)} &= \frac{\sin(\mu)}{\sqrt{1 - k^2\sin^2(\phi)}}.
\end{aligned} \tag{8.27}$$

Using equations (8.26) and (8.27) in equation (8.24), we get the differential equation (8.23). Thus, our claim holds.
Now substituting $\zeta = \cos(\mu)$ in equation (8.22), we get

$$(\cos(\phi)\cos(\psi) - \zeta)^2 = \sin^2(\phi)\sin^2(\psi)(1 - k^2(1 - \zeta^2))$$

$$\begin{aligned}
\Rightarrow\quad & \cos^2(\phi)\cos^2(\psi)+\zeta^2-2\cos(\phi)\cos(\psi)\zeta=(1-k^2)\sin^2(\phi)\sin^2(\psi)\\
&\quad +k^2\sin^2(\phi)\sin^2(\psi)\zeta^2\\
\Rightarrow\quad & (1-k^2\sin^2(\phi)\sin^2(\psi))\zeta^2-2\cos(\phi)\cos(\psi)\zeta+\cos^2(\phi)\cos^2(\psi)\\
&\quad +(k^2-1)\sin^2(\phi)\sin^2(\psi)=0.
\end{aligned}$$

Solving for ζ from the above quadratic equation, we get $\zeta = \dfrac{2\cos(\phi)\cos(\psi)\pm\sqrt{D}}{2(1-k^2\sin^2(\phi)\sin^2(\psi))}$, where

$$\begin{aligned}
D &= 4\cos^2(\phi)\cos^2(\psi)-4(1-k^2\sin^2(\phi)\sin^2(\psi))[\cos^2(\phi)\cos^2(\psi)\\
&\quad +(k^2-1)\sin^2(\phi)\sin^2(\psi)]\\
&= 4[(1-k^2)\sin^2(\phi)\sin^2(\psi)+k^2\sin^2(\phi)\sin^2(\psi)\cos^2(\phi)\cos^2(\psi)\\
&\quad +k^2(k^2-1)\sin^4(\phi)\sin^4(\psi)]\\
&= 4[\sin^2(\phi)\sin^2(\psi)-k^2\sin^2(\phi)\sin^2(\psi)\\
&\quad +k^2\sin^2(\phi)\sin^2(\psi)(1-\sin^2(\phi)-\sin^2(\psi)+\sin^2(\phi)\sin^2(\psi))\\
&\quad +k^4\sin^4(\phi)\sin^4(\psi)-k^2\sin^4(\phi)\sin^4(\psi)]\\
&= 4[\sin^2(\phi)\sin^2(\psi)-k^2\sin^2(\phi)\sin^4(\psi)-k^2\sin^4(\phi)\sin^2(\psi)\\
&\quad +k^4\sin^4(\phi)\sin^4(\psi)]\\
&= 4\sin^2(\phi)\sin^2(\psi)(1-k^2\sin^2(\psi)-k^2\sin^2(\phi)+k^4\sin^2(\phi)\sin^2(\psi))\\
&= 4\sin^2(\phi)\sin^2(\psi)(1-k^2\sin^2(\phi))(1-k^2\sin^2(\psi)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cos(\phi+\psi) &= \zeta\\
&= \frac{\cos(\phi)\cos(\psi)\pm\sin(\phi)\sin(\psi)\sqrt{(1-k^2\sin^2(\psi))}}{\sqrt{(1-k^2\sin^2(\phi))}}

Using the identity $\mathbf{sn}_k^2 + \mathbf{cn}_k^2 = 1$ in the following straightforward but lengthy manipulation, we get

$$
\begin{aligned}
&\mathbf{sn}_k^2(x+y)\\
&\quad = \quad 1 - \mathbf{cn}_k^2(x+y)^2\\
&\quad = \quad 1 - \left[\frac{\mathbf{cn}_k(x)\mathbf{cn}_k(y) - \mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}\right]^2\\
&\quad = \quad \frac{[1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)]^2 - [\mathbf{cn}_k(x)\mathbf{cn}_k(y) - \mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]^2}{Dr}\\
&\qquad (\text{ where } Dr = \left[1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\right]^2)\\
&\quad = \quad \tfrac{1}{Dr}[1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y) - \mathbf{cn}_k^2(x)\mathbf{cn}_k^2(y)\\
&\qquad -\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\mathbf{dn}_k^2(x)\mathbf{dn}_k^2(y) + 2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}[1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
&\qquad -(1 - \mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(y) + \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))\\
&\qquad -\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)(1 - k^2\mathbf{sn}_k^2(x) - k^2\mathbf{sn}_k^2(y) + k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))\\
&\qquad +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}[\mathbf{sn}_k^2(x) + \mathbf{sn}_k^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^4(y) + k^2\mathbf{sn}_k^4(x)\mathbf{sn}_k^2(y)\\
&\qquad -2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
&\qquad +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}[(\mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(y))\\
&\qquad +(\mathbf{sn}_k^2(y) - \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(x))\\
&\qquad +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}[\mathbf{sn}_k^2(x)(1 - \mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(y))\\
&\qquad +\mathbf{sn}_k^2(y)(1 - \mathbf{sn}_k^2(x))(1 - k^2\mathbf{sn}_k^2(x))\\
&\qquad +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}[\mathbf{sn}_k^2(x)\mathbf{cn}_k^2(y)\mathbf{dn}_k^2(y) + \mathbf{sn}_k^2(y)\mathbf{cn}_k^2(x)\mathbf{dn}_k^2(x)\\
&\qquad +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
&\quad = \quad \tfrac{1}{Dr}(\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) + \mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x))^2.
\end{aligned}
$$

Hence,

$$\mathbf{sn}_k(x+y) = \pm\frac{\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) + \mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}. \tag{8.28}$$

Comparing this equation with (particular case $k = 0$),

$$\mathbf{sn}_0(x+y) = \mathbf{sn}_0(x)\mathbf{cn}_0(y) + \mathbf{sn}_0(y)\mathbf{cn}_0(x),$$

we choose the sign in equation (8.28) and we get

$$\mathbf{sn}_k(x+y) = \frac{\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) + \mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}.$$

Next,

$$\begin{aligned}
&\mathbf{dn}_k^2(x+y)\\
&\quad = \quad 1 - k^2\mathbf{sn}_k^2(x+y)\\
&\quad = \quad 1 - k^2\frac{(\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) + \mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x))^2}{(1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))^2}\\
&\quad = \quad \frac{Nr}{(1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))^2},
\end{aligned}$$

where

$$\begin{aligned}
Nr \quad = \quad & 1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& -k^2[\mathbf{sn}_k^2(x)\mathbf{cn}_k^2(y)\mathbf{dn}_k^2(y) + \mathbf{sn}_k^2(y)\mathbf{cn}_k^2(x)\mathbf{dn}_k^2(x)\\
& +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
= \quad & 1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& -k^2[\mathbf{sn}_k^2(x)(1 - \mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(y)) + \mathbf{sn}_k^2(y)\\
& (1 - \mathbf{sn}_k^2(x))(1 - k^2\mathbf{sn}_k^2(x))\\
& +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
= \quad & 1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& -k^2[(\mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(y))\\
& +(\mathbf{sn}_k^2(y) - \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y))(1 - k^2\mathbf{sn}_k^2(x))\\
& +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)]\mathbf{dn}_k(x)\mathbf{dn}_k(y)]\\
= \quad & 1 + k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& -k^2[\mathbf{sn}_k^2(x) + \mathbf{sn}_k^2(y) - 2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y) - 2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& +2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)\\
= \quad & (1 + k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y) - k^2\mathbf{sn}_k^2(x) - k^2\mathbf{sn}_k^2(y))\\
& +k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y) - k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^2(y) - k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^4(y)\\
& +k^4\mathbf{sn}_k^4(x)\mathbf{sn}_k^4(y) - 2k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)\\
= \quad & (1 - k^2\mathbf{sn}_k^2(x))(1 - k^2\mathbf{sn}_k^2(y)) + k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)\\
& +k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)[1 - \mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(y) + \mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)]\\
& -2k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)\\
= \quad & \mathbf{dn}_k^2(x)\mathbf{dn}_k^2(y) + k^4\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)(1 - \mathbf{sn}_k^2(x))(1 - \mathbf{sn}_k^2(y))\\
& -2k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)\\
= \quad & (\mathbf{dn}_k(x)\mathbf{dn}_k(y) - k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y))^2.
\end{aligned}$$

Therefore,

$$\mathbf{dn}_k(x+y) = \frac{\mathbf{dn}_k(x)\mathbf{dn}_k(y) - k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}.$$

In the last expression, we have chosen the sign by considering the particular case $k = 0$ as before. □

Exercise 8.5.10 Prove that $\mathbf{sc}_k(x+y) = \dfrac{\mathbf{sc}_k(x)\mathbf{dn}_k(x) + \mathbf{sc}_k(y)\mathbf{dn}_k(y)}{1 - \mathbf{sc}_k(x)\mathbf{sc}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)}$.

Hint: Using the previous theorem, find $\dfrac{\mathbf{sn}_k(x+y)}{\mathbf{cn}_k(x+y)}$ and simplify.

The following identities are the immediate consequences of Theorem 8.5.9 and Result 8.5.2:

$$1.\quad \mathbf{cn}_k(x \pm y) = \frac{\mathbf{cn}_k(x)\mathbf{cn}_k(y) \mp \mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{dn}_k(x)\mathbf{dn}_k(y)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}$$

$$2.\quad \mathbf{sn}_k(x \pm y) = \frac{\mathbf{sn}_k(x)\mathbf{cn}_k(y)\mathbf{dn}_k(y) \pm \mathbf{sn}_k(y)\mathbf{cn}_k(y)\mathbf{dn}_k(x)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}$$

$$3.\quad \mathbf{dn}_k(x \pm y) = \frac{\mathbf{dn}_k(x)\mathbf{dn}_k(y) \mp k^2\mathbf{sn}_k(x)\mathbf{sn}_k(y)\mathbf{cn}_k(x)\mathbf{cn}_k(y)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(y)}$$

$$\begin{aligned}
4.\quad \mathbf{cn}_k(2x) &= \frac{\mathbf{cn}_k^2(x) - \mathbf{sn}_k^2(x)\mathbf{dn}_k^2(x)}{1 - k^2\mathbf{sn}_k^4(x)} \\
&= \frac{1 - \mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(x)(1 - k^2\mathbf{sn}_k^2(x))}{1 - k^2\mathbf{sn}_k^4(x)} \\
&= \frac{1 - \mathbf{sn}_k^2(x) - \mathbf{sn}_k^2(x) + k^2\mathbf{sn}_k^4(x)}{1 - k^2\mathbf{sn}_k^4(x)} \\
&= \frac{1 - 2\mathbf{sn}_k^2(x) + k^2\mathbf{sn}_k^4(x)}{1 - k^2\mathbf{sn}_k^4(x)} \\
5.\quad \mathbf{sn}_k(2x) &= \frac{2\mathbf{sn}_k(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1 - k^2\mathbf{sn}_k^4(x)} \\
6.\quad \mathbf{dn}_k(2x) &= \frac{\mathbf{dn}_k^2(x) - k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k^2(x)}{1 - k^2\mathbf{sn}_k^4(x)} \\
&= \frac{1 - k^2\mathbf{sn}_k^2(x) - k^2\mathbf{sn}_k^2(x)(1 - \mathbf{sn}_k^2(x))}{1 - k^2\mathbf{sn}_k^4(x)} \\
&= \frac{1 - 2k^2\mathbf{sn}_k^2(x) - k^2\mathbf{sn}_k^4(x)}{1 - k^2\mathbf{sn}_k^4(x)}.
\end{aligned}$$

PROBLEM 8.5.11 Prove that $\mathbf{cn}_k\left(\frac{x}{2}\right) = \sqrt{\frac{\mathbf{cn}_k(x)+\mathbf{dn}_k(x)}{1+\mathbf{dn}_k(x)}}$.

Solution: As

$$\begin{aligned}
\frac{\mathbf{cn}_k(x)+\mathbf{dn}_k(x)}{1+\mathbf{dn}_k(x)} &= \frac{\frac{\mathbf{cn}_k^2\left(\frac{x}{2}\right)-\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{dn}_k^2\left(\frac{x}{2}\right)}{1-k^2\mathbf{sn}_k^4\left(\frac{x}{2}\right)}+\frac{\mathbf{dn}_k^2\left(\frac{x}{2}\right)-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{cn}_k^2\left(\frac{x}{2}\right)}{1-k^2\mathbf{sn}_k^4\left(\frac{x}{2}\right)}}{1+\frac{\mathbf{dn}_k^2\left(\frac{x}{2}\right)-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{cn}_k^2\left(\frac{x}{2}\right)}{1-k^2\mathbf{sn}_k^4\left(\frac{x}{2}\right)}} \\
&= \frac{\mathbf{cn}_k^2\left(\frac{x}{2}\right)-\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{dn}_k^2\left(\frac{x}{2}\right)+\mathbf{dn}_k^2\left(\frac{x}{2}\right)-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{cn}_k^2\left(\frac{x}{2}\right)}{1-k^2\mathbf{sn}_k^4\left(\frac{x}{2}\right)+\mathbf{dn}_k^2\left(\frac{x}{2}\right)-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\mathbf{cn}_k^2\left(\frac{x}{2}\right)} \\
&= \frac{\mathbf{cn}_k^2\left(\frac{x}{2}\right)\left(1-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\right)+\mathbf{dn}_k^2\left(\frac{x}{2}\right)\left(1-\mathbf{sn}_k^2\left(\frac{x}{2}\right)\right)}{1-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)\left(\mathbf{sn}_k^2\left(\frac{x}{2}\right)+\mathbf{cn}_k^2\left(\frac{x}{2}\right)\right)+\mathbf{dn}_k^2\left(\frac{x}{2}\right)} \\
&= \frac{2\mathbf{cn}_k^2\left(\frac{x}{2}\right)\mathbf{dn}_k^2\left(\frac{x}{2}\right)}{1-k^2\mathbf{sn}_k^2\left(\frac{x}{2}\right)+\mathbf{dn}_k^2\left(\frac{x}{2}\right)} \\
&= \frac{2\mathbf{cn}_k^2\left(\frac{x}{2}\right)\mathbf{dn}_k^2\left(\frac{x}{2}\right)}{2\mathbf{dn}_k^2\left(\frac{x}{2}\right)} \\
&= \mathbf{cn}_k^2\left(\frac{x}{2}\right).
\end{aligned}$$

Thus, we get $\mathbf{cn}_k\left(\frac{x}{2}\right) = \sqrt{\frac{\mathbf{cn}_k(x)+\mathbf{dn}_k(x)}{1+\mathbf{dn}_k(x)}}$.

By adopting the same technique, the reader can prove the following identities.

Exercise 8.5.12 Prove the following:

$$\begin{aligned}
1.\quad \mathbf{sn}_k\left(\frac{x}{2}\right) &= \sqrt{\frac{1-\mathbf{cn}_k(x)}{1+\mathbf{dn}_k(x)}} \\
2.\quad \mathbf{dn}_k\left(\frac{x}{2}\right) &= \sqrt{\frac{k^2+\mathbf{dn}_k(x)+k^2\mathbf{cn}_k(x)}{1+\mathbf{dn}_k(x)}}.
\end{aligned}$$

By seeing the Theorem 8.5.9, we extend the Jacobi's functions $\mathbf{sn}_k$, $\mathbf{sc}_k$, and $\mathbf{dn}_k$ as follows.

Definition 8.5.13 Define the following:

$$1.\quad \mathbf{cn}_k(x+iy) = \frac{\mathbf{cn}_k(x)\mathbf{cn}_k(iy)-\mathbf{sn}_k(x)\mathbf{sn}_k(iy)\mathbf{dn}_k(x)\mathbf{dn}_k(iy)}{1-k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(iy)}$$

$$
\begin{aligned}
&= \frac{\mathbf{cn}_k(x)\dfrac{1}{\mathbf{cn}_j(y)} - \mathbf{sn}_k(x)i\dfrac{\mathbf{sn}_j(y)}{\mathbf{cn}_j(y)}\mathbf{dn}_k(x)\dfrac{\mathbf{dn}_j(y)}{\mathbf{cn}_j(y)}}{1 + k^2\mathbf{sn}_k^2(x)\dfrac{\mathbf{sn}_j^2(y)}{\mathbf{cn}_j^2(y)}} \\
&= \frac{\mathbf{cn}_k(x)\mathbf{cn}_j(y) - i\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{dn}_k(x)\mathbf{dn}_j(y)}{\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)}.
\end{aligned}
$$

2. $$
\begin{aligned}
\mathbf{sn}_k(x+iy) &= \frac{\mathbf{sn}_k(x)\mathbf{cn}_k(iy)\mathbf{dn}_k(iy) + \mathbf{sn}_k(iy)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(iy)} \\
&= \frac{\mathbf{sn}_k(x)\dfrac{1}{\mathbf{cn}_j(y)}\dfrac{\mathbf{dn}_j(y)}{\mathbf{cn}_j(y)} + i\dfrac{\mathbf{sn}_j(y)}{\mathbf{cn}_j(y)}\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{1 + k^2\mathbf{sn}_k^2(x)\dfrac{\mathbf{sn}_j^2(y)}{\mathbf{cn}_j^2(y)}} \\
&= \frac{\mathbf{sn}_k(x)\mathbf{dn}_j(y) + i\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)}.
\end{aligned}
$$

3. $$
\begin{aligned}
\mathbf{dn}_k(x+iy) &= \frac{\mathbf{dn}_k(x)\mathbf{dn}_k(iy) - k^2\mathbf{sn}_k(x)\mathbf{sn}_k(iy)\mathbf{cn}_k(x)\mathbf{cn}_k(iy)}{1 - k^2\mathbf{sn}_k^2(x)\mathbf{sn}_k^2(iy)} \\
&= \frac{\mathbf{dn}_k(x)\dfrac{\mathbf{dn}_j(y)}{\mathbf{cn}_j(y)} - ik^2\mathbf{sn}_k(x)\dfrac{\mathbf{sn}_j(y)}{\mathbf{cn}_j(y)}\mathbf{cn}_k(x)\dfrac{1}{\mathbf{cn}_j(y)}}{1 + k^2\mathbf{sn}_k^2(x)\dfrac{\mathbf{sn}_j^2(y)}{\mathbf{cn}_j^2(y)}} \\
&= \frac{\mathbf{cn}_j(y)\mathbf{dn}_k(x)\mathbf{dn}_j(y) - ik^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_k(x)}{\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)}.
\end{aligned}
$$

PROBLEM 8.5.14 Prove that if $0 < k < 1$, then $\mathbf{cn}_k$, $\mathbf{sn}_k$, and $\mathbf{dn}_k$ are elliptic functions.

Solution: First, we note that the values of the three functions, namely, $\mathbf{cn}_k$, $\mathbf{sn}_k$, and $\mathbf{dn}_k$ at $x+iy$ have the same denominator $\mathbf{cn}_j^2(y)+k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)$, and it is zero iff $\mathbf{cn}_j(y) = 0$ and $k\mathbf{sn}_k(x)\mathbf{sn}_j(y) = 0$ iff $\mathbf{cn}_j(y) = 0$ and $\mathbf{sn}_k(x) = 0$, as $\mathbf{sn}_j^2(y) = 1$.
From Theorems 8.5.2 and 8.5.3, we have

$$\mathbf{sn}_k(2mK_k) = 0,\ \forall m \in \mathbb{Z} \text{ and } \mathbf{cn}_j((2n+1)K_j) = 0,\ \forall n \in \mathbb{Z}$$

Furthermore, $F_k(0) = 0$ and $F_k(\phi + \pi) = F_k(\phi) + 2K_k$ (See equation (8.20)) imply that $F_k(\pi) = 2K_k$. Therefore, F_k maps $(0, \pi)$ onto $(0, 2K_k)$. As sin $\neq 0$ on $(0, \pi)$, $\mathbf{sn}_k \neq 0$ on $(0, 2K_k)$, and hence,

$$\mathbf{sn}_k(x) = 0 \text{ for some } x \in \mathbb{R} \text{ iff } x \in \{2mK_k : m \in \mathbb{Z}\}.$$

Similarly, using the facts (i) cos $\neq 0$ on $(0, \frac{\pi}{2})$ and (ii) F_k maps $(0, \frac{\pi}{2})$ onto $(0, K_k)$, we obtain $\mathbf{cn}_k \neq 0$ on $(0, K_k)$. Therefore,

$$\mathbf{cn}_k(x) = 0 \text{ for some } x \in \mathbb{R} \text{ iff } x \in \{(2n+1)K_j : n \in \mathbb{Z}\}.$$

Thus, if $\Omega = \mathbb{C} \setminus \{(2mK_k, (2n+1)K_j) : m, n \in \mathbb{Z}\}$, then $\mathbf{cn}_j(y) \neq 0$ and $\mathbf{sn}_k(x) \neq 0$, and hence, these three functions are finite valued on Ω.

Using Theorem 8.5.3, for $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbf{sn}_k(i(x+2K_j)) &= i\frac{\mathbf{sn}_j(x+2K_j)}{\mathbf{cn}_j(x+2K_j)} = i\left(\frac{-\mathbf{sn}_j(x)}{-\mathbf{cn}_j(x)}\right) = \mathbf{sn}_k(ix)\\
\mathbf{cn}_k(i(x+2K_j)) &= \frac{1}{\mathbf{cn}_j(x+2K_j)} = \frac{1}{-\mathbf{cn}_j(x)} = -\mathbf{cn}_k(ix)\\
\mathbf{dn}_k(i(x+2K_j)) &= \frac{\mathbf{dn}_j(x+2K_j)}{\mathbf{cn}_j(x+2K_j)} = \left(\frac{\mathbf{dn}_j(x)}{-\mathbf{cn}_j(x)}\right) = -\mathbf{dn}_k(ix).
\end{aligned}$$

Thus, replacing $x \in \mathbb{R}$ by $z \in \Omega$, we get

$$\mathbf{sn}_k(z + i2K_j) = \mathbf{sn}_k(z), \tag{8.29}$$

$$\mathbf{cn}_k(z + i2K_j) = -\mathbf{cn}_k(z), \tag{8.30}$$

$$\mathbf{dn}_k(z + i2K_j) = -\mathbf{dn}_k(z). \tag{8.31}$$

Combining the above equation with Corollary 8.5.4, we get

$$\begin{aligned}
\mathbf{cn}_k(z + 2K_k + i2K_j) &= -\mathbf{cn}_k(z + 2K_k) = \mathbf{cn}_k(z),\\
\mathbf{dn}_k(z + i4K_j) &= -\mathbf{dn}_k(z + i2K_j) = \mathbf{dn}_k(z).
\end{aligned}$$

Thus, $i2K_j$ and $4K_k$ are periods of $\mathbf{sn}_k$; $2K_k + i2K_j$ and $4K_k$ are periods of $\mathbf{cn}_k$; and $i4K_j$ and $2K_k$ are periods of $\mathbf{dn}_k$.

Next, we show that $\mathbf{sn}_k$ is analytic on Ω. We denote the real and imaginary parts of $\mathbf{sn}_k(x + iy)$ (see Definition 8.5.13.), respectively, by

$$U(x,y) = \frac{\mathbf{sn}_k(x)\mathbf{dn}_j(y)}{\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)};\ V(x,y) = \frac{\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)}{\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y)}.$$

If $Dr = (\mathbf{cn}_j^2(y) + k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y))^2$, then

$$
\begin{aligned}
&\frac{\partial U}{\partial x}(x,y)\\
&\quad = \frac{1}{Dr}\Big[(\mathbf{cn}_j^2(y)+k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y))(\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{dn}_j(y))\\
&\qquad -\mathbf{sn}_k(x)\mathbf{dn}_j(y)(2k^2\mathbf{sn}_k(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y))\Big]\\
&\quad = \frac{1}{Dr}\Big[\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)+k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)\\
&\qquad -2k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)\Big]\\
&\quad = \frac{1}{Dr}\Big[\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)-k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial V}{\partial y}(x,y)\\
&\quad = \frac{1}{Dr}\Big[(\mathbf{cn}_j^2(y)+k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y))(\mathbf{cn}_k(x)\mathbf{dn}_k(x)(\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)\\
&\qquad -\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)))-\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x)(-2\mathbf{cn}_j(y)\mathbf{sn}_j(y)\mathbf{dn}_j(y)\\
&\qquad +2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{dn}_j(y)\Big]\\
&\quad = \frac{1}{Dr}\Big[\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{cn}_j^4(y)\mathbf{dn}_j(y)-\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)\\
&\qquad +k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)-k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\\
&\qquad \mathbf{sn}_j^4(y)\mathbf{dn}_j(y)+2\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)\\
&\qquad -2k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)\Big]\\
&\quad = \frac{1}{Dr}\Big[\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)(\mathbf{cn}_j^2(y)+\mathbf{sn}_j^2(y))\\
&\qquad -k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)(\mathbf{sn}_j^2(y)+\mathbf{cn}_j^2(y))\Big]\\
&\quad = \frac{1}{Dr}\Big[\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{cn}_j^2(y)\mathbf{dn}_j(y)-k^2\mathbf{sn}_k^2(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x)\mathbf{sn}_j^2(y)\mathbf{dn}_j(y)\Big]\\
&\quad = \frac{\partial U}{\partial x}(x,y).
\end{aligned}
$$

Next, we verify the other C–R equation. Although it not as simple as the previous one, we verify it as follows:

$$
\begin{aligned}
&Dr\,\frac{\partial U}{\partial y}(x,y)\\
&\quad = (\mathbf{cn}_j^2(y)+k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y))(-j^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y))\\
&\qquad -(\mathbf{sn}_k(x)\mathbf{dn}_j(y))(-2\mathbf{cn}_j(y)\mathbf{sn}_j(y)\mathbf{dn}_j(y)+2k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{dn}_j(y))\\
&\quad = -j^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y)-j^2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{dn}_j^2(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{dn}_j^2(y)\\
&= (k^2-1)\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - (1-k^2)k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +(2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y))(1-(1-k^2)\mathbf{sn}_j^2(y))\\
&= -\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) + k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + 2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - 2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +2k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&\quad +2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&= -\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)[\mathbf{cn}_j^2(y)+\mathbf{sn}_j^2(y)] - \mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) + 2k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&= -\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - \mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&\quad [\mathbf{cn}_j^2(y)+\mathbf{sn}_j^2(y)] + k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad +2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&= \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - \mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&\quad +k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&\quad +k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&= \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)(1-\mathbf{sn}_j^2(y)) - k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)(1-\mathbf{sn}_j^2(y))\\
&\quad +k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad -k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&= \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) + k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)\\
&\quad +k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)
\end{aligned}
$$

and

$$
\begin{aligned}
&Dr\,\frac{\partial V}{\partial x}(x,y)\\
&= (\mathbf{cn}_j^2(y)+k^2\mathbf{sn}_k^2(x)\mathbf{sn}_j^2(y))\mathbf{sn}_j(y)\mathbf{cn}_j(y)[-k^2\mathbf{sn}_k(x)\mathbf{cn}_k^2(x) - \mathbf{sn}_k(x)\mathbf{dn}_k^2(x)]\\
&\quad -(\mathbf{sn}_j(y)\mathbf{cn}_j(y)\mathbf{cn}_k(x)\mathbf{dn}_k(x))(2k^2\mathbf{sn}_j^2(y)\mathbf{sn}_k(x)\mathbf{cn}_k(x)\mathbf{dn}_k(x))\\
&= -k^2\mathbf{sn}_k(x)\mathbf{cn}_k^2(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - \mathbf{sn}_k(x)\mathbf{dn}_k^2(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y)\\
&\quad -k^4\mathbf{sn}_k^3(x)\mathbf{cn}_k^2(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - k^2\mathbf{sn}_k^3(x)\mathbf{dn}_k^2(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad -2k^2\mathbf{sn}_k(x)\mathbf{cn}_k^2(x)\mathbf{dn}_k^2(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&= \Big[-k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\\
&\quad -2k^2\mathbf{sn}_k(x)(1-k^2\mathbf{sn}_k^2(x))\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\Big](1-\mathbf{sn}_k^2(x))\\
&\quad +\Big[-\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)\Big](1-k^2\mathbf{sn}_k^2(x))\\
&= -k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y)
\end{aligned}
$$

$$
\begin{aligned}
& +2k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) + k^4\mathbf{sn}_k^5(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) \\
& +2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - 2k^4\mathbf{sn}_k^5(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) - \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) \\
& -k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) + k^4\mathbf{sn}_k^5(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) \\
= \; & -k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - 2k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) \\
& +k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + 2k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) \\
= \; & -k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)[\mathbf{cn}_j^2(y) + \mathbf{sn}_j^2(y)] - k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) \\
& +k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y)[\mathbf{sn}_j^2(y) + \mathbf{cn}_j^2(y)] \\
& +k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) \\
= \; & -k^2\mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) - k^2\mathbf{sn}_k(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) + k^4\mathbf{sn}_k^3(x)\mathbf{sn}_j^3(y)\mathbf{cn}_j(y) \\
& +k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j(y) + k^2\mathbf{sn}_k^3(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y) - \mathbf{sn}_k(x)\mathbf{sn}_j(y)\mathbf{cn}_j^3(y).
\end{aligned}
$$

Therefore, the real and imaginary parts of $\mathbf{sn}_k$ satisfy the C–R equations and obviously the partial derivatives are continuous on Ω, and hence, $\mathbf{sn}_k$ is analytic on Ω and meromorphic on $\mathbb{C}$, as $\mathbf{sn}_k$ has poles on $\mathbb{C} \setminus \Omega$. Thus, $\mathbf{sn}_k$ is an elliptic function. Using Theorem 6.3.2, we have $\mathbf{cn}_k$ and $\mathbf{dn}_k$, which are meromorphic functions on Ω, and hence, they are also elliptic functions.

Exercise 8.5.15 Prove the following:

1. The fundamental periods of $\mathbf{sn}_k$ are $i2K_j$ and $4K_k$.
2. The fundamental periods of $\mathbf{cn}_k$ are $2K_k + i2K_j$ and $4K_k$.
3. The fundamental periods of $\mathbf{dn}_k$ are $i4K_j$ and $2K_k$.

Remark 8.5.16: All the identities involving the Jacobian elliptic functions holds for all $z \in \Omega$ also, using the principle of analytic continuation.

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