

INTRODUCTION TO REAL ANALYSIS



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Introduction to Real Analysis

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CHAPTER OVERVIEW

2: Real Numbers

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2.1: Basic properties

Introduction

About this book

This book is a one semester course in basic analysis. It started its life as my lecture notes for teaching Math 444 at the University of Illinois at Urbana-Champaign (UIUC) in Fall semester 2009. Later I added the metric space chapter to teach Math 521 at University of Wisconsin-Madison (UW). A prerequisite for this course is a basic proof course, using for example , , or .

It should be possible to use the book for both a basic course for students who do not necessarily wish to go to graduate school (such as UIUC 444), but also as a more advanced one-semester course that also covers topics such as metric spaces (such as UW 521). Here are my suggestions for what to cover in a semester course. For a slower course such as UIUC 444:

§0.3, §1.1–§1.4, §2.1–§2.5, §3.1–§3.4, §4.1–§4.2, §5.1–§5.3, §6.1–§6.3

For a more rigorous course covering metric spaces that runs quite a bit faster (such as UW 521):

§0.3, §1.1–§1.4, §2.1–§2.5, §3.1–§3.4, §4.1–§4.2, §5.1–§5.3, §6.1–§6.2, §7.1–§7.6

It should also be possible to run a faster course without metric spaces covering all sections of chapters 0 through 6. The approximate number of lectures given in the section notes through chapter 6 are a very rough estimate and were designed for the slower course. The first few chapters of the book can be used in an introductory proofs course as is for example done at Iowa State University Math 201, where this book is used in conjunction with Hammack's Book of Proof .

The book normally used for the class at UIUC is Bartle and Sherbert, *Introduction to Real Analysis* third edition . The structure of the beginning of the book somewhat follows the standard syllabus of UIUC Math 444 and therefore has some similarities with . A major difference is that we define the Riemann integral using Darboux sums and not tagged partitions. The Darboux approach is far more appropriate for a course of this level.

Our approach allows us to fit a course such as UIUC 444 within a semester and still spend some extra time on the interchange of limits and end with Picard's theorem on the existence and uniqueness of solutions of ordinary differential equations. This theorem is a wonderful example that uses many results proved in the book. For more advanced students, material may be covered faster so that we arrive at metric spaces and prove Picard's theorem using the fixed point theorem as is usual.

Other excellent books exist. My favorite is Rudin's excellent *Principles of Mathematical Analysis* or as it is commonly and lovingly called *baby Rudin* (to distinguish it from his other great analysis textbook). I took a lot of inspiration and ideas from Rudin. However, Rudin is a bit more advanced and ambitious than this present course. For those that wish to continue mathematics, Rudin is a fine investment. An inexpensive and somewhat simpler alternative to Rudin is Rosenlicht's *Introduction to Analysis* . There is also the freely downloadable *Introduction to Real Analysis* by William Trench .

A note about the style of some of the proofs: Many proofs traditionally done by contradiction, I prefer to do by a direct proof or by contrapositive. While the book does include proofs by contradiction, I only do so when the contrapositive statement seemed too awkward, or when contradiction follows rather quickly. In my opinion, contradiction is more likely to get beginning students into trouble, as we are talking about objects that do not exist.

I try to avoid unnecessary formalism where it is unhelpful. Furthermore, the proofs and the language get slightly less formal as we progress through the book, as more and more details are left out to avoid clutter.

As a general rule, I use $:=$ instead of $=$ to define an object rather than to simply show equality. I use this symbol rather more liberally than is usual for emphasis. I use it even when the context is "local," that is, I may simply define a function $f(x) := x^2$ for a single exercise or example.

Finally, I would like to acknowledge Jana Maříková, Glen Pugh, Paul Vojta, Frank Beatrous, Sönmez Şahutoğlu, Jim Brandt, Kenji Kozai, and Arthur Busch, for teaching with the book and giving me lots of useful feedback. Frank Beatrous wrote the University of Pittsburgh version extensions, which served as inspiration for many of the recent additions. I would also like to thank Dan Stoneham, Jeremy Sutter, Eliya Gwetta, Daniel Pimentel-Alarcón, Steve Hoerning, Yi Zhang, Nicole Cavaris, Kristopher Lee, Baoyue Bi, Hannah Lund, Trevor Mannella, Mitchel Meyer, Gregory Beauregard, Chase Meadors, Andreas Giannopoulos, an anonymous reader, and in general all the students in my classes for suggestions and finding errors and typos.

About analysis

Analysis is the branch of mathematics that deals with inequalities and limits. The present course deals with the most basic concepts in analysis. The goal of the course is to acquaint the reader with rigorous proofs in analysis and also to set a firm foundation for calculus of one variable.

Calculus has prepared you, the student, for using mathematics without telling you why what you learned is true. To use, or teach, mathematics effectively, you cannot simply know *what* is true, you must know *why* it is true. This course shows you *why* calculus is true. It is here to give you a good understanding of the concept of a limit, the derivative, and the integral.

Let us use an analogy. An auto mechanic that has learned to change the oil, fix broken headlights, and charge the battery, will only be able to do those simple tasks. He will be unable to work independently to diagnose and fix problems. A high school teacher that does not understand the definition of the Riemann integral or the derivative may not be able to properly answer all the students' questions. To this day I remember several nonsensical statements I heard from my calculus teacher in high school, who simply did not understand the concept of the limit, though he could "do" all problems in calculus.

We start with a discussion of the real number system, most importantly its completeness property, which is the basis for all that comes after. We then discuss the simplest form of a limit, the limit of a sequence. Afterwards, we study functions of one variable, continuity, and the derivative. Next, we define the Riemann integral and prove the fundamental theorem of calculus. We discuss sequences of functions and the interchange of limits. Finally, we give an introduction to metric spaces.

Let us give the most important difference between analysis and algebra. In algebra, we prove equalities directly; we prove that an object, a number perhaps, is equal to another object. In analysis, we usually prove inequalities. To illustrate the point, consider the following statement.

Let x be a real number. If $0 \leq x < \epsilon$ is true for all real numbers $\epsilon > 0$, then $x = 0$.

This statement is the general idea of what we do in analysis. If we wish to show that $x = 0$, we show that $0 \leq x < \epsilon$ for all positive ϵ .

The term *real analysis* is a little bit of a misnomer. I prefer to use simply *analysis*. The other type of analysis, *complex analysis*, really builds up on the present material, rather than being distinct. Furthermore, a more advanced course on real analysis would talk about complex numbers often. I suspect the nomenclature is historical baggage.

Let us get on with the show...

Basic set theory

Notes: 1–3 lectures (some material can be skipped or covered lightly)

Before we start talking about analysis we need to fix some language. Modern¹ analysis uses the language of sets, and therefore that is where we start. We talk about sets in a rather informal way, using the so-called "naïve set theory." Do not worry, that is what majority of mathematicians use, and it is hard to get into trouble.

We assume the reader has seen basic set theory and has had a course in basic proof writing. This section should be thought of as a refresher.

Sets

A set is a collection of objects called *elements* or *members*. A set with no objects is called the *empty set* and is denoted by $\emptyset$ (or sometimes by $\{\}$).

Think of a set as a club with a certain membership. For example, the students who play chess are members of the chess club. However, do not take the analogy too far. A set is only defined by the members that form the set; two sets that have the same members are the same set.

Most of the time we will consider sets of numbers. For example, the set

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$$S := \{0, 1, 2\}$$

is the set containing the three elements 0, 1, and 2. We write

$$1 \in S$$

to denote that the number 1 belongs to the set S . That is, 1 is a member of S . Similarly we write

$$7 \notin S$$

to denote that the number 7 is not in S . That is, 7 is not a member of S . The elements of all sets under consideration come from some set we call the *universe*. For simplicity, we often consider the universe to be the set that contains only the elements we are interested in. The universe is generally understood from context and is not explicitly mentioned. In this course, our universe will most often be the set of real numbers.

While the elements of a set are often numbers, other objects, such as other sets, can be elements of a set. A set may also contain some of the same elements as another set. For example,

$$T := \{0, 2\}$$

contains the numbers 0 and 2. In this case all elements of T also belong to S . We write $T \subset S$. More formally we make the following definition.

- i. A set A is a *subset* of a set B if $x \in A$ implies $x \in B$, and we write $A \subset B$. That is, all members of A are also members of B .
- ii. Two sets A and B are *equal* if $A \subset B$ and $B \subset A$. We write $A = B$. That is, A and B contain exactly the same elements. If it is not true that A and B are equal, then we write $A \neq B$.
- iii. A set A is a *proper subset* of B if $A \subset B$ and $A \neq B$. We write $A \subsetneq B$.

For example, for S and T defined above $T \subset S$, but $T \neq S$. So T is a proper subset of S . If $A = B$, then A and B are simply two names for the same exact set. Let us mention the *set building notation*,

$$\{x \in A : P(x)\}.$$

This notation refers to a subset of the set A containing all elements of A that satisfy the property $P(x)$. The notation is sometimes abbreviated, A is not mentioned when understood from context. Furthermore, $x \in A$ is sometimes replaced with a formula to make the notation easier to read.

The following are sets including the standard notations.

- i. The set of *natural numbers*, $\mathbb{N} := \{1, 2, 3, \dots\}$.
- ii. The set of *integers*, $\mathbb{Z} := \{0, -1, 1, -2, 2, \dots\}$.
- iii. The set of *rational numbers*, $\mathbb{Q} := \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.
- iv. The set of even natural numbers, $\{2m : m \in \mathbb{N}\}$.
- v. The set of real numbers, $\mathbb{R}$.

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

There are many operations we want to do with sets.

- i. A *union* of two sets A and B is defined as

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

- ii. An *intersection* of two sets A and B is defined as

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

- iii. A *complement of B relative to A* (or *set-theoretic difference of A and B*) is defined as

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$

- iv. We say *complement of B* and write B^c instead of $A \setminus B$ if the set A is either the entire universe or is the obvious set containing B , and is understood from context.
- v. We say sets A and B are *disjoint* if $A \cap B = \emptyset$.

The notation B^c may be a little vague at this point. If the set B is a subset of the real numbers $\mathbb{R}$, then B^c means $\mathbb{R} \setminus B$. If B is naturally a subset of the natural numbers, then B^c is $\mathbb{N} \setminus B$. If ambiguity would ever arise, we will use the set difference notation $A \setminus B$.

We illustrate the operations on the *Venn diagrams* in . Let us now establish one of most basic theorems about sets and logic.

Let A, B, C be sets. Then

$$\begin{aligned}(B \cup C)^c &= B^c \cap C^c, \\ (B \cap C)^c &= B^c \cup C^c,\end{aligned}$$

or, more generally,

$$\begin{aligned}A \setminus (B \cup C) &= (A \setminus B) \cap (A \setminus C), \\ A \setminus (B \cap C) &= (A \setminus B) \cup (A \setminus C).\end{aligned}$$

The first statement is proved by the second statement if we assume the set A is our "universe."

Let us prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. Remember the definition of equality of sets. First, we must show that if $x \in A \setminus (B \cup C)$, then $x \in (A \setminus B) \cap (A \setminus C)$. Second, we must also show that if $x \in (A \setminus B) \cap (A \setminus C)$, then $x \in A \setminus (B \cup C)$.

So let us assume $x \in A \setminus (B \cup C)$. Then x is in A , but not in B nor C . Hence x is in A and not in B , that is, $x \in A \setminus B$. Similarly $x \in A \setminus C$. Thus $x \in (A \setminus B) \cap (A \setminus C)$.

On the other hand suppose $x \in (A \setminus B) \cap (A \setminus C)$. In particular $x \in (A \setminus B)$, so $x \in A$ and $x \notin B$. Also as $x \in (A \setminus C)$, then $x \notin C$. Hence $x \in A \setminus (B \cup C)$.

The proof of the other equality is left as an exercise.

We will also need to intersect or union several sets at once. If there are only finitely many, then we simply apply the union or intersection operation several times. However, suppose we have an infinite collection of sets (a set of sets) $\{A_1, A_2, A_3, \dots\}$. We define

$$\begin{aligned}\bigcup_{n=1}^{\infty} A_n &:= \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}, \\ \bigcap_{n=1}^{\infty} A_n &:= \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}.\end{aligned}$$

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We can also have sets indexed by two integers. For example, we can have the set of sets $\{A_{1,1}, A_{1,2}, A_{2,1}, A_{1,3}, A_{2,2}, A_{3,1}, \dots\}$. Then we write

$$\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} = \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} A_{n,m} \right).$$

And similarly with intersections.

It is not hard to see that we can take the unions in any order. However, switching the order of unions and intersections is not generally permitted without proof. For example:

$$\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \{k \in \mathbb{N} : mk < n\} = \bigcup_{n=1}^{\infty} \emptyset = \emptyset.$$

However,

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{k \in \mathbb{N} : mk < n\} = \bigcap_{m=1}^{\infty} \mathbb{N} = \mathbb{N}.$$

Sometimes, the index set is not the natural numbers. In this case we need a more general notation. Suppose I is some set and for each $i \in I$, we have a set A_i . Then we define

$$\bigcup_{i \in I} A_i := \{x : x \in A_i \text{ for some } i \in I\} \quad \bigcap_{i \in I} A_i := \{x : x \in A_i \text{ for all } i \in I\}.$$

Induction

When a statement includes an arbitrary natural number, a common method of proof is the principle of induction. We start with the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, and we give them their natural ordering, that is, $1 < 2 < 3 < 4 < \dots$. By $S \subset \mathbb{N}$ having a *least element*, we mean that there exists an $x \in S$, such that for every $y \in S$, we have $x \leq y$.

The natural numbers $\mathbb{N}$ ordered in the natural way possess the so-called *well ordering property*. We take this property as an axiom; we simply assume it is true.

Every nonempty subset of $\mathbb{N}$ has a least (smallest) element.

The *principle of induction* is the following theorem, which is equivalent to the well ordering property of the natural numbers.

[induction:thm] Let $P(n)$ be a statement depending on a natural number n . Suppose that

- i. (*basis statement*) $P(1)$ is true,
- ii. (*induction step*) if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Suppose S is the set of natural numbers m for which $P(m)$ is not true. Suppose S is nonempty. Then S has a least element by the well ordering property. Let us call m the least element of S . We know $1 \notin S$ by assumption. Therefore $m > 1$ and $m-1$ is a natural number as well. Since m was the least element of S , we know that $P(m-1)$ is true. But by the induction step we see that $P(m-1+1) = P(m)$ is true, contradicting the statement that $m \in S$. Therefore S is empty and $P(n)$ is true for all $n \in \mathbb{N}$.

Sometimes it is convenient to start at a different number than 1, but all that changes is the labeling. The assumption that $P(n)$ is true in “if $P(n)$ is true, then $P(n+1)$ is true” is usually called the *induction hypothesis*.

Let us prove that for all $n \in \mathbb{N}$,

$$2^{n-1} \leq n!.$$

We let $P(n)$ be the statement that $2^{n-1} \leq n!$ is true. By plugging in $n = 1$, we see that $P(1)$ is true.

Suppose $P(n)$ is true. That is, suppose $2^{n-1} \leq n!$ holds. Multiply both sides by 2 to obtain

$$2^n \leq 2(n!).$$

As $2 \leq (n+1)$ when $n \in \mathbb{N}$, we have $2(n!) \leq (n+1)(n!) = (n+1)!$. That is,

$$2^n \leq 2(n!) \leq (n+1)!,$$

and hence $P(n+1)$ is true. By the principle of induction, we see that $P(n)$ is true for all n , and hence $2^{n-1} \leq n!$ is true for all $n \in \mathbb{N}$.

We claim that for all $c \neq 1$,

$$1 + c + c^2 + \dots + c^n = \frac{1 - c^{n+1}}{1 - c}.$$

Proof: It is easy to check that the equation holds with $n = 1$. Suppose it is true for n . Then

$$\begin{aligned} 1 + c + c^2 + \dots + c^n + c^{n+1} &= (1 + c + c^2 + \dots + c^n) + c^{n+1} \\ &= \frac{1 - c^{n+1}}{1 - c} + c^{n+1} \\ &= \frac{1 - c^{n+1} + (1 - c)c^{n+1}}{1 - c} \\ &= \frac{1 - c^{n+2}}{1 - c}. \end{aligned}$$

There is an equivalent principle called strong induction. The proof that strong induction is equivalent to induction is left as an exercise.

Let $P(n)$ be a statement depending on a natural number n . Suppose that

- i. (*basis statement*) $P(1)$ is true,
- ii. (*induction step*) if $P(k)$ is true for all $k = 1, 2, \dots, n$, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

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Functions

Informally, a *set-theoretic function* f taking a set A to a set B is a mapping that to each $x \in A$ assigns a unique $y \in B$. We write $f: A \rightarrow B$. For example, we define a function $f: S \rightarrow T$ taking $S = \{0, 1, 2\}$ to $T = \{0, 2\}$ by assigning $f(0) := 2$, $f(1) := 2$, and $f(2) := 0$. That is, a function $f: A \rightarrow B$ is a black box, into which we stick an element of A and the function spits out an element of B . Sometimes f is called a *mapping* and we say f maps A to B .

Often, functions are defined by some sort of formula, however, you should really think of a function as just a very big table of values. The subtle issue here is that a single function can have several different formulas, all giving the same function. Also, for many functions, there is no formula that expresses its values.

To define a function rigorously first let us define the Cartesian product.

Let A and B be sets. The *Cartesian product* is the set of tuples defined as

$$A \times B := \{(x, y) : x \in A, y \in B\}.$$

For example, the set $[0, 1] \times [0, 1]$ is a set in the plane bounded by a square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. When A and B are the same set we sometimes use a superscript 2 to denote such a product. For example $[0, 1]^2 = [0, 1] \times [0, 1]$, or $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (the Cartesian plane).

A function $f: A \rightarrow B$ is a subset f of $A \times B$ such that for each $x \in A$, there is a unique $(x, y) \in f$. We then write $f(x) = y$. Sometimes the set f is called the *graph* of the function rather than the function itself.

The set A is called the *domain* of f (and sometimes confusingly denoted $D(f)$). The set

$$R(f) := \{y \in B : \text{there exists an } x \text{ such that } (x, y) \in f\}$$

is called the *range* of f .

Note that $R(f)$ can possibly be a proper subset of B , while the domain of f is always equal to A . We usually assume that the domain of f is nonempty.

From calculus, you are most familiar with functions taking real numbers to real numbers. However, you saw some other types of functions as well. For example, the derivative is a function mapping the set of differentiable functions to the set of all functions. Another example is the Laplace transform, which also takes functions to functions. Yet another example is the function that takes a continuous function g defined on the interval $[0, 1]$ and returns the number $\int_0^1 g(x) dx$.

Let $f: A \rightarrow B$ be a function, and $C \subset A$. Define the *image* (or *direct image*) of C as

$$f(C) := \{f(x) \in B : x \in C\}.$$

Let $D \subset B$. Define the *inverse image* as

$$f^{-1}(D) := \{x \in A : f(x) \in D\}.$$

Define the function $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) := \sin(\pi x)$. Then $f([0, \frac{1}{2}]) = [0, 1]$, $f^{-1}(\{0\}) = \mathbb{Z}$, etc....

[st:propinv] Let $f: A \rightarrow B$. Let C, D be subsets of B . Then

$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\ f^{-1}(C^c) &= (f^{-1}(C))^c. \end{aligned}$$

Read the last line as $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

Let us start with the union. Suppose $x \in f^{-1}(C \cup D)$. That means x maps to C or D . Thus $f^{-1}(C \cup D) \subset f^{-1}(C) \cup f^{-1}(D)$. Conversely if $x \in f^{-1}(C)$, then $x \in f^{-1}(C \cup D)$. Similarly for $x \in f^{-1}(D)$. Hence $f^{-1}(C \cup D) \supset f^{-1}(C) \cup f^{-1}(D)$, and we have equality.

The rest of the proof is left as an exercise.

The proposition does not hold for direct images. We do have the following weaker result.

[st:propfor] Let $f: A \rightarrow B$. Let C, D be subsets of A . Then

$$\begin{aligned} f(C \cup D) &= f(C) \cup f(D), \\ f(C \cap D) &\subset f(C) \cap f(D). \end{aligned}$$

The proof is left as an exercise.

Let $f: A \rightarrow B$ be a function. The function f is said to be *injective* or *one-to-one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. In other words, for all $y \in B$ the set $f^{-1}(\{y\})$ is empty or consists of a single element. We call such an f an *injection*.

The function f is said to be *surjective* or *onto* if $f(A) = B$. We call such an f a *surjection*.

A function f that is both an injection and a surjection is said to be *bijective*, and we say f is a *bijection*.

When $f: A \rightarrow B$ is a bijection, then $f^{-1}(\{y\})$ is always a unique element of A , and we can consider f^{-1} as a function $f^{-1}: B \rightarrow A$. In this case, we call f^{-1} the *inverse function* of f . For example, for the bijection $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) := x^3$ we have $f^{-1}(x) = \sqrt[3]{x}$.

A final piece of notation for functions that we need is the *composition of functions*.

Let $f: A \rightarrow B$, $g: B \rightarrow C$. The function $g \circ f: A \rightarrow C$ is defined as

$$(g \circ f)(x) := g(f(x)).$$

Cardinality

A subtle issue in set theory and one generating a considerable amount of confusion among students is that of cardinality, or “size” of sets. The concept of cardinality is important in modern mathematics in general and in analysis in particular. In this section, we will see the first really unexpected theorem.

Let A and B be sets. We say A and B have the same *cardinality* when there exists a bijection $f: A \rightarrow B$. We denote by $|A|$ the equivalence class of all sets with the same cardinality as A and we simply call $|A|$ the cardinality of A .

Note that A has the same cardinality as the empty set if and only if A itself is the empty set. We then write $|A| := 0$.

Suppose A has the same cardinality as $\{1, 2, 3, \dots, n\}$ for some $n \in \mathbb{N}$. We then write $|A| := n$, and we say A is *finite*. When A is the empty set, we also call A finite.

We say A is *infinite* or “of infinite cardinality” if A is not finite.

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That the notation $|A| = n$ is justified we leave as an exercise. That is, for each nonempty finite set A , there exists a unique natural number n such that there exists a bijection from A to $\{1, 2, 3, \dots, n\}$.

We can order sets by size.

[def:comparecards] We write

$$|A| \leq |B|$$

if there exists an injection from A to B . We write $|A| = |B|$ if A and B have the same cardinality. We write $|A| < |B|$ if $|A| \leq |B|$, but A and B do not have the same cardinality.

We state without proof that $|A| = |B|$ have the same cardinality if and only if $|A| \leq |B|$ and $|B| \leq |A|$. This is the so-called Cantor-Bernstein-Schroeder theorem. Furthermore, if A and B are any two sets, we can always write $|A| \leq |B|$ or $|B| \leq |A|$. The issues surrounding this last statement are very subtle. As we do not require either of these two statements, we omit proofs.

The truly interesting cases of cardinality are infinite sets. We start with the following definition.

If $|\{A\}| = |\mathbb{N}|$ then A is said to be *countably infinite*. If A is finite or countably infinite, then we say A is *countable*. If A is not countable, then A is said to be *uncountable*.

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then A is said to be *countably infinite*. If A is finite or countably infinite, then we say A is *countable*. If A is not countable, then A is said to be *uncountable*.

The cardinality of $\mathbb{N}$ is usually denoted as $\aleph_0$ (read as aleph-naught)³.

The set of even natural numbers has the same cardinality as $\mathbb{N}$. Proof: Given an even natural number, write it as $2n$ for some $n \in \mathbb{N}$. Then create a bijection taking $2n$ to n .

In fact, let us mention without proof the following characterization of infinite sets: *A set is infinite if and only if it is in one-to-one correspondence with a proper subset of itself.*

$\mathbb{N} \times \mathbb{N}$ is a countably infinite set. Proof: Arrange the elements of $\mathbb{N} \times \mathbb{N}$ as follows $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \dots$. That is, always write down first all the elements whose two entries sum to k , then write down all the elements whose entries sum to $k + 1$ and so on. Then define a bijection with $\mathbb{N}$ by letting 1 go to $(1, 1)$, 2 go to $(1, 2)$ and so on.

The set of rational numbers is countable. Proof: (informal) Follow the same procedure as in the previous example, writing $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \dots$. However, leave out any fraction (such as $\frac{2}{2}$) that has already appeared.

For completeness we mention the following statement. *If $A \subset B$ and B is countable, then A is countable. Similarly if A is uncountable, then B is uncountable.* As we will not need this statement in the sequel, and as the proof requires the Cantor-Bernstein-Schroeder theorem mentioned above, we will not give it here.

We give the first truly striking result. First, we need a notation for the set of all subsets of a set.

If A is a set, we define the *power set* of A , denoted by $P(A)$, to be the set of all subsets of A .

For example, if $A := \{1, 2\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. For a finite set A of cardinality n , the cardinality of $P(A)$ is 2^n . This fact is left as an exercise. Hence, for finite sets the cardinality of $P(A)$ is strictly larger than the cardinality of A . What is an unexpected and striking fact is that this statement is still true for infinite sets.

$|\{A\}| = |\mathbb{N}|$ then A is said to be *countably infinite*.

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$(A) \setminus$.

There exists an injection $f: A \rightarrow P(A)$. For any $x \in A$, define $f(x) := \{x\}$. Therefore $|A| \leq |P(A)|$.

To finish the proof, we must show that no function $f: A \rightarrow P(A)$ is a surjection. Suppose $f: A \rightarrow P(A)$ is a function. So for $x \in A$, $f(x)$ is a subset of A . Define the set

$$B := \{x \in A : x \notin f(x)\}.$$

We claim that B is not in the range of f and hence f is not a surjection. Suppose there exists an x_0 such that $f(x_0) = B$. Either $x_0 \in B$ or $x_0 \notin B$. If $x_0 \in B$, then $x_0 \notin f(x_0) = B$, which is a contradiction. If $x_0 \notin B$, then $x_0 \in f(x_0) = B$, which is again a contradiction. Thus such an x_0 does not exist. Therefore, B is not in the range of f , and f is not a surjection. As f was an arbitrary function, no surjection exists.

One particular consequence of this theorem is that there do exist uncountable sets, as $P(\mathbb{N})$ must be uncountable. A related fact is that the set of real numbers (which we study in the next chapter) is uncountable. The existence of uncountable sets may seem unintuitive, and the theorem caused quite a controversy at the time it was announced. The theorem not only says that uncountable sets exist, but that there in fact exist progressively larger and larger infinite sets $\mathbb{N}, P(\mathbb{N}), P(P(\mathbb{N})), P(P(P(\mathbb{N})))$, etc....

Exercises

Show $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Prove that the principle of strong induction is equivalent to the standard induction.

Finish the proof of .

a. Prove .

b. Find an example for which equality of sets in $f(C \cap D) \subset f(C) \cap f(D)$ fails. That is, find an f, A, B, C , and D such that $f(C \cap D)$ is a proper subset of $f(C) \cap f(D)$.

Prove that if A is finite, then there exists a unique number n such that there exists a bijection between A and $\{1, 2, 3, \dots, n\}$. In other words, the notation $|A| := n$ is justified. Hint: Show that if $n > m$, then there is no injection from $\{1, 2, 3, \dots, n\}$ to $\{1, 2, 3, \dots, m\}$.

Prove

a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $A \Delta B$ denote the *symmetric difference*, that is, the set of all elements that belong to either A or B , but not to both A and B .

a. Draw a Venn diagram for $A \Delta B$.

b. Show $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

c. Show $A \Delta B = (A \cup B) \setminus (A \cap B)$.

For each $n \in \mathbb{N}$, let $A_n := \{(n+1)k : k \in \mathbb{N}\}$.

a. Find $A_1 \cap A_2$.

b. Find $\bigcup_{n=1}^{\infty} A_n$.

c. Find $\bigcap_{n=1}^{\infty} A_n$.

Determine $P(S)$ (the power set) for each of the following:

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- $S = \emptyset$,
- $S = \{1\}$,
- $S = \{1, 2\}$,
- $S = \{1, 2, 3, 4\}$.

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions.

- Prove that if $g \circ f$ is injective, then f is injective.
- Prove that if $g \circ f$ is surjective, then g is surjective.
- Find an explicit example where $g \circ f$ is bijective, but neither f nor g are bijective.

Prove that $n < 2^n$ by induction.

Show that for a finite set A of cardinality n , the cardinality of $P(A)$ is 2^n .

Prove $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Prove $1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \in \mathbb{N}$.

Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

Find the smallest $n \in \mathbb{N}$ such that $2(n+5)^2 < n^3$ and call it n_0 . Show that $2(n+5)^2 < n^3$ for all $n \geq n_0$.

Find all $n \in \mathbb{N}$ such that $n^2 < 2^n$.

Finish the proof that the is equivalent to the [well ordering property](#) of $\mathbb{N}$. That is, prove the well ordering property for $\mathbb{N}$ using the principle of induction.

Give an example of a countable collection of finite sets $A_1, A_2, \dots$, whose union is not a finite set.

Give an example of a countable collection of infinite sets $A_1, A_2, \dots$, with $A_j \cap A_k$ being infinite for all j and k , such that $\bigcap_{j=1}^{\infty} A_j$ is nonempty and finite.

Real Numbers

Basic properties

Note: 1.5 lectures

The main object we work with in analysis is the set of real numbers. As this set is so fundamental, often much time is spent on formally constructing the set of real numbers. However, we take an easier approach here and just assume that a set with the correct properties exists. We need to start with the definitions of those properties.

An *ordered set* is a set S , together with a relation $<$ such that

- For any $x, y \in S$, exactly one of $x < y$, $x = y$, or $y < x$ holds.
- If $x < y$ and $y < z$, then $x < z$.

We write $x \leq y$ if $x < y$ or $x = y$. We define $>$ and $\geq$ in the obvious way.

For example, the set of rational numbers $\mathbb{Q}$ is an ordered set by letting $x < y$ if and only if $y - x$ is a positive rational number, that is if $y - x = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Similarly, $\mathbb{N}$ and $\mathbb{Z}$ are also ordered sets.

There are other ordered sets than sets of numbers. For example, the set of countries can be ordered by landmass, so for example India $>$ Lichtenstein. Any time you sort a set in some way, you are making an ordered set. A typical ordered set that you have used since primary school is the dictionary. It is the ordered set of words where the order is the so-called lexicographic ordering. Such ordered sets appear often for example in computer science. In this class we will mostly be interested in ordered set of numbers however.

Let $E \subset S$, where S is an ordered set.

- If there exists a $b \in S$ such that $x \leq b$ for all $x \in E$, then we say E is *bounded above* and b is an *upper bound* of E .
- If there exists a $b \in S$ such that $x \geq b$ for all $x \in E$, then we say E is *bounded below* and b is a *lower bound* of E .
- If there exists an upper bound b_0 of E such that whenever b is any upper bound for E we have $b_0 \leq b$, then b_0 is called the *least upper bound* or the *supremum* of E . We write

$$\sup E := b_0.$$

- Similarly, if there exists a lower bound b_0 of E such that whenever b is any lower bound for E we have $b_0 \geq b$, then b_0 is called the *greatest lower bound* or the *infimum* of E . We write

$$\inf E := b_0.$$

When a set E is both bounded above and bounded below, we say simply that E is *bounded*.

A supremum or infimum for E (even if they exist) need not be in E . For example, the set $E := \{x \in \mathbb{Q} : x < 1\}$ has a least upper bound of 1, but 1 is not in the set E itself. On the other hand, if we take $G := \{x \in \mathbb{Q} : x \leq 1\}$, then the least upper bound of G is clearly also 1, and in this case $1 \in G$. On the other hand, the set $P := \{x \in \mathbb{Q} : x \geq 0\}$ has no upper bound (why?) and therefore it can have no least upper bound. On the other hand 0 is the greatest lower bound of P .

[defn:lub] An ordered set S has the *least-upper-bound property* if every nonempty subset $E \subset S$ that is bounded above has a least upper bound, that is $\sup E$ exists in S .

The *least-upper-bound property* is sometimes called the *completeness property* or the *Dedekind completeness property*.

The set $\mathbb{Q}$ of rational numbers does not have the least-upper-bound property. The subset $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum in $\mathbb{Q}$. The obvious supremum $\sqrt{2}$ is not rational. Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. Write $x = \frac{m}{n}$ in lowest terms. So $(\frac{m}{n})^2 = 2$ or $m^2 = 2n^2$. Hence m^2 is divisible by 2 and so m is divisible by 2. Write $m = 2k$ and so $(2k)^2 = 2n^2$. Divide by 2 and note that $2k^2 = n^2$, and hence n is divisible by 2. But that is a contradiction as $\frac{m}{n}$ was in lowest terms.

That $\mathbb{Q}$ does not have the least-upper-bound property is one of the most important reasons why we work with $\mathbb{R}$ in analysis. The set $\mathbb{Q}$ is just fine for algebraists. But analysts require the least-upper-bound property to do any work. We also require our real numbers to have many algebraic properties. In particular, we require that they are a field.

A set F is called a *field* if it has two operations defined on it, addition $+$ and multiplication $\cdot$, and if it satisfies the following axioms.

- If $x \in F$ and $y \in F$, then $x + y \in F$.
- (commutativity of addition) $x + y = y + x$ for all $x, y \in F$.
- (associativity of addition) $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- There exists an element $0 \in F$ such that $0 + x = x$ for all $x \in F$.
- For every element $x \in F$ there exists an element $-x \in F$ such that $x + (-x) = 0$.

- If $x \in F$ and $y \in F$, then $xy \in F$.
- (commutativity of multiplication) $xy = yx$ for all $x, y \in F$.

Processing math: 52% $(xy)z = x(yz)$ for all $x, y, z \in F$.

4. There exists an element $1 \in F$ (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.
5. For every $x \in F$ such that $x \neq 0$ there exists an element $\frac{1}{x} \in F$ such that $x(\frac{1}{x}) = 1$.
6. (distributive law) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

The set Q of rational numbers is a field. On the other hand Z is not a field, as it does not contain multiplicative inverses. For example, there is no $x \in Z$ such that $2x = 1$, so (M5) is not satisfied. You can check that (M5) is the only property that fails³.

We will assume the basic facts about fields that are easily proved from the axioms. For example, $0x = 0$ is easily proved by noting that $xx = (0 + x)x = 0x + xx$, using (A4), (D), and (M2). Then using (A5) on xx , along with (A2), (A3), and (A4), we obtain $0 = 0x$.

A field F is said to be an *ordered field* if F is also an ordered set such that:

- i. [defn:ordfield:i] For $x, y, z \in F$, $x < y$ implies $x + z < y + z$.
- ii. [defn:ordfield:ii] For $x, y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$.

If $x > 0$, we say x is *positive*. If $x < 0$, we say x is *negative*. We also say x is *nonnegative* if $x \geq 0$, and x is *nonpositive* if $x \leq 0$.

For example, it can be checked that the rational numbers Q with the standard ordering is an ordered field.

[prop:bordfield] Let F be an ordered field and $x, y, z \in F$. Then:

- i. [prop:bordfield:i] If $x > 0$, then $-x < 0$ (and vice-versa).
- ii. [prop:bordfield:ii] If $x > 0$ and $y < z$, then $xy < xz$.
- iii. [prop:bordfield:iii] If $x < 0$ and $y < z$, then $xy > xz$.
- iv. [prop:bordfield:iv] If $x \neq 0$, then $x^2 > 0$.
- v. [prop:bordfield:v] If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Note that [prop:bordfield:iv] implies in particular that $1 > 0$.

Let us prove [prop:bordfield:i]. The inequality $x > 0$ implies by item [defn:ordfield:i] of definition of ordered field that $x + (-x) > 0 + (-x)$. Now apply the algebraic properties of fields to obtain $0 > -x$. The “vice-versa” follows by similar calculation.

For [prop:bordfield:ii], first notice that $y < z$ implies $0 < z - y$ by applying item [defn:ordfield:i] of the definition of ordered fields. Now apply item [defn:ordfield:ii] of the definition of ordered fields to obtain $0 < x(z - y)$. By algebraic properties we get $0 < xz - xy$, and again applying item [defn:ordfield:i] of the definition we obtain $xy < xz$.

Part [prop:bordfield:iii] is left as an exercise.

To prove part [prop:bordfield:iv] first suppose $x > 0$. Then by item [defn:ordfield:ii] of the definition of ordered fields we obtain that $x^2 > 0$ (use $y = x$). If $x < 0$, we use part [prop:bordfield:iii] of this proposition. Plug in $y = x$ and $z = 0$.

Finally to prove part [prop:bordfield:v], notice that $\frac{1}{x}$ cannot be equal to zero (why?). Suppose $\frac{1}{x} < 0$, then $\frac{1}{-x} > 0$ by [prop:bordfield:i]. Then apply part [prop:bordfield:ii] (as $x > 0$) to obtain $x(\frac{1}{-x}) > 0x$ or $-1 > 0$, which contradicts $1 > 0$ by using part [prop:bordfield:i] again. Hence $\frac{1}{x} > 0$. Similarly $\frac{1}{y} > 0$. Thus $(\frac{1}{x})(\frac{1}{y}) > 0$ by definition of ordered field and by part [prop:bordfield:ii]

$$(\frac{1}{x})(\frac{1}{y})x < (\frac{1}{x})(\frac{1}{y})y.$$

By algebraic properties we get $\frac{1}{y} < \frac{1}{x}$.

Product of two positive numbers (elements of an ordered field) is positive. However, it is not true that if the product is positive, then each of the two factors must be positive.

Let $x, y \in F$ where F is an ordered field. Suppose $xy > 0$. Then either both x and y are positive, or both are negative.

Clearly both of the conclusions can happen. If either x and y are zero, then xy is zero and hence not positive. Hence we assume that x and y are nonzero, and we simply need to show that if they have opposite signs, then $xy < 0$. Without loss of generality suppose $x > 0$ and $y < 0$. Multiply $y < 0$ by x to get $xy < 0x = 0$. The result follows by contrapositive.

The reader may also know about the *complex numbers*, usually denoted by C . That is, C is the set of numbers of the form $x + iy$, where x and y are real numbers, and i is the imaginary number, a number such that $i^2 = -1$. The reader may remember from algebra that C is also a field, however, it is not an ordered field. While one can make C into an ordered set in some way, it can be proved that it is not possible to put an order on C that will make it an ordered field.

Exercises

Prove part [prop:bordfield:iii] of .

[exercise:finitesethasminmax] Let S be an ordered set. Let $A \subset S$ be a nonempty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and is in A and $\sup A$ exists and is in A . Hint: Use .

Let $x, y \in F$, where F is an ordered field. Suppose $0 < x < y$. Show that $x^2 < y^2$.

Let S be an ordered set. Let $B \subset S$ be bounded (above and below). Let $A \subset B$ be a nonempty subset. Suppose all the $\inf$'s and $\sup$'s exist. Show that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A . Suppose $b \in A$. Show that $b = \sup A$.

Let S be an ordered set. Let $A \subset S$ be a nonempty subset that is bounded above. Suppose $\sup A$ exists and $\sup A \notin A$. Show that A contains a countably infinite subset. In particular, A is infinite.

Find a (nonstandard) ordering of the set of natural numbers N such that there exists a nonempty proper subset $A \subsetneq N$ and such that $\sup A$ exists in N , but $\sup A \notin A$.

Let $F = \{0, 1, 2\}$. a) Prove that there is exactly one way to define addition and multiplication so that F is a field if 0 and 1 have their usual meaning of (A4) and (M4). b) Show that F cannot be an ordered field.

[exercise:dominatingb] Let S be an ordered set and A is a nonempty subset such that $\sup A$ exists. Suppose there is a $B \subset A$ such that whenever $x \in A$ there is a $y \in B$ such that $x \leq y$. Show that $\sup B$ exists and $\sup B = \sup A$.

Let D be the ordered set of all possible words (not just English words, all strings of letters of arbitrary length) using the Latin alphabet using only lower case letters. The order is the lexicographic order as in a dictionary (e.g. $aaa < dog < door$). Let A be the subset of D containing the words whose first letter is ‘a’ (e.g. $a \in A, abcd \in A$). Show that A has a supremum and find what it is.

The set of real numbers

Note: 2 lectures, the extended real numbers are optional

The set of real numbers

We finally get to the real number system. To simplify matters, instead of constructing the real number set from the rational numbers, we simply state their existence as a theorem without proof. Notice that Q is an ordered field.

There exists a unique⁴ ordered field R with the such that $Q \subset R$.

Note that also $N \subset Q$. We saw that $1 > 0$. By (exercise) we can prove that $n > 0$ for all $n \in N$. Similarly we can easily verify all the statements we know about rational numbers and their natural ordering.

Processing math: 52% the most basic but useful results about the real numbers. The following proposition is essentially how an analyst proves that a number is zero.

If $x \in \mathbb{R}$ is such that $x \geq 0$ and $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x = 0$.

If $x > 0$, then $0 < \frac{x}{2} < x$ (why?). Taking $\epsilon = \frac{x}{2}$ obtains a contradiction. Thus $x = 0$.

A related simple fact is that any time we have two real numbers $a < b$, then there is another real number c such that $a < c < b$. Just take for example $c = \frac{a+b}{2}$ (why?). In fact, there are infinitely many real numbers between a and b .

The most useful property of $\mathbb{R}$ for analysts is not just that it is an ordered field, but that it has the . Essentially we want $\mathbb{Q}$, but we also want to take suprema (and infima) willy-nilly. So what we do is to throw in enough numbers to obtain $\mathbb{R}$.

We mentioned already that $\mathbb{R}$ must contain elements that are not in $\mathbb{Q}$ because of the . We saw there is no rational square root of two. The set $\{x \in \mathbb{Q} : x^2 < 2\}$ implies the existence of the real number $\sqrt{2}$, although this fact requires a bit of work.

[example:sqrt2] Claim: There exists a unique positive real number r such that $r^2 = 2$. We denote r by $\sqrt{2}$.

Take the set $A := \{x \in \mathbb{R} : x^2 < 2\}$. First if $x^2 < 2$, then $x < 2$. To see this fact, note that $x \geq 2$ implies $x^2 \geq 4$ (use , we will not explicitly mention its use from now on), hence any number x such that $x \geq 2$ is not in A . Thus A is bounded above. On the other hand, $1 \in A$, so A is nonempty.

Let us define $r := \sup A$. We will show that $r^2 = 2$ by showing that $r^2 \geq 2$ and $r^2 \leq 2$. This is the way analysts show equality, by showing two inequalities. We already know that $r \geq 1 > 0$.

In the following, it may seem we are pulling certain expressions out of a hat. When writing a proof such as this we would, of course, come up with the expressions only after playing around with what we wish to prove. The order in which we write the proof is not necessarily the order in which we come up with the proof.

Let us first show that $r^2 \geq 2$. Take a positive number s such that $s^2 < 2$. We wish to find an $h > 0$ such that $(s+h)^2 < 2$. As $2 - s^2 > 0$, we have $\frac{2-s^2}{2s+1} > 0$. We choose an $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$. Furthermore, we assume $h < 1$.

$$\begin{aligned} (s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) && \text{(since } h < 1\text{)} \\ &< 2 - s^2 && \left(\text{since } h < \frac{2-s^2}{2s+1} \right). \end{aligned}$$

Therefore, $(s+h)^2 < 2$. Hence $s+h \in A$, but as $h > 0$ we have $s+h > s$. So $s < r = \sup A$. As s was an arbitrary positive number such that $s^2 < 2$, it follows that $r^2 \geq 2$.

Now take a positive number s such that $s^2 > 2$. We wish to find an $h > 0$ such that $(s-h)^2 > 2$. As $s^2 - 2 > 0$ we have $\frac{s^2-2}{2s} > 0$. We choose an $h \in \mathbb{R}$ such that $0 < h < \frac{s^2-2}{2s}$ and $h < s$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \\ &< s^2 - 2 && \left(\text{since } h < \frac{s^2-2}{2s} \right). \end{aligned}$$

By subtracting s^2 from both sides and multiplying by -1 , we find $(s-h)^2 > 2$. Therefore $s-h \notin A$.

Furthermore, if $x \geq s-h$, then $x^2 \geq (s-h)^2 > 2$ (as $x > 0$ and $s-h > 0$) and so $x \notin A$. Thus $s-h$ is an upper bound for A . However, $s-h < s$, or in other words $s > r = \sup A$. Thus $r^2 \leq 2$.

Together, $r^2 \geq 2$ and $r^2 \leq 2$ imply $r^2 = 2$. The existence part is finished. We still need to handle uniqueness. Suppose $s \in \mathbb{R}$ such that $s^2 = 2$ and $s > 0$. Thus $s^2 = r^2$. However, if $0 < s < r$, then $s^2 < r^2$. Similarly $0 < r < s$ implies $r^2 < s^2$. Hence $s = r$.

The number $\sqrt{2} \notin \mathbb{Q}$. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational* numbers. We just saw that $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. Not only is it nonempty, we will see later that it is very large indeed.

Using the same technique as above, we can show that a positive real number $x^{1/n}$ exists for all $n \in \mathbb{N}$ and all $x > 0$. That is, for each $x > 0$, there exists a unique positive real number r such that $r^n = x$. The proof is left as an exercise.

Archimedean property

As we have seen, there are plenty of real numbers in any interval. But there are also infinitely many rational numbers in any interval. The following is one of the fundamental facts about the real numbers. The two parts of the next theorem are actually equivalent, even though it may not seem like that at first sight.

[thm:arch]

i. [thm:arch:i] (*Archimedean property*) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that

$$nx > y.$$

ii. [thm:arch:ii] (*$\mathbb{Q}$ is dense in $\mathbb{R}$*) If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.

Let us prove [thm:arch:i]. We divide through by x and then [thm:arch:i] says that for any real number $t := \frac{y}{x}$, we can find natural number n such that $n > t$. In other words, [thm:arch:i] says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Let $b := \sup \mathbb{N}$. The number $b-1$ cannot possibly be an upper bound for $\mathbb{N}$ as it is strictly less than b (the supremum). Thus there exists an $m \in \mathbb{N}$ such that $m > b-1$. We add one to obtain $m+1 > b$, which contradicts b being an upper bound.

Let us tackle [thm:arch:ii]. First assume $x \geq 0$. Note that $y-x > 0$. By [thm:arch:i], there exists an $n \in \mathbb{N}$ such that

$$n(y-x) > 1.$$

Also by [thm:arch:i] the set $A := \{k \in \mathbb{N} : k > nx\}$ is nonempty. By the [well ordering property](#) of $\mathbb{N}$, A has a least element m . As $m \in A$, then $m > nx$. We divide through by n to get $x < \frac{m}{n}$. As m is the least element of A , $m-1 \notin A$. If $m > 1$, then $m-1 \in \mathbb{N}$, but $m-1 \notin A$ and so $m-1 \leq nx$. If $m = 1$, then $m-1 = 0$, and $m-1 \leq nx$ still holds as $x \geq 0$. In other words,

$$m-1 \leq nx \quad \text{or} \quad m \leq nx+1.$$

On the other hand from $n(y-x) > 1$ we obtain $ny > 1+nx$. Hence $ny > 1+nx \geq m$, and therefore $y > \frac{m}{n}$. Putting everything together we obtain $x < \frac{m}{n} < y$. So let $r = \frac{m}{n}$.

Now assume $x < 0$. If $y > 0$, then we just take $r = 0$. If $y \leq 0$, then $0 \leq -y < -x$, and we find a rational q such that $-y < q < -x$. Then take $r = -q$.

Let us state and prove a simple but useful corollary of the .

$$\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0.$$

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Let $A := \{\frac{1}{n} : n \in \mathbb{N}\}$. Obviously A is not empty. Furthermore, $\frac{1}{n} > 0$ and so 0 is a lower bound, and $b := \inf A$ exists. As 0 is a lower bound, then $b \geq 0$. Now take an arbitrary $a > 0$. By the there exists an n such that $na > 1$, or in other words $a > \frac{1}{n} \in A$. Therefore a cannot be a lower bound for A . Hence $b = 0$.

Using supremum and infimum

We want to make sure that suprema and infima are compatible with algebraic operations. For a set $A \subset \mathbb{R}$ and a number $x \in \mathbb{R}$ define

$$x + A := \{x + y \in \mathbb{R} : y \in A\}, \\ xA := \{xy \in \mathbb{R} : y \in A\}.$$

[prop:supinfalg] Let $A \subset \mathbb{R}$ be nonempty.

- If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$.
- If $x \in \mathbb{R}$ and A is bounded below, then $\inf(x + A) = x + \inf A$.
- If $x > 0$ and A is bounded above, then $\sup(xA) = x(\sup A)$.
- If $x > 0$ and A is bounded below, then $\inf(xA) = x(\inf A)$.
- If $x < 0$ and A is bounded below, then $\sup(xA) = x(\inf A)$.
- If $x < 0$ and A is bounded above, then $\inf(xA) = x(\sup A)$.

Do note that multiplying a set by a negative number switches supremum for an infimum and vice-versa. Also, as the proposition implies that supremum (resp. infimum) of $x + A$ or xA exists, it also implies that $x + A$ or xA is nonempty and bounded from above (resp. from below).

Let us only prove the first statement. The rest are left as exercises.

Suppose b is an upper bound for A . That is, $y \leq b$ for all $y \in A$. Then $x + y \leq x + b$ for all $y \in A$, and so $x + b$ is an upper bound for $x + A$. In particular, if $b = \sup A$, then

$$\sup(x + A) \leq x + b = x + \sup A.$$

The other direction is similar. If b is an upper bound for $x + A$, then $x + y \leq b$ for all $y \in A$ and so $y \leq b - x$ for all $y \in A$. So $b - x$ is an upper bound for A . If $b = \sup(x + A)$, then

$$\sup A \leq b - x = \sup(x + A) - x.$$

And the result follows.

Sometimes we need to apply supremum or infimum twice. Here is an example.

[infsupineq:prop] Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Any $x \in A$ is a lower bound for B . Therefore $x \leq \inf B$ for all $x \in A$, so $\inf B$ is an upper bound for A . Hence, $\sup A \leq \inf B$.

We must be careful about strict inequalities and taking suprema and infima. Note that $x < y$ whenever $x \in A$ and $y \in B$ still only implies $\sup A \leq \inf B$, and not a strict inequality. This is an important subtle point that comes up often. For example, take $A := \{0\}$ and take $B := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$. However, $\sup A = 0$ and $\inf B = 0$.

The proof of the following often used elementary fact is left to the reader. A similar statement holds for infima.

[prop:existsxepsfromsup] If $S \subset \mathbb{R}$ is a nonempty set, bounded from above, then for every $\epsilon > 0$ there exists $x \in S$ such that $(\sup S) - \epsilon < x \leq \sup S$.

To make using suprema and infima even easier, we may want to write $\sup A$ and $\inf A$ without worrying about A being bounded and nonempty. We make the following natural definitions.

Let $A \subset \mathbb{R}$ be a set.

- If A is empty, then $\sup A := -\infty$.
- If A is not bounded above, then $\sup A := \infty$.
- If A is empty, then $\inf A := \infty$.
- If A is not bounded below, then $\inf A := -\infty$.

For convenience, ∞ and $-\infty$ are sometimes treated as if they were numbers, except we do not allow arbitrary arithmetic with them. We make $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ into an ordered set by letting

$$-\infty < \infty \quad \text{and} \quad -\infty < x \quad \text{and} \quad x < \infty \quad \text{for all } x \in \mathbb{R}.$$

The set $\mathbb{R}^*$ is called the set of *extended real numbers*. It is possible to define some arithmetic on $\mathbb{R}^*$. Most operations are extended in an obvious way, but we must leave $\infty - \infty$, $0 \cdot (\pm \infty)$, and $\frac{\pm \infty}{\pm \infty}$ undefined. We refrain from using this arithmetic, it leads to easy mistakes as $\mathbb{R}^*$ is not a field. Now we can take suprema and infima without fear of emptiness or unboundedness. In this book we mostly avoid using $\mathbb{R}^*$ outside of exercises, and leave such generalizations to the interested reader.

Maxima and minima

By we know a finite set of numbers always has a supremum or an infimum that is contained in the set itself. In this case we usually do not use the words supremum or infimum.

When a set A of real numbers is bounded above, such that $\sup A \in A$, then we can use the word *maximum* and the notation $\max A$ to denote the supremum. Similarly for infimum; when a set A is bounded below and $\inf A \in A$, then we can use the word *minimum* and the notation $\min A$. For example,

$$\max \{1, 2.4, \pi, 100\} = 100, \\ \min \{1, 2.4, \pi, 100\} = 1.$$

While writing $\sup$ and $\inf$ may be technically correct in this situation, $\max$ and $\min$ are generally used to emphasize that the supremum or infimum is in the set itself.

Exercises

Prove that if $t > 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n^2} < t$.

Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Finish the proof of .

Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.

Show that $\sqrt{3}$ is irrational.

Let $n \in \mathbb{N}$. Show that either $\sqrt{n}$ is either an integer or it is irrational.

Prove the *arithmetic-geometric mean inequality*. That is, for two positive real numbers x, y we have

$$\sqrt{xy} \leq \frac{x + y}{2}.$$

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Furthermore, equality occurs if and only if $x = y$.

Show that for any two real numbers x and y such that $x < y$, there exists an irrational number s such that $x < s < y$. Hint: Apply the density of $\mathbb{Q}$ to $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$.

[exercise:supofsum] Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = \sup A + \sup B \quad \text{and} \quad \inf C = \inf A + \inf B.$$

Let A and B be two nonempty bounded sets of nonnegative real numbers. Define the set $C := \{ab : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = (\sup A)(\sup B) \quad \text{and} \quad \inf C = (\inf A)(\inf B).$$

Given $x > 0$ and $n \in \mathbb{N}$, show that there exists a unique positive real number r such that $x = r^n$. Usually r is denoted by $x^{1/n}$.

Prove .

[exercise:bernoulliineq] Prove the so-called *Bernoulli's inequality* ⁵ : If $1 + x > 0$ then for all $n \in \mathbb{N}$ we have $(1 + x)^n \geq 1 + nx$.

Absolute value

Note: 0.5–1 lecture

A concept we will encounter over and over is the concept of *absolute value*. You want to think of the absolute value as the “size” of a real number. Let us give a formal definition.

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Let us give the main features of the absolute value as a proposition.

[prop:absbas]

- [prop:absbas:i] $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$.
- [prop:absbas:ii] $|-x| = |x|$ for all $x \in \mathbb{R}$.
- [prop:absbas:iii] $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.
- [prop:absbas:iv] $|x|^2 = x^2$ for all $x \in \mathbb{R}$.
- [prop:absbas:v] $|x| \leq y$ if and only if $-y \leq x \leq y$.
- [prop:absbas:vi] $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$.

[prop:absbas:i]: This statement is not difficult to see from the definition.

[prop:absbas:ii]: Suppose $x > 0$, then $|-x| = -(-x) = x = |x|$. Similarly when $x < 0$, or $x = 0$.

[prop:absbas:iii]: If x or y is zero, then the result is obvious. When x and y are both positive, then $|x||y| = xy$. xy is also positive and hence $xy = |xy|$. If x and y are both negative then xy is still positive and $xy = |xy|$, and $|x||y| = (-x)(-y) = xy$. Next assume $x > 0$ and $y < 0$. Then $|x||y| = x(-y) = -(xy)$. Now xy is negative and hence $|xy| = -(xy)$. Similarly if $x < 0$ and $y > 0$.

[prop:absbas:iv]: Obvious if $x \geq 0$. If $x < 0$, then $|x|^2 = (-x)^2 = x^2$.

[prop:absbas:v]: Suppose $|x| \leq y$. If $x \geq 0$, then $x \leq y$. Obviously $y \geq 0$ and hence $-y \leq 0 \leq x$ so $-y \leq x \leq y$ holds. If $x < 0$, then $|x| \leq y$ means $-x \leq y$. Negating both sides we get $x \geq -y$. Again $y \geq 0$ and so $y \geq 0 > x$. Hence, $-y \leq x \leq y$.

On the other hand, suppose $-y \leq x \leq y$ is true. If $x \geq 0$, then $x \leq y$ is equivalent to $|x| \leq y$. If $x < 0$, then $-y \leq x$ implies $(-x) \leq y$, which is equivalent to $|x| \leq y$.

[prop:absbas:vi]: Apply [prop:absbas:v] with $y = |x|$.

A property used frequently enough to give it a name is the so-called *triangle inequality*.

$|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

From we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. We add these two inequalities to obtain

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Again by we have $|x + y| \leq |x| + |y|$.

There are other often applied versions of the triangle inequality.

Let $x, y \in \mathbb{R}$

- (reverse triangle inequality) $||x| - |y|| \leq |x - y|$.
- $|x - y| \leq |x| + |y|$.

Let us plug in $x = a - b$ and $y = b$ into the standard triangle inequality to obtain

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

or $|a| - |b| \leq |a - b|$. Switching the roles of a and b we obtain or $|b| - |a| \leq |b - a| = |a - b|$. Now applying again we obtain the reverse triangle inequality.

The second version of the triangle inequality is obtained from the standard one by just replacing y with $-y$ and noting again that $|-y| = |y|$.

Let $x_1, x_2, \dots, x_n \in \mathbb{R}$. Then

$$|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|.$$

We proceed by . The conclusion holds trivially for $n = 1$, and for $n = 2$ it is the standard triangle inequality. Suppose the corollary holds for n . Take $n + 1$ numbers $x_1, x_2, \dots, x_{n+1}$ and first use the standard triangle inequality, then the induction hypothesis

$$\begin{aligned} |x_1 + x_2 + \dots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \dots + x_n| + |x_{n+1}| \\ &\leq |x_1| + |x_2| + \dots + |x_n| + |x_{n+1}|. \quad \text{\textcolor{red}{qedhere}} \end{aligned}$$

Let us see an example of the use of the triangle inequality.

Find a number M such that $|x^2 - 9x + 1| \leq M$ for all $-1 \leq x \leq 5$.

Using the triangle inequality, write

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$$|x^2 - 9x + 1| \leq |x^2| + |9x| + |1| = |x|^2 + 9|x| + 1.$$

It is obvious that $|x|^2 + 9|x| + 1$ is largest when $|x|$ is largest. In the interval provided, $|x|$ is largest when $x = 5$ and so $|x| = 5$. One possibility for M is

$$M = 5^2 + 9(5) + 1 = 71.$$

There are, of course, other M that work. The bound of 71 is much higher than it need be, but we didn't ask for the best possible M , just one that works.

The last example leads us to the concept of bounded functions.

Suppose $f: D \rightarrow \mathbb{R}$ is a function. We say f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for all $x \in D$.

In the example we proved $x^2 - 9x + 1$ is bounded when considered as a function on $D = \{x: -1 \leq x \leq 5\}$. On the other hand, if we consider the same polynomial as a function on the whole real line $\mathbb{R}$, then it is not bounded.

For a function $f: D \rightarrow \mathbb{R}$ we write

$$\begin{aligned}\sup_{x \in D} f(x) &:= \sup f(D), \\ \inf_{x \in D} f(x) &:= \inf f(D).\end{aligned}$$

We also sometimes replace the " $x \in D$ " with an expression. For example if, as before, $f(x) = x^2 - 9x + 1$, for $-1 \leq x \leq 5$, a little bit of calculus shows

$$\sup_{x \in D} f(x) = \sup_{-1 \leq x \leq 5} (x^2 - 9x + 1) = 11, \quad \inf_{x \in D} f(x) = \inf_{-1 \leq x \leq 5} (x^2 - 9x + 1) = -\frac{17}{4}.$$

[prop:funccsupinf] If $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ (D nonempty) are bounded⁶ functions and

$$f(x) \leq g(x) \quad \text{for all } x \in D,$$

then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \quad \text{and} \quad \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

You should be careful with the variables. The x on the left side of the inequality in [prop:funccsupinf:eq] is different from the x on the right. You should really think of the first inequality as

$$\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y).$$

Let us prove this inequality. If b is an upper bound for $g(D)$, then $f(x) \leq g(x) \leq b$ for all $x \in D$, and hence b is an upper bound for $f(D)$. Taking the least upper bound we get that for all $x \in D$

$$f(x) \leq \sup_{y \in D} g(y).$$

Therefore $\sup_{y \in D} g(y)$ is an upper bound for $f(D)$ and thus greater than or equal to the least upper bound of $f(D)$.

$$\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y).$$

The second inequality (the statement about the $\inf$) is left as an exercise.

A common mistake is to conclude

$$\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y).$$

The inequality [rm:av:ltnottrue] is not true given the hypothesis of the claim above. For this stronger inequality we need the stronger hypothesis

$$f(x) \leq g(y) \quad \text{for all } x \in D \text{ and } y \in D.$$

The proof as well as a counterexample is left as an exercise.

Exercises

Show that $|x - y| < \epsilon$ if and only if $x - \epsilon < y < x + \epsilon$.

Show that

$$\begin{aligned}\text{a. } \max\{x, y\} &= \frac{x+y+|x-y|}{2} \\ \text{b. } \min\{x, y\} &= \frac{x+y-|x-y|}{2}\end{aligned}$$

Find a number M such that $|x^3 - x^2 + 8x| \leq M$ for all $-2 \leq x \leq 10$.

Finish the proof of . That is, prove that given any set D , and two bounded functions $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ such that $f(x) \leq g(x)$ for all $x \in D$, then

$$\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

Let $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ be functions (D nonempty).

a. Suppose $f(x) \leq g(y)$ for all $x \in D$ and $y \in D$. Show that

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

b. Find a specific D , f , and g , such that $f(x) \leq g(x)$ for all $x \in D$, but

$$\sup_{x \in D} f(x) > \inf_{x \in D} g(x).$$

Prove without the assumption that the functions are bounded. Hint: You need to use the extended real numbers.

Let D be a nonempty set. Suppose $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ are bounded functions. a) Show

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$$\sup_{x \in D} (f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x) \quad \text{and} \quad \inf_{x \in D} (f(x) + g(x)) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x).$$

b) Find examples where we obtain strict inequalities.

Intervals and the size of $\mathbb{R}$

Note: 0.5–1 lecture (proof of uncountability of $\mathbb{R}$ can be optional)

You surely saw the notation for intervals before, but let us give a formal definition here. For $a, b \in \mathbb{R}$ such that $a < b$ we define

$$\begin{aligned} [a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, \\ (a, b) &:= \{x \in \mathbb{R} : a < x < b\}, \\ [a, b) &:= \{x \in \mathbb{R} : a \leq x < b\}, \\ (a, b] &:= \{x \in \mathbb{R} : a < x \leq b\}. \end{aligned}$$

The interval $[a, b]$ is called a *closed interval* and (a, b) is called an *open interval*. The intervals of the form $(a, b]$ and $[a, b)$ are called *half-open intervals*.

The above intervals were all *bounded intervals*, since both a and b were real numbers. We define *unbounded intervals*,

$$\begin{aligned} [a, \infty) &:= \{x \in \mathbb{R} : a \leq x\}, \\ (a, \infty) &:= \{x \in \mathbb{R} : a < x\}, \\ (-\infty, b] &:= \{x \in \mathbb{R} : x \leq b\}, \\ (-\infty, b) &:= \{x \in \mathbb{R} : x < b\}. \end{aligned}$$

For completeness we define $(-\infty, \infty) := \mathbb{R}$.

In short, an interval is a set $I \subset \mathbb{R}$ with at least 2 elements, such that if $a < b < c$ and $a, c \in I$, then $b \in I$. See .

We have already seen that any open interval (a, b) (where $a < b$ of course) must be nonempty. For example, it contains the number $\frac{a+b}{2}$. An unexpected fact is that from a set-theoretic perspective, all intervals have the same “size,” that is, they all have the same cardinality. For example the map $f(x) := 2x$ takes the interval $[0, 1]$ bijectively to the interval $[0, 2]$.

Maybe more interestingly, the function $f(x) := \tan(x)$ is a bijective map from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. Hence the bounded interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ has the same cardinality as $\mathbb{R}$. It is not completely straightforward to construct a bijective map from $[0, 1]$ to say $(0, 1)$, but it is possible.

And do not worry, there does exist a way to measure the “size” of subsets of real numbers that “sees” the difference between $[0, 1]$ and $[0, 2]$. However, its proper definition requires much more machinery than we have right now.

Let us say more about the cardinality of intervals and hence about the cardinality of $\mathbb{R}$. We have seen that there exist irrational numbers, that is $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. The question is: How many irrational numbers are there? It turns out there are a lot more irrational numbers than rational numbers. We have seen that $\mathbb{Q}$ is countable, and we will show that $\mathbb{R}$ is uncountable. In fact, the cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathcal{P}(\mathbb{N})$, although we will not prove this claim here.

$\mathbb{R}$ is uncountable.

We give a modified version of Cantor’s original proof from 1874 as this proof requires the least setup. Normally this proof is stated as a contradiction proof, but a proof by contrapositive is easier to understand.

Let $X \subset \mathbb{R}$ be a countably infinite subset such that for any two real numbers $a < b$, there is an $x \in X$ such that $a < x < b$. Were $\mathbb{R}$ countable, then we could take $X = \mathbb{R}$. If we show that X is necessarily a proper subset, then X cannot equal $\mathbb{R}$, and $\mathbb{R}$ must be uncountable.

As X is countably infinite, there is a bijection from $\mathbb{N}$ to X . Consequently, we write X as a sequence of real numbers $x_1, x_2, x_3, \dots$, such that each number in X is given by x_j for some $j \in \mathbb{N}$.

Let us inductively construct two sequences of real numbers $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$. Let $a_1 := x_1$ and $b_1 := x_1 + 1$. Note that $a_1 < b_1$ and $x_1 \notin (a_1, b_1)$. For $k > 1$, suppose a_{k-1} and b_{k-1} has been defined. Let us also suppose (a_{k-1}, b_{k-1}) does not contain any x_j for any $j = 1, \dots, k-1$.

i. Define $a_k := x_j$, where j is the smallest $j \in \mathbb{N}$ such that $x_j \in (a_{k-1}, b_{k-1})$. Such an x_j exists by our assumption on X .

ii. Next, define $b_k := x_j$ where j is the smallest $j \in \mathbb{N}$ such that $x_j \in (a_k, b_{k-1})$.

Notice that $a_k < b_k$ and $a_{k-1} < a_k < b_k < b_{k-1}$. Also notice that (a_k, b_k) does not contain x_k and hence does not contain any x_j for $j = 1, \dots, k$.

Claim: $a_j < b_k$ for all j and k in $\mathbb{N}$. Let us first assume $j < k$. Then $a_j < a_{j+1} < \dots < a_{k-1} < a_k < b_k$. Similarly for $j > k$. The claim follows.

Let $A = \{a_j : j \in \mathbb{N}\}$ and $B = \{b_j : j \in \mathbb{N}\}$. By and the claim above we have

$$\sup A \leq \inf B.$$

Define $y := \sup A$. The number y cannot be a member of A . If $y = a_j$ for some j , then $y < a_{j+1}$, which is impossible. Similarly y cannot be a member of B . Therefore, $a_j < y$ for all $j \in \mathbb{N}$ and $y < b_j$ for all $j \in \mathbb{N}$. In other words $y \in (a_j, b_j)$ for all $j \in \mathbb{N}$.

Finally we must show that $y \notin X$. If we do so, then we will have constructed a real number not in X showing that X must have been a proper subset. Take any $x_k \in X$. By the above construction $x_k \notin (a_k, b_k)$, so $x_k \neq y$ as $y \in (a_k, b_k)$.

Therefore, the sequence $x_1, x_2, \dots$ cannot contain all elements of $\mathbb{R}$ and thus $\mathbb{R}$ is uncountable.

Exercises

For $a < b$, construct an explicit bijection from $(a, b]$ to $(0, 1]$.

Suppose $f: [0, 1] \rightarrow (0, 1)$ is a bijection. Using f , construct a bijection from $[-1, 1]$ to $\mathbb{R}$.

[exercise:intervaldef] Suppose $I \subset \mathbb{R}$ is a subset with at least 2 elements such that if $a < b < c$ and $a, c \in I$, then it is one of the nine types of intervals explicitly given in this section. Furthermore, prove that the intervals given in this section all satisfy this property.

Construct an explicit bijection from $(0, 1]$ to $(0, 1)$. Hint: One approach is as follows: First map $(\frac{1}{2}, 1]$ to $(0, \frac{1}{2})$, then map $(\frac{1}{4}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{3}{4})$, etc.... Write down the map explicitly, that is, write down an algorithm that tells you exactly what number goes where. Then prove that the map is a bijection.

Construct an explicit bijection from $[0, 1]$ to $(0, 1)$.

a) Show that every closed interval $[a, b]$ is the intersection of countably many open intervals. b) Show that every open interval (a, b) is a countable union of closed intervals. c) Show that an intersection of a possibly infinite family of closed intervals is either empty, a single point, or a closed interval.

Suppose S is a set of disjoint open intervals in $\mathbb{R}$. That is, if $(a, b) \in S$ and $(c, d) \in S$, then either $(a, b) = (c, d)$ or $(a, b) \cap (c, d) = \emptyset$. Prove S is a countable set.

Prove that the cardinality of $[0, 1]$ is the same as the cardinality of $(0, 1)$ by showing that $|(0, 1]| \leq |(0, 1)|$ and $|(0, 1)| \leq |[0, 1]|$. See . Note that this requires the Cantor-Bernstein-Schroeder theorem we stated without proof. Also note that this proof does not give you an explicit bijection.

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A number x is *algebraic* if x is a root of a polynomial with integer coefficients, in other words, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ where all $a_n \in \mathbb{Z}$. a) Show that there are only countably many algebraic numbers. b) Show that there exist non-algebraic numbers (follow in the footsteps of Cantor, use uncountability of $\mathbb{R}$). Hint: Feel free to use the fact that a polynomial of degree n has at most n real roots.

Decimal representation of the reals

Note: 1 lecture (optional)

We often think of real numbers as their *decimal representation*. For a positive integer n , we find the digits $d_K, d_{K-1}, \dots, d_2, d_1, d_0$ for some K , where each d_j is an integer between 0 and 9, then

$$n = d_K 10^K + d_{K-1} 10^{K-1} + \dots + d_2 10^2 + d_1 10 + d_0.$$

We often assume $d_K \neq 0$. To represent n we write the sequence of digits: $n = d_K d_{K-1} \dots d_2 d_1 d_0$. By a (decimal) *digit*, we mean an integer between 0 and 9.

Similarly we represent some rational numbers. That is, for certain numbers x , we can find negative integer $-M$, a positive integer K , and digits $d_K, d_{K-1}, \dots, d_1, d_0, d_{-1}, \dots, d_{-M}$, such that

$$x = d_K 10^K + d_{K-1} 10^{K-1} + \dots + d_2 10^2 + d_1 10 + d_0 + d_{-1} 10^{-1} + d_{-2} 10^{-2} + \dots + d_{-M} 10^{-M}.$$

We write $x = d_K d_{K-1} \dots d_1 d_0 . d_{-1} d_{-2} \dots d_{-M}$.

Not every real number has such a representation, even the simple rational number $\frac{1}{3}$ does not. The irrational number $\sqrt{2}$ does not have such a representation either. To get a representation for all real numbers we must allow infinitely many digits.

Let us from now on consider only real numbers in the interval $(0, 1]$. If we find a representation for these, we simply add integers to them to obtain a representation for all real numbers. Suppose we take an infinite sequence of decimal digits:

$$0.d_1 d_2 d_3 \dots$$

That is, we have a digit d_j for every $j \in \mathbb{N}$. We have renumbered the digits to avoid the negative signs. We say this sequence of digits represents a real number x if

$$x = \sup_{n \in \mathbb{N}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \dots + \frac{d_n}{10^n} \right).$$

We call

$$D_n := \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \dots + \frac{d_n}{10^n}$$

the truncation of x to n decimal digits.

[prop:decimalprop]

- Every infinite sequence of digits $0.d_1 d_2 d_3 \dots$ represents a unique real number $x \in [0, 1]$.
- For every $x \in (0, 1]$ there exists an infinite sequence of digits $0.d_1 d_2 d_3 \dots$ that represents x . There exists a unique representation such that

$$D_n < x \leq D_n + \frac{1}{10^n} \quad \text{for all } n \in \mathbb{N}.$$

Let us start with the first item. Suppose there is an infinite sequence of digits $0.d_1 d_2 d_3 \dots$. We use the geometric sum formula to write $D_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}$.

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$\{1 - \frac{1}{10^n}\} \rightarrow 1$. In particular, $D_n < 1$ for all n . As $D_n \geq 0$ is obvious, we obtain

$$0 \leq \sup_{n \in \mathbb{N}} D_n \leq 1,$$

and therefore $0.d_1 d_2 d_3 \dots$ represents a unique number $x \in [0, 1]$.

We move on to the second item. Take $x \in (0, 1]$. First let us tackle the existence. By convention define $D_0 := 0$, then automatically we obtain $D_0 < x \leq D_0 + 10^{-0}$. Suppose for induction that we defined all the digits $d_1, d_2, \dots, d_n$, and that $D_n < x \leq D_n + 10^{-n}$. We need to define d_{n+1} .

By the of the real numbers we find an integer j such that $x - D_n \leq j 10^{-(n+1)}$. We take the least such j and obtain

$$(j-1)10^{-(n+1)} < x - D_n \leq j 10^{-(n+1)}.$$

Let $d_{n+1} := j$. As $D_n < x$, then $d_{n+1} = j - 1 \geq 0$. On the other hand since $x - D_n \leq 10^{-n}$ we have that j is at most 10, and therefore $d_{n+1} \leq 9$. So d_{n+1} is a decimal digit. Since $D_{n+1} = D_n + d_{n+1} 10^{-(n+1)}$ we add D_n to the inequality [eq:theDnjineq] above:

$$D_{n+1} = D_n + (j-1)10^{-(n+1)} < x \leq$$

And so $D_{n+1} < x \leq D_{n+1} + 10^{-(n+1)}$ holds. We have inductively defined an infinite sequence of digits $0.d_1 d_2 d_3 \dots$. As $D_n < x$ for all n , then $\sup \{D_n : n \in \mathbb{N}\} \leq x$. As $x - 10^{-n} \leq D_n$, then $x - 10^{-n} \leq \sup \{D_m : m \in \mathbb{N}\}$ for all n . The two inequalities together imply $\sup \{D_n : n \in \mathbb{N}\} = x$.

What is left to show is the uniqueness. Suppose $0.e_1 e_2 e_3 \dots$ is another representation of x . Let E_n be the n -digit truncation of $0.e_1 e_2 e_3 \dots$, and suppose $E_n < x \leq E_n + 10^{-n}$ for all $n \in \mathbb{N}$. Suppose for some $K \in \mathbb{N}$, $e_n = d_n$ for all $n < K$, so $D_{K-1} = E_{K-1}$. Then

$$E_K = D_{K-1} + e_K 10^{-K} < x \leq E_K + 10^{-K} = D_{K-1} + e_K 10^{-K} + 10^{-K}.$$

Subtracting D_{K-1} and multiplying by 10^K we get

$$e_K < (x - D_{K-1})10^K \leq e_K + 1.$$

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Similarly we obtain

$$d_K < (x - D_{K-1})10^K \leq d_K + 1.$$

Hence, both e_K and d_K are the largest integer j such that $j < (x - D_{K-1})10^K$, and therefore $e_K = d_K$. That is, the representation is unique.

The representation is not unique if we do not require the extra condition in the proposition. For example, for the number $\frac{1}{2}$ the method in the proof obtains the representation

$$0.49999\dots$$

However, we also have the representation $0.5000\dots$. The key requirement that makes the representation unique is $D_n < x$ for all n . The inequality $x \leq D_n + 10^{-n}$ is true for every representation by the computation in the beginning of the proof.

The only numbers that have nonunique representations are ones that end either in an infinite sequence of 0s or 9s, because the only representation for which $D_n = x$ is one where all digits past n th one are zero. In this case there are exactly two representations of x (see the exercises).

Let us give another proof of the uncountability of the reals using decimal representations. This is Cantor's second proof, and is probably more well known. While this proof may seem shorter, it is because we have already done the hard part above and we are left with a slick trick to prove that $\mathbb{R}$ is uncountable. This trick is called *Cantor diagonalization* and finds use in other proofs as well.

The set $(0, 1]$ is uncountable.

Let $X := \{x_1, x_2, x_3, \dots\}$ be any countable subset of real numbers in $(0, 1]$. We will construct a real number not in X . Let

$$x_n = 0.d_1^n d_2^n d_3^n \dots$$

be the unique representation from the proposition, that is d_j^n is the j th digit of the n th number. Let $e_n := 1$ if $d_n^n \neq 1$, and let $e_n := 2$ if $d_n^n = 1$. Let E_n be the n -digit truncation of $y = 0.e_1 e_2 e_3 \dots$. Because all the digits are nonzero we get that $E_n < E_{n+1} \leq y$. Therefore

$$E_n < y \leq E_n + 10^{-n}$$

for all n , and the representation is the unique one for y from the proposition. But for every n , the n th digit of y is different from the n th digit of x_n , so $y \neq x_n$. Therefore $y \notin X$, and as X was an arbitrary countable subset, $(0, 1]$ must be uncountable.

Using decimal digits we can also find lots of numbers that are not rational. The following proposition is true for every rational number, but we give it only for $x \in (0, 1]$ for simplicity.

[prop:rationaldecimal] If $x \in (0, 1]$ is a rational number and $x = 0.d_1 d_2 d_3 \dots$, then the decimal digits eventually start repeating. That is, there are positive integers N and P , such that for all $n \geq N$, $d_n = d_{n+P}$.

Let $x = \frac{p}{q}$ for positive integers p and q . Let us suppose x is a number with a unique representation, as otherwise we have seen above that both its representations are repeating.

To compute the first digit we take $10p$ and divide by q . The quotient is the first digit d_1 and the remainder r is some integer between 0 and $q - 1$. That is, d_1 is the largest integer such that $d_1 q \leq 10p$ and then $r = 10p - d_1 q$.

The next digit is computed by dividing $10r$ by q , and so on. We notice that at each step there are at most q possible remainders and hence at some point the process must start repeating. In fact we see that P is at most q .

The converse of the proposition is also true and is left as an exercise.

The number

$$x = 0.101001000100001000001\dots,$$

is irrational. That is, the digits are n zeros, then a one, then $n + 1$ zeros, then a one, and so on and so forth. The fact that x is irrational follows from the proposition; the digits never start repeating. For every P , if we go far enough, we find a 1 that is followed by at least $P + 1$ zeros.

Exercises

What is the decimal representation of $\frac{1}{2}$ guaranteed by ? Make sure to show that it does satisfy the condition.

Prove the converse of , that is, if the digits in the decimal representation of x are eventually repeating, then x must be rational.

Show that real numbers $x \in (0, 1)$ with nonunique decimal representation are exactly the rational numbers that can be written as $\frac{m}{10^n}$ for some integers m and n . In this case show that there exist exactly two representations of x .

Let $b \geq 2$ be an integer. Define a representation of a real number in $[0, 1]$ in terms of base b rather than base 10 and prove for base b .

Using the previous exercise with $b = 2$ (binary), show that cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathbb{P}(\mathbb{N})$, obtaining yet another (though related) proof that $\mathbb{R}$ is uncountable. Hint: Construct two injections, one from $[0, 1]$ to $\mathbb{P}(\mathbb{N})$ and one from $\mathbb{P}(\mathbb{N})$ to $[0, 1]$. Hint 2: Given a set $A \subset \mathbb{N}$, let the n th binary digit of x be 1 if $n \in A$.

Construct a bijection between $[0, 1]$ and $[0, 1] \times [0, 1]$. Hint: consider even and odd digits, and be careful about the uniqueness of representation.

Sequences and Series

Sequences and limits

Note: 2.5 lectures

Analysis is essentially about taking limits. The most basic type of a limit is a limit of a sequence of real numbers. We have already seen sequences used informally. Let us give the formal definition.

A *sequence* (of real numbers) is a function $x: \mathbb{N} \rightarrow \mathbb{R}$. Instead of $x(n)$ we usually denote the n th element in the sequence by x_n . We use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty},$$

to denote a sequence.

A sequence $\{x_n\}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that

$$|x_n| \leq B \quad \text{for all } n \in \mathbb{N}.$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n: n \in \mathbb{N}\}$ is bounded.

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When we need to give a concrete sequence we often give each term as a formula in terms of n . For example, $\{\frac{1}{n}\}_{n=1}^{\infty}$, or simply $\{\frac{1}{n}\}$, stands for the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$. The sequence $\{\frac{1}{n}\}$ is a bounded sequence ($B = 1$ will suffice). On the other hand the sequence $\{n\}$ stands for $1, 2, 3, 4, \dots$, and this sequence is not bounded (why?).

While the notation for a sequence is similar⁷ to that of a set, the notions are distinct. For example, the sequence $\{(-1)^n\}$ is the sequence $-1, 1, -1, 1, -1, 1, \dots$, whereas the set of values, the *range of the sequence*, is just the set $\{-1, 1\}$. We can write this set as $\{(-1)^n : n \in \mathbb{N}\}$. When ambiguity can arise, we use the words *sequence* or *set* to distinguish the two concepts.

Another example of a sequence is the so-called *constant sequence*. That is a sequence $\{c\} = c, c, c, \dots$ consisting of a single constant $c \in \mathbb{R}$ repeating indefinitely.

We now get to the idea of a *limit of a sequence*. We will see in that the notation below is well defined. That is, if a limit exists, then it is unique. So it makes sense to talk about *the* limit of a sequence.

A sequence $\{x_n\}$ is said to *converge* to a number $x \in \mathbb{R}$, if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq M$. The number x is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n \rightarrow \infty} x_n = x.$$

A sequence that converges is said to be *convergent*. Otherwise, the sequence is said to be *divergent*.

It is good to know intuitively what a limit means. It means that eventually every number in the sequence is close to the number x . More precisely, we can get arbitrarily close to the limit, provided we go far enough in the sequence. It does not mean we ever reach the limit. It is possible, and quite common, that there is no x_n in the sequence that equals the limit x . We illustrate the concept in . In the figure we first think of the sequence as a graph, as it is a function of $\mathbb{N}$. Secondly we also plot it as a sequence of labeled points on the real line.

When we write $\lim_{n \rightarrow \infty} x_n = x$ for some real number x , we are saying two things. First, that $\{x_n\}$ is convergent, and second that the limit is x .

The above definition is one of the most important definitions in analysis, and it is necessary to understand it perfectly. The key point in the definition is that given any $\epsilon > 0$, we can find an M . The M can depend on ϵ , so we only pick an M once we know ϵ . Let us illustrate this concept on a few examples.

The constant sequence $1, 1, 1, \dots$ is convergent and the limit is 1 . For every $\epsilon > 0$, we pick $M = 1$.

Claim: The sequence $\{\frac{1}{n}\}$ is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Proof: Given an $\epsilon > 0$, we find an $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \epsilon$ (at work). Then for all $n \geq M$ we have that

$$|x_n - 0| = \left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{M} < \epsilon.$$

The sequence $\{(-1)^n\}$ is divergent. Proof: If there were a limit x , then for $\epsilon = \frac{1}{2}$ we expect an M that satisfies the definition. Suppose such an M exists, then for an even $n \geq M$ we compute

$$\frac{1}{2} > |x_n - x| = |1 - x| \quad \text{and} \quad \frac{1}{2} > |x_{n+1} - x| = |-1 - x|.$$

But

$$2 = |1 - x - (-1 - x)| \leq |1 - x| + |-1 - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

and that is a contradiction.

[prop:limisunique] A convergent sequence has a unique limit.

The proof of this proposition exhibits a useful technique in analysis. Many proofs follow the same general scheme. We want to show a certain quantity is zero. We write the quantity using the triangle inequality as two quantities, and we estimate each one by arbitrarily small numbers.

Suppose the sequence $\{x_n\}$ has the limit x and the limit y . Take an arbitrary $\epsilon > 0$. From the definition find an M_1 such that for all $n \geq M_1$, $|x_n - x| < \frac{\epsilon}{2}$. Similarly find an M_2 such that for all $n \geq M_2$ we have $|x_n - y| < \frac{\epsilon}{2}$. Take $M := \max\{M_1, M_2\}$. For $n \geq M$ (so that both $n \geq M_1$ and $n \geq M_2$) we have

$$\begin{aligned} |y - x| &= |x_n - x - (x_n - y)| \\ &\leq |x_n - x| + |x_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As $|y - x| < \epsilon$ for all $\epsilon > 0$, then $|y - x| = 0$ and $y = x$. Hence the limit (if it exists) is unique.

A convergent sequence $\{x_n\}$ is bounded.

Suppose $\{x_n\}$ converges to x . Thus there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $|x_n - x| < 1$. Let $B_1 := |x| + 1$ and note that for $n \geq M$ we have

$$\begin{aligned} |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x| = B_1. \end{aligned}$$

The set $\{|x_1|, |x_2|, \dots, |x_{M-1}|\}$ is a finite set and hence let

$$B_2 := \max\{|x_1|, |x_2|, \dots, |x_{M-1}|\}.$$

Let $B := \max\{B_1, B_2\}$. Then for all $n \in \mathbb{N}$ we have

$$|x_n| \leq B. \quad \text{qedhere}$$

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The sequence $\{(-1)^n\}$ shows that the converse does not hold. A bounded sequence is not necessarily convergent.

Let us show the sequence $\left\{\frac{n^2+1}{n^2+n}\right\}$ converges and

$$\lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+n} = 1.$$

Given $\epsilon > 0$, find $M \in \mathbb{N}$ such that $\frac{1}{M+1} < \epsilon$. Then for any $n \geq M$ we have

$$\left| \frac{n^2+1}{n^2+n} - 1 \right| = \left| \frac{n^2+1 - (n^2+n)}{n^2+n} \right| = \left| \frac{-n+1}{n^2+n} \right| \leq \frac{n}{n^2+n} = \frac{1}{n+1} < \frac{1}{M+1} < \epsilon.$$

Therefore, $\lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+n} = 1$.

Monotone sequences

The simplest type of a sequence is a monotone sequence. Checking that a monotone sequence converges is as easy as checking that it is bounded. It is also easy to find the limit for a convergent monotone sequence, provided we can find the supremum or infimum of a countable set of numbers.

A sequence $\{x_n\}$ is *monotone increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}$ is *monotone decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If a sequence is either monotone increasing or monotone decreasing, we can simply say the sequence is *monotone*. Some authors also use the word *monotonic*.

For example, $\{\frac{1}{n}\}$ is monotone decreasing, the constant sequence $\{1\}$ is both monotone increasing and monotone decreasing, and $\{(-1)^n\}$ is not monotone. First few terms of a sample monotone increasing sequence are shown in .

[thm:monotoneconv] A monotone sequence $\{x_n\}$ is bounded if and only if it is convergent.

Furthermore, if $\{x_n\}$ is monotone increasing and bounded, then

$$\lim_{n \rightarrow \infty} x_n = \sup \{x_n : n \in \mathbb{N}\}.$$

If $\{x_n\}$ is monotone decreasing and bounded, then

$$\lim_{n \rightarrow \infty} x_n = \inf \{x_n : n \in \mathbb{N}\}.$$

Let us suppose the sequence is monotone increasing. Suppose the sequence is bounded, so there exists a B such that $x_n \leq B$ for all n , that is the set $\{x_n : n \in \mathbb{N}\}$ is bounded from above. Let

$$x := \sup \{x_n : n \in \mathbb{N}\}.$$

Let $\epsilon > 0$ be arbitrary. As x is the supremum, then there must be at least one $M \in \mathbb{N}$ such that $x_M > x - \epsilon$ (because x is the supremum). As $\{x_n\}$ is monotone increasing, then it is easy to see (by) that $x_n \geq x_M$ for all $n \geq M$. Hence

$$|x_n - x| = x - x_n \leq x - x_M < \epsilon.$$

Therefore the sequence converges to x . We already know that a convergent sequence is bounded, which completes the other direction of the implication.

The proof for monotone decreasing sequences is left as an exercise.

Take the sequence $\{\frac{1}{\sqrt{n}}\}$.

First $\frac{1}{\sqrt{n}} > 0$ for all $n \in \mathbb{N}$, and hence the sequence is bounded from below. Let us show that it is monotone decreasing. We start with $\sqrt{n+1} \geq \sqrt{n}$ (why is that true?). From this inequality we obtain

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}.$$

So the sequence is monotone decreasing and bounded from below (hence bounded). We apply the theorem to note that the sequence is convergent and in fact

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = \inf \left\{ \frac{1}{\sqrt{n}} : n \in \mathbb{N} \right\}.$$

We already know that the infimum is greater than or equal to 0, as 0 is a lower bound. Take a number $b \geq 0$ such that $b \leq \frac{1}{\sqrt{n}}$ for all n . We square both sides to obtain

$$b^2 \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

We have seen before that this implies that $b^2 \leq 0$ (a consequence of the). As we also have $b^2 \geq 0$, then $b^2 = 0$ and so $b = 0$. Hence $b = 0$ is the greatest lower bound, and $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$.

A word of caution: We must show that a monotone sequence is bounded in order to use . For example, the sequence $\{1 + \frac{1}{n^2} + \dots + \frac{1}{n^2}\}$ is a monotone increasing sequence that grows very slowly. We will see, once we get to series, that this sequence has no upper bound and so does not converge. It is not at all obvious that this sequence has no upper bound.

A common example of where monotone sequences arise is the following proposition. The proof is left as an exercise.

[prop:supinfseq] Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in S$ and

$$\sup S = \lim_{n \rightarrow \infty} x_n \quad \text{and} \quad \inf S = \lim_{n \rightarrow \infty} y_n.$$

Tail of a sequence

For a sequence $\{x_n\}$, the *K-tail* (where $K \in \mathbb{N}$) or just the *tail* of the sequence is the sequence starting at $K + 1$, usually written as

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$$\{x_{n+K}\}_{n=1}^{\infty} \quad \text{or} \quad \{x_n\}_{n=K+1}^{\infty}.$$

The main result about the tail of a sequence is the following proposition.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then the following statements are equivalent:

- The sequence $\{x_n\}_{n=1}^{\infty}$ converges.
- The K -tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.
- The K -tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

Furthermore, if any (and hence all) of the limits exist, then for any $K \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+K}.$$

It is clear that [\[prop:ktail:ii\]](#) implies [\[prop:ktail:iii\]](#). We will therefore show first that [\[prop:ktail:i\]](#) implies [\[prop:ktail:ii\]](#), and then we will show that [\[prop:ktail:iii\]](#) implies [\[prop:ktail:i\]](#). In the process we will also show that the limits are equal.

Let us start with [\[prop:ktail:i\]](#) implies [\[prop:ktail:ii\]](#). Suppose $\{x_n\}$ converges to some $x \in \mathbb{R}$. Let $K \in \mathbb{N}$ be arbitrary. Define $y_n := x_{n+K}$, we wish to show that $\{y_n\}$ converges to x . That is, given an $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x - x_n| < \epsilon$ for all $n \geq M$. Note that $n \geq M$ implies $n + K \geq M$. Therefore, it is true that for all $n \geq M$ we have that

$$|x - y_n| = |x - x_{n+K}| < \epsilon.$$

Therefore $\{y_n\}$ converges to x .

Let us move to [\[prop:ktail:iii\]](#) implies [\[prop:ktail:i\]](#). Let $K \in \mathbb{N}$ be given, define $y_n := x_{n+K}$, and suppose that $\{y_n\}$ converges to $x \in \mathbb{R}$. That is, given an $\epsilon > 0$, there exists an $M' \in \mathbb{N}$ such that $|x - y_n| < \epsilon$ for all $n \geq M'$. Let $M := M' + K$. Then $n \geq M$ implies $n - K \geq M'$. Thus, whenever $n \geq M$ we have

$$|x - x_n| = |x - y_{n-K}| < \epsilon.$$

Therefore $\{x_n\}$ converges to x .

Essentially, the limit does not care about how the sequence begins, it only cares about the tail of the sequence. That is, the beginning of the sequence may be arbitrary.

For example, the sequence defined by $x_n := \frac{n}{n^2 + 16}$ is decreasing if we start at $n = 4$ (it is increasing before). That is, $\{x_n\} = \frac{1}{17}, \frac{1}{10}, \frac{1}{25}, \frac{1}{18}, \frac{1}{54}, \frac{1}{32}, \frac{1}{76}, \frac{1}{110}, \frac{1}{99}, \frac{1}{58}, \dots$, and

$$\frac{1}{17} < \frac{1}{10} < \frac{1}{25} < \frac{1}{18} > \frac{1}{54} > \frac{1}{32} > \frac{1}{76} > \frac{1}{110} > \frac{1}{99} > \frac{1}{58} > \dots$$

That is if we throw away the first 3 terms and look at the 3 tail it is decreasing. The proof is left as an exercise. Since the 3-tail is monotone and bounded below by zero, it is convergent, and therefore the sequence is convergent.

Subsequences

A very useful concept related to sequences is that of a subsequence. A subsequence of $\{x_n\}$ is a sequence that contains only some of the numbers from $\{x_n\}$ in the same order.

Let $\{x_n\}$ be a sequence. Let $\{n_i\}$ be a strictly increasing sequence of natural numbers (that is $n_1 < n_2 < n_3 < \dots$). The sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a *subsequence* of $\{x_n\}$.

For example, take the sequence $\{\frac{1}{n}\}$. The sequence $\{\frac{1}{3n}\}$ is a subsequence. To see how these two sequences fit in the definition, take $n_i := 3i$. The numbers in the subsequence must come from the original sequence, so $1, 0, \frac{1}{3}, 0, \frac{1}{15}, \dots$ is not a subsequence of $\{\frac{1}{n}\}$. Similarly order must be preserved, so the sequence $1, \frac{1}{13}, \frac{1}{12}, \frac{1}{15}, \dots$ is not a subsequence of $\{\frac{1}{n}\}$.

A tail of a sequence is one special type of a subsequence. For an arbitrary subsequence, we have the following proposition about convergence.

[\[prop:seqtosubseq\]](#) If $\{x_n\}$ is a convergent sequence, then any subsequence $\{x_{n_i}\}$ is also convergent and

$$\lim_{n \rightarrow \infty} x_n = \lim_{i \rightarrow \infty} x_{n_i}.$$

Suppose $\lim_{n \rightarrow \infty} x_n = x$. That means that for every $\epsilon > 0$ we have an $M \in \mathbb{N}$ such that for all $n \geq M$

$$|x_n - x| < \epsilon.$$

It is not hard to prove (do it!) by that $n_i \geq i$. Hence $i \geq M$ implies $n_i \geq M$. Thus, for all $i \geq M$ we have

$$|x_{n_i} - x| < \epsilon,$$

and we are done.

Existence of a convergent subsequence does not imply convergence of the sequence itself. Take the sequence $0, 1, 0, 1, 0, 1, \dots$. That is, $x_n = 0$ if n is odd, and $x_n = 1$ if n is even. The sequence $\{x_n\}$ is divergent, however, the subsequence $\{x_{2n}\}$ converges to 1 and the subsequence $\{x_{2n+1}\}$ converges to 0. Compare .

Exercises

In the following exercises, feel free to use what you know from calculus to find the limit, if it exists. But you must *prove* that you found the correct limit, or prove that the series is divergent.

Is the sequence $\{3n\}$ bounded? Prove or disprove.

Is the sequence $\{n\}$ convergent? If so, what is the limit?

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Is the sequence $\left\{ \frac{(-1)^n}{2n} \right\}$ convergent? If so, what is the limit?

Is the sequence $\{2^{-n}\}$ convergent? If so, what is the limit?

Is the sequence $\left\{ \frac{n}{n+1} \right\}$ convergent? If so, what is the limit?

Is the sequence $\left\{ \frac{n}{n^2+1} \right\}$ convergent? If so, what is the limit?

[exercise:absconv] Let $\{x_n\}$ be a sequence.

- Show that $\lim x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim |x_n| = 0$.
- Find an example such that $\{|x_n|\}$ converges and $\{x_n\}$ diverges.

Is the sequence $\left\{ \frac{2^n}{n!} \right\}$ convergent? If so, what is the limit?

Show that the sequence $\left\{ \frac{1}{\sqrt[n]{n}} \right\}$ is monotone, bounded, and use to find the limit.

Show that the sequence $\left\{ \frac{n+1}{n} \right\}$ is monotone, bounded, and use to find the limit.

Finish the proof of for monotone decreasing sequences.

Prove .

Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

Find a convergent subsequence of the sequence $\{(-1)^n\}$.

Let $\{x_n\}$ be a sequence defined by

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ \lfloor n/2 \rfloor & \text{if } n \text{ is even.} \end{cases}$$

- Is the sequence bounded? (prove or disprove)
- Is there a convergent subsequence? If so, find it.

Let $\{x_n\}$ be a sequence. Suppose there are two convergent subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$. Suppose

$$\lim_{i \rightarrow \infty} x_{n_i} = a \quad \text{and} \quad \lim_{i \rightarrow \infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}$ is not convergent, without using .

Find a sequence $\{x_n\}$ such that for any $y \in \mathbb{R}$, there exists a subsequence $\{x_{n_i}\}$ converging to y .

Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$. Suppose for any $\epsilon > 0$, there is an M such that for all $n \geq M$, $|x_n - x| \leq \epsilon$. Show that $\lim x_n = x$.

Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$ such that there exists a $k \in \mathbb{N}$ such that for all $n \geq k$, $x_n = x$. Prove that $\{x_n\}$ converges to x .

Let $\{x_n\}$ be a sequence and define a sequence $\{y_n\}$ by $y_{2k} := x_{k^2}$ and $y_{2k-1} = x_k$ for all $k \in \mathbb{N}$. Prove that $\{x_n\}$ converges if and only if $\{y_n\}$ converges. Furthermore, prove that if they converge, then $\lim x_n = \lim y_n$.

Show that the 3-tail of the sequence defined by $x_n := \frac{n}{n^2+16}$ is monotone decreasing. Hint: Suppose $n \geq m \geq 4$ and consider the numerator of the expression $x_n - x_m$.

Suppose that $\{x_n\}$ is a sequence such that the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$, and $\{x_{3n}\}$ all converge. Show that $\{x_n\}$ is convergent.

Facts about limits of sequences

Note: 2–2.5 lectures, recursively defined sequences can safely be skipped

In this section we go over some basic results about the limits of sequences. We start by looking at how sequences interact with inequalities.

Limits and inequalities

A basic lemma about limits and inequalities is the so-called squeeze lemma. It allows us to show convergence of sequences in difficult cases if we find two other simpler convergent sequences that “squeeze” the original sequence.

[squeeze:lemma] Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that

$$a_n \leq x_n \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

Suppose $\{a_n\}$ and $\{b_n\}$ converge and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

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Then $\{x_n\}$ converges and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

The intuitive idea of the proof is illustrated in . If x is the limit of a_n and b_n , then if they are both within $\frac{\epsilon}{3}$ of x , then the distance between a_n and b_n is at most $\frac{2\epsilon}{3}$. As x_n is between a_n and b_n it is at most $\frac{2\epsilon}{3}$ from a_n . Since a_n is at most $\frac{\epsilon}{3}$ away from x , then x_n must be at most ϵ away from x . Let us follow through on this intuition rigorously.

Let $x := \lim a_n = \lim b_n$. Let $\epsilon > 0$ be given.

Find an M_1 such that for all $n \geq M_1$ we have that $|a_n - x| < \frac{\epsilon}{3}$, and an M_2 such that for all $n \geq M_2$ we have $|b_n - x| < \frac{\epsilon}{3}$. Set $M := \max\{M_1, M_2\}$. Suppose $n \geq M$. We compute

$$\begin{aligned} |x_n - a_n| &= x_n - a_n \leq b_n - a_n \\ &= |b_n - x + x - a_n| \\ &\leq |b_n - x| + |x - a_n| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}. \end{aligned}$$

Armed with this information we estimate

$$\begin{aligned} |x_n - x| &= |x_n - x + a_n - a_n| \\ &\leq |x_n - a_n| + |a_n - x| \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

And we are done.

One application of the is to compute limits of sequences using limits that are already known. For example, suppose we have the sequence $\{\frac{1}{n\sqrt{n}}\}$. Since $\sqrt{n} \geq 1$ for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. We already know $\lim \frac{1}{n} = 0$. Hence, using the constant sequence $\{0\}$ and the sequence $\{\frac{1}{n\sqrt{n}}\}$ in the squeeze lemma, we conclude

$$\lim_{n \rightarrow \infty} \frac{1}{n\sqrt{n}} = 0.$$

Limits also preserve inequalities.

[limandineq:lemma] Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences and

$$x_n \leq y_n,$$

for all $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n.$$

Let $x := \lim x_n$ and $y := \lim y_n$. Let $\epsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$ we have $|x_n - x| < \frac{\epsilon}{2}$. Find an M_2 such that for all $n \geq M_2$ we have $|y_n - y| < \frac{\epsilon}{2}$. In particular, for some $n \geq \max\{M_1, M_2\}$ we have $x - x_n < \frac{\epsilon}{2}$ and $y_n - y < \frac{\epsilon}{2}$. We add these inequalities to obtain

$$y_n - x_n + x - y < \epsilon, \quad \text{or} \quad y_n - x_n < y - x + \epsilon.$$

Since $x_n \leq y_n$ we have $0 \leq y_n - x_n$ and hence $0 < y - x + \epsilon$. In other words

$$x - y < \epsilon.$$

Because $\epsilon > 0$ was arbitrary we obtain $x - y \leq 0$, as we have seen that a nonnegative number less than any positive ϵ is zero. Therefore $x \leq y$.

An easy corollary is proved using constant sequences in . The proof is left as an exercise.

[limandineq:cor]

i. Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$, then

$$\lim_{n \rightarrow \infty} x_n \geq 0.$$

ii. Let $a, b \in \mathbb{R}$ and let $\{x_n\}$ be a convergent sequence such that

$$a \leq x_n \leq b,$$

for all $n \in \mathbb{N}$. Then

$$a \leq \lim_{n \rightarrow \infty} x_n \leq b.$$

In and we cannot simply replace all the non-strict inequalities with strict inequalities. For example, let $x_n := \frac{1}{n}$ and $y_n := \frac{1}{n\sqrt{n}}$. Then $x_n < y_n$, $x_n < 0$, and $y_n > 0$ for all n . However, these inequalities are not preserved by the limit operation as we have $\lim x_n = \lim y_n = 0$. The moral of this example is that strict inequalities may become non-strict inequalities when limits are applied; if we know $x_n < y_n$ for all n , we may only conclude

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$$\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n.$$

This issue is a common source of errors.

[Continuity of algebraic operations](#)

Limits interact nicely with algebraic operations.

[prop:contalg] Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.

i. [prop:contalg:i] The sequence $\{z_n\}$, where $z_n := x_n + y_n$, converges and

$$\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n.$$

ii. [prop:contalg:ii] The sequence $\{z_n\}$, where $z_n := x_n - y_n$, converges and

$$\lim_{n \rightarrow \infty} (x_n - y_n) = \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} x_n - \lim_{n \rightarrow \infty} y_n.$$

iii. [prop:contalg:iii] The sequence $\{z_n\}$, where $z_n := x_n y_n$, converges and

$$\lim_{n \rightarrow \infty} (x_n y_n) = \lim_{n \rightarrow \infty} z_n = \left(\lim_{n \rightarrow \infty} x_n \right) \left(\lim_{n \rightarrow \infty} y_n \right).$$

iv. [prop:contalg:iv] If $\lim_{n \rightarrow \infty} y_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{z_n\}$, where $z_n := \frac{x_n}{y_n}$, converges and

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} z_n =$$

Let us start with [prop:contalg:i]. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n + y_n$. Let $x := \lim_{n \rightarrow \infty} x_n$, $y := \lim_{n \rightarrow \infty} y_n$, and $z := x + y$.

Let $\epsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$ we have $|x_n - x| < \frac{\epsilon}{2}$. Find an M_2 such that for all $n \geq M_2$ we have $|y_n - y| < \frac{\epsilon}{2}$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$ we have

$$\begin{aligned} |z_n - z| &= |(x_n + y_n) - (x + y)| = |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore [prop:contalg:i] is proved. Proof of [prop:contalg:ii] is almost identical and is left as an exercise.

Let us tackle [prop:contalg:iii]. Suppose again that $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n y_n$. Let $x := \lim_{n \rightarrow \infty} x_n$, $y := \lim_{n \rightarrow \infty} y_n$, and $z := xy$.

Let $\epsilon > 0$ be given. As $\{x_n\}$ is convergent, it is bounded. Therefore, find a $B > 0$ such that $|x_n| \leq B$ for all $n \in \mathbb{N}$. Find an M_1 such that for all $n \geq M_1$ we have $|x_n - x| < \frac{\epsilon}{2(|y| + 1)}$. Find an M_2 such that for all $n \geq M_2$ we have $|y_n - y| < \frac{\epsilon}{2B}$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$ we have

$$\begin{aligned} |z_n - z| &= |(x_n y_n) - (xy)| \\ &= |x_n y_n - (x + x_n - x_n)y| \\ &= |x_n(y_n - y) + (x_n - x)y| \\ &\leq |x_n(y_n - y)| + |(x_n - x)y| \\ &= |x_n| |y_n - y| + |x_n - x| |y| \\ &\leq B |y_n - y| + |x_n - x| |y| \\ &< B \frac{\epsilon}{2B} + \frac{\epsilon}{2(|y| + 1)} |y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Finally let us tackle [prop:contalg:iv]. Instead of proving [prop:contalg:iv] directly, we prove the following simpler claim:

Claim: If $\{y_n\}$ is a convergent sequence such that $\lim_{n \rightarrow \infty} y_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{y_n} = \frac{1}{\lim_{n \rightarrow \infty} y_n}.$$

Once the claim is proved, we take the sequence $\{\frac{1}{y_n}\}$, multiply it by the sequence $\{x_n\}$ and apply item [prop:contalg:iii].

Proof of claim: Let $\epsilon > 0$ be given. Let $y := \lim_{n \rightarrow \infty} y_n$. Find an M such that for all $n \geq M$ we have

$$|y_n - y| < \min \left\{ |y|^2 \frac{\epsilon}{2}, \frac{|y|}{2} \right\}.$$

Notice that we can make this claim as the right hand side is positive because $|y| \neq 0$. Therefore for all $n \geq M$ we have $|y - y_n| < \frac{|y|}{2}$, and so

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$$|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \frac{|y|}{2} + |y_n|.$$

Subtracting $\frac{|y|}{2}$ from both sides we obtain $\frac{|y|}{2} < |y_n|$, or in other words,

$$\frac{1}{|y_n|} < \frac{2}{|y|}.$$

Now we finish the proof of the claim:

$$\begin{aligned} \left| \frac{1}{y_n} - \frac{1}{y} \right| &= \left| \frac{y - y_n}{yy_n} \right| \\ &= \frac{|y - y_n|}{|y||y_n|} \\ &< \frac{|y - y_n|}{|y|} \frac{2}{|y|} \\ &< \frac{|y|^2 \frac{\epsilon}{2}}{|y|} \frac{2}{|y|} = \epsilon. \end{aligned}$$

And we are done.

By plugging in constant sequences, we get several easy corollaries. If $c \in \mathbb{R}$ and $\{x_n\}$ is a convergent sequence, then for example

$$\lim_{n \rightarrow \infty} cx_n = c \left(\lim_{n \rightarrow \infty} x_n \right) \quad \text{and} \quad \lim_{n \rightarrow \infty} (c + x_n) = c + \lim_{n \rightarrow \infty} x_n.$$

Similarly with constant subtraction and division.

As we can take limits past multiplication we can show (exercise) that $\lim_{n \rightarrow \infty} x_n^k = (\lim_{n \rightarrow \infty} x_n)^k$ for all $k \in \mathbb{N}$. That is, we can take limits past powers. Let us see if we can do the same with roots.

Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$. Then

$$\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\lim_{n \rightarrow \infty} x_n}.$$

Of course to even make this statement, we need to apply to show that $\lim_{n \rightarrow \infty} x_n \geq 0$, so that we can take the square root without worry.

Let $\{x_n\}$ be a convergent sequence and let $x := \lim_{n \rightarrow \infty} x_n$.

First suppose $x = 0$. Let $\epsilon > 0$ be given. Then there is an M such that for all $n \geq M$ we have $x_n = |x_n| < \epsilon^2$, or in other words $\sqrt{x_n} < \epsilon$. Hence

$$|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} < \epsilon.$$

Now suppose $x > 0$ (and hence $\sqrt{x} > 0$).

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{1}{\sqrt{x_n} + \sqrt{x}} |x_n - x| \\ &\leq \frac{1}{\sqrt{x}} |x_n - x|. \end{aligned}$$

We leave the rest of the proof to the reader.

A similar proof works for the k th root. That is, we also obtain $\lim_{n \rightarrow \infty} x_n^{1/k} = (\lim_{n \rightarrow \infty} x_n)^{1/k}$. We leave this to the reader as a challenging exercise.

We may also want to take the limit past the absolute value sign. The converse of this proposition is not true, see part b).

If $\{x_n\}$ is a convergent sequence, then $\{|x_n|\}$ is convergent and

$$\lim_{n \rightarrow \infty} |x_n| = \left| \lim_{n \rightarrow \infty} x_n \right|.$$

We simply note the reverse triangle inequality

$$||x_n| - |x|| \leq |x_n - x|.$$

Hence if $|x_n - x|$ can be made arbitrarily small, so can $||x_n| - |x||$. Details are left to the reader.

Let us see an example putting the above propositions together. Since we know that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, then

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$$\lim_{n \rightarrow \infty} \left| \sqrt{1 + \frac{1}{n}} - \frac{1}{\sqrt{1 + \frac{1}{n}}} \right| = \left| \sqrt{1 + \lim_{n \rightarrow \infty} \frac{1}{n}} - \frac{1}{\sqrt{1 + \lim_{n \rightarrow \infty} \frac{1}{n}}} \right| = 1.$$

That is, the limit on the left hand side exists because the right hand side exists. You really should read the above equality from right to left.

Recursively defined sequences

Now that we know we can interchange limits and algebraic operations, we can compute the limits of many sequences. One such class are recursively defined sequences, that is, sequences where the next number in the sequence computed using a formula from a fixed number of preceding elements in the sequence.

Let $\{x_n\}$ be defined by $x_1 := 2$ and

$$x_{n+1} := x_n - \frac{x_n^2 - 2}{2x_n}.$$

We must first find out if this sequence is well defined; we must show we never divide by zero. Then we must find out if the sequence converges. Only then can we attempt to find the limit.

First let us prove x_n exists and $x_n > 0$ for all n (so the sequence is well defined and bounded below). Let us show this by . We know that $x_1 = 2 > 0$. For the induction step, suppose $x_n > 0$. Then

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

If $x_n > 0$, then $x_n^2 + 2 > 0$ and hence $x_{n+1} > 0$.

Next let us show that the sequence is monotone decreasing. If we show that $x_n^2 - 2 \geq 0$ for all n , then $x_{n+1} \leq x_n$ for all n . Obviously $x_1^2 - 2 = 4 - 2 = 2 > 0$. For an arbitrary n we have

$$x_{n+1}^2 - 2 = \left(\frac{x_n^2 + 2}{2x_n} \right)^2 - 2 = \frac{x_n^4 + 4x_n^2 + 4 - 8x_n^2}{4x_n^2} = \frac{x_n^4 - 4x_n^2 + 4}{4x_n^2} = \frac{(x_n^2 - 2)^2}{4x_n^2}.$$

Since any number squared is nonnegative, we have that $x_{n+1}^2 - 2 \geq 0$ for all n . Therefore, $\{x_n\}$ is monotone decreasing and bounded ($x_n > 0$ for all n), and the limit exists. It remains to find the limit.

Let us write

$$2x_n x_{n+1} = x_n^2 + 2.$$

Since $\{x_{n+1}\}$ is the 1-tail of $\{x_n\}$, it converges to the same limit. Let us define $x := \lim_{n \rightarrow \infty} x_n$. We take the limit of both sides to obtain

$$2x^2 = x^2 + 2,$$

or $x^2 = 2$. As $x_n > 0$ for all n we get $x \geq 0$, and therefore $x = \sqrt{2}$.

You may have seen the above sequence before. It is the *Newton's method* ⁸ for finding the square root of 2. This method comes up very often in practice and converges very rapidly. Notice that we have used the fact that $x_1^2 - 2 > 0$, although it was not strictly needed to show convergence by considering a tail of the sequence. In fact the sequence converges as long as $x_1 \neq 0$, although with a negative x_1 we would arrive at $x = -\sqrt{2}$. By replacing the 2 in the numerator we obtain the square root of any positive number. These statements are left as an exercise.

You should, however, be careful. Before taking any limits, you must make sure the sequence converges. Let us see an example.

Suppose $x_1 := 1$ and $x_{n+1} := x_n^2 + x_n$. If we blindly assumed that the limit exists (call it x), then we would get the equation $x = x^2 + x$, from which we might conclude $x = 0$. However, it is not hard to show that $\{x_n\}$ is unbounded and therefore does not converge.

The thing to notice in this example is that the method still works, but it depends on the initial value x_1 . If we set $x_1 := 0$, then the sequence converges and the limit really is 0. An entire branch of mathematics, called dynamics, deals precisely with these issues.

Some convergence tests

It is not always necessary to go back to the definition of convergence to prove that a sequence is convergent. We first give a simple convergence test. The main idea is that $\{x_n\}$ converges to x if and only if $\{|x_n - x|\}$ converges to zero.

[converzero:prop] Let $\{x_n\}$ be a sequence. Suppose there is an $x \in \mathbb{R}$ and a convergent sequence $\{a_n\}$ such that

$$\lim_{n \rightarrow \infty} a_n = 0$$

and

$$|x_n - x| \leq a_n$$

for all n . Then $\{x_n\}$ converges and $\lim_{n \rightarrow \infty} x_n = x$.

Let $\epsilon > 0$ be given. Note that $a_n \geq 0$ for all n . Find an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $a_n = |a_n - 0| < \epsilon$. Then, for all $n \geq M$ we have

$$|x_n - x| \leq a_n < \epsilon. \quad \text{qedhere}$$

As the proposition shows, to study when a sequence has a limit is the same as studying when another sequence goes to zero. In general it may be hard to decide if a sequence converges, but for certain sequences there exist easy to apply tests that tell us if the sequence converges or not. Let us see one such test. First let us compute the limit of a very specific sequence.

Let $c > 0$.

i. If $c < 1$, then

$$\lim_{n \rightarrow \infty} c^n = 0.$$

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ii. If $c > 1$, then $\{c^n\}$ is unbounded.

First let us suppose $c > 1$. We write $c = 1 + r$ for some $r > 0$. By (or using the binomial theorem if you know it) we have Bernoulli's inequality (see also):

$$c^n = (1 + r)^n \geq 1 + nr.$$

By the of the real numbers, the sequence $\{1 + nr\}$ is unbounded (for any number B , we find an $n \in \mathbb{N}$ such that $nr \geq B - 1$). Therefore c^n is unbounded.

Now let $c < 1$. Write $c = \frac{1}{1+r}$, where $r > 0$. Then

$$c^n = \frac{1}{(1+r)^n} \leq \frac{1}{1+nr} \leq \frac{1}{r} \frac{1}{n}.$$

As $\{\frac{1}{n}\}$ converges to zero, so does $\{\frac{1}{r} \frac{1}{n}\}$. Hence, $\{c^n\}$ converges to zero.

If we look at the above proposition, we note that the ratio of the $(n+1)$ th term and the n th term is c . We generalize this simple result to a larger class of sequences. The following lemma will come up again once we get to series.

[seq:ratiotest] Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n and such that the limit

$$L := \lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|}$$

exists.

i. If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$.

ii. If $L > 1$, then $\{x_n\}$ is unbounded (hence diverges).

If L exists, but $L = 1$, the lemma says nothing. We cannot make any conclusion based on that information alone. For example, the sequence $\{\frac{1}{n}\}$ converges to zero, but $L = 1$. The constant sequence $\{1\}$ converges to 1, not zero, and also $L = 1$. The sequence $\{(-1)^n\}$ does not converge at all, and $L = 1$. Finally the sequence $\{\ln n\}$ is unbounded, yet again $L = 1$.

Suppose $L < 1$. As $\frac{|x_{n+1}|}{|x_n|} \geq 0$, we have that $L \geq 0$. Pick r such that $L < r < 1$. We wish to compare the sequence to the sequence r^n . The idea is that while the sequence is not going to be less than L eventually, it will eventually be less than r , which is still less than 1. The intuitive idea of the proof is illustrated in .

As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M + 1$) we write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| r r \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}$ converges to zero and hence $|x_M| r^{-M} r^n$ converges to zero. By , the M -tail of $\{x_n\}$ converges to zero and therefore $\{x_n\}$ converges to zero.

Now suppose $L > 1$. Pick r such that $1 < r < L$. As $L - r > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < L - r.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} > r.$$

Again for $n > M$ we write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} > |x_M| r r \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}$ is unbounded (since $r > 1$), and therefore $\{x_n\}$ cannot be bounded (if $|x_n| \leq B$ for all n , then $r^n < \frac{B}{|x_M|} r^M$ for all n , which is impossible). Consequently, $\{x_n\}$ cannot converge.

A simple application of the above lemma is to prove that

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$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0.$$

Proof: We find that

$$\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \frac{2}{n+1}.$$

It is not hard to see that $\{\frac{2}{n+1}\}$ converges to zero. The conclusion follows by the lemma.

Exercises

Prove . Hint: Use constant sequences and .

Prove part [prop:contalg:ii] of .

Prove that if $\{x_n\}$ is a convergent sequence, $k \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} x_n^k = \left(\lim_{n \rightarrow \infty} x_n \right)^k.$$

Hint: Use .

Suppose $x_1 := \frac{1}{2}$ and $x_{n+1} := x_n^2$. Show that $\{x_n\}$ converges and find $\lim x_n$. Hint: You cannot divide by zero!

Let $x_n := \frac{n - \cos(n)}{n}$. Use the to show that $\{x_n\}$ converges and find the limit.

Let $x_n := \frac{1}{n^2}$ and $y_n := \frac{1}{n}$. Define $z_n := \frac{x_n}{y_n}$ and $w_n := \frac{y_n}{x_n}$. Do $\{z_n\}$ and $\{w_n\}$ converge? What are the limits? Can you apply ? Why or why not?

True or false, prove or find a counterexample. If $\{x_n\}$ is a sequence such that $\{x_n^2\}$ converges, then $\{x_n\}$ converges.

Show that

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = 0.$$

Suppose $\{x_n\}$ is a sequence and suppose for some $x \in \mathbb{R}$, the limit

$$L := \lim_{n \rightarrow \infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Show that $\{x_n\}$ converges to x .

Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ and $k \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} x_n^{1/k} = \left(\lim_{n \rightarrow \infty} x_n \right)^{1/k}.$$

Hint: Find an expression q such that $\frac{x_n^{1/k} - x^{1/k}}{x_n - x} = \frac{1}{q}$.

Let $r > 0$. Show that starting with any $x_1 \neq 0$, the sequence defined by

$$x_{n+1} := x_n - \frac{x_n^2 - r}{2x_n}$$

converges to $\sqrt{r}$ if $x_1 > 0$ and $-\sqrt{r}$ if $x_1 < 0$.

- Suppose $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0. Show that $\{a_n b_n\}$ converges to 0.
- Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_n b_n\}$ is not convergent.
- Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_n b_n\}$ is not convergent.

Limit superior, limit inferior, and Bolzano-Weierstrass

Note: 1–2 lectures, alternative proof of BW optional

In this section we study bounded sequences and their subsequences. In particular we define the so-called limit superior and limit inferior of a bounded sequence and talk about limits of subsequences. Furthermore, we prove the Bolzano-Weierstrass theorem ⁹, which is an indispensable tool in analysis.

We have seen that every convergent sequence is bounded, although there exist many bounded divergent sequences. For example, the sequence $\{(-1)^n\}$ is bounded, but it is divergent. All is not lost however and we can still compute certain limits with a bounded divergent sequence.

Upper and lower limits

There are ways of creating monotone sequences out of any sequence, and in this fashion we get the so-called *limit superior* and *limit inferior*. These limits always exist for bounded sequences.

If a sequence $\{x_n\}$ is bounded, then the set $\{x_k : k \in \mathbb{N}\}$ is bounded. Then for every n the set $\{x_k : k \geq n\}$ is also bounded (as it is a subset).

[liminf limsup: def] Let $\{x_n\}$ be a bounded sequence. Let $a_n := \sup \{x_k : k \geq n\}$ and $b_n := \inf \{x_k : k \geq n\}$. Define

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &:= \lim_{n \rightarrow \infty} a_n, \\ \liminf_{n \rightarrow \infty} x_n &:= \lim_{n \rightarrow \infty} b_n. \end{aligned}$$

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For a bounded sequence, $\liminf$ and $\limsup$ always exist (see below). It is possible to define $\liminf$ and $\limsup$ for unbounded sequences if we allow ∞ and $-\infty$. It is not hard to generalize the following results to include unbounded sequences, however, we first restrict our attention to bounded ones.

Let $\{x_n\}$ be a bounded sequence. Let a_n and b_n be as in the definition above.

- The sequence $\{a_n\}$ is bounded monotone decreasing and $\{b_n\}$ is bounded monotone increasing. In particular, $\liminf x_n$ and $\limsup x_n$ exist.
- $\limsup_{n \rightarrow \infty} x_n = \inf_{n \rightarrow \infty} \{a_n : n \in \mathbb{N}\}$ and $\liminf_{n \rightarrow \infty} x_n = \sup_{n \rightarrow \infty} \{b_n : n \in \mathbb{N}\}$.
- $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$.

Let us see why $\{a_n\}$ is a decreasing sequence. As a_n is the least upper bound for $\{x_k : k \geq n\}$, it is also an upper bound for the subset $\{x_k : k \geq (n+1)\}$. Therefore a_{n+1} , the least upper bound for $\{x_k : k \geq (n+1)\}$, has to be less than or equal to a_n , that is, $a_n \geq a_{n+1}$. Similarly (an exercise), b_n is an increasing sequence. It is left as an exercise to show that if x_n is bounded, then a_n and b_n must be bounded.

The second item in the proposition follows as the sequences $\{a_n\}$ and $\{b_n\}$ are monotone.

For the third item, we note that $b_n \leq a_n$, as the $\inf$ of a set is less than or equal to its $\sup$. We know that $\{a_n\}$ and $\{b_n\}$ converge to the $\limsup$ and the $\liminf$ (respectively). We apply to obtain

$$\lim_{n \rightarrow \infty} b_n \leq \lim_{n \rightarrow \infty} a_n. \quad \text{qedhere}$$

Let $\{x_n\}$ be defined by

$$x_n := \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Let us compute the $\liminf$ and $\limsup$ of this sequence. First the limit inferior:

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf \{x_k : k \geq n\} \right) = \lim_{n \rightarrow \infty} 0 = 0.$$

For the limit superior we write

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup \{x_k : k \geq n\} \right).$$

It is not hard to see that

$$\sup \{x_k : k \geq n\} = \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ \frac{n+2}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

We leave it to the reader to show that the limit is 1. That is,

$$\limsup_{n \rightarrow \infty} x_n = 1.$$

Do note that the sequence $\{x_n\}$ is not a convergent sequence.

We associate with $\limsup$ and $\liminf$ certain subsequences.

[subseqlimsupinf:thm] If $\{x_n\}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k \rightarrow \infty} x_{n_k} = \limsup_{n \rightarrow \infty} x_n.$$

Similarly, there exists a (perhaps different) subsequence $\{x_{m_k}\}$ such that

$$\lim_{k \rightarrow \infty} x_{m_k} = \liminf_{n \rightarrow \infty} x_n.$$

Define $a_n := \sup \{x_k : k \geq n\}$. Write $x := \limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n$. Define the subsequence as follows. Pick $n_1 := 1$ and work inductively. Suppose we have defined the subsequence until n_k for some k . Now pick some $m > n_k$ such that

$$a_{(n_k+1)} - x_m < \frac{1}{k+1}.$$

We can do this as $a_{(n_k+1)}$ is a supremum of the set $\{x_n : n \geq n_k + 1\}$ and hence there are elements of the sequence arbitrarily close (or even possibly equal) to the supremum. Set $n_{k+1} := m$. The subsequence $\{x_{n_k}\}$ is defined. Next we need to prove that it converges and has the right limit.

Note that $a_{(n_{k-1}+1)} \geq a_{n_k}$ (why?) and that $a_{n_k} \geq x_{n_k}$. Therefore, for every $k > 1$ we have

$$\begin{aligned} |a_{n_k} - x_{n_k}| &= a_{n_k} - x_{n_k} \\ &\leq a_{(n_{k-1}+1)} - x_{n_k} \\ &< \frac{1}{k}. \end{aligned}$$

Let us show that $\{x_{n_k}\}$ converges to x . Note that the subsequence need not be monotone. Let $\epsilon > 0$ be given. As $\{a_n\}$ converges to x , then the subsequence $\{a_{n_k}\}$ converges to x . Thus there exists an $M_1 \in \mathbb{N}$ such that for all $k \geq M_1$ we have

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$$|a_{n_k} - x| < \frac{\epsilon}{2}.$$

Find an $M_2 \in \mathbb{N}$ such that

$$\frac{1}{M_2} \leq \frac{\epsilon}{2}.$$

Take $M := \max \{M_1, M_2\}$ and compute. For all $k \geq M$ we have

$$\begin{aligned} |x - x_{n_k}| &= |a_{n_k} - x_{n_k} + x - a_{n_k}| \\ &\leq |a_{n_k} - x_{n_k}| + |x - a_{n_k}| \\ &< \frac{1}{k} + \frac{\epsilon}{2} \\ &\leq \frac{1}{M_2} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We leave the statement for $\liminf$ as an exercise.

Using limit inferior and limit superior

The advantage of $\liminf$ and $\limsup$ is that we can always write them down for any (bounded) sequence. If we could somehow compute them, we could also compute the limit of the sequence if it exists, or show that the sequence diverges. Working with $\liminf$ and $\limsup$ is a little bit like working with limits, although there are subtle differences.

[liminf-sup-conv:thm] Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ converges if and only if

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n.$$

Furthermore, if $\{x_n\}$ converges, then

$$\lim_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n.$$

Define a_n and b_n as in . Note that

$$b_n \leq x_n \leq a_n.$$

If $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$, then we know that $\{a_n\}$ and $\{b_n\}$ have limits and that these two limits are the same. By the squeeze lemma (), $\{x_n\}$ converges and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n.$$

Now suppose $\{x_n\}$ converges to x . We know by that there exists a subsequence $\{x_{n_k}\}$ that converges to $\limsup_{n \rightarrow \infty} x_n$. As $\{x_n\}$ converges to x , every subsequence converges to x and therefore $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n_k} = x$. Similarly $\liminf_{n \rightarrow \infty} x_n = x$.

Limit superior and limit inferior behave nicely with subsequences.

[prop:subseq-lim-sup-inf] Suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Then

$$\liminf_{n \rightarrow \infty} x_n \leq \liminf_{k \rightarrow \infty} x_{n_k} \leq \limsup_{k \rightarrow \infty} x_{n_k} \leq \limsup_{n \rightarrow \infty} x_n.$$

The middle inequality has been proved already. We will prove the third inequality, and leave the first inequality as an exercise.

We want to prove that $\limsup_{n \rightarrow \infty} x_{n_k} \leq \limsup_{n \rightarrow \infty} x_n$. Define $a_j := \sup \{x_k : k \geq j\}$ as usual. Also define $c_j := \sup \{x_{n_k} : k \geq j\}$. It is not true that c_j is necessarily a subsequence of a_j . However, as $n_k \geq k$ for all k , we have that $\{x_{n_k} : k \geq j\} \subset \{x_k : k \geq j\}$. A supremum of a subset is less than or equal to the supremum of the set and therefore

$$c_j \leq a_j.$$

We apply to conclude

$$\lim_{j \rightarrow \infty} c_j \leq \lim_{j \rightarrow \infty} a_j,$$

which is the desired conclusion.

Limit superior and limit inferior are the largest and smallest subsequential limits. If the subsequence in the previous proposition is convergent, then we have that $\liminf_{n \rightarrow \infty} x_{n_k} = \lim_{n \rightarrow \infty} x_{n_k} = \limsup_{n \rightarrow \infty} x_{n_k}$. Therefore,

$$\liminf_{n \rightarrow \infty} x_n \leq \lim_{k \rightarrow \infty} x_{n_k} \leq \limsup_{n \rightarrow \infty} x_n.$$

Similarly we get the following useful test for convergence of a bounded sequence. We leave the proof as an exercise.

[seq-conv-subseq-conv:thm] A bounded sequence $\{x_n\}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}$ converges to x .

Bolzano-Weierstrass theorem

While it is not true that a bounded sequence is convergent, the Bolzano-Weierstrass theorem tells us that we can at least find a convergent subsequence. The version of Bolzano-Weierstrass that we present in this section is the Bolzano-Weierstrass for sequences.

[thm:bwseq] Suppose a sequence $\{x_n\}$ of real numbers is bounded. Then there exists a convergent subsequence $\{x_{n_i}\}$.

We use . It says that there exists a subsequence whose limit is $\limsup_{n \rightarrow \infty} x_n$.

The reader might complain right now that is strictly stronger than the Bolzano-Weierstrass theorem as presented above. That is true. However, only applies to the real line, but Bolzano-Weierstrass

Processing math: 52% eral contexts (that is, in $\mathbb{R}^n$) with pretty much the exact same statement.

As the theorem is so important to analysis, we present an explicit proof. The following proof generalizes more easily to different contexts.

As the sequence is bounded, then there exist two numbers $a_1 < b_1$ such that $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$.

We will define a subsequence $\{x_{n_i}\}$ and two sequences $\{a_i\}$ and $\{b_i\}$, such that $\{a_i\}$ is monotone increasing, $\{b_i\}$ is monotone decreasing, $a_i \leq x_{n_i} \leq b_i$ and such that $\lim a_i = \lim b_i$. That x_{n_i} converges follows by the .

We define the sequences inductively. We will always have that $a_i < b_i$, and that $x_n \in [a_i, b_i]$ for infinitely many $n \in \mathbb{N}$. We have already defined a_1 and b_1 . We take $n_1 := 1$, that is $x_{n_1} = x_1$.

Now suppose that up to some $k \in \mathbb{N}$ we have defined the subsequence $x_{n_1}, x_{n_2}, \dots, x_{n_k}$, and the sequences $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_k$. Let $y := \frac{a_k + b_k}{2}$. Clearly $a_k < y < b_k$. If there exist infinitely many $j \in \mathbb{N}$ such that $x_j \in [a_k, y]$, then set $a_{k+1} := a_k$, $b_{k+1} := y$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [a_k, y]$. If there are not infinitely many j such that $x_j \in [a_k, y]$, then it must be true that there are infinitely many $j \in \mathbb{N}$ such that $x_j \in [y, b_k]$. In this case pick $a_{k+1} := y$, $b_{k+1} := b_k$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [y, b_k]$.

Now we have the sequences defined. What is left to prove is that $\lim a_i = \lim b_i$. Obviously the limits exist as the sequences are monotone. From the construction, it is obvious that $b_i - a_i$ is cut in half in each step. Therefore $b_{i+1} - a_{i+1} = \frac{b_i - a_i}{2}$. By , we obtain that

$$b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

Let $x := \lim a_i$. As $\{a_i\}$ is monotone we have that

$$x = \sup \{a_i : i \in \mathbb{N}\}$$

Now let $y := \lim b_i = \inf \{b_i : i \in \mathbb{N}\}$. Obviously $y \leq x$ as $a_i < b_i$ for all i . As the sequences are monotone, then for any i we have (why?)

$$y - x \leq b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

As $\frac{b_1 - a_1}{2^{i-1}}$ is arbitrarily small and $y - x \geq 0$, we have that $y - x = 0$. We finish by the .

Yet another proof of the Bolzano-Weierstrass theorem is to show the following claim, which is left as a challenging exercise. *Claim: Every sequence has a monotone subsequence.*

[Infinite limits](#)

If we allow $\liminf$ and $\limsup$ to take on the values ∞ and $-\infty$, we can apply $\liminf$ and $\limsup$ to all sequences, not just bounded ones. For any sequence, we write

$$\limsup x_n := \inf \{a_n : n \in \mathbb{N}\}, \quad \text{and} \quad \liminf x_n := \sup \{b_n : n \in \mathbb{N}\},$$

where $a_n := \sup \{x_k : k \geq n\}$ and $b_n := \inf \{x_k : k \geq n\}$ as before.

We also often define infinite limits for certain divergent sequences.

We say $\{x_n\}$ *diverges to infinity* ¹⁰ if for every $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n > M$. In this case we write $\lim x_n := \infty$. Similarly if for every $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n < M$, we say $\{x_n\}$ *diverges to minus infinity* and we write $\lim x_n := -\infty$.

This definition behaves as expected with $\limsup$ and $\liminf$, see exercises [\[exercise:infseqlimex\]](#) and [\[exercise:infseqlimlims\]](#).

If $x_n := 0$ for odd n and $x_n := n$ for even n then

$$\lim_{n \rightarrow \infty} n = \infty, \quad \lim_{n \rightarrow \infty} x_n \text{ does not exist}, \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

Exercises

Suppose $\{x_n\}$ is a bounded sequence. Define a_n and b_n as in . Show that $\{a_n\}$ and $\{b_n\}$ are bounded.

Suppose $\{x_n\}$ is a bounded sequence. Define b_n as in . Show that $\{b_n\}$ is an increasing sequence.

Finish the proof of . That is, suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Prove $\liminf_{n \rightarrow \infty} x_n \leq \liminf_{k \rightarrow \infty} x_{n_k}$.

Prove .

a. Let $x_n := \frac{(-1)^n}{n}$, find $\limsup x_n$ and $\liminf x_n$.

b. Let $x_n := \frac{(n-1)(-1)^n}{n}$, find $\limsup x_n$ and $\liminf x_n$.

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences such that $x_n \leq y_n$ for all n . Then show that

$$\limsup_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} y_n$$

and

$$\liminf_{n \rightarrow \infty} x_n \leq \liminf_{n \rightarrow \infty} y_n.$$

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.

a. Show that $\{x_n + y_n\}$ is bounded.

b. Show that

$$(\liminf_{n \rightarrow \infty} x_n) + (\liminf_{n \rightarrow \infty} y_n) \leq \liminf_{n \rightarrow \infty} (x_n + y_n).$$

Hint: Find a subsequence $\{x_{n_i} + y_{n_i}\}$ of $\{x_n + y_n\}$ that converges. Then find a subsequence $\{x_{n_{m_i}}\}$ of $\{x_{n_i}\}$ that converges. Then apply what you know about limits.

c. Find an explicit $\{x_n\}$ and $\{y_n\}$ such that

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$$(\liminf_{n \rightarrow \infty} x_n) + (\liminf_{n \rightarrow \infty} y_n) < \liminf_{n \rightarrow \infty} (x_n + y_n).$$

Hint: Look for examples that do not have a limit.

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences (from the previous exercise we know that $\{x_n + y_n\}$ is bounded).

a. Show that

$$(\limsup_{n \rightarrow \infty} x_n) + (\limsup_{n \rightarrow \infty} y_n) \geq \limsup_{n \rightarrow \infty} (x_n + y_n).$$

Hint: See previous exercise.

b. Find an explicit $\{x_n\}$ and $\{y_n\}$ such that

$$(\limsup_{n \rightarrow \infty} x_n) + (\limsup_{n \rightarrow \infty} y_n) > \limsup_{n \rightarrow \infty} (x_n + y_n).$$

Hint: See previous exercise.

If $S \subset \mathbb{R}$ is a set, then $x \in \mathbb{R}$ is a *cluster point* if for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$ is not empty. That is, if there are points of S arbitrarily close to x . For example, $S := \{\frac{1}{n} : n \in \mathbb{N}\}$ has a unique (only one) cluster point 0, but $0 \notin S$. Prove the following version of the Bolzano-Weierstrass theorem:

Theorem. Let $S \subset \mathbb{R}$ be a bounded infinite set, then there exists at least one cluster point of S .

Hint: If S is infinite, then S contains a countably infinite subset. That is, there is a sequence $\{x_n\}$ of distinct numbers in S .

a) Prove that any sequence contains a monotone subsequence. Hint: Call $n \in \mathbb{N}$ a *peak* if $a_m \leq a_n$ for all $m \geq n$. There are two possibilities: either the sequence has at most finitely many peaks, or it has infinitely many peaks.

b) Conclude the Bolzano-Weierstrass theorem.

Let us prove a stronger version of . Suppose $\{x_n\}$ is a sequence such that every subsequence $\{x_{n_i}\}$ has a subsequence $\{x_{n_{i_j}}\}$ that converges to x . a) First show that $\{x_n\}$ is bounded. b) Now show that $\{x_n\}$ converges to x .

Let $\{x_n\}$ be a bounded sequence.

a) Prove that there exists an s such that for any $r > s$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $x_n < r$.

b) If s is a number as in a), then prove $\limsup x_n \leq s$.

c) Show that if S is the set of all s as in a), then $\limsup x_n = \inf S$.

[exercise:infseqlimex] Suppose $\{x_n\}$ is such that $\liminf x_n = -\infty$, $\limsup x_n = \infty$. a) Show that $\{x_n\}$ is not convergent, and also that neither $\lim x_n = \infty$ nor $\lim x_n = -\infty$ is true. b) Find an example of such a sequence.

[exercise:infseqlims] Given a sequence $\{x_n\}$. a) Show that $\lim x_n = \infty$ if and only if $\liminf x_n = \infty$. b) Then show that $\lim x_n = -\infty$ if and only if $\limsup x_n = -\infty$. c) If $\{x_n\}$ is monotone increasing, show that either $\lim x_n$ exists and is finite or $\lim x_n = \infty$.

Cauchy sequences

Note: 0.5–1 lecture

Often we wish to describe a certain number by a sequence that converges to it. In this case, it is impossible to use the number itself in the proof that the sequence converges. It would be nice if we could check for convergence without knowing the limit.

A sequence $\{x_n\}$ is a *Cauchy sequence* ¹¹ if for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$ we have

$$|x_n - x_k| < \epsilon.$$

Intuitively what it means is that the terms of the sequence are eventually arbitrarily close to each other. We would expect such a sequence to be convergent. It turns out that is true because $\mathbb{R}$ has the . First, let us look at some examples.

The sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Proof: Given $\epsilon > 0$, find M such that $M > \frac{2}{\epsilon}$. Then for $n, k \geq M$ we have that $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{k} < \frac{\epsilon}{2}$. Therefore for $n, k \geq M$ we have

$$\left| \frac{1}{n} - \frac{1}{k} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{k} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The sequence $\{\frac{n+1}{n}\}$ is a Cauchy sequence.

Proof: Given $\epsilon > 0$, find M such that $M > \frac{2}{\epsilon}$. Then for $n, k \geq M$ we have that $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{k} < \frac{\epsilon}{2}$. Therefore for $n, k \geq M$ we have

$$\begin{aligned} \left| \frac{n+1}{n} - \frac{k+1}{k} \right| &= \left| \frac{k(n+1) - n(k+1)}{nk} \right| \\ &= \left| \frac{kn + k - nk - n}{nk} \right| \\ &= \left| \frac{k - n}{nk} \right| \\ &\leq \left| \frac{k}{nk} \right| + \left| \frac{-n}{nk} \right| \\ &= \frac{1}{n} + \frac{1}{k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

A Cauchy sequence is bounded.

Suppose $\{x_n\}$ is Cauchy. Pick M such that for all $n, k \geq M$ we have $|x_n - x_k| < 1$. In particular, we have that for all $n \geq M$

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$$|x_n - x_M| < 1.$$

Or by the reverse triangle inequality, $|x_n| - |x_M| \leq |x_n - x_M| < 1$. Hence for $n \geq M$ we have

$$|x_n| < 1 + |x_M|.$$

Let

$$B := \max \{ |x_1|, |x_2|, \dots, |x_{M-1}|, 1 + |x_M| \}.$$

Then $|x_n| \leq B$ for all $n \in \mathbb{N}$.

A sequence of real numbers is Cauchy if and only if it converges.

Let $\epsilon > 0$ be given and suppose $\{x_n\}$ converges to x . Then there exists an M such that for $n \geq M$ we have

$$|x_n - x| < \frac{\epsilon}{2}.$$

Hence for $n \geq M$ and $k \geq M$ we have

$$|x_n - x_k| = |x_n - x + x - x_k| \leq |x_n - x| + |x - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Alright, that direction was easy. Now suppose $\{x_n\}$ is Cauchy. We have shown that $\{x_n\}$ is bounded. If we show that

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n,$$

then $\{x_n\}$ must be convergent by . Assuming that liminf and limsup exist is where we use the .

Define $a := \limsup_{n \rightarrow \infty} x_n$ and $b := \liminf_{n \rightarrow \infty} x_n$. By , there exist subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$, such that

$$\lim_{i \rightarrow \infty} x_{n_i} = a \quad \text{and} \quad \lim_{i \rightarrow \infty} x_{m_i} = b.$$

Given an $\epsilon > 0$, there exists an M_1 such that for all $i \geq M_1$ we have $|x_{n_i} - a| < \frac{\epsilon}{3}$ and an M_2 such that for all $i \geq M_2$ we have $|x_{m_i} - b| < \frac{\epsilon}{3}$. There also exists an M_3 such that for all $n, k \geq M_3$ we have $|x_n - x_k| < \frac{\epsilon}{3}$. Let $M := \max \{M_1, M_2, M_3\}$. Note that if $i \geq M$, then $n_i \geq M$ and $m_i \geq M$. Hence

$$\begin{aligned} |a - b| &= |a - x_{n_i} + x_{n_i} - x_{m_i} + x_{m_i} - b| \\ &\leq |a - x_{n_i}| + |x_{n_i} - x_{m_i}| + |x_{m_i} - b| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

As $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$ and the sequence converges.

The statement of this proposition is sometimes used to define the completeness property of the real numbers. We say a set is *Cauchy-complete* (or sometimes just *complete*) if every Cauchy sequence converges. Above we proved that $\mathbb{R}$ has the , then $\mathbb{R}$ is Cauchy-complete. We can “complete” $\mathbb{Q}$ by “throwing in” just enough points to make all Cauchy sequences converge (we omit the details). The resulting field has the least-upper-bound property. The advantage of using Cauchy sequences to define completeness is that this idea generalizes to more abstract settings.

It should be noted that the Cauchy criterion is stronger than just $|x_{n+1} - x_n|$ (or $|x_{n+j} - x_n|$ for a fixed j) going to zero as n goes to infinity. In fact, when we get to the partial sums of the harmonic series (see in the next section), we will have a sequence such that $x_{n+1} - x_n = \frac{1}{n+1}$, yet $\{x_n\}$ is divergent. In fact, for that sequence it is true that $\lim_{n \rightarrow \infty} |x_{n+j} - x_n| = 0$ for any $j \in \mathbb{N}$ (confer). The key point in the definition of Cauchy is that n and k vary independently and can be arbitrarily far apart.

Exercises

Prove that $\{\frac{n^2-1}{n^2}\}$ is Cauchy using directly the definition of Cauchy sequences.

Let $\{x_n\}$ be a sequence such that there exists a $0 < C < 1$ such that

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|.$$

Prove that $\{x_n\}$ is Cauchy. Hint: You can freely use the formula (for $C \neq 1$)

$$1 + C + C^2 + \dots + C^n = \frac{1 - C^{n+1}}{1 - C}.$$

Suppose F is an ordered field that contains the rational numbers $\mathbb{Q}$, such that $\mathbb{Q}$ is dense, that is: whenever $x, y \in F$ are such that $x < y$, then there exists a $q \in \mathbb{Q}$ such that $x < q < y$. Say a sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers is Cauchy if given any $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$, there exists an M such that for all $n, k \geq M$ we have $|x_n - x_k| < \epsilon$. Suppose any Cauchy sequence of rational numbers has a limit in F . Prove that F has the .

Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $\lim_{n \rightarrow \infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$ we have

$$|x_m - x_k| \leq y_k.$$

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Show that $\{x_n\}$ is Cauchy.

Suppose a Cauchy sequence $\{x_n\}$ is such that for every $M \in \mathbb{N}$, there exists a $k \geq M$ and an $n \geq M$ such that $x_k < 0$ and $x_n > 0$. Using simply the definition of a Cauchy sequence and of a convergent sequence, show that the sequence converges to 0.

Suppose $|x_n - x_k| \leq \frac{1}{n^2 k^2}$ for all n and k . Show that $\{x_n\}$ is Cauchy.

Suppose $\{x_n\}$ is a Cauchy sequence such that for infinitely many n , $x_n = c$. Using only the definition of Cauchy sequence prove that $\lim x_n = c$.

True/False prove or find a counterexample: If $\{x_n\}$ is a Cauchy sequence then there exists an M such that for all $n \geq M$ we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$.

Series

Note: 2 lectures

A fundamental object in mathematics is that of a series. In fact, when foundations of analysis were being developed, the motivation was to understand series. Understanding series is very important in applications of analysis. For example, solving differential equations often includes series, and differential equations are the basis for understanding almost all of modern science.

Definition

Given a sequence $\{x_n\}$, we write the formal object

$$\sum_{n=1}^{\infty} x_n \quad \text{or sometimes just} \quad \sum x_n$$

and call it a *series*. A series *converges*, if the sequence $\{s_k\}$ defined by

$$s_k := \sum_{n=1}^k x_n = x_1 + x_2 + \cdots + x_k,$$

converges. The numbers s_k are called *partial sums*. If $x := \lim s_k$, we write

$$\sum_{n=1}^{\infty} x_n = x.$$

In this case, we cheat a little and treat $\sum_{n=1}^{\infty} x_n$ as a number.

On the other hand, if the sequence $\{s_k\}$ diverges, we say the series is *divergent*. In this case, $\sum x_n$ is simply a formal object and not a number.

In other words, for a convergent series we have

$$\sum_{n=1}^{\infty} x_n = \lim_{k \rightarrow \infty} \sum_{n=1}^k x_n.$$

We should be careful to only use this equality if the limit on the right actually exists. That is, the right-hand side does not make sense (the limit does not exist) if the series does not converge.

Before going further, let us remark that it is sometimes convenient to start the series at an index different from 1. That is, for example we can write

$$\sum_{n=0}^{\infty} r^n = \sum_{n=1}^{\infty} r^{n-1}.$$

The left-hand side is more convenient to write. The idea is the same as the notation for the tail of a sequence.

It is common to write the series $\sum x_n$ as

$$x_1 + x_2 + x_3 + \cdots$$

with the understanding that the ellipsis indicates a series and not a simple sum. We do not use this notation as it often leads to mistakes in proofs.

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges and the limit is 1. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{1}{2^n} = 1.$$

Proof: First we prove the following equality

$$\left(\sum_{n=1}^k \frac{1}{2^n} \right) + \frac{1}{2^k} = 1.$$

The equality is easy to see when $k = 1$. The proof for general k follows by induction, which we leave to the reader. Let s_k be the partial sum. We write

$$|1 - s_k| = \left| 1 - \sum_{n=1}^k \frac{1}{2^n} \right| = \left| \frac{1}{2^k} \right| = \frac{1}{2^k}.$$

The sequence $\frac{1}{2^k}$ and therefore $|1 - s_k|$ converges to zero. So, $\{s_k\}$ converges to 1.

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For $-1 < r < 1$, the *geometric series*

$$\sum_{n=0}^{\infty} r^n$$

converges. In fact, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. The proof is left as an exercise to the reader. The proof consists of showing

$$\sum_{n=0}^{k-1} r^n = \frac{1-r^k}{1-r},$$

and then taking the limit as k goes to ∞ .

A fact we often use is the following analogue of looking at the tail of a sequence.

Let $\sum x_n$ be a series. Let $M \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} x_n \text{ converges if and only if } \sum_{n=M}^{\infty} x_n \text{ converges.}$$

We look at partial sums of the two series (for $k \geq M$)

$$\sum_{n=1}^k x_n = \left(\sum_{n=1}^{M-1} x_n \right) + \sum_{n=M}^k x_n.$$

Note that $\sum_{n=1}^{M-1} x_n$ is a fixed number. Now use to finish the proof.

Cauchy series

A series $\sum x_n$ is said to be *Cauchy* or a *Cauchy series*, if the sequence of partial sums $\{s_n\}$ is a Cauchy sequence.

A sequence of real numbers converges if and only if it is Cauchy. Therefore a series is convergent if and only if it is Cauchy.

The series $\sum x_n$ is Cauchy if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$, such that for every $n \geq M$ and $k \geq M$ we have

$$\left| \left(\sum_{j=1}^k x_j \right) - \left(\sum_{j=1}^n x_j \right) \right| < \epsilon.$$

Without loss of generality we assume $n < k$. Then we write

$$\left| \left(\sum_{j=1}^k x_j \right) - \left(\sum_{j=1}^n x_j \right) \right| = \left| \sum_{j=n+1}^k x_j \right| < \epsilon.$$

We have proved the following simple proposition.

[prop:cahyser] The series $\sum x_n$ is Cauchy if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$ we have

$$\left| \sum_{j=n+1}^k x_j \right| < \epsilon.$$

Basic properties

Let $\sum x_n$ be a convergent series. Then the sequence $\{x_n\}$ is convergent and

$$\lim_{n \rightarrow \infty} x_n = 0.$$

Let $\epsilon > 0$ be given. As $\sum x_n$ is convergent, it is Cauchy. Thus we find an M such that for every $n \geq M$ we have

$$\epsilon > \left| \sum_{j=n+1}^{n+1} x_j \right| = |x_{n+1}|.$$

Hence for every $n \geq M+1$ we have $|x_n| < \epsilon$.

So if a series converges, the terms of the series go to zero. The implication, however, goes only one way. Let us give an example.

[example:harmonicseries] The series $\sum \frac{1}{n}$ diverges (despite the fact that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$). This is the famous *harmonic series* ¹².

Proof: We will show that the sequence of partial sums is unbounded, and hence cannot converge. Write the partial sums s_n for $n = 2^k$ as:

$$\begin{aligned}
 s_1 &= 1, \\
 s_2 &= (1) + \left(\frac{1}{2}\right), \\
 s_4 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right), \\
 s_8 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right), \\
 &\vdots \\
 s_{2^k} &= 1 + \sum_{j=1}^k \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right).
 \end{aligned}$$

We note that $\frac{1}{2^k} \geq \frac{1}{2^{k-1}} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$ and $\frac{1}{2^{k-1}} + \frac{1}{2^k} \geq \frac{1}{2^{k-1}} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$. More generally

$$\sum_{m=2^{k-1}+1}^{2^k} \frac{1}{m} \geq \sum_{m=2^{k-1}+1}^{2^k} \frac{1}{2^k} = (2^{k-1}) \frac{1}{2^k} = \frac{1}{2}.$$

Therefore

$$s_{2^k} = 1 + \sum_{j=1}^k \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right) \geq 1 + \sum_{j=1}^k \frac{1}{2} = 1 + \frac{k}{2}.$$

As $\{ \frac{k}{2} \}$ is unbounded by the, that means that $\{s_{2^k}\}$ is unbounded, and therefore $\{s_n\}$ is unbounded. Hence $\{s_n\}$ diverges, and consequently $\sum \frac{1}{n}$ diverges.

Convergent series are linear. That is, we can multiply them by constants and add them and these operations are done term by term.

Let $\alpha \in \mathbb{R}$ and $\sum x_n$ and $\sum y_n$ be convergent series. Then

i. $\sum \alpha x_n$ is a convergent series and

$$\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n.$$

ii. $\sum (x_n + y_n)$ is a convergent series and

$$\sum_{n=1}^{\infty} (x_n + y_n) = \left(\sum_{n=1}^{\infty} x_n \right) + \left(\sum_{n=1}^{\infty} y_n \right).$$

For the first item, we simply write the k th partial sum

$$\sum_{n=1}^k \alpha x_n = \alpha \left(\sum_{n=1}^k x_n \right).$$

We look at the right-hand side and note that the constant multiple of a convergent sequence is convergent. Hence, we simply take the limit of both sides to obtain the result.

For the second item we also look at the k th partial sum

$$\sum_{n=1}^k (x_n + y_n) = \left(\sum_{n=1}^k x_n \right) + \left(\sum_{n=1}^k y_n \right).$$

We look at the right-hand side and note that the sum of convergent sequences is convergent. Hence, we simply take the limit of both sides to obtain the proposition.

Note that multiplying series is not as simple as adding, see the next section. It is not true, of course, that we can multiply term by term, since that strategy does not work even for finite sums. For example, $(a+b)(c+d) \neq ac+bd$.

Absolute convergence

Since monotone sequences are easier to work with than arbitrary sequences, it is generally easier to work with series $\sum x_n$ where $x_n \geq 0$ for all n . Then the sequence of partial sums is monotone increasing and converges if it is bounded from above. Let us formalize this statement as a proposition.

If $x_n \geq 0$ for all n , then $\sum x_n$ converges if and only if the sequence of partial sums is bounded from above.

As the limit of a monotone increasing sequence is the supremum, have the inequality

$$\sum_{n=1}^k x_n \leq \sum_{n=1}^{\infty} x_n.$$

The following criterion often gives a convenient way to test for convergence of a series.

A series $\sum x_n$ converges absolutely if the series $\sum |x_n|$ converges. If a series converges, but does not converge absolutely, we say it is *conditionally convergent*.

If the series $\sum x_n$ converges absolutely, then it converges.

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A series is convergent if and only if it is Cauchy. Hence suppose $\sum |x_n|$ is Cauchy. That is, for every $\epsilon > 0$, there exists an M such that for all $k \geq M$ and $n > k$ we have

$$\sum_{j=k+1}^n |x_j| = \left| \sum_{j=k+1}^n |x_j| \right| < \epsilon.$$

We apply the triangle inequality for a finite sum to obtain

$$\left| \sum_{j=k+1}^n x_j \right| \leq \sum_{j=k+1}^n |x_j| < \epsilon.$$

Hence $\sum x_n$ is Cauchy and therefore it converges.

Of course, if $\sum x_n$ converges absolutely, the limits of $\sum x_n$ and $\sum |x_n|$ are different. Computing one does not help us compute the other.

Absolutely convergent series have many wonderful properties. For example, absolutely convergent series can be rearranged arbitrarily, or we can multiply such series together easily. Conditionally convergent series on the other hand do not often behave as one would expect. See the next section.

We leave as an exercise to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, although the reader should finish this section before trying. On the other hand we proved

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Therefore $\sum \frac{(-1)^n}{n}$ is a conditionally convergent subsequence.

Comparison test and the p-series

We have noted above that for a series to converge the terms not only have to go to zero, but they have to go to zero “fast enough.” If we know about convergence of a certain series we can use the following comparison test to see if the terms of another series go to zero “fast enough.”

Let $\sum x_n$ and $\sum y_n$ be series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$.

- i. If $\sum y_n$ converges, then so does $\sum x_n$.
- ii. If $\sum x_n$ diverges, then so does $\sum y_n$.

Since the terms of the series are all nonnegative, the sequences of partial sums are both monotone increasing. Since $x_n \leq y_n$ for all n , the partial sums satisfy for all k

$$\sum_{n=1}^k x_n \leq \sum_{n=1}^k y_n.$$

If the series $\sum y_n$ converges the partial sums for the series are bounded. Therefore the right-hand side of [comptest:eq] is bounded for all k . Hence the partial sums for $\sum x_n$ are also bounded. Since the partial sums are a monotone increasing sequence they are convergent. The first item is thus proved.

On the other hand if $\sum x_n$ diverges, the sequence of partial sums must be unbounded since it is monotone increasing. That is, the partial sums for $\sum x_n$ are eventually bigger than any real number. Putting this together with [comptest:eq] we see that for any $B \in \mathbb{R}$, there is a k such that

$$B \leq \sum_{n=1}^k x_n \leq \sum_{n=1}^k y_n.$$

Hence the partial sums for $\sum y_n$ are also unbounded, and $\sum y_n$ also diverges.

A useful series to use with the comparison test is the p -series.

For $p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

First suppose $p \leq 1$. As $n \geq 1$, we have $\frac{1}{n^p} \geq \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, we see that the $\sum \frac{1}{n^p}$ must diverge for all $p \leq 1$ by the comparison test.

Now suppose $p > 1$. We proceed in a similar fashion as we did in the case of the harmonic series, but instead of showing that the sequence of partial sums is unbounded we show that it is bounded. Since the terms of the series are positive, the sequence of partial sums is monotone increasing and will converge if we show that it is bounded above. Let s_n denote the n th partial sum.

$$\begin{aligned}
 s_1 &= 1, \\
 s_3 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p} \right), \\
 s_7 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right), \\
 &\vdots \\
 s_{2^k-1} &= 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right).
 \end{aligned}$$

Instead of estimating from below, we estimate from above. In particular, as p is positive, then $2^p < 3^p$, and hence $\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$. Similarly $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$. Therefore $\left[\begin{array}{l} s_{2^k-1} \\ \end{array} \right] &= 1 + \sum_{j=1}^k \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right) &< 1 + \sum_{j=1}^k \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{2^{jp}} \right)$

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< 1). Then by using the result of , we note that

$$\sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j$$

converges. Therefore

$$s_{2^k-1} < 1 + \sum_{j=1}^k \left(\frac{1}{2^{p-1}} \right)^j \leq 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j.$$

As $\{s_n\}$ is a monotone sequence, then all $s_n \leq s_{2^k-1}$ for all $n \leq 2^k - 1$. Thus for all n ,

$$s_n < 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j.$$

The sequence of partial sums is bounded and hence converges.

Note that neither the p -series test nor the comparison test tell us what the sum converges to. They only tell us that a limit of the partial sums exists. For example, while we know that $\sum \frac{1}{n^2}$ converges it is far harder to find¹³ that the limit is $\frac{\pi^2}{6}$. If we treat $\sum \frac{1}{n^p}$ as a function of p , we get the so-called Riemann ζ function. Understanding the behavior of this function contains one of the most famous unsolved problems in mathematics today and has applications in seemingly unrelated areas such as modern cryptography.

The series $\sum \frac{1}{n^2+1}$ converges.

Proof: First note that $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Note that $\sum \frac{1}{n^2}$ converges by the p -series test. Therefore, by the comparison test, $\sum \frac{1}{n^2+1}$ converges.

[Ratio test](#)

Let $\sum x_n$ be a series such that

$$L := \lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. Then

- i. If $L < 1$, then $\sum x_n$ converges absolutely.
- ii. If $L > 1$, then $\sum x_n$ diverges.

From we note that if $L > 1$, then x_n diverges. Since it is a necessary condition for the convergence of series that the terms go to zero, we know that $\sum x_n$ must diverge.

Thus suppose $L < 1$. We will argue that $\sum |x_n|$ must converge. The proof is similar to that of . Of course $L \geq 0$. Pick r such that $L < r < 1$. As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M + 1$) write

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$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \dots \frac{|x_n|}{|x_{n-1}|} < |x_M| r \dots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

For $k > M$ we write the partial sum as

$$\begin{aligned} \sum_{n=1}^k |x_n| &= \left(\sum_{n=1}^M |x_n| \right) + \left(\sum_{n=M+1}^k |x_n| \right) \\ &\leq \left(\sum_{n=1}^M |x_n| \right) + \left(\sum_{n=M+1}^k (|x_M| r^{-M}) r^n \right) \\ &\leq \left(\sum_{n=1}^M |x_n| \right) + (|x_M| r^{-M}) \left(\sum_{n=M+1}^k r^n \right). \end{aligned}$$

As $0 < r < 1$ the geometric series $\sum_{n=0}^{\infty} r^n$ converges, so $\sum_{n=M+1}^{\infty} r^n$ converges as well (why?). We take the limit as k goes to infinity on the right-hand side above to obtain

$$\begin{aligned} \sum_{n=1}^k |x_n| &\leq \left(\sum_{n=1}^M |x_n| \right) + (|x_M| r^{-M}) \left(\sum_{n=M+1}^k r^n \right) \\ &\leq \left(\sum_{n=1}^M |x_n| \right) + (|x_M| r^{-M}) \left(\sum_{n=M+1}^{\infty} r^n \right). \end{aligned}$$

The right-hand side is a number that does not depend on n . Hence the sequence of partial sums of $\sum |x_n|$ is bounded and $\sum |x_n|$ is convergent. Thus $\sum x_n$ is absolutely convergent.

The series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges absolutely.

Proof: We write

$$\lim_{n \rightarrow \infty} \frac{2^{(n+1)} / (n+1)!}{2^n / n!} = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0.$$

Therefore, the series converges absolutely by the ratio test.

Exercises

For $r \neq 1$, prove

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

Hint: Let $s := \sum_{k=0}^{n-1} r^k$, then compute $s(1-r) = s - rs$, and solve for s .

[geometric:exr] Prove that for $-1 < r < 1$ we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Hint: Use the previous exercise.

Decide the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{3}{9n+1}$		b) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$		c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	
d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$		e) $\sum_{n=1}^{\infty} n e^{-n^2}$			

a. Prove that if $\sum_{n=1}^{\infty} x_n$ converges, then $\sum_{n=1}^{\infty} (x_{2n} + x_{2n+1})$ also converges.

b. Find an explicit example where the converse does not hold.

For $j = 1, 2, \dots, n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ denote n sequences. Suppose that for each j

$$\sum_{k=1}^{\infty} x_{j,k}$$

is convergent. Then show

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$$\sum_{j=1}^n \left(\sum_{k=1}^{\infty} x_{j,k} \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^n x_{j,k} \right).$$

Prove the following stronger version of the ratio test: Let $\sum x_n$ be a series.

a. If there is an N and a $\rho < 1$ such that for all $n \geq N$ we have $\frac{|x_{n+1}|}{|x_n|} < \rho$, then the series converges absolutely.

b. If there is an N such that for all $n \geq N$ we have $\frac{|x_{n+1}|}{|x_n|} \geq 1$, then the series diverges.

Let $\{x_n\}$ be a decreasing sequence such that $\sum x_n$ converges. Show that $\lim_{n \rightarrow \infty} nx_n = 0$.

Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Hint: consider the sum of two subsequent entries.

a. Prove that if $\sum x_n$ and $\sum y_n$ converge absolutely, then $\sum x_n y_n$ converges absolutely.

b. Find an explicit example where the converse does not hold.

c. Find an explicit example where all three series are absolutely convergent, are not just finite sums, and $(\sum x_n)(\sum y_n) \neq \sum x_n y_n$. That is, show that series are not multiplied term-by-term.

Prove the triangle inequality for series, that is if $\sum x_n$ converges absolutely then

$$\left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n|.$$

Prove the *limit comparison test*. That is, prove that if $a_n > 0$ and $b_n > 0$ for all n , and

$$0 < \lim_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty,$$

then either $\sum a_n$ and $\sum b_n$ both converge or both diverge.

[exercise:badnocauchy] Let $x_n = \sum_{j=1}^n \frac{1}{j}$. Show that for every k we have $\lim_{n \rightarrow \infty} |x_{n+k} - x_n| = 0$, yet $\{x_n\}$ is not Cauchy.

Let s_k be the k th partial sum of $\sum x_n$.

a) Suppose that there exists a $m \in \mathbb{N}$ such that $\lim_{k \rightarrow \infty} s_{mk}$ exists and $\lim_{n \rightarrow \infty} x_n = 0$. Show that $\sum x_n$ converges.

b) Find an example where $\lim_{k \rightarrow \infty} s_{2k}$ exists and $\lim_{n \rightarrow \infty} x_n \neq 0$ (and therefore $\sum x_n$ diverges).

c) (Challenging) Find an example where $\lim_{n \rightarrow \infty} x_n = 0$, and there exists a subsequence $\{s_{k_j}\}$ such that $\lim_{j \rightarrow \infty} s_{k_j}$ exists, but $\sum x_n$ still diverges.

More on series

Note: up to 2–3 lectures (optional, can safely be skipped or covered partially)

Root test

We have seen the ratio test before. There is one more similar test called the *root test*. In fact, the proof of this test is similar and somewhat easier.

Let $\sum x_n$ be a series and let

$$L := \limsup_{n \rightarrow \infty} |x_n|^{1/n}.$$

Then

i. If $L < 1$ then $\sum x_n$ converges absolutely.

ii. If $L > 1$ then $\sum x_n$ diverges.

If $L > 1$, then there exists a subsequence $\{x_{n_k}\}$ such that $L = \lim_{k \rightarrow \infty} |x_{n_k}|^{1/n_k}$. Let r be such that $L > r > 1$. There exists an M such that for all $k \geq M$, we have $|x_{n_k}|^{1/n_k} > r > 1$, or in other words

$|x_{n_k}| > r^{n_k} > 1$. The subsequence $\{|x_{n_k}|\}$, and therefore also $\{|x_n|\}$, cannot possibly converge to zero, and so the series diverges.

Now suppose $L < 1$. Pick r such that $L < r < 1$. By definition of limit supremum, pick M such that for all $n \geq M$ we have

$$\sup \{|x_k|^{1/k} : k \geq n\} < r.$$

Therefore, for all $n \geq M$ we have

$$|x_n|^{1/n} < r, \quad \text{or in other words} \quad |x_n| < r^n.$$

Let $k > M$ and let us estimate the k th partial sum

$$\sum_{n=1}^k |x_n| = \left(\sum_{n=1}^M |x_n| \right) + \left(\sum_{n=M+1}^k |x_n| \right) \leq \left(\sum_{n=1}^M |x_n| \right) + \left(\sum_{n=M+1}^k r^n \right).$$

As $0 < r < 1$ the geometric series $\sum_{n=M+1}^{\infty} r^n$ converges to $\frac{r^{M+1}}{1-r}$. As everything is positive we have

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$$\sum_{n=1}^k |x_n| \leq \left(\sum_{n=1}^M |x_n| \right) + \frac{r^{M+1}}{1-r}.$$

Thus the sequence of partial sums of $\sum |x_n|$ is bounded, and so the series converges. Therefore $\sum x_n$ converges absolutely.

Alternating series test

The tests we have so far only addressed absolute convergence. The following test gives a large supply of conditionally convergent series.

Let $\{x_n\}$ be a monotone decreasing sequence of positive real numbers such that $\lim x_n = 0$. Then

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

converges.

Write $s_m := \sum_{k=1}^m (-1)^k x_k$ be the m th partial sum. Then write

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k x_k = (-x_1 + x_2) + \cdots + (-x_{2n-1} + x_{2n}) = \sum_{k=1}^n (-x_{2k-1} + x_{2k}).$$

The sequence $\{x_k\}$ is decreasing and so $(-x_{2k-1} + x_{2k}) \leq 0$ for all k . Therefore the subsequence $\{s_{2n}\}$ of partial sums is a decreasing sequence. Similarly, $(x_{2k} - x_{2k+1}) \geq 0$, and so

$$s_{2n} = -x_1 + (x_2 - x_3) + \cdots + (x_{2n-2} - x_{2n-1}) + x_{2n} \geq -x_1.$$

The sequence $\{s_{2n}\}$ is decreasing and bounded below, so it converges. Let $a := \lim s_{2n}$.

We wish to show that $\lim s_m = a$ (not just for the subsequence). Notice

$$s_{2n+1} = s_{2n} + x_{2n+1}.$$

Given $\epsilon > 0$, pick M such that $|s_{2n} - a| < \frac{\epsilon}{2}$ whenever $2n \geq M$. Since $\lim x_n = 0$, we also make M possibly larger to obtain $x_{2n+1} < \frac{\epsilon}{2}$ whenever $2n \geq M$. If $2n \geq M$, we have $|s_{2n} - a| < \frac{\epsilon}{2} < \epsilon$, so we just need to check the situation for s_{2n+1} :

$$|s_{2n+1} - a| = |s_{2n} - a + x_{2n+1}| \leq |s_{2n} - a| + x_{2n+1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \text{qedhere}$$

In particular, there exist conditionally convergent series where the absolute values of the terms go to zero arbitrarily slowly. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

converges for arbitrarily small $p > 0$, but it does not converge absolutely when $p \leq 1$.

Rearrangements

Generally, absolutely convergent series behave as we imagine they should. For example, absolutely convergent series can be summed in any order whatsoever. Nothing of the sort holds for conditionally convergent series (see and).

Take a series

$$\sum_{n=1}^{\infty} x_n.$$

Given a bijective function $\sigma: \mathbb{N} \rightarrow \mathbb{N}$, the corresponding rearrangement is the following series:

$$\sum_{k=1}^{\infty} x_{\sigma(k)}.$$

We simply sum the series in a different order.

Let $\sum x_n$ be an absolutely convergent series converging to a number x . Let $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then $\sum x_{\sigma(n)}$ is absolutely convergent and converges to x .

In other words, a rearrangement of an absolutely convergent series converges (absolutely) to the same number.

Let $\epsilon > 0$ be given. Then take M to be such that

$$\left| \left(\sum_{n=1}^M x_n \right) - x \right| < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{n=M+1}^{\infty} |x_n| < \frac{\epsilon}{2}.$$

As σ is a bijection, there exists a number K such that for each $n \leq M$, there exists $k \leq K$ such that $\sigma(k) = n$. In other words $\{1, 2, \dots, M\} \subset \sigma(\{1, 2, \dots, K\})$.

Then for any $N \geq K$, let $Q := \max \sigma(\{1, 2, \dots, K\})$ and compute

$$\begin{aligned} \left| \left(\sum_{n=1}^N x_{\sigma(n)} \right) - x \right| &= \left| \left(\sum_{n=1}^M x_n + \sum_{\substack{n=1 \\ \sigma(n) > M}}^N x_{\sigma(n)} \right) - x \right| \\ &\leq \left| \left(\sum_{n=1}^M x_n \right) - x \right| + \sum_{\substack{n=1 \\ \sigma(n) > M}}^N |x_{\sigma(n)}| \\ &\leq \left| \left(\sum_{n=1}^M x_n \right) - x \right| + \sum_{n=M+1}^Q |x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So $\sum x_{\sigma(n)}$ converges to x . To see that the convergence is absolute, we apply the above argument to $\sum |x_n|$ to show that $\sum |x_{\sigma(n)}|$ converges.

[example:harmonsumanything] Let us show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$

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$\sum \frac{(-1)^{n+1}}{n}$), which does not converge absolutely, can be rearranged to converge to anything. The odd terms and the even terms both diverge to infinity (prove this!):

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{2n} = \infty.$$

Let $a_n := \frac{(-1)^{n+1}}{n}$

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$\sum a_n$) for simplicity, let an arbitrary number $L \in \mathbb{R}$ be given, and set $\sigma(1) := 1$. Suppose we have defined $\sigma(n)$ for all $n \leq N$. If

$$\sum_{n=1}^N a_{\sigma(n)} \leq L,$$

then let $\sigma(N+1) := k$ be the smallest odd $k \in \mathbb{N}$ that we have not used yet, that is $\sigma(n) \neq k$ for all $n \leq N$. Otherwise let $\sigma(N+1) := k$ be the smallest even k that we have not yet used.

By construction $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ is one-to-one. It is also onto, because if we keep adding either odd (resp. even) terms, eventually we will pass L and switch to the evens (resp. odds). So we switch infinitely many times.

Finally, let N be the N where we just pass L and switch. For example suppose we have just switched from odd to even (so we start subtracting), and let $N' > N$ be where we first switch back from even to odd. Then

$$L + \frac{1}{\sigma(N)} \geq \sum_{n=1}^{N-1} a_{\sigma(n)} > \sum_{n=1}^{N'-1} a_{\sigma(n)} > L - \frac{1}{\sigma(N')}.$$

And similarly for switching in the other direction. Therefore, the sum up to $N' - 1$ is within $\frac{1}{\min\{\sigma(N), \sigma(N')\}}$ of L . As we switch infinitely many times we obtain that $\sigma(N) \rightarrow \infty$ and $\sigma(N') \rightarrow \infty$, and hence $\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n$.

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$\sum a_n = L$.

Here is an example to illustrate the proof. Suppose $L = 1.2$, then the order is

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} - \frac{1}{12} + \dots$$

At this point we are no more than $\frac{1}{18}$ from the limit.

Multiplication of series

As we have already mentioned, multiplication of series is somewhat harder than addition. If we have that at least one of the series converges absolutely, then we can use the following theorem. For this result it is convenient to start the series at 0, rather than at 1.

Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series, converging to A and B respectively. If at least one of the series converges absolutely, then the series $\sum_{n=0}^{\infty} c_n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{j=0}^n a_j b_{n-j},$$

converges to AB .

The series $\sum c_n$ is called the *Cauchy product* of $\sum a_n$ and $\sum b_n$.

Suppose $\sum a_n$ converges absolutely, and let $\epsilon > 0$ be given. In this proof instead of picking complicated estimates just to make the final estimate come out as less than ϵ , let us simply obtain an estimate that depends on ϵ and can be made arbitrarily small.

Write

$$A_m := \sum_{n=0}^m a_n, \quad B_m := \sum_{n=0}^m b_n.$$

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We rearrange the m th partial sum of $\sum c_n$:

$$\begin{aligned} \left| \left(\sum_{n=0}^m c_n \right) - AB \right| &= \left| \left(\sum_{n=0}^m \sum_{j=0}^n a_j b_{n-j} \right) - AB \right| \\ &= \left| \left(\sum_{n=0}^m B_n a_{m-n} \right) - AB \right| \\ &= \left| \left(\sum_{n=0}^m (B_n - B) a_{m-n} \right) + BA_m - AB \right| \\ &\leq \left(\sum_{n=0}^m |B_n - B| |a_{m-n}| \right) + |B| |A_m - A| \end{aligned}$$

We can surely make the second term on the right hand side go to zero. The trick is to handle the first term. Pick K such that for all $m \geq K$ we have $|A_m - A| < \epsilon$ and also $|B_m - B| < \epsilon$. Finally, as $\sum a_n$ converges absolutely, make sure that K is large enough such that for all $m \geq K$,

$$\sum_{n=K}^m |a_n| < \epsilon.$$

As $\sum b_n$ converges, then we have that $B_{\max} := \sup \{ |B_n - B| : n = 0, 1, 2, \dots \}$ is finite. Take $m \geq 2K$, then in particular $m - K + 1 > K$. So

Therefore, for $m \geq 2K$ we have

$$\begin{aligned} \left| \left(\sum_{n=0}^m c_n \right) - AB \right| &\leq \left(\sum_{n=0}^m |B_n - B| |a_{m-n}| \right) + |B| |A_m - A| \\ &< \epsilon B_{\max} + \epsilon \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \epsilon = \epsilon \left(B_{\max} + \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \right). \end{aligned}$$

The expression in the parenthesis on the right hand side is a fixed number. Hence, we can make the right hand side arbitrarily small by picking a small enough $\epsilon > 0$. So $\sum_{n=0}^{\infty} c_n$ converges to AB .

If both series are only conditionally convergent, the Cauchy product series need not even converge. Suppose we take $a_n = b_n = (-1)^n \frac{1}{\sqrt{n+1}}$. The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$ converges by the alternating series test, however, it does not converge absolutely as can be seen from the p -test. Let us look at the Cauchy product.

$$c_n = \sum_{j=0}^n (-1)^j \left(\frac{1}{\sqrt{j+1}} + \frac{1}{\sqrt{2n-j+1}} \right) = (-1)^n \sum_{j=0}^n \frac{1}{\sqrt{j+1}}$$

Therefore

$$|c_n| = \sum_{j=0}^n \frac{1}{\sqrt{(j+1)(n-j+1)}} \geq \sum_{j=0}^n \frac{1}{\sqrt{(n+1)(n+1)}} = 1.$$

The terms do not go to zero and hence $\sum c_n$ cannot converge.

[Power series](#)

Fix $x_0 \in \mathbb{R}$. A *power series* about x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

A power series is really a function of x , and many important functions in analysis can be written as a power series.

We say that a power series is *convergent* if there is at least one $x \neq x_0$ that makes the series converge. Note that it is trivial to see that if $x = x_0$ then the series always converges since all terms except the first are zero. If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

[ps:expex] The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

is absolutely convergent for all $x \in \mathbb{R}$. This can be seen using the ratio test: For any x notice that

$$\lim_{n \rightarrow \infty} \frac{(1/(n+1)!) x^{n+1}}{(1/n!) x^n} = \lim_{n \rightarrow \infty} \frac{x}{n+1} = 0.$$

In fact, you may recall from calculus that this series converges to e^x .

[ps:1kex] The series

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$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

converges absolutely for all $x \in (-1, 1)$ via the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(1/(n+1))x^{n+1}}{(1/n)x^n} \right| = \lim_{n \rightarrow \infty} |x| \frac{n}{n+1} = |x| < 1.$$

It converges at $x = -1$, as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test. But the power series does not converge absolutely at $x = -1$, because $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. The series diverges at $x = 1$. When $|x| > 1$, then the series diverges via the ratio test.

[ps:divergeex] The series

$$\sum_{n=1}^{\infty} n^n x^n$$

diverges for all $x \neq 0$. Let us apply the root test

$$\limsup_{n \rightarrow \infty} \left| n^n x^n \right|^{1/n} = \limsup_{n \rightarrow \infty} n|x| = \infty.$$

Therefore the series diverges for all $x \neq 0$.

In fact, convergence of power series in general always works analogously to one of the three examples above.

Let $\sum a_n(x - x_0)^n$ be a power series. If the series is convergent, then either it converges at all $x \in \mathbb{R}$, or there exists a number ρ , such that the series converges absolutely on the interval $(x_0 - \rho, x_0 + \rho)$ and diverges when $x < x_0 - \rho$ or $x > x_0 + \rho$.

The number ρ is called the *radius of convergence* of the power series. We write $\rho = \infty$ if the series converges for all x , and we write $\rho = 0$ if the series is divergent. See . In the radius of convergence is $\rho = 1$. In the radius of convergence is $\rho = \infty$, and in the radius of convergence is $\rho = 0$.

Write

$$R := \limsup_{n \rightarrow \infty} \left| a_n \right|^{1/n}.$$

We use the root test to prove the proposition:

$$L = \limsup_{n \rightarrow \infty} \left| a_n(x - x_0)^n \right|^{1/n} = |x - x_0| \limsup_{n \rightarrow \infty} \left| a_n \right|^{1/n} = |x - x_0| R.$$

In particular if $R = \infty$, then $L = \infty$ for any $x \neq x_0$, and the series diverges by the root test. On the other hand if $R = 0$, then $L = 0$ for any x , and the series converges absolutely for all x .

Suppose $0 < R < \infty$. The series converges absolutely if $1 > L = R|x - x_0|$, or in other words when

$$|x - x_0| < \frac{1}{R}.$$

The series diverges when $1 < L = R|x - x_0|$, or

$$|x - x_0| > \frac{1}{R}.$$

Letting $\rho = \frac{1}{R}$ completes the proof.

It may be useful to restate what we have learned in the proof as a separate proposition.

Let $\sum a_n(x - x_0)^n$ be a power series, and let

$$R := \limsup_{n \rightarrow \infty} \left| a_n \right|^{1/n}.$$

If $R = \infty$, the power series is divergent. If $R = 0$, then the power series converges everywhere. Otherwise the radius of convergence $\rho = \frac{1}{R}$.

Often, radius of convergence is written as $\rho = \frac{1}{R}$ in all three cases, with the obvious understanding of what ρ should be if $R = 0$ or $R = \infty$.

Convergent power series can be added and multiplied together, and multiplied by constants. The proposition has an easy proof using what we know about series in general, and power series in particular. We leave the proof to the reader.

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ be two convergent power series with radius of convergence at least $\rho > 0$ and $\alpha \in \mathbb{R}$. Then for all x such that $|x - x_0| < \rho$, we have

$$\left(\sum_{n=0}^{\infty} a_n(x - x_0)^n \right) + \left(\sum_{n=0}^{\infty} b_n(x - x_0)^n \right) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n,$$

$$\alpha \left(\sum_{n=0}^{\infty} a_n(x - x_0)^n \right) = \sum_{n=0}^{\infty} \alpha a_n(x - x_0)^n,$$

and

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$$\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n \right) \left(\sum_{n=0}^{\infty} b_n (x - x_0)^n \right) = \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.

That is, after performing the algebraic operations, the radius of convergence of the resulting series is at least ρ . For all x with $|x - x_0| < \rho$, we have two convergent series so their term by term addition and multiplication by constants follows by what we learned in the last section. For multiplication of two power series, the series are absolutely convergent inside the radius of convergence and that is why for those x we can apply Mertens' theorem. Note that after applying an algebraic operation the radius of convergence could increase. See the exercises.

Let us look at some examples of power series. Polynomials are simply finite power series. That is, a polynomial is a power series where the a_n are zero for all n large enough. We expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x - x_0)$. For example, $2x^2 - 3x + 4$ as a power series around $x_0 = 1$ is

$$2x^2 - 3x + 4 = 3 + (x - 1) + 2(x - 1)^2.$$

We can also expand *rational functions*, that is, ratios of polynomials as power series, although we will not completely prove this fact here. Notice that a series for a rational function only defines the function on an interval even if the function is defined elsewhere. For example, for the *geometric series* we have that for $x \in (-1, 1)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

The series diverges when $|x| > 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions as power series around x_0 , as long as the denominator is not zero at x_0 . We state without proof that this is always possible, and we give an example of such a computation using the geometric series.

Let us expand $\frac{x}{1+2x+x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

Write $1 + 2x + x^2 = (1+x)^2 = (1 - (-x))^2$, and suppose $|x| < 1$. Compute

$$\begin{aligned} \frac{x}{1+2x+x^2} &= x \left(\frac{1}{1-(-x)} \right)^2 \\ &= x \left(\sum_{n=0}^{\infty} (-1)^n x^n \right)^2 \\ &= x \left(\sum_{n=0}^{\infty} c_n x^n \right) \\ &= \sum_{n=0}^{\infty} c_n x^{n+1}, \end{aligned}$$

where using the formula for the product of series we obtain, $c_0 = 1$, $c_1 = -1 - 1 = -2$, $c_2 = 1 + 1 + 1 = 3$, etc.... Therefore we get that for $|x| < 1$,

$$\frac{x}{1+2x+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^n.$$

The radius of convergence is at least 1. We leave it to the reader to verify that the radius of convergence is exactly equal to 1.

You can use the method of partial fractions you know from calculus. For example, to find the power series for $\frac{x^3+x}{x^2-1}$ at 0, write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n.$$

Exercises

Decide the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}}$	b) $\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{n}$	c) $\sum_{n=1}^{\infty} \frac{1}{n}$ ParseError: EOF expected (click for details)	d) $\sum_{n=1}^{\infty} \frac{1}{n^n}$ ParseError: EOF expected (click for details)
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Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Show that the product series, $\sum_{n=0}^{\infty} c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$, also converges absolutely.

[exercise:seriesconvergestoanything] Let $\sum a_n$ be conditionally convergent. Show that given any number x there exists a rearrangement of $\sum a_n$ such that the rearranged series converges to x . Hint: See .

a) Show that the alternating harmonic series $\sum \frac{(-1)^n}{n}$

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$\{s_n\}$ has a rearrangement such that for any $x < y$, there exists a partial sum s_n of the rearranged series such that $x < s_n < y$. b) Show that the rearrangement you found does not converge. See . c) Show that for any $x \in \mathbb{R}$, there exists a subsequence of partial sums $\{s_{n_k}\}$ of your rearrangement such that $\lim s_{n_k} = x$.

For the following power series, find if they are convergent or not, and if so find their radius of convergence.

a) $\sum_{n=0}^{\infty} 2^n x^n$	b) $\sum_{n=0}^{\infty} nx^n$	c) $\sum_{n=0}^{\infty} n! x^n$	d) $\sum_{n=0}^{\infty} \frac{1}{(2k)!} (x-10)^n$	e) $\sum_{n=0}^{\infty} x^{2n}$	f) $\sum_{n=0}^{\infty} n! x^{n!}$
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Suppose $\sum a_n x^n$ converges for $x = 1$. a) What can you say about the radius of convergence? b) If you further know that at $x = 1$ the convergence is not absolute, what can you say?

Expand $\frac{x}{4-x^2}$ as a power series around $x_0 = 0$ and compute its radius of convergence.

a) Find an example where the radius of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are 1, but the radius of convergence of the sum of the two series is infinite. b) (Trickier) Find an example where the radius of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are 1, but the radius of convergence of the product of the two series is infinite.

Figure out how to compute the radius of convergence using the ratio test. That is, suppose $\sum a_n x^n$ is a power series and $R := \lim \frac{|a_{n+1}|}{|a_n|}$ exists or is ∞ . Find the radius of convergence and prove your claim.

a) Prove that $\lim n^{1/n} = 1$. Hint: Write $n^{1/n} = 1 + b_n$ and note $b_n > 0$. Then show that $(1 + b_n)^n \geq \frac{n(n-1)}{2} b_n^2$ and use this to show that $\lim b_n = 0$. b) Use the result of part a) to show that if $\sum a_n x^n$ is a convergent power series with radius of convergence R , then $\sum na_n x^n$ is also convergent with the same radius of convergence.

There are different notions of summability (convergence) of a series than just the one we have seen. A common one is *Cesàro summability*¹⁴. Let $\sum a_n$ be a series and let s_n be the n th partial sum. The series is said to be Cesàro summable to a if

$$a = \lim_{n \rightarrow \infty} \frac{s_1 + s_2 + \dots + s_n}{n}.$$

a) If $\sum a_n$ is convergent to a (in the usual sense), show that $\sum a_n$ is Cesàro summable to a . b) Show that in the sense of Cesàro $\sum (-1)^n$ is summable to $\frac{1}{2}$. c) Let $a_n := k$ when $n = k^3$ for some $k \in \mathbb{N}$, $a_n := -k$ when $n = k^3 + 1$ for some $k \in \mathbb{N}$, otherwise let $a_n := 0$. Show that $\sum a_n$ diverges in the usual sense, (partial sums are unbounded), but it is Cesàro summable to 0 (seems a little paradoxical at first sight).

Show that the monotonicity in the alternating series test is necessary. That is, find a sequence of positive real numbers $\{x_n\}$ with $\lim x_n = 0$ but such that $\sum (-1)^n x_n$ diverges.

Continuous Functions

Limits of functions

Note: 2–3 lectures

Before we define continuity of functions, we need to visit a somewhat more general notion of a limit. That is, given a function $f: S \rightarrow \mathbb{R}$, we want to see how $f(x)$ behaves as x tends to a certain point.

Cluster points

First, let us return to a concept we have previously seen in an exercise.

Let $S \subset \mathbb{R}$ be a set. A number $x \in \mathbb{R}$ is called a *cluster point* of S if for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$ is not empty.

That is, x is a cluster point of S if there are points of S arbitrarily close to x . Another way of phrasing the definition is to say that x is a cluster point of S if for every $\epsilon > 0$, there exists a $y \in S$ such that $y \neq x$ and $|x - y| < \epsilon$. Note that a cluster point of S need not lie in S .

Let us see some examples.

- The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ has a unique cluster point zero.
- The cluster points of the open interval $(0, 1)$ are all points in the closed interval $[0, 1]$.
- For the set $\mathbb{Q}$, the set of cluster points is the whole real line $\mathbb{R}$.
- For the set $[0, 1) \cup \{2\}$, the set of cluster points is the interval $[0, 1]$.
- The set $\mathbb{N}$ has no cluster points in $\mathbb{R}$.

Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}$ such that $x_n \neq x$, $x_n \in S$, and $\lim x_n = x$.

First suppose x is a cluster point of S . For any $n \in \mathbb{N}$, we pick x_n to be an arbitrary point of $(x - \frac{1}{n}, x + \frac{1}{n}) \cap S \setminus \{x\}$, which we know is nonempty because x is a cluster point of S . Then x_n is within $\frac{1}{n}$ of x , that is,

$$|x - x_n| < \frac{1}{n}.$$

As $\frac{1}{n}$ converges to zero, $\{x_n\}$ converges to x .

On the other hand, if we start with a sequence of numbers $\{x_n\}$ in S converging to x such that $x_n \neq x$ for all n , then for every $\epsilon > 0$ there is an M such that in particular $|x_M - x| < \epsilon$. That is, $x_M \in (x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$.

Limits of functions

If a function f is defined on a set S and c is a cluster point of S , then we can define the limit of $f(x)$ as x gets close to c . Do note that it is irrelevant for the definition if f is defined at c or not. Furthermore, even if the function is defined at c , the limit of the function as x goes to c could very well be different from $f(c)$.

Let $f: S \rightarrow \mathbb{R}$ be a function and c a cluster point of S . Suppose there exists an $L \in \mathbb{R}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then

$$|f(x) - L| < \epsilon.$$

In this case we say $f(x)$ converges to L as x goes to c . We say L is the *limit* of $f(x)$ as x goes to c . We write

$$\lim_{x \rightarrow c} f(x) := L,$$

or

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$$f(x) \rightarrow L \text{ as } x \rightarrow c.$$

If no such L exists, then we say that the limit does not exist or that f diverges at c .

Again the notation and language we are using above assumes the limit is unique even though we have not yet proved that. Let us do that now.

Let c be a cluster point of $S \subset \mathbb{R}$ and let $f: S \rightarrow \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c . Then the limit of $f(x)$ as x goes to c is unique.

Let L_1 and L_2 be two numbers that both satisfy the definition. Take an $\epsilon > 0$ and find a $\delta_1 > 0$ such that $|f(x) - L_1| < \frac{\epsilon}{2}$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_1$. Also find $\delta_2 > 0$ such that $|f(x) - L_2| < \frac{\epsilon}{2}$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_2$. Put $\delta := \min\{\delta_1, \delta_2\}$. Suppose $x \in S$, $|x - c| < \delta$, and $x \neq c$. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As $|L_1 - L_2| < \epsilon$ for arbitrary $\epsilon > 0$, then $L_1 = L_2$.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) := x^2$. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x^2 = c^2.$$

Proof: First let c be fixed. Let $\epsilon > 0$ be given. Take

$$\delta := \min\left\{1, \frac{\epsilon}{2|c| + 1}\right\}.$$

Take $x \neq c$ such that $|x - c| < \delta$. In particular, $|x - c| < 1$. Then by reverse triangle inequality we get

$$|x| - |c| \leq |x - c| < 1.$$

Adding $2|c|$ to both sides we obtain $|x| + |c| < 2|c| + 1$. We compute

$$\begin{aligned} |f(x) - c^2| &= |x^2 - c^2| \\ &= |(x + c)(x - c)| \\ &= |x + c||x - c| \\ &\leq (|x| + |c|)|x - c| \\ &< (2|c| + 1)|x - c| \\ &< (2|c| + 1)\frac{\epsilon}{2|c| + 1} = \epsilon. \end{aligned}$$

Define $f: [0, 1) \rightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then

$$\lim_{x \rightarrow 0} f(x) = 0,$$

even though $f(0) = 1$.

Proof: Let $\epsilon > 0$ be given. Let $\delta := \epsilon$. Then for $x \in [0, 1)$, $x \neq 0$, and $|x - 0| < \delta$ we get

$$|f(x) - 0| = |x| < \delta = \epsilon.$$

Sequential limits

Let us connect the limit as defined above with limits of sequences.

[seqflimit:lemma] Let $S \subset \mathbb{R}$ and c be a cluster point of S . Let $f: S \rightarrow \mathbb{R}$ be a function.

Then $f(x) \rightarrow L$ as $x \rightarrow c$, if and only if for every sequence $\{x_n\}$ of numbers such that $x_n \in S \setminus \{c\}$ for all n , and such that $\lim x_n = c$, we have that the sequence $\{f(x_n)\}$ converges to L .

Suppose $f(x) \rightarrow L$ as $x \rightarrow c$, and $\{x_n\}$ is a sequence such that $x_n \in S \setminus \{c\}$ and $\lim x_n = c$. We wish to show that $\{f(x_n)\}$ converges to L . Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that if $x \in S \setminus \{c\}$

and $|x - c| < \delta$, then $|f(x) - L| < \epsilon$. As $\{x_n\}$ converges to c , find an M such that for $n \geq M$ we have that $|x_n - c| < \delta$. Therefore, for $n \geq M$,

$$|f(x_n) - L| < \epsilon.$$

Thus $\{f(x_n)\}$ converges to L .

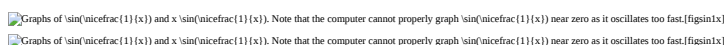
For the other direction, we use proof by contrapositive. Suppose it is not true that $f(x) \rightarrow L$ as $x \rightarrow c$. The negation of the definition is that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists an $x \in S \setminus \{c\}$, where $|x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

Let us use $\frac{1}{n}$ for δ in the above statement to construct a sequence $\{x_n\}$. We have that there exists an $\epsilon > 0$ such that for every n , there exists a point $x_n \in S \setminus \{c\}$, where

$|x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$. The sequence $\{x_n\}$ just constructed converges to c , but the sequence $\{f(x_n)\}$ does not converge to L . And we are done.

It is possible to strengthen the reverse direction of the lemma by simply stating that $\{f(x_n)\}$ converges without requiring a specific limit. See .

$\lim_{x \rightarrow 0} \sin(\frac{1}{x})$ does not exist, but $\lim_{x \rightarrow 0} x \sin(\frac{1}{x}) = 0$. See .



Proof: Let us work with $\sin(\frac{1}{x})$ first. Let us define the sequence $x_n := \frac{1}{n\pi + \frac{\pi}{2}}$. It is not hard to see that $\lim x_n = 0$. Furthermore,

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$$\sin(\frac{1}{x_n}) = \sin(\pi n + \frac{1}{x_n}) = (-1)^n.$$

Therefore, $\{\sin(\frac{1}{x_n})\}$ does not converge. Thus, by , $\lim_{x \rightarrow 0} \sin(\frac{1}{x})$ does not exist.

Now let us look at $x \sin(\frac{1}{x})$. Let x_n be a sequence such that $x_n \neq 0$ for all n and such that $\lim x_n = 0$. Notice that $|\sin(t)| \leq 1$ for any $t \in \mathbb{R}$. Therefore,

$$|x_n \sin(\frac{1}{x_n}) - 0| = |x_n| |\sin(\frac{1}{x_n})| \leq |x_n|.$$

As x_n goes to 0, then $|x_n|$ goes to zero, and hence $\{x_n \sin(\frac{1}{x_n})\}$ converges to zero. By , $\lim_{x \rightarrow 0} x \sin(\frac{1}{x}) = 0$.

Keep in mind the phrase “for every sequence” in the lemma. For example, take $\sin(\frac{1}{x})$ and the sequence $x_n = \frac{1}{n\pi}$. Then $\{\sin(\frac{1}{x_n})\}$ is the constant zero sequence, and therefore converges to zero.

Using , we can start applying everything we know about sequential limits to limits of functions. Let us give a few important examples.

Let $S \subset \mathbb{R}$ and c be a cluster point of S . Let $f: S \rightarrow \mathbb{R}$ and $g: S \rightarrow \mathbb{R}$ be functions. Suppose the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and that

$$f(x) \leq g(x) \quad \text{for all } x \in S.$$

Then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

Take $\{x_n\}$ be a sequence of numbers in $S \setminus \{c\}$ that converges to c . Let

$$L_1 := \lim_{x \rightarrow c} f(x), \quad \text{and} \quad L_2 := \lim_{x \rightarrow c} g(x).$$

By we know $\{f(x_n)\}$ converges to L_1 and $\{g(x_n)\}$ converges to L_2 . We also have $f(x_n) \leq g(x_n)$. We obtain $L_1 \leq L_2$ using .

By applying constant functions, we get the following corollary. The proof is left as an exercise.

[fconstineq:cor] Let $S \subset \mathbb{R}$ and c be a cluster point of S . Let $f: S \rightarrow \mathbb{R}$ be a function. And suppose the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that

$$a \leq f(x) \leq b \quad \text{for all } x \in S.$$

Then

$$a \leq \lim_{x \rightarrow c} f(x) \leq b.$$

Using in the same way as above we also get the following corollaries, whose proofs are again left as an exercise.

[fsqueeze:cor] Let $S \subset \mathbb{R}$ and c be a cluster point of S . Let $f: S \rightarrow \mathbb{R}$, $g: S \rightarrow \mathbb{R}$, and $h: S \rightarrow \mathbb{R}$ be functions. Suppose

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in S,$$

and the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x).$$

[falg:cor] Let $S \subset \mathbb{R}$ and c be a cluster point of S . Let $f: S \rightarrow \mathbb{R}$ and $g: S \rightarrow \mathbb{R}$ be functions. Suppose limits of $f(x)$ and $g(x)$ as x goes to c both exist. Then

- $\lim_{x \rightarrow c} (f(x) + g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) + \left(\lim_{x \rightarrow c} g(x) \right).$
- $\lim_{x \rightarrow c} (f(x) - g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) - \left(\lim_{x \rightarrow c} g(x) \right).$
- $\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right).$
- [falg:cor:iv] If $\lim_{x \rightarrow c} g(x) \neq 0$, and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}.$$

Limits of restrictions and one-sided limits

Sometimes we work with the function defined on a subset.

Let $f: S \rightarrow \mathbb{R}$ be a function. Let $A \subset S$. Define the function $f|_A: A \rightarrow \mathbb{R}$ by

$$f|_A(x) := f(x) \quad \text{for } x \in A.$$

The function $f|_A$ is called the *restriction* of f to A .

The function $f|_A$ is simply the function f taken on a smaller domain. The following proposition is the analogue of taking a tail of a sequence.

[prop:limrest] Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and let $f: S \rightarrow \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ such that $A \cap (c - \alpha, c + \alpha) = S \cap (c - \alpha, c + \alpha)$.

- The point c is a cluster point of A if and only if c is a cluster point of S .
- Supposing c is a cluster point of S , then $f(x) \rightarrow L$ as $x \rightarrow c$ if and only if $f|_A(x) \rightarrow L$ as $x \rightarrow c$.

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First, let c be a cluster point of A . Since $A \subset S$, then if $(A \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$ is nonempty for every $\epsilon > 0$, then $(S \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$ is nonempty for every $\epsilon > 0$. Thus c is a cluster point of S . Second, suppose c is a cluster point of S . Then for $\epsilon > 0$ such that $\epsilon < \alpha$ we get that $(A \setminus \{c\}) \cap (c - \epsilon, c + \epsilon) = (S \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$, which is nonempty. This is true for all $\epsilon < \alpha$ and hence $(A \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$ must be nonempty for all $\epsilon > 0$. Thus c is a cluster point of A .

Now suppose $f(x) \rightarrow L$ as $x \rightarrow c$. That is, for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \epsilon$. Because $A \subset S$, if x is in $A \setminus \{c\}$, then x is in $S \setminus \{c\}$, and hence $f|_A(x) \rightarrow L$ as $x \rightarrow c$.

Finally suppose $f|_A(x) \rightarrow L$ as $x \rightarrow c$. Hence for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in A \setminus \{c\}$ and $|x - c| < \delta$, then $|f|_A(x) - L| < \epsilon$. Without loss of generality assume $\delta \leq \alpha$. If $|x - c| < \delta$, then $x \in S \setminus \{c\}$ if and only if $x \in A \setminus \{c\}$. Thus $|f(x) - L| = |f|_A(x) - L| < \epsilon$.

The hypothesis of the proposition is necessary. For an arbitrary restriction we generally only get implication in only one direction, see .

A common use of restriction with respect to limits are *one-sided limits*.

[defn:onesidedlimits] Let $f: S \rightarrow \mathbb{R}$ be function and let c be a cluster point of $S \cap (c, \infty)$. Then if the limit of the restriction of f to $S \cap (c, \infty)$ as $x \rightarrow c$ exists, we define

$$\lim_{x \rightarrow c^+} f(x) := \lim_{x \rightarrow c} f|_{S \cap (c, \infty)}(x).$$

Similarly if c is a cluster point of $S \cap (-\infty, c)$ and the limit of the restriction as $x \rightarrow c$ exists, we define

$$\lim_{x \rightarrow c^-} f(x) := \lim_{x \rightarrow c} f|_{S \cap (-\infty, c)}(x).$$

The proposition above does not apply to one-sided limits. It is possible to have one-sided limits, but no limit at a point. For example, define $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) := 1$ for $x < 0$ and $f(x) := 0$ for $x \geq 0$. We leave it to the reader to verify that

$$\lim_{x \rightarrow 0^-} f(x) = 1, \quad \lim_{x \rightarrow 0^+} f(x) = 0, \quad \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

We have the following replacement.

[prop:onesidedlimits] Let $S \subset \mathbb{R}$ be a set such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, and let $f: S \rightarrow \mathbb{R}$ be a function. Then

$$\lim_{x \rightarrow c} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L.$$

That is, a limit exists if both one-sided limits exist and are equal, and vice-versa. The proof is a straightforward application of the definition of limit and is left as an exercise. The key point is that $(S \cap (-\infty, c)) \cup (S \cap (c, \infty)) = S \setminus \{c\}$.

Exercises

Find the limit or prove that the limit does not exist

a) $\lim_{x \rightarrow c} \sqrt{x}$, for $c \geq 0$	b) $\lim_{x \rightarrow c} x^2 + x + 1$, for any $c \in \mathbb{R}$	c) $\lim_{x \rightarrow 0} x^2 \cos(\frac{1}{x})$
d) $\lim_{x \rightarrow 0} \sin(\frac{1}{x}) \cos(\frac{1}{x})$	e) $\lim_{x \rightarrow 0} \sin(x) \cos(\frac{1}{x})$	

Prove .

Prove .

Prove .

Let $A \subset S$. Show that if c is a cluster point of A , then c is a cluster point of S . Note the difference from .

[exercise:restrictionlimitexercise] Let $A \subset S$. Suppose c is a cluster point of A and it is also a cluster point of S . Let $f: S \rightarrow \mathbb{R}$ be a function. Show that if $f(x) \rightarrow L$ as $x \rightarrow c$, then $f|_A(x) \rightarrow L$ as $x \rightarrow c$. Note the difference from .

Find an example of a function $f: [-1, 1] \rightarrow \mathbb{R}$ such that for $A := [0, 1]$, the restriction $f|_A(x) \rightarrow 0$ as $x \rightarrow 0$, but the limit of $f(x)$ as $x \rightarrow 0$ does not exist. Note why you cannot apply .

Find example functions f and g such that the limit of neither $f(x)$ nor $g(x)$ exists as $x \rightarrow 0$, but such that the limit of $f(x) + g(x)$ exists as $x \rightarrow 0$.

[exercise:contlimitcomposition] Let c_1 be a cluster point of $A \subset \mathbb{R}$ and c_2 be a cluster point of $B \subset \mathbb{R}$. Suppose $f: A \rightarrow B$ and $g: B \rightarrow \mathbb{R}$ are functions such that $f(x) \rightarrow c_2$ as $x \rightarrow c_1$ and $g(y) \rightarrow L$ as $y \rightarrow c_2$. If $c_2 \in B$ also suppose that $g(c_2) = L$. Let $h(x) := g(f(x))$ and show $h(x) \rightarrow L$ as $x \rightarrow c_1$. Hint: note that $f(x)$ could equal c_2 for many $x \in A$, see also .

Let c be a cluster point of $A \subset \mathbb{R}$, and $f: A \rightarrow \mathbb{R}$ be a function. Suppose for every sequence $\{x_n\}$ in A , such that $\lim x_n = c$, the sequence $\{f(x_n)\}_{n=1}^\infty$ is Cauchy. Prove that $\lim_{x \rightarrow c} f(x)$ exists.

[exercise:seqflimlalt] Prove the following stronger version of one direction of : Let $S \subset \mathbb{R}$, c be a cluster point of S , and $f: S \rightarrow \mathbb{R}$ be a function. Suppose that for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that $\lim x_n = c$ the sequence $\{f(x_n)\}$ is convergent. Then show $f(x) \rightarrow L$ as $x \rightarrow c$ for some $L \in \mathbb{R}$.

Prove .

Suppose $S \subset \mathbb{R}$ and c is a cluster point of S . Suppose $f: S \rightarrow \mathbb{R}$ is bounded. Show that there exists a sequence $\{x_n\}$ with $x_n \in S \setminus \{c\}$ and $\lim x_n = c$ such that $\{f(x_n)\}$ converges.

[exercise:contlimitbadcomposition] Show that the hypothesis that $g(c_2) = L$ in is necessary. That is, find f and g such that $f(x) \rightarrow c_2$ as $x \rightarrow c_1$ and $g(y) \rightarrow L$ as $y \rightarrow c_2$, but $g(f(x))$ does not go to L as $x \rightarrow c_1$.

Continuous functions

Note: 2–2.5 lectures

You undoubtedly heard of continuous functions in your schooling. A high-school criterion for this concept is that a function is continuous if we can draw its graph without lifting the pen from the paper. While that intuitive concept may be useful in simple situations, we require rigor. The following definition took three great mathematicians (Bolzano, Cauchy, and finally Weierstrass) to get correctly and its final form dates only to the late 1800s.

Definition and basic properties

Let $S \subset \mathbb{R}$, $c \in S$, and let $f: S \rightarrow \mathbb{R}$ be a function. We say that f is *continuous at c* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

When $f: S \rightarrow \mathbb{R}$ is continuous at all $c \in S$, then we simply say f is a *continuous function*.

If f is continuous for all $c \in A$, we say f is continuous on $A \subset S$. It is left as an easy exercise to show that this implies that $f|_A$ is continuous, although the converse does not hold.

Continuity may be the most important definition to understand in analysis, and it is not an easy one. See . Note that δ not only depends on ϵ , but also on c ; we need not pick one δ for all $c \in S$. It is no accident that the definition of continuity is similar to the definition of a limit of a function. The main feature of continuous functions is that these are precisely the functions that behave nicely with

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limits.

[contbasic:prop] Suppose $f: S \rightarrow \mathbb{R}$ is a function and $c \in S$. Then

- If c is not a cluster point of S , then f is continuous at c .
- If c is a cluster point of S , then f is continuous at c if and only if the limit of $f(x)$ as $x \rightarrow c$ exists and

$$\lim_{x \rightarrow c} f(x) = f(c).$$

- f is continuous at c if and only if for every sequence $\{x_n\}$ where $x_n \in S$ and $\lim_{n \rightarrow \infty} x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Let us start with the first item. Suppose c is not a cluster point of S . Then there exists a $\delta > 0$ such that $S \cap (c - \delta, c + \delta) = \{c\}$. Therefore, for any $\epsilon > 0$, simply pick this given delta. The only $x \in S$ such that $|x - c| < \delta$ is $x = c$. Then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.

Let us move to the second item. Suppose c is a cluster point of S . Let us first suppose that $\lim_{x \rightarrow c} f(x) = f(c)$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. As $|f(c) - f(c)| = 0 < \epsilon$, then the definition of continuity at c is satisfied. On the other hand, suppose f is continuous at c . For every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x \in S$ where $|x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$. Then the statement is, of course, still true if $x \in S \setminus \{c\} \subset S$. Therefore $\lim_{x \rightarrow c} f(x) = f(c)$.

For the third item, suppose f is continuous at c . Let $\{x_n\}$ be a sequence such that $x_n \in S$ and $\lim_{n \rightarrow \infty} x_n = c$. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in S$ where $|x - c| < \delta$.

Find an $M \in \mathbb{N}$ such that for $n \geq M$ we have $|x_n - c| < \delta$. Then for $n \geq M$ we have that $|f(x_n) - f(c)| < \epsilon$, so $\{f(x_n)\}$ converges to $f(c)$.

Let us prove the converse of the third item by contrapositive. Suppose f is not continuous at c . Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists an $x \in S$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$. Let us define a sequence $\{x_n\}$ as follows. Let $x_n \in S$ be such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \epsilon$. Now $\{x_n\}$ is a sequence of numbers in S such that $\lim_{n \rightarrow \infty} x_n = c$ and such that $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ does not converge to $f(c)$. It may or may not converge, but it definitely does not converge to $f(c)$.

The last item in the proposition is particularly powerful. It allows us to quickly apply what we know about limits of sequences to continuous functions and even to prove that certain functions are continuous. It can also be strengthened, see .

$f: (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is continuous.

Proof: Fix $c \in (0, \infty)$. Let $\{x_n\}$ be a sequence in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} x_n = c$. Then we know that

$$f(c) = \frac{1}{c} = \frac{1}{\lim_{n \rightarrow \infty} x_n} = \lim_{n \rightarrow \infty} \frac{1}{x_n} = \lim_{n \rightarrow \infty} f(x_n).$$

Thus f is continuous at c . As f is continuous at all $c \in (0, \infty)$, f is continuous.

We have previously shown $\lim_{x \rightarrow c} x^2 = c^2$ directly. Therefore the function x^2 is continuous. We can use the continuity of algebraic operations with respect to limits of sequences, which we proved in the previous chapter, to prove a much more general result.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a *polynomial*. That is

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0,$$

for some constants $a_0, a_1, \dots, a_d$. Then f is continuous.

Fix $c \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $\lim_{n \rightarrow \infty} x_n = c$. Then

$$\begin{aligned} f(c) &= a_d c^d + a_{d-1} c^{d-1} + \dots + a_1 c + a_0 \\ &= a_d (\lim_{n \rightarrow \infty} x_n)^d + a_{d-1} (\lim_{n \rightarrow \infty} x_n)^{d-1} + \dots + a_1 (\lim_{n \rightarrow \infty} x_n) + a_0 \\ &= \lim_{n \rightarrow \infty} (a_d x_n^d + a_{d-1} x_n^{d-1} + \dots + a_1 x_n + a_0) = \lim_{n \rightarrow \infty} f(x_n). \end{aligned}$$

Thus f is continuous at c . As f is continuous at all $c \in \mathbb{R}$, f is continuous.

By similar reasoning, or by appealing to , we can prove the following. The details of the proof are left as an exercise.

[contalg:prop] Let $f: S \rightarrow \mathbb{R}$ and $g: S \rightarrow \mathbb{R}$ be functions continuous at $c \in S$.

- The function $h: S \rightarrow \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continuous at c .
- The function $h: S \rightarrow \mathbb{R}$ defined by $h(x) := f(x) - g(x)$ is continuous at c .
- The function $h: S \rightarrow \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c .
- If $g(x) \neq 0$ for all $x \in S$, the function $h: S \rightarrow \mathbb{R}$ defined by $h(x) := \frac{f(x)}{g(x)}$ is continuous at c .

[sincos:example] The functions $\sin(x)$ and $\cos(x)$ are continuous. In the following computations we use the sum-to-product trigonometric identities. We also use the simple facts that $|\sin(x)| \leq |x|$, $|\cos(x)| \leq 1$, and $|\sin(x)| \leq 1$.

$$\begin{aligned} |\sin(x) - \sin(c)| &= \left| 2 \sin\left(\frac{x-c}{2}\right) \cos\left(\frac{x+c}{2}\right) \right| \\ &= 2 \left| \sin\left(\frac{x-c}{2}\right) \right| \left| \cos\left(\frac{x+c}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x-c}{2}\right) \right| \\ &\leq 2 \left| \frac{x-c}{2} \right| = |x-c| \end{aligned}$$

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$$\begin{aligned}
 |\cos(x) - \cos(c)| &= \left| -2\sin\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right) \right| \\
 &= 2 \left| \sin\left(\frac{x+c}{2}\right) \right| \left| \sin\left(\frac{x-c}{2}\right) \right| \\
 &\leq 2 \left| \sin\left(\frac{x-c}{2}\right) \right| \\
 &\leq 2 \left| \frac{x-c}{2} \right| = |x-c|
 \end{aligned}$$

The claim that $\sin$ and $\cos$ are continuous follows by taking an arbitrary sequence $\{x_n\}$ converging to c , or by applying the definition of continuity directly. Details are left to the reader.

Composition of continuous functions

You have probably already realized that one of the basic tools in constructing complicated functions out of simple ones is composition. A useful property of continuous functions is that compositions of continuous functions are again continuous. Recall that for two functions f and g , the composition $f \circ g$ is defined by $(f \circ g)(x) := f(g(x))$.

Let $A, B \subset \mathbb{R}$ and $f: B \rightarrow \mathbb{R}$ and $g: A \rightarrow B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g: A \rightarrow \mathbb{R}$ is continuous at c .

Let $\{x_n\}$ be a sequence in A such that $\lim x_n = c$. Then as g is continuous at c , then $\{g(x_n)\}$ converges to $g(c)$. As f is continuous at $g(c)$, then $\{f(g(x_n))\}$ converges to $f(g(c))$. Thus $f \circ g$ is continuous at c .

Claim: $(\sin(\frac{1}{x}))^2$ is a continuous function on $(0, \infty)$.

Proof: First note that $\frac{1}{x}$ is a continuous function on $(0, \infty)$ and $\sin(x)$ is a continuous function on $(0, \infty)$ (actually on all of $\mathbb{R}$, but $(0, \infty)$ is the range for $\frac{1}{x}$). Hence the composition $\sin(\frac{1}{x})$ is continuous. We also know that x^2 is continuous on the interval $(-1, 1)$ (the range of $\sin$). Thus the composition $(\sin(\frac{1}{x}))^2$ is also continuous on $(0, \infty)$.

Discontinuous functions

When f is not continuous at c , we say f is *discontinuous* at c , or that it has a *discontinuity* at c . If we state the contrapositive of the third item of as a separate claim we get an easy to use test for discontinuities.

Let $f: S \rightarrow \mathbb{R}$ be a function. Suppose that for some $c \in S$, there exists a sequence $\{x_n\}$, $x_n \in S$, and $\lim x_n = c$ such that $\{f(x_n)\}$ does not converge to $f(c)$ (or does not converge at all), then f is not continuous at c .

[example:stepdiscont] The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

is not continuous at 0.

Proof: Take the sequence $\{-\frac{1}{n}\}$. Then $f(-\frac{1}{n}) = -1$ and so $\lim f(-\frac{1}{n}) = -1$, but $f(0) = 1$.

For an extreme example we take the so-called *Dirichlet function*.

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

The function f is discontinuous at all $c \in \mathbb{R}$.

Proof: Suppose c is rational. Take a sequence $\{x_n\}$ of irrational numbers such that $\lim x_n = c$ (why can we?). Then $f(x_n) = 0$ and so $\lim f(x_n) = 0$, but $f(c) = 1$. If c is irrational, take a sequence of rational numbers $\{x_n\}$ that converges to c (why can we?). Then $\lim f(x_n) = 1$, but $f(c) = 0$.

Let us yet again test the limits of your intuition. Can there exist a function that is continuous on all irrational numbers, but discontinuous at all rational numbers? There are rational numbers arbitrarily close to any irrational number. Perhaps strangely, the answer is yes. The following example is called the *Thomae function*¹⁵ or the *popcorn function*.

[popcornfunction:example] Let $f: (0, 1) \rightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \frac{1}{k} & \text{if } x = \frac{m}{k} \text{ where } m, k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at all irrational $c \in (0, 1)$ and discontinuous at all rational c . See the graph of f in .

Graph of the "popcorn function."

Graph of the "popcorn function."

Proof: Suppose $c = \frac{m}{k}$ is rational. Take a sequence of irrational numbers $\{x_n\}$ such that $\lim x_n = c$. Then $\lim f(x_n) = \lim 0 = 0$, but $f(c) = \frac{1}{k} \neq 0$. So f is discontinuous at c .

Now let c be irrational, so $f(c) = 0$. Take a sequence $\{x_n\}$ of numbers in $(0, 1)$ such that $\lim x_n = c$. Given $\epsilon > 0$, find $K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$ by the . If $\frac{m}{k} \in (0, 1)$ is lowest terms (no common divisors), then $m < k$. So there are only finitely many rational numbers in $(0, 1)$ whose denominator k in lowest terms is less than K . Hence there is an M such that for $n \geq M$, all the numbers x_n that are rational have a denominator larger than or equal to K . Thus for $n \geq M$

$$|f(x_n) - 0| = f(x_n) \leq \frac{1}{K} < \epsilon.$$

Therefore f is continuous at irrational c .

Let us end on an easier example.

Define $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) := 0$ if $x \neq 0$ and $g(0) := 1$. Then g is not continuous at zero, but continuous everywhere else (why?). The point $x = 0$ is called a *removable discontinuity*. That is because if we would change the definition of g , by insisting that $g(0)$ be 0, we would obtain a continuous function. On the other hand let f be the function of example . Then f does not have a removable discontinuity at 0. No matter how we would define $f(0)$ the function will still fail to be continuous. The difference is that $\lim_{x \rightarrow 0} g(x)$ exists while $\lim_{x \rightarrow 0} f(x)$ does not.

Let us stay with this example but show another phenomenon. Let $A = \{0\}$, then $g|_A$ is continuous (why?), while g is not continuous on A .

Exercises

Using the definition of continuity directly prove that $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) := x^2$ is continuous.

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Using the definition of continuity directly prove that $f: (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is continuous.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Prove .

Prove the following statement. Let $S \subset \mathbb{R}$ and $A \subset S$. Let $f: S \rightarrow \mathbb{R}$ be a continuous function. Then the restriction $f|_A$ is continuous.

Suppose $S \subset \mathbb{R}$. Suppose for some $c \in \mathbb{R}$ and $\alpha > 0$, we have $A = (c - \alpha, c + \alpha) \subset S$. Let $f: S \rightarrow \mathbb{R}$ be a function. Prove that if $f|_A$ is continuous at c , then f is continuous at c .

Give an example of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ such that the function h defined by $h(x) := f(x) + g(x)$ is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r , $f(r) = g(r)$. Show that $f(x) = g(x)$ for all x .

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$ we have $f(x) > 0$.

Let $f: \mathbb{Z} \rightarrow \mathbb{R}$ be a function. Show that f is continuous.

[exercise:contseqalt] Let $f: S \rightarrow \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}$ in S with $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges. Show that f is continuous at c .

Suppose $f: [-1, 0] \rightarrow \mathbb{R}$ and $g: [0, 1] \rightarrow \mathbb{R}$ are continuous and $f(0) = g(0)$. Define $h: [-1, 1] \rightarrow \mathbb{R}$ by $h(x) := f(x)$ if $x \leq 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.

Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x - y)$ for all x and y . Show that f is continuous.

Suppose $f(x + y) = f(x) + f(y)$ for some $f: \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous at 0. Show that $f(x) = ax$ for some $a \in \mathbb{R}$. Hint: Show that $f(nx) = nf(x)$, then show f is continuous on $\mathbb{R}$. Then show that $\frac{f(x)}{x} = f(1)$ for all rational x .

Min-max and intermediate value theorems

Note: 1.5 lectures

Continuous functions defined on closed and bounded intervals have some interesting and very useful properties.

Min-max theorem

Recall a function $f: [a, b] \rightarrow \mathbb{R}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$. We have the following lemma.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f is bounded.

Let us prove this claim by contrapositive. Suppose f is not bounded, then for each $n \in \mathbb{N}$, there is an $x_n \in [a, b]$, such that

$$|f(x_n)| \geq n.$$

Now $\{x_n\}$ is a bounded sequence as $a \leq x_n \leq b$. By the , there is a convergent subsequence $\{x_{n_i}\}$. Let $x := \lim_{i \rightarrow \infty} x_{n_i}$. Since $a \leq x_{n_i} \leq b$ for all i , then $a \leq x \leq b$. The limit $\lim_{i \rightarrow \infty} f(x_{n_i})$ does not exist as the sequence is not bounded as $|f(x_{n_i})| \geq n_i \geq i$. On the other hand $f(x)$ is a finite number and

$$f(x) = f\left(\lim_{i \rightarrow \infty} x_{n_i}\right).$$

Thus f is not continuous at x .

In fact, for a continuous f , we will see that the minimum and the maximum are actually achieved. Recall from calculus that $f: S \rightarrow \mathbb{R}$ achieves an *absolute minimum* at $c \in S$ if

$$f(x) \geq f(c) \quad \text{for all } x \in S.$$

On the other hand, f achieves an *absolute maximum* at $c \in S$ if

$$f(x) \leq f(c) \quad \text{for all } x \in S.$$

We say f achieves an absolute minimum or an absolute maximum on S if such a $c \in S$ exists. If S is a closed and bounded interval, then a continuous f must have an absolute minimum and an absolute maximum on S .

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f achieves both an absolute minimum and an absolute maximum on $[a, b]$.

We have shown that f is bounded by the lemma. Therefore, the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ has a supremum and an infimum. From what we know about suprema and infima, there exist sequences in the set $f([a, b])$ that approach them. That is, there are sequences $\{f(x_n)\}$ and $\{f(y_n)\}$, where x_n, y_n are in $[a, b]$, such that

$$\lim_{n \rightarrow \infty} f(x_n) = \inf f([a, b]) \quad \text{and} \quad \lim_{n \rightarrow \infty} f(y_n) = \sup f([a, b]).$$

We are not done yet, we need to find where the minimum and the maxima are. The problem is that the sequences $\{x_n\}$ and $\{y_n\}$ need not converge. We know $\{x_n\}$ and $\{y_n\}$ are bounded (their

Processing math: 52% a bounded interval $[a, b]$). We apply the . Hence there exist convergent subsequences $\{x_{n_i}\}$ and $\{y_{m_i}\}$. Let

$$x := \lim_{i \rightarrow \infty} x_{n_i} \quad \text{and} \quad y := \lim_{i \rightarrow \infty} y_{m_i}.$$

Then as $a \leq x_{n_i} \leq b$, we have that $a \leq x \leq b$. Similarly $a \leq y \leq b$, so x and y are in $[a, b]$. We apply that a limit of a subsequence is the same as the limit of the sequence, and we apply the continuity of f to obtain

$$\inf f([a, b]) = \lim_{n \rightarrow \infty} f(x_{n_i}) = \lim_{i \rightarrow \infty} f(x_{n_i}) = f\left(\lim_{i \rightarrow \infty} x_{n_i}\right) = f(x).$$

Similarly,

$$\sup f([a, b]) = \lim_{n \rightarrow \infty} f(m_{n_i}) = \lim_{i \rightarrow \infty} f(m_{n_i}) = f\left(\lim_{i \rightarrow \infty} y_{m_i}\right) = f(y).$$

Therefore, f achieves an absolute minimum at x and f achieves an absolute maximum at y .

The function $f(x) := x^2 + 1$ defined on the interval $[-1, 2]$ achieves a minimum at $x = 0$ when $f(0) = 1$. It achieves a maximum at $x = 2$ where $f(2) = 5$. Do note that the domain of definition matters. If we instead took the domain to be $[-10, 10]$, then $x = 2$ would no longer be a maximum of f . Instead the maximum would be achieved at either $x = 10$ or $x = -10$.

Let us show by examples that the different hypotheses of the theorem are truly needed.

The function $f(x) := x$, defined on the whole real line, achieves neither a minimum, nor a maximum. So it is important that we are looking at a bounded interval.

The function $f(x) := \frac{1}{x}$, defined on $(0, 1)$ achieves neither a minimum, nor a maximum. The values of the function are unbounded as we approach 0. Also as we approach $x = 1$, the values of the function approach 1, but $f(x) > 1$ for all $x \in (0, 1)$. There is no $x \in (0, 1)$ such that $f(x) = 1$. So it is important that we are looking at a closed interval.

Continuity is important. Define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(x) := \frac{1}{x}$ for $x > 0$ and let $f(0) := 0$. Then the function does not achieve a maximum. The problem is that the function is not continuous at 0.

[Bolzano's intermediate value theorem](#)

Bolzano's intermediate value theorem is one of the cornerstones of analysis. It is sometimes called only intermediate value theorem, or just Bolzano's theorem. To prove Bolzano's theorem we prove the following simpler lemma.

[IVT:lemma] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

We define two sequences $\{a_n\}$ and $\{b_n\}$ inductively:

- Let $a_1 := a$ and $b_1 := b$.
- If $f\left(\frac{a_n + b_n}{2}\right) \geq 0$, let $a_{n+1} := a_n$ and $b_{n+1} := \frac{a_n + b_n}{2}$.
- If $f\left(\frac{a_n + b_n}{2}\right) < 0$, let $a_{n+1} := \frac{a_n + b_n}{2}$ and $b_{n+1} := b_n$.

See for an example defining the first five steps. From the definition of the two sequences it is obvious that if $a_n < b_n$, then $a_{n+1} < b_{n+1}$. Thus by $a_n < b_n$ for all n . Furthermore, $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n , that is the sequences are monotone. As $a_n < b_n \leq b_1 = b$ and $b_n > a_n \geq a_1 = a$ for all n , the sequences are also bounded. Therefore, the sequences converge. Let $c := \lim a_n$ and $d := \lim b_n$. We now want to show that $c = d$. We notice

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}.$$

By we see that

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = 2^{1-n}(b - a).$$

As $2^{1-n}(b - a)$ converges to zero, we take the limit as n goes to infinity to get

$$d - c = \lim_{n \rightarrow \infty} (b_n - a_n) = \lim_{n \rightarrow \infty} 2^{1-n}(b - a) = 0.$$

In other words $d = c$.

By construction, for all n we have

$$f(a_n) < 0 \quad \text{and} \quad f(b_n) \geq 0.$$

We use the fact that $\lim a_n = \lim b_n = c$ and the continuity of f to take limits in those inequalities to get

$$f(c) = \lim_{n \rightarrow \infty} f(a_n) \leq 0 \quad \text{and} \quad f(c) = \lim_{n \rightarrow \infty} f(b_n) \geq 0.$$

As $f(c) \geq 0$ and $f(c) \leq 0$, we conclude $f(c) = 0$. Obviously, $a < c < b$.

Notice that the proof tells us how to find the c . The proof is not only useful for us pure mathematicians, but it is a useful idea in applied mathematics.

[IVT:thm] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Suppose there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists a $c \in (a, b)$ such that $f(c) = y$.

The theorem says that a continuous function on a closed interval achieves all the values between the values at the endpoints.

If $f(a) < y < f(b)$, then define $g(x) := f(x) - y$. Then we see that $g(a) < 0$ and $g(b) > 0$ and we can apply to g . If $g(c) = 0$, then $f(c) = y$.

Similarly if $f(a) > y > f(b)$, then define $g(x) := y - f(x)$. Then again $g(a) < 0$ and $g(b) > 0$ and we can apply . Again if $g(c) = 0$, then $f(c) = y$.

If a function is continuous, then the restriction to a subset is continuous. So if $f: S \rightarrow \mathbb{R}$ is continuous and $[a, b] \subset S$, then $f|_{[a, b]}$ is also continuous. Hence, we generally apply the theorem to a function continuous on some large set S , but we restrict attention to an interval.

The polynomial $f(x) := x^3 - 2x^2 + x - 1$ has a real root in $(1, 2)$. We simply notice that $f(1) = -1$ and $f(2) = 1$. Hence there must exist a point $c \in (1, 2)$ such that $f(c) = 0$. To find a better approximation of the root we could follow the proof of . For example, next we would look at 1.5 and find that $f(1.5) = -0.625$. Therefore, there is a root of the equation in $(1.5, 2)$. Next we look at

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1.75 and note that $f(1.75) \approx -0.016$. Hence there is a root of f in $(1.75, 2)$. Next we look at 1.875 and find that $f(1.875) \approx 0.44$, thus there is a root in $(1.75, 1.875)$. We follow this procedure until we gain sufficient precision.

The technique above is the simplest method of finding roots of polynomials. Finding roots of polynomials is perhaps the most common problem in applied mathematics. In general it is hard to do quickly, precisely and automatically. We can use the intermediate value theorem to find roots for any continuous function, not just a polynomial.

There are better and faster methods of finding roots of equations, such as Newton's method. One advantage of the above method is its simplicity. The moment we find an initial interval where the intermediate value theorem applies, we are guaranteed to find a root up to a desired precision in finitely many steps. Furthermore, the method only requires a continuous function.

The theorem guarantees at least one c such that $f(c) = y$, but there may be many different roots of the equation $f(c) = y$. If we follow the procedure of the proof, we are guaranteed to find approximations to one such root. We need to work harder to find any other roots.

Polynomials of even degree may not have any real roots. For example, there is no real number x such that $x^2 + 1 = 0$. Odd polynomials, on the other hand, always have at least one real root.

Let $f(x)$ be a polynomial of odd degree. Then f has a real root.

Suppose f is a polynomial of odd degree d . We write

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0,$$

where $a_d \neq 0$. We divide by a_d to obtain a polynomial

$$g(x) := x^d + b_{d-1} x^{d-1} + \dots + b_1 x + b_0,$$

where $b_k = \frac{a_k}{a_d}$. Let us show that $g(n)$ is positive for some large $n \in \mathbb{N}$.

$$\begin{aligned} \left| \frac{b_{d-1} n^{d-1} + \dots + b_1 n + b_0}{n^d} \right| &= \frac{|b_{d-1} n^{d-1} + \dots + b_1 n + b_0|}{n^d} \\ &\leq \frac{|b_{d-1}| n^{d-1} + \dots + |b_1| n + |b_0|}{n^d} \\ &\leq \frac{|b_{d-1}| n^{d-1} + \dots + |b_1| n^{d-1} + |b_0| n^{d-1}}{n^d} \\ &= \frac{n^{d-1} (|b_{d-1}| + \dots + |b_1| + |b_0|)}{n^d} \\ &= \frac{1}{n} (|b_{d-1}| + \dots + |b_1| + |b_0|). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{b_{d-1} n^{d-1} + \dots + b_1 n + b_0}{n^d} = 0.$$

Thus there exists an $M \in \mathbb{N}$ such that

$$\left| \frac{b_{d-1} M^{d-1} + \dots + b_1 M + b_0}{M^d} \right| < 1,$$

which implies

$$-(b_{d-1} M^{d-1} + \dots + b_1 M + b_0) < M^d.$$

Therefore $g(M) > 0$.

Next we look at $g(-n)$ for $n \in \mathbb{N}$. By a similar argument (exercise) we find that there exists some $K \in \mathbb{N}$ such that $b_{d-1}(-K)^{d-1} + \dots + b_1(-K) + b_0 < K^d$ and therefore $g(-K) < 0$ (why?). In the proof make sure you use the fact that d is odd. In particular, if d is odd then $(-n)^d = -(n^d)$.

We appeal to the intermediate value theorem, to find a $c \in [-K, M]$ such that $g(c) = 0$. As $g(x) = \frac{f(x)}{a_d}$, we see that $f(c) = 0$, and the proof is done.

An interesting fact is that there do exist discontinuous functions that have the intermediate value property. The function

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0, however, it has the intermediate value property. That is, for any $a < b$, and any y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, there exists a c such that $f(y) = c$. Proof is left as an exercise.

Exercises

Find an example of a discontinuous function $f: [0, 1] \rightarrow \mathbb{R}$ where the intermediate value theorem fails.

Find an example of a *bounded* discontinuous function $f: [0, 1] \rightarrow \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.

Let $f: (0, 1) \rightarrow \mathbb{R}$ be a continuous function such that $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0, 1)$ (but perhaps not both).

Let

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

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Show that f has the intermediate value property. That is, for any $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = y$.

Suppose $g(x)$ is a polynomial of odd degree d such that

$$g(x) = x^d + b_{d-1}x^{d-1} + \dots + b_1x + b_0,$$

for some real numbers $b_0, b_1, \dots, b_{d-1}$. Show that there exists a $K \in \mathbb{N}$ such that $g(-K) < 0$. Hint: Make sure to use the fact that d is odd. You will have to use that $(-n)^d = -(n^d)$.

Suppose $g(x)$ is a polynomial of positive even degree d such that

$$g(x) = x^d + b_{d-1}x^{d-1} + \dots + b_1x + b_0,$$

for some real numbers $b_0, b_1, \dots, b_{d-1}$. Suppose $g(0) < 0$. Show that g has at least two distinct real roots.

[exercise:imageofinterval] Suppose $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function. Prove that the direct image $f([a, b])$ is a closed and bounded interval or a single number.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and periodic with period $P > 0$. That is, $f(x + P) = f(x)$ for all $x \in \mathbb{R}$. Show that f achieves an absolute minimum and an absolute maximum.

Suppose $f(x)$ is a bounded polynomial, in other words, there is an M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f must be a constant.

Suppose $f: [0, 1] \rightarrow [0, 1]$ is continuous. Show that f has a fixed point, in other words, show that there exists an $x \in [0, 1]$ such that $f(x) = x$.

Find an example of a bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ that does not achieve an absolute minimum nor an absolute maximum on $\mathbb{R}$.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $x \leq f(x) \leq x + 1$ for all $x \in \mathbb{R}$. Find $f(\mathbb{R})$.

True/False, prove or find a counterexample. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $\forall f|_{\mathbb{R}}$

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$\forall$ is bounded, then f is bounded.

[Uniform continuity](#)

Note: 1.5–2 lectures (Continuous extension and Lipschitz can be optional)

[Uniform continuity](#)

We made a fuss of saying that the δ in the definition of continuity depended on the point c . There are situations when it is advantageous to have a δ independent of any point. Let us give a name to this concept.

Let $S \subset \mathbb{R}$, and let $f: S \rightarrow \mathbb{R}$ be a function. Suppose for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Then we say f is *uniformly continuous*.

It is not hard to see that a uniformly continuous function must be continuous. The only difference in the definitions is that for a given $\epsilon > 0$ we pick a $\delta > 0$ that works for all $c \in S$. That is, δ can no longer depend on c , it only depends on ϵ . The domain of definition of the function makes a difference now. A function that is not uniformly continuous on a larger set, may be uniformly continuous when restricted to a smaller set.

The function $f: (0, 1) \rightarrow \mathbb{R}$, defined by $f(x) := \frac{1}{x}$ is not uniformly continuous, but it is continuous.

Proof: Given $\epsilon > 0$, then for $\epsilon > |\frac{1}{x} - \frac{1}{y}|$ to hold we must have

$$\epsilon > \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|y - x|}{|xy|} = \frac{|y - x|}{xy},$$

or

$$|x - y| < xy\epsilon.$$

Therefore, to satisfy the definition of uniform continuity we would have to have $\delta \leq xy\epsilon$ for all x, y in $(0, 1)$, but that would mean that $\delta \leq 0$. Therefore there is no single $\delta > 0$.

$f: [0, 1] \rightarrow \mathbb{R}$, defined by $f(x) := x^2$ is uniformly continuous.

Proof: Note that $0 \leq x, c \leq 1$. Then

$$\left| x^2 - c^2 \right| = |x + c||x - c| \leq (|x| + |c|)|x - c| \leq (1 + 1)|x - c|.$$

Therefore given $\epsilon > 0$, let $\delta := \frac{\epsilon}{2}$. If $|x - c| < \delta$, then $\left| x^2 - c^2 \right| < \epsilon$.

On the other hand, $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) := x^2$ is not uniformly continuous.

Proof: Suppose it is uniformly continuous, then for all $\epsilon > 0$, there would exist a $\delta > 0$ such that if $|x - c| < \delta$, then $\left| x^2 - c^2 \right| < \epsilon$. Take $x > 0$ and let $c := x + \frac{\epsilon}{2}$. Write

$$\epsilon > \left| x^2 - c^2 \right| = |x + c||x - c| = (2x + \frac{\epsilon}{2})\frac{\epsilon}{2} \geq \delta x.$$

Therefore $x < \frac{\epsilon}{2\delta}$ for all $x > 0$, which is a contradiction.

We have seen that if f is defined on an interval that is either not closed or not bounded, then f can be continuous, but not uniformly continuous. For a closed and bounded interval $[a, b]$, we can, however, make the following statement.

[unifcont:thm] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous.

We prove the statement by contrapositive. Suppose f is not uniformly continuous. We will prove that there is some $c \in [a, b]$ where f is not continuous. Let us negate the definition of uniformly continuous. There exists an $\epsilon > 0$ such that for every $\delta > 0$, there exist points x, y in S with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

So for the $\epsilon > 0$ above, we find sequences $\{x_n\}$ and $\{y_n\}$ such that $|x_n - y_n| < \frac{1}{n}$ and such that $|f(x_n) - f(y_n)| \geq \epsilon$. By , there exists a convergent subsequence $\{x_{n_k}\}$. Let $c := \lim x_{n_k}$. As $a \leq x_{n_k} \leq b$, then $a \leq c \leq b$. Write

$$\left| y_{n_k} - c \right| = \left| y_{n_k} - x_{n_k} + x_{n_k} - c \right| \leq \left| y_{n_k} - x_{n_k} \right| + \left| x_{n_k} - c \right| < \frac{1}{n_k} + \left| x_{n_k} - c \right|.$$

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As $\frac{1}{n_k}$ and $|x_{n_k} - c|$ both go to zero when k goes to infinity, $\{y_{n_k}\}$ converges and the limit is c . We now show that f is not continuous at c . We estimate

$$\begin{aligned} |f(x_{n_k}) - f(c)| &= |f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c)| \\ &\geq |f(x_{n_k}) - f(y_{n_k})| - |f(y_{n_k}) - f(c)| \\ &\geq \epsilon - |f(y_{n_k}) - f(c)|. \end{aligned}$$

Or in other words

$$|f(x_{n_k}) - f(c)| + |f(y_{n_k}) - f(c)| \geq \epsilon.$$

At least one of the sequences $\{f(x_{n_k})\}$ or $\{f(y_{n_k})\}$ cannot converge to $f(c)$, otherwise the left hand side of the inequality would go to zero while the right-hand side is positive. Thus f cannot be continuous at c .

Continuous extension

Before we get to uniform extension, we show the following useful lemma. It says that uniformly continuous functions behave nicely with respect to Cauchy sequences. The new issue here is that for a Cauchy sequence we no longer know where the limit ends up; it may not end up in the domain of the function.

[unifcauchycauchy:lemma] Let $f: S \rightarrow \mathbb{R}$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence in S . Then $\{f(x_n)\}$ is Cauchy.

Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Now find an $M \in \mathbb{N}$ such that for all $n, k \geq M$ we have $|x_n - x_k| < \delta$. Then for all $n, k \geq M$ we have $|f(x_n) - f(x_k)| < \epsilon$.

An application of the above lemma is the following theorem. It says that a function on an open interval is uniformly continuous if and only if it can be extended to a continuous function on the closed interval.

[context:thm] A function $f: (a, b) \rightarrow \mathbb{R}$ is uniformly continuous if and only if the limits

$$L_a := \lim_{x \rightarrow a} f(x) \quad \text{and} \quad L_b := \lim_{x \rightarrow b} f(x)$$

exist and the function $\tilde{f}: [a, b] \rightarrow \mathbb{R}$ defined by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in (a, b), \\ L_a & \text{if } x = a, \\ L_b & \text{if } x = b, \end{cases}$$

is continuous.

One direction is not hard to prove. If $\tilde{f}$ is continuous, then it is uniformly continuous by . As f is the restriction of $\tilde{f}$ to (a, b) , then f is also uniformly continuous (easy exercise).

Now suppose f is uniformly continuous. We must first show that the limits L_a and L_b exist. Let us concentrate on L_a . Take a sequence $\{x_n\}$ in (a, b) such that $\lim x_n = a$. The sequence is a Cauchy sequence and hence by , the sequence $\{f(x_n)\}$ is Cauchy and therefore convergent. We have some number $L_1 := \lim f(x_n)$. Take another sequence $\{y_n\}$ in (a, b) such that $\lim y_n = a$. By the same reasoning we get $L_2 := \lim f(y_n)$. If we show that $L_1 = L_2$, then the limit $L_a = \lim_{x \rightarrow a} f(x)$ exists. Let $\epsilon > 0$ be given, find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{3}$. Find $M \in \mathbb{N}$ such that for $n \geq M$ we have $|a - x_n| < \frac{\delta}{2}$, $|a - y_n| < \frac{\delta}{2}$, $|f(x_n) - L_1| < \frac{\epsilon}{3}$, and $|f(y_n) - L_2| < \frac{\epsilon}{3}$. Then for $n \geq M$ we have

$$|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

So

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x_n) + f(x_n) - f(y_n) + f(y_n) - L_2| \\ &\leq |L_1 - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - L_2| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore $L_1 = L_2$. Thus L_a exists. To show that L_b exists is left as an exercise.

Now that we know that the limits L_a and L_b exist, we are done. If $\lim_{x \rightarrow a} f(x)$ exists, then $\lim_{x \rightarrow a} \tilde{f}(x)$ exists (See). Similarly with L_b . Hence $\tilde{f}$ is continuous at a and b . And since f is continuous at $c \in (a, b)$, then $\tilde{f}$ is continuous at $c \in (a, b)$.

Lipschitz continuous functions

Let $f: S \rightarrow \mathbb{R}$ be a function such that there exists a number K such that for all x and y in S we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Then f is said to be *Lipschitz continuous* ¹⁶.

A large class of functions is Lipschitz continuous. Be careful, just as for uniformly continuous functions, the domain of definition of the function is important. See the examples below and the exercises. First we justify the use of the word *continuous*.

A Lipschitz continuous function is uniformly continuous.

Let $f: S \rightarrow \mathbb{R}$ be a function and let K be a constant such that for all x, y in S we have $|f(x) - f(y)| \leq K|x - y|$.

Let $\epsilon > 0$ be given. Take $\delta := \frac{\epsilon}{K}$. For any x and y in S such that $|x - y| < \delta$ we have that

$$|f(x) - f(y)| \leq K|x - y| < K\delta = K \frac{\epsilon}{K} = \epsilon.$$

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Therefore f is uniformly continuous.

We interpret Lipschitz continuity geometrically. If f is a Lipschitz continuous function with some constant K . We rewrite the inequality to say that for $x \neq y$ we have

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq K.$$

The quantity $\frac{f(x) - f(y)}{x - y}$ is the slope of the line between the points $(x, f(x))$ and $(y, f(y))$, that is, a *secant line*. Therefore, f is Lipschitz continuous if and only if every line that intersects the graph of f in at least two distinct points has slope less than or equal to K . See .

The functions $\sin(x)$ and $\cos(x)$ are Lipschitz continuous. We have seen () the following two inequalities.

$$|\sin(x) - \sin(y)| \leq |x - y| \quad \text{and} \quad |\cos(x) - \cos(y)| \leq |x - y|.$$

Hence $\sin$ and $\cos$ are Lipschitz continuous with $K = 1$.

The function $f: [1, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is Lipschitz continuous. Proof:

$$\left| \sqrt{x} - \sqrt{y} \right| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}}.$$

As $x \geq 1$ and $y \geq 1$, we see that $\frac{1}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}$. Therefore

$$\left| \sqrt{x} - \sqrt{y} \right| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq \frac{1}{2} |x - y|.$$

On the other hand $f: [0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is not Lipschitz continuous. Let us see why: Suppose we have

$$\left| \sqrt{x} - \sqrt{y} \right| \leq K|x - y|,$$

for some K . Let $y = 0$ to obtain $\sqrt{x} \leq Kx$. If $K > 0$, then for $x > 0$ we then get $\frac{1}{K} \leq \sqrt{x}$. This cannot possibly be true for all $x > 0$. Thus no such $K > 0$ exists and f is not Lipschitz continuous.

The last example is a function that is uniformly continuous but not Lipschitz continuous. To see that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$ note that it is uniformly continuous on $[0, 1]$ by . It is also Lipschitz (and therefore uniformly continuous) on $[1, \infty)$. It is not hard (exercise) to show that this means that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Exercises

Let $f: S \rightarrow \mathbb{R}$ be uniformly continuous. Let $A \subset S$. Then the restriction $f|_A$ is uniformly continuous.

Let $f: (a, b) \rightarrow \mathbb{R}$ be a uniformly continuous function. Finish the proof of by showing that the limit $\lim_{x \rightarrow b} f(x)$ exists.

Show that $f: (c, \infty) \rightarrow \mathbb{R}$ for some $c > 0$ and defined by $f(x) := \frac{1}{x}$ is Lipschitz continuous.

Show that $f: (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is not Lipschitz continuous.

Let A, B be intervals. Let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be uniformly continuous functions such that $f(x) = g(x)$ for $x \in A \cap B$. Define the function $h: A \cup B \rightarrow \mathbb{R}$ by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B \setminus A$. a) Prove that if $A \cap B \neq \emptyset$, then h is uniformly continuous. b) Find an example where $A \cap B = \emptyset$ and h is not even continuous.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree $d \geq 2$. Show that f is not Lipschitz continuous.

Let $f: (0, 1) \rightarrow \mathbb{R}$ be a bounded continuous function. Show that the function $g(x) := x(1 - x)f(x)$ is uniformly continuous.

Show that $f: (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) := \sin(\frac{1}{x})$ is not uniformly continuous.

Let $f: \mathbb{Q} \rightarrow \mathbb{R}$ be a uniformly continuous function. Show that there exists a uniformly continuous function $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \tilde{f}(x)$ for all $x \in \mathbb{Q}$.

a) Find a continuous $f: (0, 1) \rightarrow \mathbb{R}$ and a sequence $\{x_n\}$ in $(0, 1)$ that is Cauchy, but such that $\{f(x_n)\}$ is not Cauchy. b) Prove that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

a) If $f: S \rightarrow \mathbb{R}$ and $g: S \rightarrow \mathbb{R}$ are uniformly continuous, then show that $h: S \rightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f: S \rightarrow \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then show that $h: S \rightarrow \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

a) If $f: S \rightarrow \mathbb{R}$ and $g: S \rightarrow \mathbb{R}$ are Lipschitz, then show that $h: S \rightarrow \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz.

b) If $f: S \rightarrow \mathbb{R}$ is Lipschitz and $a \in \mathbb{R}$, then show that $h: S \rightarrow \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz.

a) If $f: [0, 1] \rightarrow \mathbb{R}$ is given by $f(x) := x^m$ for an integer $m \geq 0$, show f is Lipschitz and find the best (the smallest) Lipschitz constant K (depending on m of course). Hint: $(x - y)(x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \dots + xy^{m-2} + y^{m-1}) = x^m - y^m$.

b) Using the previous exercise, show that if $f: [0, 1] \rightarrow \mathbb{R}$ is a polynomial, that is, $f(x) := a_mx^m + a_{m-1}x^{m-1} + \dots + a_0$, then f is Lipschitz.

Suppose for $f: [0, 1] \rightarrow \mathbb{R}$ we have $|f(x) - f(y)| \leq K|x - y|$, and $f(0) = f(1) = 0$. Prove that $|f(x)| \leq \frac{K}{2}$. Further show by example that $\frac{K}{2}$ is the best possible, that is, there exists such a continuous function for which $|f(x)| = \frac{K}{2}$ for some $x \in [0, 1]$.

Limits at infinity

Note: less than 1 lecture (optional, can safely be omitted unless or is also covered)

Limits at infinity

As for sequences, a continuous variable can also approach infinity. Let us make this notion precise.

We say ∞ is a cluster point of $S \subset \mathbb{R}$, if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \geq M$. Similarly $-\infty$ is a cluster point of $S \subset \mathbb{R}$, if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \leq M$.

Let $f: S \rightarrow \mathbb{R}$ be a function, where ∞ is a cluster point of S . If there exists an $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

whenever $x \geq M$, then we say $f(x)$ converges to L as x goes to ∞ . We call L the *limit* and write

$$\lim_{x \rightarrow \infty} f(x) := L.$$

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Alternatively we write $f(x) \rightarrow L$ as $x \rightarrow \infty$.

Similarly, if $-\infty$ is a cluster point of S and there exists an $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \epsilon$$

whenever $x \leq M$, then we say $f(x)$ converges to L as x goes to $-\infty$. We call L the *limit* and write

$$\lim_{x \rightarrow -\infty} f(x) := L.$$

Alternatively we write $f(x) \rightarrow L$ as $x \rightarrow -\infty$.

We cheated a little bit again and said *the* limit. We leave it as an exercise for the reader to prove the following proposition.

[liminfy:unique] The limit at ∞ or $-\infty$ as defined above is unique if it exists.

Let $f(x) := \frac{1}{|x|+1}$. Then

$$\lim_{x \rightarrow \infty} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 0.$$

Proof: Let $\epsilon > 0$ be given. Find $M > 0$ large enough so that $\frac{1}{M+1} < \epsilon$. If $x \geq M$, then $\frac{1}{x+1} \leq \frac{1}{M+1} < \epsilon$. Since $\frac{1}{|x|+1} > 0$ for all x the first limit is proved. The proof for $-\infty$ is left to the reader.

Let $f(x) := \sin(\pi x)$. Then $\lim_{x \rightarrow \infty} f(x)$ does not exist. To prove this fact note that if $x = 2n + \frac{1}{2}$ for some $n \in \mathbb{N}$ then $f(x) = 1$, while if $x = 2n + \frac{3}{2}$ then $f(x) = -1$, so they cannot both be within a small ϵ of a single real number.

We must be careful not to confuse continuous limits with limits of sequences. For $f(x) = \sin(\pi x)$ we could say

$$\lim_{n \rightarrow \infty} f(n) = 0, \quad \text{but} \quad \lim_{x \rightarrow \infty} f(x) \text{ does not exist.}$$

Of course the notation is ambiguous. We are simply using the convention that $n \in \mathbb{N}$, while $x \in \mathbb{R}$. When the notation is not clear, it is good to explicitly mention where the variable lives, or what kind of limit are you using.

There is a connection of continuous limits to limits of sequences, but we must take all sequences going to infinity, just as before in .

[seqlimitinf:lemma] Suppose $f: S \rightarrow \mathbb{R}$ is a function, ∞ is a cluster point of $S \subset \mathbb{R}$, and $L \in \mathbb{R}$. Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

if and only if

$$\lim_{n \rightarrow \infty} f(x_n) = L$$

for all sequences $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = \infty$.

The lemma holds for the limit as $x \rightarrow -\infty$. Its proof is almost identical and is left as an exercise.

First suppose $f(x) \rightarrow L$ as $x \rightarrow \infty$. Given an $\epsilon > 0$, there exists an M such that for all $x \geq M$ we have $|f(x) - L| < \epsilon$. Let $\{x_n\}$ be a sequence in S such that $\lim_{n \rightarrow \infty} x_n = \infty$. Then there exists an N such that for all $n \geq N$ we have $x_n \geq M$. And thus $|f(x_n) - L| < \epsilon$.

We prove the converse by contrapositive. Suppose $f(x)$ does not go to L as $x \rightarrow \infty$. This means that there exists an $\epsilon > 0$, such that for every $M \in \mathbb{N}$, there exists an $x \in S$, $x \geq M$, let us call it x_M , such that $|f(x_M) - L| \geq \epsilon$. Consider the sequence $\{x_n\}$. Clearly $\{f(x_n)\}$ does not converge to L . It remains to note that $\lim_{n \rightarrow \infty} x_n = \infty$, because $x_n \geq n$ for all n .

Using the lemma, we again translate results about sequential limits into results about continuous limits as x goes to infinity. That is, we have almost immediate analogues of the corollaries in . We simply allow the cluster point c to be either ∞ or $-\infty$, in addition to a real number. We leave it to the student to verify these statements.

Infinite limit

Just as for sequences, it is often convenient to distinguish certain divergent sequences, and talk about limits being infinite almost as if the limits existed.

Let $f: S \rightarrow \mathbb{R}$ be a function and suppose S has ∞ as a cluster point. We say $f(x)$ *diverges to infinity* as x goes to ∞ , if for every $N \in \mathbb{R}$ there exists an $M \in \mathbb{R}$ such that

$$f(x) > N$$

whenever $x \in S$ and $x \geq M$. We write

$$\lim_{x \rightarrow \infty} f(x) := \infty,$$

or we say that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$.

A similar definition can be made for limits as $x \rightarrow -\infty$ or as $x \rightarrow c$ for a finite c . Also similar definitions can be made for limits being $-\infty$. Stating these definitions is left as an exercise. Note that sometimes *converges to infinity* is used. We can again use sequential limits, and an analogue of is left as an exercise.

Let us show that $\lim_{x \rightarrow \infty} \frac{1+x^2}{1+x} = \infty$.

Proof: For $x \geq 1$ we have

$$\frac{1+x^2}{1+x} \geq \frac{x^2}{x+x} = \frac{x}{2}.$$

Given $N \in \mathbb{R}$, take $M = \max \{2N + 1, 1\}$. If $x \geq M$, then $x \geq 1$ and $\frac{x}{2} > N$. So

$$\frac{1+x^2}{1+x} \geq \frac{x}{2} > N.$$

Compositions

Finally, just as for limits at finite numbers we can compose functions easily.

[prop:inlimcompositions] Suppose $f: A \rightarrow B$, $g: B \rightarrow \mathbb{R}$, $A, B \subset \mathbb{R}$, $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of A , and $b \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of B . Suppose

$$\lim_{x \rightarrow a} f(x) = b \quad \text{and} \quad \lim_{y \rightarrow b} g(y) = c$$

for some $c \in \mathbb{R} \cup \{-\infty, \infty\}$. If $b \in B$, then suppose $g(b) = c$. Then

$$\lim_{x \rightarrow a} g(f(x)) = c.$$

The proof is straightforward, and left as an exercise. We already know the proposition when $a, b, c \in \mathbb{R}$, see Exercises [exercise:contlimitcomposition] and [exercise:contlimitbadcomposition]. Again the requirement that g is continuous at b , if $b \in B$, is necessary.

Let $h(x) := e^{-x^2+x}$. Then

$$\lim_{x \rightarrow \infty} h(x) = 0.$$

Proof: The claim follows once we know

$$\lim_{x \rightarrow \infty} -x^2 + x = -\infty$$

and

$$\lim_{y \rightarrow -\infty} e^y = 0,$$

which is usually proved when the exponential function is defined.

Exercises

Prove .

Let $f: [1, \infty) \rightarrow \mathbb{R}$ be a function. Define $g: (0, 1] \rightarrow \mathbb{R}$ via $g(x) := f(\frac{1}{x})$. Using the definitions of limits directly, show that $\lim_{x \rightarrow 0^+} g(x)$ exists if and only if $\lim_{x \rightarrow \infty} f(x)$ exists, in which case they are equal.

Prove .

Let us justify terminology. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\lim_{x \rightarrow \infty} f(x) = \infty$ (diverges to infinity). Show that $f(x)$ diverges (i.e. does not converge) as $x \rightarrow \infty$.

Come up with the definitions for limits of $f(x)$ going to $-\infty$ as $x \rightarrow \infty$, $x \rightarrow -\infty$, and as $x \rightarrow c$ for a finite $c \in \mathbb{R}$. Then state the definitions for limits of $f(x)$ going to ∞ as $x \rightarrow -\infty$, and as $x \rightarrow c$ for a finite $c \in \mathbb{R}$.

Suppose $P(x) := x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a *monic polynomial* of degree $n \geq 1$ (monic means that the coefficient of x^n is 1). a) Show that if n is even then $\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow -\infty} P(x) = \infty$. b) Show that if n is odd then $\lim_{x \rightarrow \infty} P(x) = \infty$ and $\lim_{x \rightarrow -\infty} P(x) = -\infty$ (see previous exercise).

Let $\{x_n\}$ be a sequence. Consider $S := \mathbb{N} \subset \mathbb{R}$, and $f: S \rightarrow \mathbb{R}$ defined by $f(n) := x_n$. Show that the two notions of limit,

$$\lim_{n \rightarrow \infty} x_n \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x)$$

are equivalent. That is, show that if one exists so does the other one, and in this case they are equal.

Extend as follows. Suppose $S \subset \mathbb{R}$ has a cluster point $c \in \mathbb{R}$, $c = \infty$, or $c = -\infty$. Let $f: S \rightarrow \mathbb{R}$ be a function and let $L = \infty$ or $L = -\infty$. Show that

$$\lim_{x \rightarrow c} f(x) = L \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} f(x_n) = L \quad \text{for all sequences } \{x_n\} \text{ such that } \lim_{n \rightarrow \infty} x_n = c.$$

Monotone functions and continuity

Note: 1 lecture (optional, can safely be omitted unless is also covered, requires)

Let $S \subset \mathbb{R}$. We say $f: S \rightarrow \mathbb{R}$ is *increasing* (resp. *strictly increasing*) if $x, y \in S$ with $x < y$ implies $f(x) \leq f(y)$ (resp. $f(x) < f(y)$). We define *decreasing* and *strictly decreasing* in the same way by switching the inequalities for f .

If a function is either increasing or decreasing we say it is *monotone*. If it is strictly increasing or strictly decreasing we say it is *strictly monotone*.

Sometimes *nondecreasing* (resp. *nonincreasing*) is used for increasing (resp. decreasing) function to emphasize it is not strictly increasing (resp. strictly decreasing).

Continuity of monotone functions

It is easy to compute one-sided limits for monotone functions.

[prop:monotlimits] Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and $f: S \rightarrow \mathbb{R}$ be increasing. If c is a cluster point of $S \cap (-\infty, c)$, then

$$\lim_{x \rightarrow c^-} f(x) = \sup \{f(x) : x < c, x \in S\},$$

and if c is a cluster point of $S \cap (c, \infty)$, then

$$\lim_{x \rightarrow c^+} f(x) = \inf \{f(x) : x > c, x \in S\}.$$

Similarly, if f is decreasing and c is a cluster point of $S \cap (-\infty, c)$, then

$$\lim_{x \rightarrow c^-} f(x) = \inf \{f(x) : x < c, x \in S\},$$

and if c is a cluster point of $S \cap (c, \infty)$, then

$$\lim_{x \rightarrow c^+} f(x) = \sup \{f(x) : x > c, x \in S\}.$$

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In particular all the one-sided limits exist whenever they make sense. If we from now on say that say a the left hand limit $x \rightarrow c^-$ exists we mean that c is a cluster point of $S \cap (-\infty, c)$.

Let us assume f is increasing, and we will show the first equality. The rest of the proof is very similar and is left as an exercise.

Let $a := \sup \{f(x) : x < c, x \in S\}$. If $a = \infty$, then for every $M \in \mathbb{R}$, there exists an x_M such that $f(x_M) > M$. As f is increasing, $f(x) \geq f(x_M) > M$ for all $x \in S$ with $x > x_M$. If we take $\delta = c - x_M$ we obtain the definition of the limit going to infinity.

So assume $a < \infty$. Let $\epsilon > 0$ be given. Because a is the supremum, there exists an $x_\epsilon < c$, $x_\epsilon \in S$, such that $f(x_\epsilon) > a - \epsilon$. As f is increasing, if $x \in S$ and $x_\epsilon < x < c$, we have $a - \epsilon < f(x_\epsilon) \leq f(x) \leq a$. Let $\delta := c - x_\epsilon$. Then for $x \in S \cap (-\infty, c)$ with $|x - c| < \delta$, we have $|f(x) - a| < \epsilon$.

Suppose $f: S \rightarrow \mathbb{R}$, $c \in S$ and that both one-sided limits exist. Since $f(x) \leq f(c) \leq f(y)$ whenever $x < c < y$, taking the limits we obtain

$$\lim_{x \rightarrow c^-} f(x) \leq f(c) \leq \lim_{x \rightarrow c^+} f(x).$$

Then f is continuous at c if and only if both limits are equal to each other (and hence equal to $f(c)$). See also . See to get an idea of what a discontinuity looks like.

[cor:continterval] If $I \subset \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is monotone and not constant, then $f(I)$ is an interval if and only if f is continuous.

Assuming f is not constant is to avoid the technicality that $f(I)$ is a single point in that case; $f(I)$ is a single point if and only if f is constant. A constant function is continuous.

If f is continuous then $f(I)$ being an interval is a consequence of . See also .

Let us prove the reverse direction by contrapositive. Suppose f is not continuous at $c \in I$, and that c is not an endpoint of I . Without loss of generality suppose f is increasing. Let

$$a := \lim_{x \rightarrow c^-} f(x) = \sup \{f(x) : x \in I, x < c\}, \quad b := \lim_{x \rightarrow c^+} f(x) = \inf \{f(x) : x \in I, x > c\}.$$

As c is a discontinuity, $a < b$. If $x < c$, then $f(x) \leq a$, and if $x > c$, then $f(x) \geq b$. Therefore any point in $(a, b) \setminus \{f(c)\}$ is not in $f(I)$. However there exists $x_1 \in S$, $x_1 < c$ so $f(x_1) \leq a$, and there exists $x_2 \in S$, $x_2 > c$ so $f(x_2) \geq b$. Both $f(x_1)$ and $f(x_2)$ are in $f(I)$, and so $f(I)$ is not an interval. See .

When $c \in I$ is an endpoint, the proof is similar and is left as an exercise.

A striking property of monotone functions is that they cannot have too many discontinuities.

[cor:monotcount] Let $I \subset \mathbb{R}$ be an interval and $f: I \rightarrow \mathbb{R}$ be monotone. Then f has at most countably many discontinuities.

Let $E \subset I$ be the set of all discontinuities that are not endpoints of I . As there are only two endpoints, it is enough to show that E is countable. Without loss of generality, suppose f is increasing. We will define an injection $h: E \rightarrow \mathbb{Q}$. For each $c \in E$ the one-sided limits of f both exist as c is not an endpoint. Let $a := \lim_{x \rightarrow c^-} f(x) = \sup \{f(x) : x \in I, x < c\}$, $b := \lim_{x \rightarrow c^+} f(x) = \inf \{f(x) : x \in I, x > c\}$. As c is a discontinuity, we have $a < b$. There exists a rational number $q \in (a, b)$, so let $h(c) := q$. Because f is increasing, q cannot correspond to any other discontinuity, so after making this choice for all $c \in E$, we have that h is one-to-one (injective). Therefore, E is countable.

[example:countdiscont] By $\lfloor x \rfloor$ denote the largest integer less than or equal to x . Define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(x) := x + \sum_{n=0}^{\lfloor 1/(1-x) \rfloor} 2^{-n}$, for $x < 1$ and $f(1) = 3$. It is left as an exercise to show that f is strictly increasing, bounded, and has a discontinuity at all points $1 - \frac{1}{k}$ for $k \in \mathbb{N}$. In particular, there are countably many discontinuities, but the function is bounded and defined on a closed bounded interval.

Continuity of inverse functions

A strictly monotone function f is one-to-one (injective). To see this notice that if $x \neq y$ then we can assume $x < y$. Then either $f(x) < f(y)$ if f is strictly increasing or $f(x) > f(y)$ if f is strictly decreasing, so $f(x) \neq f(y)$. Hence, it must have an inverse f^{-1} defined on its range.

[prop:invcont] If $I \subset \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is strictly monotone. Then the inverse $f^{-1}: f(I) \rightarrow I$ is continuous.

Let us suppose f is strictly increasing. The proof is almost identical for a strictly decreasing function. Since f is strictly increasing, so is f^{-1} . That is, if $f(x) < f(y)$, then we must have $x < y$ and therefore $f^{-1}(f(x)) < f^{-1}(f(y))$.

Take $c \in f(I)$. If c is not a cluster point of $f(I)$, then f^{-1} is continuous at c automatically. So let c be a cluster point of $f(I)$. Suppose both of the following one-sided limits exist:
$$\begin{aligned} x_0 &:= \lim_{y \rightarrow c^-} f^{-1}(y) = \sup \{f^{-1}(y) : y < c, y \in f(I)\} = \sup \{x : f(x) < c, x \in I\}, \\ x_1 &:= \lim_{y \rightarrow c^+} f^{-1}(y) = \inf \{f^{-1}(y) : y > c, y \in f(I)\} = \inf \{x : f(x) > c, x \in I\}. \end{aligned}$$
 We have $x_0 \leq x_1$ as f^{-1} is increasing. For all $x > x_0$ with $x \in I$, we have $f(x) \geq c$. As f is strictly increasing, we must have $f(x) > c$ for all $x > x_0$, $x \in I$. Therefore, $\{x : x > x_0, x \in I\} \subset \{x : f(x) > c, x \in I\}$. The infimum of the left hand set is x_0 and the infimum of the right hand set is x_1 , so we obtain $x_0 \geq x_1$. So $x_1 = x_0$, and f^{-1} is continuous at c .

If one of the one-sided limits does not exist the argument is similar and is left as an exercise.

The proposition does not require f itself to be continuous. For example, let $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) := \begin{cases} x & \text{if } x < 0, \\ x+1 & \text{if } x \geq 0. \end{cases}$ The function f is not continuous at 0. The image of $I = \mathbb{R}$ is the set $(-\infty, 0) \cup [1, \infty)$, not an interval. Then $f^{-1}: (-\infty, 0) \cup [1, \infty) \rightarrow \mathbb{R}$ can be written as $f^{-1}(x) = \begin{cases} x & \text{if } x < 0, \\ x-1 & \text{if } x \geq 1. \end{cases}$ It is not difficult to see that f^{-1} is a continuous function.

Notice what happens with the proposition if $f(I)$ is an interval. In that case we could simply apply to both f and f^{-1} . That is, if $f: I \rightarrow J$ is an onto strictly monotone function and I and J are intervals, then both f and f^{-1} are continuous. Furthermore $f(I)$ is an interval precisely when f is continuous.

Exercises

Suppose $f: [0, 1] \rightarrow \mathbb{R}$ is monotone. Prove f is bounded.

Finish the proof of .

Finish the proof of .

Prove the claims in .

Finish the proof of .

Suppose $S \subset \mathbb{R}$, and $f: S \rightarrow \mathbb{R}$ is an increasing function. a) If c is a cluster point of $S \cap (c, \infty)$ show that $\lim_{x \rightarrow c^+} f(x) < \infty$. b) If c is a cluster point of $S \cap (-\infty, c)$ and $\lim_{x \rightarrow c^-} f(x) = \infty$, prove that $S \cap (-\infty, c) = \emptyset$.

Suppose $I \subset \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is a function such that for each $c \in I$, there exist $a, b \in \mathbb{R}$ with $a > 0$ such that $f(x) \geq a + b$ for all $x \in I$ and $f(c) = a + b$. Show that f is strictly increasing.

Suppose $f: I \rightarrow J$ is a continuous, bijective (one-to-one and onto) function for two intervals I and J . Show that f is strictly monotone.

Consider a monotone function $f: I \rightarrow \mathbb{R}$ on an interval I . Prove that there exists a function $g: I \rightarrow \mathbb{R}$ such that $\lim_{x \rightarrow c^-} g(x) = g(c)$ for all $c \in I$, except the smaller (left) endpoint of I , and such that $g(x) = f(x)$ for all but countably many x .

a) Let $S \subset \mathbb{R}$ be any subset. If $f: S \rightarrow \mathbb{R}$ is increasing, then show that there exists an increasing $F: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = F(x)$ for all $x \in S$. b) Find an example of a strictly increasing $f: S \rightarrow \mathbb{R}$ such that an increasing F as above is never strictly increasing.

[exercise:increasingfuncdiscontQ] Find an example of an increasing function $f: [0, 1] \rightarrow \mathbb{R}$ that has a discontinuity at each rational number. Then show that the image $f([0, 1])$ contains no interval. Hint: Enumerate the rational numbers and define the function with a series.

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The Derivative

The derivative

Note: 1 lecture

The idea of a derivative is the following. Let us suppose a graph of a function looks locally like a straight line. We can then talk about the slope of this line. The slope tells us the rate at which the value of the function changing at the particular point. Of course, we are leaving out any function that has corners or discontinuities. Let us be precise.

Definition and basic properties

Let I be an interval, let $f: I \rightarrow \mathbb{R}$ be a function, and let $c \in I$. If the limit $L := \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists, then we say f is *differentiable* at c , that L is the *derivative* of f at c , and write $f'(c) := L$.

If f is differentiable at all $c \in I$, then we simply say that f is *differentiable*, and then we obtain a function $f': I \rightarrow \mathbb{R}$.

The expression $\frac{f(x) - f(c)}{x - c}$ is called the *difference quotient*.

The graphical interpretation of the derivative is depicted in . The left-hand plot gives the line through $(c, f(c))$ and $(x, f(x))$ with slope $\frac{f(x) - f(c)}{x - c}$, that is, the so-called *secant line*. When we take the limit as x goes to c , we get the right-hand plot, where we see that the derivative of the function at the point c is the slope of the line tangent to the graph of f at the point $(c, f(c))$.

We allow I to be a closed interval and we allow c to be an endpoint of I . Some calculus books do not allow c to be an endpoint of an interval, but all the theory still works by allowing it, and it makes our work easier.

Let $f(x) := x^2$ defined on the whole real line. We find that $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \rightarrow c} (x + c) = 2c$. Therefore $f'(c) = 2c$.

The function $f(x) := \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is not differentiable at the origin. When $x > 0$, then $\frac{f(x) - f(0)}{x - 0} = \frac{x}{x} = 1$, and when $x < 0$ we have $\frac{f(x) - f(0)}{x - 0} = \frac{0}{x} = 0$.

A famous example of Weierstrass shows that there exists a continuous function that is not differentiable at *any* point. The construction of this function is beyond the scope of this book. On the other hand, a differentiable function is always continuous.

Let $f: I \rightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c .

We know the limits $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c)$ and $\lim_{x \rightarrow c} (x - c) = 0$ exist. Furthermore, $f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c)$. Therefore the limit of $f(x) - f(c)$ exists and $\lim_{x \rightarrow c} (f(x) - f(c)) = \left(\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \rightarrow c} (x - c) \right) = f'(c) \cdot 0 = 0$. Hence $\lim_{x \rightarrow c} f(x) = f(c)$, and f is continuous at c .

An important property of the derivative is linearity. The derivative is the approximation of a function by a straight line. The slope of a line through two points changes linearly when the y -coordinates are changed linearly. By taking the limit, it makes sense that the derivative is linear.

Let I be an interval, let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be differentiable at $c \in I$, and let $\alpha \in \mathbb{R}$.

- Define $h: I \rightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.
- Define $h: I \rightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.

First, let $h(x) := \alpha f(x)$. For $x \in I$, $x \neq c$ we have $\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}$. The limit as x goes to c exists on the right by . We get $\lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \alpha f'(c)$. Therefore h is differentiable at c , and the derivative is computed as given.

Next, define $h(x) := f(x) + g(x)$. For $x \in I$, $x \neq c$ we have $\frac{h(x) - h(c)}{x - c} = \frac{f(x) + g(x) - f(c) - g(c)}{x - c} = \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}$. The limit as x goes to c exists on the right by . We get $\lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} = f'(c) + g'(c)$. Therefore h is differentiable at c and the derivative is computed as given.

It is not true that the derivative of a multiple of two functions is the multiple of the derivatives. Instead we get the so-called *product rule* or the *Leibniz rule* ¹⁷.

Let I be an interval, let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions differentiable at c . If $h: I \rightarrow \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and $h'(c) = f(c)g'(c) + f'(c)g(c)$.

The proof of the product rule is left as an exercise. The key is to use the identity $f(x)g(x) - f(c)g(c) = f(x)(g(x) - g(c)) + g(c)(f(x) - f(c))$.

Let I be an interval, let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be differentiable at c and $g(x) \neq 0$ for all $x \in I$. If $h: I \rightarrow \mathbb{R}$ is defined by $h(x) := \frac{f(x)}{g(x)}$, then h is differentiable at c and $h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$.

Again the proof is left as an exercise.

Chain rule

A useful rule for computing derivatives is the chain rule.

Let I_1, I_2 be intervals, let $g: I_1 \rightarrow I_2$ be differentiable at $c \in I_1$, and $f: I_2 \rightarrow \mathbb{R}$ be differentiable at $g(c)$. If $h: I_1 \rightarrow \mathbb{R}$ is defined by $h(x) := f(g(x))$, then h is differentiable at c and $h'(c) = f'(g(c))g'(c)$.

Let $d := g(c)$. Define $u: I_2 \rightarrow \mathbb{R}$ and $v: I_1 \rightarrow \mathbb{R}$ by
$$u(y) := \begin{cases} \frac{f(y) - f(d)}{y - d} & \text{if } y \neq d \\ f'(d) & \text{if } y = d \end{cases} \quad \text{and} \quad v(x) := \begin{cases} \frac{g(x) - g(c)}{x - c} & \text{if } x \neq c \\ g'(c) & \text{if } x = c \end{cases}$$
 We note that $f(y) - f(d) = u(y)(y - d)$ and $g(x) - g(c) = v(x)(x - c)$. We plug in to obtain $h(x) - h(c) = f(g(x)) - f(g(c)) = u(g(x))(g(x) - g(c)) = u(g(x))v(x)(x - c)$. Therefore, $\frac{h(x) - h(c)}{x - c} = u(g(x))v(x)$. We compute the limits $\lim_{y \rightarrow d} u(y) = f'(d) = f'(g(c))$ and $\lim_{x \rightarrow c} v(x) = g'(c)$. That is, the functions u and v are continuous at $d = g(c)$ and c respectively. Furthermore the function g is continuous at c . Hence the limit of the right-hand side of [eq:chainruleeq] as x goes to c exists and is equal to $f'(g(c))g'(c)$. Thus h is differentiable at c and the limit is $f'(g(c))g'(c)$.

Exercises

Prove the product rule. Hint: Use $f(x)g(x) - f(c)g(c) = f(x)(g(x) - g(c)) + g(c)(f(x) - f(c))$.

Prove the quotient rule. Hint: You can do this directly, but it may be easier to find the derivative of $\frac{1}{x}$ and then use the chain rule and the product rule.

[exercise:diffboxn] For $n \in \mathbb{Z}$, prove that x^n is differentiable and find the derivative, unless, of course, $n < 0$ and $x = 0$. Hint: Use the product rule.

Prove that a polynomial is differentiable and find the derivative. Hint: Use the previous exercise.

Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ Prove that f is differentiable at 0, but discontinuous at all points except 0.

Assume the inequality $|\sin(x)| \leq x^2$. Prove that $\sin$ is differentiable at 0, and find the derivative at 0.

Using the previous exercise, prove that $\sin$ is differentiable at all x and that the derivative is $\cos(x)$. Hint: Use the sum-to-product trigonometric identity as we did before.

Let $f: I \rightarrow \mathbb{R}$ be differentiable. Given $n \in \mathbb{Z}$, define f^n be the function defined by $f^n(x) := (f(x))^n$. If $n < 0$ assume $f(x) \neq 0$. Prove that $(f^n)'(x) = n(f(x))^{n-1} f'(x)$.

Suppose $f: I \rightarrow \mathbb{R}$ is a differentiable Lipschitz continuous function. Prove that f' is a bounded function.

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Let I_1, I_2 be intervals. Let $f: I_1 \rightarrow I_2$ be a bijective function and $g: I_2 \rightarrow I_1$ be the inverse. Suppose that both f is differentiable at $c \in I_1$ and $f'(c) \neq 0$ and g is differentiable at $f(c)$. Use the chain rule to find a formula for $g'(f(c))$ (in terms of $f'(c)$).

[exercise:boundediff] Suppose $f: I \rightarrow \mathbb{R}$ is a bounded function and $g: I \rightarrow \mathbb{R}$ is a function differentiable at $c \in I$ and $g(c) = g'(c) = 0$. Show that $h(x) := f(x)g(x)$ is differentiable at c . Hint: Note that you cannot apply the product rule.

[exercise:difffsqueeze] Suppose $f: I \rightarrow \mathbb{R}$, $g: I \rightarrow \mathbb{R}$, and $h: I \rightarrow \mathbb{R}$ are functions. Suppose $c \in I$ is such that $f(c) = g(c) = h(c)$, g and h are differentiable at c , and $g'(c) = h'(c)$. Furthermore suppose $h(x) \leq f(x) \leq g(x)$ for all $x \in I$. Prove f is differentiable at c and $f'(c) = g'(c) = h'(c)$.

Mean value theorem

Note: 2 lectures (some applications may be skipped)

Relative minima and maxima

We talked about absolute maxima and minima. These are the tallest peaks and lowest valleys in the whole mountain range. We might also want to talk about peaks of individual mountains and valleys.

Let $S \subset \mathbb{R}$ be a set and let $f: S \rightarrow \mathbb{R}$ be a function. The function f is said to have a *relative maximum* at $c \in S$ if there exists a $\delta > 0$ such that for all $x \in S$ such that $|x - c| < \delta$ we have $f(x) \leq f(c)$. The definition of *relative minimum* is analogous.

[relminmax:thm] Let $f: [a, b] \rightarrow \mathbb{R}$ be a function differentiable at $c \in (a, b)$, and c is a relative minimum or a relative maximum of f . Then $f'(c) = 0$.

We prove the statement for a maximum. For a minimum the statement follows by considering the function $-f$.

Let c be a relative maximum of f . In particular as long as $|x - c| < \delta$ we have $f(x) - f(c) \leq 0$. Then we look at the difference quotient. If $x > c$ we note that $\frac{f(x) - f(c)}{x - c} \leq 0$, and if $y < c$ we have $\frac{f(y) - f(c)}{y - c} \geq 0$. See for an illustration.

We now take sequences $\{x_n\}$ and $\{y_n\}$, such that $x_n > c$, and $y_n < c$ for all $n \in \mathbb{N}$, and such that $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = c$. Since f is differentiable at c we know $0 \leq \lim_{n \rightarrow \infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c) = \lim_{n \rightarrow \infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0$. $\square$

For a differentiable function, a point where $f'(c) = 0$ is called a *critical point*. When f is not differentiable at some points, it is common to also say c is a critical point if $f'(c)$ does not exist. The theorem says that a relative minimum or maximum at an interior point of an interval must be a critical point. As you remember from calculus, finding minima and maxima of a function can be done by finding all the critical points together with the endpoints of the interval and simply checking where is the function biggest or smallest.

Rolle's theorem

Suppose a function is zero at both endpoints of an interval. Intuitively it ought to attain a minimum or a maximum in the interior of the interval, then at such a minimum or a maximum, the derivative should be zero. See for the geometric idea. This is the content of the so-called Rolle's theorem ¹⁸.

[thm:rolle] Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous function differentiable on (a, b) such that $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

As f is continuous on $[a, b]$ it attains an absolute minimum and an absolute maximum in $[a, b]$. We wish to apply and so we need a minimum or maximum at some $c \in (a, b)$. Write $K := f(a) = f(b)$. If there exists an x such that $f(x) > K$, then the absolute maximum is bigger than K and hence occurs at $c \in (a, b)$, and therefore we get $f'(c) = 0$. On the other hand if there exists an x such that $f(x) < K$, then the absolute minimum occurs at some $c \in (a, b)$ and we have that $f'(c) = 0$. If there is no x such that $f(x) > K$ or $f(x) < K$, then we have that $f(x) = K$ for all x and then $f'(x) = 0$ for all $x \in [a, b]$, so any c will work.

Note that it is absolutely necessary for the derivative to exist for all $x \in (a, b)$. For example take the function $f(x) = |x|$ on $[-1, 1]$. Clearly $f(-1) = f(1)$, but there is no point where $f'(c) = 0$.

Mean value theorem

We extend to functions that attain different values at the endpoints.

[thm:mvt] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function differentiable on (a, b) . Then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

The theorem follows from . Define the function $g: [a, b] \rightarrow \mathbb{R}$ by $g(x) := f(x) - f(b) + \frac{f(b) - f(a)}{b - a}(x - b)$. The function g is a differentiable on (a, b) , continuous on $[a, b]$, such that $g(a) = 0$ and $g(b) = 0$. Thus there exists $c \in (a, b)$ such that $g'(c) = 0$. $0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$. Or in other words $f'(c)(b - a) = f(b) - f(a)$.

For a geometric interpretation of the mean value theorem, see . The idea is that the value $\frac{f(b) - f(a)}{b - a}$ is the slope of the line between the points $(a, f(a))$ and $(b, f(b))$. Then c is the point such that $f'(c) = \frac{f(b) - f(a)}{b - a}$, that is, the tangent line at the point $(c, f(c))$ has the same slope as the line between $(a, f(a))$ and $(b, f(b))$.

Applications

We now solve our very first differential equation.

[prop:derzeroconst] Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in I$. Then f is constant.

Take arbitrary $x, y \in I$ with $x < y$. Then f restricted to $[x, y]$ satisfies the hypotheses of the . Therefore there is a $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. As $f'(c) = 0$, we have $f(y) = f(x)$. Therefore, the function is constant.

Now that we know what it means for the function to stay constant, let us look at increasing and decreasing functions. We say $f: I \rightarrow \mathbb{R}$ is *increasing* (resp. *strictly increasing*) if $x < y$ implies $f(x) \leq f(y)$ (resp. $f(x) < f(y)$). We define *decreasing* and *strictly decreasing* in the same way by switching the inequalities for f .

[incdec:diffprop] Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be a differentiable function.

- f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.
- f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

Let us prove the first item. Suppose f is increasing, then for all x and c in I we have $\frac{f(x) - f(c)}{x - c} \geq 0$. Taking a limit as x goes to c we see that $f'(c) \geq 0$.

For the other direction, suppose $f'(x) \geq 0$ for all $x \in I$. Take any $x, y \in I$ where $x < y$. By the there is some $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$. As $f'(c) \geq 0$, and $x - y < 0$, then $f(x) - f(y) \leq 0$ or $f(x) \leq f(y)$ and so f is increasing.

We leave the decreasing part to the reader as exercise.

We can make a similar but weaker statement about strictly increasing and decreasing functions. If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing. The proof is left as an exercise. The converse is not true. For example, $f(x) := x^3$ is a strictly increasing function, but $f'(0) = 0$.

Another application of the is the following result about location of extrema. The theorem is stated for an absolute minimum and maximum, but the way it is applied to find relative minima and maxima is to restrict f to an interval $(c - \delta, c + \delta)$.

[firstderminmaxtest] Let $f: (a, b) \rightarrow \mathbb{R}$ be continuous. Let $c \in (a, b)$ and suppose f is differentiable on (a, c) and (c, b) .

- If $f'(x) \leq 0$ for $x \in (a, c)$ and $f'(x) \geq 0$ for $x \in (c, b)$, then f has an absolute minimum at c .
- If $f'(x) \geq 0$ for $x \in (a, c)$ and $f'(x) \leq 0$ for $x \in (c, b)$, then f has an absolute maximum at c .

Let us prove the first item. The second is left to the reader. Let x be in (a, c) and $\{y_n\}$ a sequence such that $x < y_n < c$ and $\lim_{n \rightarrow \infty} y_n = c$. By the previous proposition, the function is decreasing on (a, c) so $f(x) \geq f(y_n)$. The function is continuous at c so we can take the limit to get $f(x) \geq f(c)$ for all $x \in (a, c)$.

Similarly take $x \in (c, b)$ and $\{y_n\}$ a sequence such that $c < y_n < x$ and $\lim_{n \rightarrow \infty} y_n = c$. The function is increasing on (c, b) so $f(x) \leq f(y_n)$. By continuity of f we get $f(x) \leq f(c)$ for all $x \in (c, b)$. Thus $f(x) \geq f(c)$ for all $x \in (a, b)$.

The converse of the proposition does not hold. See below.

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Continuity of derivatives and the intermediate value theorem

Derivatives of functions satisfy an intermediate value property. The result is usually called Darboux's theorem.

[thm:darboux] Let $f \colon [a, b] \rightarrow \mathbb{R}$ be differentiable. Suppose that there exists a $y \in \mathbb{R}$ such that $f'(a) < y < f'(b)$ or $f'(a) > y > f'(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = y$.

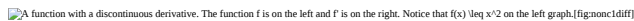
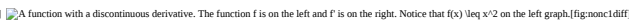
Suppose without loss of generality that $f'(a) < y < f'(b)$. Define $g(x) := yx - f(x)$. As g is continuous on $[a, b]$, then g attains a maximum at some $c \in [a, b]$.

Now compute $g'(x) = y - f'(x)$. Thus $g'(a) > 0$. As the derivative is the limit of difference quotients and is positive, there must be some difference quotient that is positive. That is, there must exist an $x > a$ such that $\frac{g(x) - g(a)}{x - a} > 0$, or $g(x) > g(a)$. Thus a cannot possibly be a maximum of g . Similarly as $g'(b) < 0$, we find an $x < b$ (a different x) such that $\frac{g(x) - g(b)}{x - b} < 0$ or that $g(x) > g(b)$, thus b cannot possibly be a maximum.

Therefore $c \in (a, b)$. Then as c is a maximum of g we find $g'(c) = 0$ and $f'(c) = y$.

We have seen already that there exist discontinuous functions that have the intermediate value property. While it is hard to imagine at first, there also exist functions that are differentiable everywhere and the derivative is not continuous.

[baddiffunc:example] Let $f \colon \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) := \begin{cases} \bigl(x \sin(\frac{1}{x}) \bigr)^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ We claim that f is differentiable everywhere, but $f \colon \mathbb{R} \rightarrow \mathbb{R}$ is not continuous at the origin. Furthermore, f has a minimum at 0, but the derivative changes sign infinitely often near the origin. See .

Proof: It is easy to see from the definition that f has an absolute minimum at 0: we know $f(x) \geq 0$ for all x and $f(0) = 0$.

The function f is differentiable for $x \neq 0$ and the derivative is $2 \sin(\frac{1}{x}) \bigl(x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) \bigr)$. As an exercise show that for $x_n = \frac{4}{(8n+1)\pi}$ we have $\lim_{n \rightarrow \infty} f'(x_n) = -1$, and for $y_n = \frac{4}{(8n+3)\pi}$ we have $\lim_{n \rightarrow \infty} f'(y_n) = 1$. Hence if f' exists at 0, then it cannot be continuous.

Let us show that f' exists at 0. We claim that the derivative is zero. In other words $\frac{f(x) - f(0)}{x - 0} \rightarrow 0$ as $x \rightarrow 0$. For $x \neq 0$ we have $\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \frac{x^2 \sin^2(\frac{1}{x})}{x} = x \sin^2(\frac{1}{x}) \rightarrow 0$ as $x \rightarrow 0$. And, of course, as x tends to zero, then $\frac{f(x) - f(0)}{x - 0} \rightarrow 0$ and hence $\frac{f(x) - f(0)}{x - 0} \rightarrow 0$. Therefore, f is differentiable at 0 and the derivative at 0 is 0. A key point in the above calculation is that $\frac{f(x) - f(0)}{x - 0} \leq x^2$, see also Exercises [exercise:bndmuldiff] and [exercise:diffsqueeze].

It is sometimes useful to assume the derivative of a differentiable function is continuous. If $f \colon I \rightarrow \mathbb{R}$ is differentiable and the derivative f' is continuous on I , then we say f is *continuously differentiable*. It is common to write $C^1(I)$ for the set of continuously differentiable functions on I .

Exercises

Finish the proof of .

Finish the proof of .

Suppose $f \colon \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that f' is a bounded function. Prove f is a Lipschitz continuous function.

Suppose $f \colon [a, b] \rightarrow \mathbb{R}$ is differentiable and $c \in [a, b]$. Then show there exists a sequence $\{x_n\}$ converging to c , $x_n \neq c$ for all n , such that $f'(c) = \lim_{n \rightarrow \infty} f'(x_n)$. Do note this does *not* imply that f' is continuous (why?).

Suppose $f \colon \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(y)}{x - y}^2$ for all x and y . Show that $f(x) = C$ for some constant C . Hint: Show that f is differentiable at all points and compute the derivative.

[exercise:posderincr] Suppose I is an interval and $f \colon I \rightarrow \mathbb{R}$ is a differentiable function. If $f'(x) > 0$ for all $x \in I$, show that f is strictly increasing.

Suppose $f \colon (a, b) \rightarrow \mathbb{R}$ is a differentiable function such that $f'(x) \neq 0$ for all $x \in (a, b)$. Suppose there exists a point $c \in (a, b)$ such that $f'(c) > 0$. Prove $f'(x) > 0$ for all $x \in (a, b)$.

[exercise:samediffconst] Suppose $f \colon (a, b) \rightarrow \mathbb{R}$ and $g \colon (a, b) \rightarrow \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in (a, b)$, then show that there exists a constant C such that $f(x) = g(x) + C$.

Prove the following version of L'Hopital's rule. Suppose $f \colon (a, b) \rightarrow \mathbb{R}$ and $g \colon (a, b) \rightarrow \mathbb{R}$ are differentiable functions. Suppose that at $c \in (a, b)$, $f(c) = 0$, $g(c) = 0$, and that the limit of $\frac{f'(x)}{g'(x)}$ as x goes to c exists. Show that $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$.

Let $f \colon (a, b) \rightarrow \mathbb{R}$ be an unbounded differentiable function. Show $f' \colon (a, b) \rightarrow \mathbb{R}$ is unbounded.

Prove the theorem Rolle actually proved in 1691: If f is a polynomial, $f'(a) = f'(b) = 0$ for some $a < b$, and there is no $c \in (a, b)$ such that $f'(c) = 0$, then there is at most one root of f in (a, b) , that is at most one $x \in (a, b)$ such that $f(x) = 0$. In other words, between any two consecutive roots of f' is at most one root of f . Hint: suppose there are two roots and see what happens.

Suppose $a, b \in \mathbb{R}$ and $f \colon \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, $f'(x) = a$ for all x , and $f(0) = b$. Find f and prove that it is the unique differentiable function with this property.

Taylor's theorem

Note: half a lecture (optional section)

Derivatives of higher orders

When $f \colon I \rightarrow \mathbb{R}$ is differentiable, we obtain a function $f' \colon I \rightarrow \mathbb{R}$. The function f' is called the *first derivative* of f . If f' is differentiable, we denote by $f'' \colon I \rightarrow \mathbb{R}$ the derivative of f' . The function f'' is called the *second derivative* of f . We similarly obtain f''' , $f^{(4)}$, and so on. With a larger number of derivatives the notation would get out of hand; we denote by $f^{(n)}$ the *n*th derivative of f .

When f possesses n derivatives, we say f is *n times differentiable*.

Taylor's theorem

Taylor's theorem ¹⁹ is a generalization of the . Mean value theorem says that up to a small error $f(x)$ for x near x_0 can be approximated by $f(x_0)$, that is $f(x) = f(x_0) + f'(c)(x - x_0)$, where the "error" is measured in terms of the first derivative at some point c between x and x_0 . Taylor's theorem generalizes this result to higher derivatives. It tells us that up to a small error, any n times differentiable function can be approximated at a point x_0 by a polynomial. The error of this approximation behaves like $\frac{(x - x_0)^{n+1}}{(n+1)!}$ near the point x_0 . To see why this is a good approximation notice that for a big n , $\frac{(x - x_0)^{n+1}}{(n+1)!}$ is very small in a small interval around x_0 .

For an n times differentiable function f defined near a point $x_0 \in \mathbb{R}$, define the *n*th Taylor polynomial for f at x_0 as $\begin{cases} P_n^{x_0}(x) & := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \\ & = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \frac{f^{(3)}(x_0)}{6!} (x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \end{cases}$

Taylor's theorem says a function behaves like its *n*th Taylor polynomial. The is really Taylor's theorem for the first derivative.

[thm:taylor] Suppose $f \colon [a, b] \rightarrow \mathbb{R}$ is a function with n continuous derivatives on $[a, b]$ and such that $f^{(n+1)}$ exists on (a, b) . Given distinct points x_0 and x in $[a, b]$, we can find a point c between x_0 and x such that $f(x) = P_n^{x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$.

The term $R_n^{x_0}(x) := \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$ is called the *remainder term*. This form of the remainder term is called the *Lagrange form* of the remainder. There are other ways to write the remainder term, but we skip those. Note that c depends on both x and x_0 .

Find a number M_{x, x_0} (depending on x and x_0) solving the equation $f(x) = P_n^{x_0}(x) + M_{x, x_0} \frac{(x - x_0)^{n+1}}{(n+1)!}$. Define a function $g(s)$ by $g(s) := f(s) - P_n^{x_0}(s) - M_{x, x_0} \frac{(s - x_0)^{n+1}}{(n+1)!}$. We compute the *k*th derivative at x_0 of the Taylor polynomial $(P_n^{x_0}(x))^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, 2, \dots, n$ (the zeroth derivative corresponds to the function itself).

Processing math: 52% $g'(x_0) = g''(x_0) = \cdots = g^{(n)}(x_0) = 0$. In particular $g(x_0) = 0$. On the other hand $g(x) = 0$. By the there exists an x_1 between x_0 and x such that $g'(x_1) = 0$. Applying

the to g' we obtain that there exists x_2 between x_0 and x_1 (and therefore between x_0 and x) such that $g''(x_2) = 0$. We repeat the argument $n+1$ times to obtain a number x_{n+1} between x_0 and x_n (and therefore between x_0 and x) such that $g^{(n+1)}(x_{n+1}) = 0$.

Let $c := x_{n+1}$. We compute the $(n+1)$ th derivative of g to find $g^{(n+1)}(s) = f^{(n+1)}(s) - (n+1)! M_{x,x_0}$. Plugging in c for s we obtain $M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$, and we are done.

In the proof we have computed $(P_n^{x_0})^{(k)}(x_0) = f^{(k)}(x_0)$ for $k=0,1,2,\dots,n$. Therefore the Taylor polynomial has the same derivatives as f at x_0 up to the n th derivative. That is why the Taylor polynomial is a good approximation to f .

The definition of derivative says that a function is differentiable if it is locally approximated by a line. Similarly we mention in passing that there exists a converse to Taylor's theorem, which we will neither state nor prove, saying that if a function is locally approximated in a certain way by a polynomial of degree d , then it has d derivatives.

Exercises

Compute the n th Taylor Polynomial at 0 for the exponential function.

Suppose p is a polynomial of degree d . Given any $x_0 \in \mathbb{R}$, show that the $(d+1)$ th Taylor polynomial for p at x_0 is equal to p .

Let $f(x) := \left\lfloor \sqrt[3]{x} \right\rfloor$. Compute $f(x)$ and $f'(x)$ for all x , but show that $f^{(3)}(0)$ does not exist.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ has n continuous derivatives. Show that for any $x_0 \in \mathbb{R}$, there exist polynomials P and Q of degree n and an $\epsilon > 0$ such that $P(x) \leq f(x) \leq Q(x)$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$ and $Q(x) - P(x) = \lambda (x - x_0)^n$ for some $\lambda \geq 0$.

If $f: [a,b] \rightarrow \mathbb{R}$ has $n+1$ continuous derivatives and $x_0 \in [a,b]$, prove $\lim_{x \rightarrow x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{(x - x_0)^n} = 0$.

Suppose $f: [a,b] \rightarrow \mathbb{R}$ has $n+1$ continuous derivatives and $x_0 \in (a,b)$. Show that $f^{(k)}(x_0) = 0$ for all $k = 0, 1, 2, \dots, n$ if and only if $g(x) := \frac{f(x)}{(x - x_0)^{n+1}}$.

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f is continuous at x_0 .

Suppose $a, b, c \in \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, $f'(x) = a$ for all x , $f(0) = b$, and $f(1) = c$. Find f and prove that it is the unique differentiable function with this property.

Show that a simple converse to Taylor's theorem does not hold. Find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ with no second derivative at $x=0$ such that $\left\lfloor \sqrt[3]{f(x)} \right\rfloor \leq \left\lfloor \sqrt[3]{x^3} \right\rfloor$, that is, f goes to zero at 0 faster than x^3 , and while $f'(0)$ exists, $f''(0)$ does not.

Inverse function theorem

Note: less than 1 lecture (optional section, needed for , requires)

Inverse function theorem

The main idea of differentiating inverse functions is the following lemma.

[lemma:ift] Let $I, J \subset \mathbb{R}$ be intervals. If $f: I \rightarrow J$ is strictly monotone (hence one-to-one), onto ($f(I) = J$), differentiable at x , and $f'(x) \neq 0$, then the inverse f^{-1} is differentiable at $y = f(x)$ and $(f^{-1})'(y) = \frac{1}{f'(x)}$. If f is continuously differentiable and f' is never zero, then f^{-1} is continuously differentiable.

By f has a continuous inverse. Let us call the inverse $g: J \rightarrow I$ for convenience. Let x, y be as in the statement, take $t \in I$ to be arbitrary and let $s := f(t)$. Then $\frac{g(s) - g(y)}{s - y} = \frac{g(f(t)) - g(f(y))}{f(t) - f(y)} = \frac{f(t) - f(y)}{f(t) - f(y)} = 1$. As f is differentiable at x and $f'(x) \neq 0$, then $\frac{f(t) - f(x)}{t - x} \rightarrow f'(x)$ as $t \rightarrow x$. Because $g(s) \rightarrow g(y)$ as $s \rightarrow y$, we can plug in $g(s)$ for t , and $g(y)$ for x and take the limit as s goes to y , that is, the limit exists. In other words, $\lim_{s \rightarrow y} \frac{g(s) - g(y)}{s - y} = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{f(t) - f(x)} = \frac{1}{f'(x)}$. See for the geometric idea.

If both f and g are continuous, f' is nonzero at all x , then the lemma applies at all points $x \in I$ and the resulting function $g'(y) = \frac{1}{f'(g(y))}$ must be continuous.

What is usually called the inverse function theorem is the following result.

Let $f: (a,b) \rightarrow \mathbb{R}$ be a continuously differentiable function, $x_0 \in (a,b)$ a point where $f'(x_0) \neq 0$. Then there exists an interval $I \subset (a,b)$ with $x_0 \in I$, the restriction $f|_I$ is injective with an inverse $g: J \rightarrow I$ defined on $J := f(I)$, which is continuously differentiable and $g'(y) = \frac{1}{f'(g(y))}$, $\forall y \in J$.

Without loss of generality, suppose $f'(x_0) > 0$. As f' is continuous, there must exist an interval I with $x_0 \in I$ such that $f'(x) > 0$ for all $x \in I$.

By f is strictly increasing on I , and hence the restriction $f|_I$ bijective onto $J := f(I)$. As f is continuous, then by the (see also), $f(I)$ is an interval. Now apply .

If you tried to prove the existence of roots directly as in you may have seen how difficult that endeavor is. However, with the machinery we have built for inverse functions it becomes an almost trivial exercise, and with the inverse function theorem we prove far more than mere existence.

Given any $n \in \mathbb{N}$ and any $x \geq 0$ there exists a unique number $y \geq 0$ (denoted $x^{1/n} := y$), such that $y^n = x$. Furthermore, the function $g: (0, \infty) \rightarrow (0, \infty)$ defined by $g(x) := x^{1/n}$ is continuously differentiable and $g'(x) = \frac{1}{n} x^{(1/n)-1} = \frac{1}{n} x^{1/n} \cdot x^{-1} = \frac{1}{n} x^{1/n-1}$, using the convention $x^{a/m} := (x^{1/m})^a$.

For $x=0$ the existence of a unique root is trivial.

Let $f(x) := x^n$. Using product rule, f is continuously differentiable and $f'(x) = nx^{n-1}$, see . For $x > 0$ the derivative f' is strictly positive and so again by , f is strictly increasing (this can also be proved directly). It is also easy to see that the image of f is the entire interval $(0, \infty)$. We obtain a unique inverse g and so the existence and uniqueness of positive n th roots. We apply to obtain the derivative.

The corollary provides a good example of where the inverse function theorem gives us an interval smaller than (a,b) . Take $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) := x^2$. Then $f'(x) \neq 0$ as long as $x \neq 0$. If $x_0 > 0$, we can take $I = (0, \infty)$, but no larger.

Another useful example is $f(x) := x^{1/3}$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is one-to-one and onto, so $f^{-1}(x) = x^{1/3}$ exists on the entire real line including zero and negative x . The function f has a continuous derivative, but f^{-1} has no derivative at the origin. The point is that $f'(0) = 0$. See also .

Exercises

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable such that $f'(x) > 0$ for all x . Show that f is invertible on the interval $J = f(\mathbb{R})$, the inverse is continuously differentiable, and $(f^{-1})'(y) > 0$ for all $y \in J$.

Suppose I, J are intervals and a monotone onto $f: I \rightarrow J$ has an inverse $g: J \rightarrow I$. Suppose you already know that both f and g are differentiable everywhere and f' is never zero. Using chain rule but not prove the formula $g'(y) = \frac{1}{f'(g(y))}$.

Let $n \in \mathbb{N}$ be even. Prove that every $x > 0$ has a unique negative n th root. That is, there exists a negative number y such that $y^n = x$. Compute the derivative of the function $g(x) := y$.

[exercise:oddroot] Let $n \in \mathbb{N}$ be odd and $n \geq 3$. Prove that every x has a unique n th root. That is, there exists a number y such that $y^n = x$. Prove that the function defined by $g(x) := y$ is differentiable except at $x=0$ and compute the derivative. Prove that g is not differentiable at $x=0$.

[3] Show that if in the inverse function theorem f has k continuous derivatives, then the inverse function g also has k continuous derivatives.

Let $f(x) := x + 2x^{1/2}$ for $x \geq 0$ and $f(0) = 0$. Show that f is differentiable at all x , that $f'(0) > 0$, but that f is not invertible on any interval containing the origin.

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[intbound:prop] Let $f: [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then $m(b-a) \leq \int_a^b f \leq M(b-a)$.

Often we use a weaker form of this proposition. That is, if $f: [a, b] \rightarrow \mathbb{R}$ for all $x \in [a, b]$, then $\int_a^b f \leq M(b-a)$.

We integrate constant functions using . If $f(x) := c$ for some constant c , then we take $m = M = c$. In inequality [intbound:eq] all the inequalities must be equalities. Thus f is integrable on $[a, b]$ and $\int_a^b f = c(b-a)$.

Let $f: [0, 2] \rightarrow \mathbb{R}$ be defined by $f(x) := \begin{cases} 1 & \text{if } x < 1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$ We claim f is Riemann integrable and $\int_0^2 f = 1$.

Proof: Let $0 < \epsilon < 1$ be arbitrary. Let $P := \{0, 1-\epsilon, 1+\epsilon, 2\}$ be a partition. We use the notation from the definition of the Darboux sums. Then
$$\begin{aligned} m_1 &= \inf_{x \in [0, 1-\epsilon]} f(x) = 1, & M_1 &= \sup_{x \in [0, 1-\epsilon]} f(x) = 1, & m_2 &= \inf_{x \in [1-\epsilon, 1+\epsilon]} f(x) = 0, & M_2 &= \sup_{x \in [1-\epsilon, 1+\epsilon]} f(x) = 1, \\ m_3 &= \inf_{x \in [1+\epsilon, 2]} f(x) = 0, & M_3 &= \sup_{x \in [1+\epsilon, 2]} f(x) = 0. \end{aligned}$$
 Furthermore, $\Delta x_1 = 1-\epsilon$, $\Delta x_2 = 2\epsilon$ and $\Delta x_3 = 1-\epsilon$. We compute
$$\begin{aligned} L(P, f) &= \sum_{i=1}^3 m_i \Delta x_i = 1 \cdot (1-\epsilon) + 0 \cdot 2\epsilon + 0 \cdot (1-\epsilon) = 1-\epsilon, \\ U(P, f) &= \sum_{i=1}^3 M_i \Delta x_i = 1 \cdot (1-\epsilon) + 1 \cdot 2\epsilon + 0 \cdot (1-\epsilon) = 1+\epsilon. \end{aligned}$$
 Thus,
$$\overline{\int_0^2 f} - \underline{\int_0^2 f} = U(P, f) - L(P, f) = (1+\epsilon) - (1-\epsilon) = 2\epsilon.$$
 By we have $\overline{\int_0^2 f} \leq \overline{\int_0^2 f} \leq \underline{\int_0^2 f}$. As ϵ was arbitrary we see $\overline{\int_0^2 f} = \underline{\int_0^2 f} = f$. So f is Riemann integrable. Finally, $1-\epsilon \leq \int_0^2 f \leq 1+\epsilon$. Hence, $|\int_0^2 f - 1| \leq \epsilon$. As ϵ was arbitrary, we have $\int_0^2 f = 1$.

It may be worthwhile to extract part of the technique of the example into a proposition.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then f is Riemann integrable if for every $\epsilon > 0$, there exists a partition P such that $U(P, f) - L(P, f) < \epsilon$.

If for every $\epsilon > 0$, a P exists we have: $0 \leq \overline{\int_a^b f} - \underline{\int_a^b f} \leq U(P, f) - L(P, f) < \epsilon$. Therefore, $\overline{\int_a^b f} = \underline{\int_a^b f}$, and f is integrable.

Let us show $\frac{1}{1+x}$ is integrable on $[0, b]$ for any $b > 0$. We will see later that all continuous functions are integrable, but let us demonstrate how we do it directly.

Let $\epsilon > 0$ be given. Take $n \in \mathbb{N}$ and pick $x_j := \frac{j}{n}$, to form the partition $P := \{x_0, x_1, \dots, x_n\}$ of $[0, b]$. We have $\Delta x_j = \frac{b}{n}$ for all j . As f is decreasing, for any subinterval $[x_{j-1}, x_j]$ we obtain $m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) = f(x_j)$, $M_j = \sup_{x \in [x_{j-1}, x_j]} f(x) = f(x_{j-1})$. Then we have
$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{j=1}^n (M_j - m_j) \Delta x_j = \sum_{j=1}^n (f(x_{j-1}) - f(x_j)) \frac{b}{n} \\ &= \frac{b}{n} \sum_{j=1}^n (f(x_{j-1}) - f(x_j)) = \frac{b}{n} (f(x_0) - f(x_n)) = \frac{b}{n} (1 - \frac{1}{1+b}). \end{aligned}$$
 The sum telescopes, the terms successively cancel each other, something we have seen before. Picking n to be such that $\frac{b}{n} (1 - \frac{1}{1+b}) < \epsilon$ the proposition is satisfied and the function is integrable.

More notation

When $f: S \rightarrow \mathbb{R}$ is defined on a larger set S and $[a, b] \subset S$, we say f is Riemann integrable on $[a, b]$ if the restriction of f to $[a, b]$ is Riemann integrable. In this case, we say $f|_{[a, b]}$ is Riemann integrable, and we write $\int_a^b f$ to mean the Riemann integral of the restriction of f to $[a, b]$.

It is useful to define the integral $\int_a^b f$ even if $a \not\leq b$. Suppose $b < a$ and $f|_{[b, a]}$ is Riemann integrable, then define $\int_a^b f := -\int_b^a f$. For any function f we define $\int_a^a f := 0$.

At times, the variable x may already have some other meaning. When we need to write down the variable of integration, we may simply use a different letter. For example, $\int_a^b f(x) dx := \int_a^b f$.

Exercises

Let $f: [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) := x^3$ and let $P := \{0, 0.1, 0.4, 1\}$. Compute $L(P, f)$ and $U(P, f)$.

Let $f: [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) := x$. Show that $f|_{[0, 1]}$ is Riemann integrable and compute $\int_0^1 f$ using the definition of the integral (but feel free to use the propositions of this section).

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}$ of $[a, b]$ such that $\lim_{k \rightarrow \infty} (U(P_k, f) - L(P_k, f)) = 0$. Show that f is Riemann integrable and that $\int_a^b f = \lim_{k \rightarrow \infty} U(P_k, f) = \lim_{k \rightarrow \infty} L(P_k, f)$.

Finish the proof of .

Suppose $f: [-1, 1] \rightarrow \mathbb{R}$ is defined as $f(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$ Prove that $f|_{[-1, 1]}$ is not Riemann integrable and compute $\int_{-1}^1 f$ using the definition of the integral (but feel free to use the propositions of this section).

Let $c \in (a, b)$ and let $d \in \mathbb{R}$. Define $f: [a, b] \rightarrow \mathbb{R}$ as $f(x) := \begin{cases} d & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$ Prove that $f|_{[a, b]}$ is Riemann integrable and compute $\int_a^b f$ using the definition of the integral (but feel free to use the propositions of this section).

[exercise:taggedpartition] Suppose $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable. Let $\epsilon > 0$ be given. Then show that there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ such that if we pick any set of numbers $\{c_1, c_2, \dots, c_n\}$ with $c_k \in [x_{k-1}, x_k]$ for all k , then $|\int_a^b f - \sum_{k=1}^n f(c_k) \Delta x_k| < \epsilon$.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then define $g(x) := f(\alpha x + \beta)$ on the interval $I = [\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}]$. Show that g is Riemann integrable on I .

Suppose $f: [0, 1] \rightarrow \mathbb{R}$ and $g: [0, 1] \rightarrow \mathbb{R}$ are such that for all $x \in (0, 1)$ we have $f(x) = g(x)$. Suppose f is Riemann integrable. Prove g is Riemann integrable and $\int_0^1 f = \int_0^1 g$.

Let $f: [0, 1] \rightarrow \mathbb{R}$ be a bounded function. Let $P_n = \{x_0, x_1, \dots, x_n\}$ be a uniform partition of $[0, 1]$, that is, $x_j := \frac{j}{n}$. Is $\{L(P_n, f)\}_{n=1}^\infty$ always monotone? Yes/No: Prove or find a counterexample.

For a bounded function $f: [0, 1] \rightarrow \mathbb{R}$ let $R_n := \sum_{j=1}^n (M_j - m_j) \Delta x_j$ (the uniform right hand rule). a) If f is Riemann integrable show $\int_0^1 f = \lim_{n \rightarrow \infty} R_n$. b) Find an f that is not Riemann integrable, but $\lim_{n \rightarrow \infty} R_n$ exists.

[exercise:riemannintdarboux] Generalize the previous exercise. Show that $f|_{[a, b]}$ is Riemann integrable if and only if there exists an $I \subset \mathbb{R}$, such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if P is a partition with $\Delta x_i < \delta$ for all i , then $|L(P, f) - I| < \epsilon$ and $|U(P, f) - I| < \epsilon$. If $f|_{[a, b]}$ is Riemann integrable, then $I = \int_a^b f$.

[exercise:riemannintdarboux2] Using and the idea of the proof in , show that Darboux integral is the same as the standard definition of Riemann integral, which you have most likely seen in calculus. That is, show that $f|_{[a, b]}$ is Riemann integrable if and only if there exists an $I \in \mathbb{R}$, such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, x_1, \dots, x_n\}$ is a partition with $\Delta x_i < \delta$ for all i , then $|\sum_{i=1}^n f(c_i) \Delta x_i - I| < \epsilon$ for any set $\{c_1, c_2, \dots, c_n\}$ with $c_i \in [x_{i-1}, x_i]$. If $f|_{[a, b]}$ is Riemann integrable, then $I = \int_a^b f$.

Find an example of functions $f: [0, 1] \rightarrow \mathbb{R}$ which is Riemann integrable, and $g: [0, 1] \rightarrow [0, 1]$ which is one-to-one and onto, such that the composition $f \circ g$ is not Riemann integrable.

Properties of the integral

Note: 2 lectures, integrability of functions with discontinuities can safely be skipped

Additivity

The next result we prove is usually referred to as the additive property of the integral. First we prove the additivity property for the lower and upper Darboux integrals.

[lemma:darbouxadd] Suppose $a < b < c$ and $f: [a, c] \rightarrow \mathbb{R}$ is a bounded function. Then $\underline{\int_a^c f} = \underline{\int_a^b f} + \underline{\int_b^c f}$ and $\overline{\int_a^c f} = \overline{\int_a^b f} + \overline{\int_b^c f}$.

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If we have partitions $P_1 = \{x_0, x_1, \dots, x_k\}$ of $[a, b]$ and $P_2 = \{x_k, x_{k+1}, \dots, x_n\}$ of $[b, c]$, then the set $P := P_1 \cup P_2 = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, c]$. Then $L(P, f) = \sum_{j=1}^n m_j \Delta x_j = \sum_{j=1}^k m_j \Delta x_j + \sum_{j=k+1}^n m_j \Delta x_j = L(P_1, f) + L(P_2, f)$. When we take the supremum of the right hand side over all P_1 and P_2 , we are taking a supremum of the left hand side over all partitions P of $[a, c]$ that contain b . If Q is any partition of $[a, c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $L(Q, f) \leq L(P, f)$. Therefore, taking a supremum only over the P that contain b is sufficient to find the supremum of $L(P, f)$ over all partitions P , see . Finally recall to compute $\begin{aligned} \sup \{ L(P, f) : \text{``$P$ is a partition of $[a, c]$''} \} &= \sup \{ L(P, f) : \text{``P is a partition of $[a, c]$, $\$b \in P\$''} \} \\ &= \sup \{ L(P_1, f) + L(P_2, f) : \text{``$\$P_1\$ is a partition of $\$[a, b]$, $\$P_2\$ is a partition of $\$[b, c]\$''} \} \\ &= \sup \{ L(P_1, f) : \text{``$\$P_1\$ is a partition of $\$[a, b]\$''} \} + \sup \{ L(P_2, f) : \text{``$\$P_2\$ is a partition of $\$[b, c]\$''} \} \\ &= \underbrace{\sup_{a \leq x < b} f(x)}_{\text{``$\int_a^b f$''}} + \underbrace{\sup_{b \leq x \leq c} f(x)}_{\text{``$\int_b^c f$''}}. \end{aligned}$

Similarly, for P_1 and P_2 as above we obtain $U(P, f) = \sum_{j=1}^n M_j \Delta x_j = \sum_{j=1}^k M_j \Delta x_j + \sum_{j=k+1}^n M_j \Delta x_j = U(P_1, f) + U(P_2, f)$. We wish to take the infimum on the right over all P_1 and P_2 , and so we are taking the infimum over all partitions P of $[a, c]$ that contain b . If Q is any partition of $[a, c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $U(Q, f) \geq U(P, f)$. Therefore, taking an infimum only over the P that contain b is sufficient to find the infimum of $U(P, f)$ for all P . We obtain $\overline{\int_a^c f} = \overline{\int_a^b f} + \overline{\int_b^c f}$.

Let $a < b < c$. A function f $\colon [a, c] \rightarrow \mathbb{R}$ is Riemann integrable if and only if f is Riemann integrable on $[a, b]$ and $[b, c]$. If f is Riemann integrable, then $\int_a^c f = \int_a^b f + \int_b^c f$.

Suppose $f \in \mathcal{R}[a, c]$, then $\overline{\int_a^c} f = \underline{\int_a^c} f = \int_a^c f$. We apply the lemma to get $\int_a^c f = \overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f \leq \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f = \int_a^c f$. Thus the inequality is an equality and $\underline{\int_a^b} f + \underline{\int_b^c} f = \underline{\int_a^c} f = \int_a^c f = \overline{\int_a^b} f + \overline{\int_b^c} f$. As we also know $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ and $\underline{\int_b^c} f \leq \overline{\int_b^c} f$, we conclude $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$ and $\underline{\int_b^c} f = \overline{\int_b^c} f = \int_b^c f$. Thus f is Riemann integrable on $[a, b]$ and $[b, c]$ and the desired formula holds.

Now assume the restrictions of f to $[a,b]$ and to $[b,c]$ are Riemann integrable. We again apply the lemma to get $\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f$. Therefore f is Riemann integrable on $[a,c]$, and the integral is computed as indicated.

An easy consequence of the additivity is the following corollary. We leave the details to the reader as an exercise.

[intsubcor] If $f \in \{\mathcal{R}\}[a,b]$ and $[c,d] \subset [a,b]$, then the restriction $f|_{[c,d]}$ is in $\{\mathcal{R}\}[c,d]$.

Linearity and monotonicity

Let f and g be in $\mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$.

- i. αf is in $\{\mathcal{M}\}[a,b]$ and $\int_a^b \alpha f(x) \sim dx = \alpha \int_a^b f(x) \sim dx$.
- ii. $f+g$ is in $\{\mathcal{M}\}[a,b]$ and $\int_a^b |f|g| f(x)+g(x) |g| \sim dx = \int_a^b f(x) \sim dx + \int_a^b g(x) \sim dx$

Let us prove the first item. First suppose $\alpha \geq 0$. Let P be a partition of $[a, b]$. Let $m_i := \inf \{ f(x) : x \in [x_{i-1}, x_i] \}$ as usual. Since α is nonnegative, we can move the multiplication by α past the infimum, $\inf \{ \alpha f(x) : x \in [x_{i-1}, x_i] \} = \alpha \inf \{ f(x) : x \in [x_{i-1}, x_i] \} = \alpha m_i$. Therefore $L(P, \alpha f) = \sum_{i=1}^n \alpha m_i \Delta x_i = \alpha L(P, f)$. Similarly $U(P, \alpha f) = \alpha U(P, f)$. Again, as $\alpha \geq 0$ we may move multiplication by α past the supremum. Hence, $\begin{aligned} \underline{\int_a^b} \alpha f(x) dx &= \sup \{ L(P, \alpha f) : P \text{ a partition of } [a, b] \} \\ &= \alpha \sup \{ L(P, f) : P \text{ a partition of } [a, b] \} \\ &= \alpha \underline{\int_a^b} f(x) dx. \end{aligned}$ Similarly we show $\overline{\int_a^b} \alpha f(x) dx = \alpha \overline{\int_a^b} f(x) dx$. The conclusion now follows for $\alpha \geq 0$.

To finish the proof of the first item, we need to show that $-f$ is Riemann integrable and $\int_a^b -f(x) \sim dx = - \int_a^b f(x) \sim dx$. The proof of this fact is left as an exercise.

The proof of the second item in the proposition is also left as an exercise. It is not as trivial as it may appear at first glance.

We should note that the second item in the proposition does not hold with equality for the Darboux integrals. For arbitrary bounded functions f and g we only obtain $\overline{\int_{\text{a}}^{\text{b}} (f(x)+g(x)) \, dx} \leq \overline{\int_{\text{a}}^{\text{b}} f(x) \, dx} + \overline{\int_{\text{a}}^{\text{b}} g(x) \, dx}$ and $\underline{\int_{\text{a}}^{\text{b}} (f(x)+g(x)) \, dx} \geq \underline{\int_{\text{a}}^{\text{b}} f(x) \, dx} + \underline{\int_{\text{a}}^{\text{b}} g(x) \, dx}$.

Let f and g be in $\{\mathcal{R}\}[a,b]$ and let $f(x) \leq g(x)$ for all $x \in [a,b]$. Then $\int_a^b f \leq \int_a^b g$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \leq \widetilde{m}_i := \inf\{g(x) : x \in [x_{i-1}, x_i]\}$. As $f(x) \leq g(x)$, then $m_i \leq \widetilde{m}_i$. Therefore, $L(P, f) = \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n \widetilde{m}_i \Delta x_i = L(P, g)$. We take the supremum over all P (see) to obtain $\underline{\int_a^b} f \leq \underline{\int_a^b} g$. As f and g are Riemann integrable, the conclusion follows.

Continuous functions

Let us show that continuous functions are Riemann integrable. In fact we will show we can even allow some discontinuities. We start with a function continuous on the whole closed interval $[a, b]$.

[lemma:contint] If $f \colon [a,b] \rightarrow \mathbb{R}$ is a continuous function, then $f \in \mathcal{R}[a,b]$.

As f is continuous on a closed bounded interval, it is uniformly continuous. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$ for all $i = 1, 2, \dots, n$. For example, take n such that $\frac{b-a}{n} < \delta$ and let $x_i = \frac{b-a}{n}i + a$. Then for all x, y in $[x_{i-1}, x_i]$ we have $|f(x) - f(y)| \leq \Delta x_i < \delta$ and so $f(x) - f(y) \leq \delta$ and $f(x) - f(y) \geq -\delta$. As f is continuous on $[x_{i-1}, x_i]$, it attains a maximum and a minimum on this interval. Let M_i be a point where f attains the maximum and m_i be a point where f attains the minimum. Then $f(x) = M_i$ and $f(y) = m_i$ in the notation from the definition of the integral. Therefore, $M_i - m_i = f(x) - f(y) < \delta$. And so $\sum_{i=1}^n (M_i - m_i) \Delta x_i < \delta \sum_{i=1}^n \Delta x_i = \delta(b-a)$. As δ is arbitrary, $\sum_{i=1}^n (M_i - m_i) \Delta x_i = 0$. And so $\sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n m_i \Delta x_i$. And so $\sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n m_i \Delta x_i = \int_a^b f(x) dx$. And so f is Riemann integrable on $[a, b]$.

The second lemma says that we need the function to only be “Riemann integrable inside the interval,” as long as it is bounded. It also tells us how to compute the integral.

[lemma:boundedimpriemann] Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function that is Riemann integrable on $[a', b']$ for all a', b' such that $a < a' < b' < b$. Then f is Riemann integrable on $[a, b]$. Furthermore, if $a < a_n < b_n < b$ are such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$, then $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} f(x) dx$.

Let $M > 0$ be a real number such that $\left| f(x) \right| \leq M$. Pick two sequences of numbers $a < a_n < b_n < b$ such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. Note $M > 0$ and $(b-a) \geq (b_n - a_n)$. Thus $-M(b-a) \leq -M(b_n - a_n) \leq \inf_{a_n \leq x \leq b_n} f(x) \leq M(b_n - a_n) \leq M(b-a)$. Therefore the sequence of numbers $\{ \inf_{a_n \leq x \leq b_n} f(x) \}_{n=1}^{\infty}$ is bounded and by has a convergent subsequence indexed by n_k . Let us call L the limit of the subsequence $\{ \inf_{a_{n_k} \leq x \leq b_{n_k}} f(x) \}_{k=1}^{\infty}$.

says that the lower and upper integral are additive and the hypothesis says that f is integrable on $[a_{-n_k}, b_{-n_k}]$. Therefore $\underline{\int} f = \underline{\int} f + \underline{\int} f \wedge b_{-n_k}$ and $\underline{\int} f = \underline{\int} f + \underline{\int} f \wedge b_{-n_k}$. We take the limit as k goes to ∞ on the right-hand side,

Next we use additivity of the upper integral, $\overline{\int} (a+b) = \overline{\int} a + \overline{\int} b$. We take the same subsequence $\{ \int_{a_k}^{b_k} f \} (k=1)^\infty$ and take the limit to obtain $\overline{\int} (a+b) \leq \overline{\int} a + \overline{\int} b$. Thus $\overline{\int} (a+b) = \overline{\int} a + \overline{\int} b$. In particular, no matter what sequences $\{ a_n \}$ and $\{ b_n \}$ we started with and what subsequence we chose, the L is the same number.

To prove the final statement of the lemma we use ϵ . We have shown that every convergent subsequence $\{ \langle a_{n_k} \rangle^b_{b_{n_k}} \}$ converges to $L = \langle a \rangle^b_b$. Therefore, the sequence $\{ \langle a_n \rangle^b_{b_n} \}$ is convergent and converges to L .

We say a function $f \colon [a,b] \rightarrow \mathbb{R}$ has *finitely many discontinuities* if there exists a finite set $S := \{x_1, x_2, \dots, x_n\} \subset [a,b]$, and f is continuous at all points of $[a,b] \setminus S$.

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Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function with finitely many discontinuities. Then f is Riemann integrable.

We divide the interval into finitely many intervals $[a_i, b_i]$ so that f is continuous on the interior (a_i, b_i) . If f is continuous on (a_i, b_i) , then it is continuous and hence integrable on $[c_i, d_i]$ whenever $a_i < c_i < d_i < b_i$. By the restriction of f to $[a_i, b_i]$ is integrable. By additivity of the integral (and f is integrable on the union of the intervals).

Sometimes it is convenient (or necessary) to change certain values of a function and then integrate. The next result says that if we change the values only at finitely many points, the integral does not change.

Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable. Let $g: [a, b] \rightarrow \mathbb{R}$ be a function such that $f(x) = g(x)$ for all $x \in [a, b] \setminus S$, where S is a finite set. Then g is a Riemann integrable function and $\int_a^b g = \int_a^b f$.

Using additivity of the integral, we split up the interval $[a, b]$ into smaller intervals such that $f(x) = g(x)$ holds for all x except at the endpoints (details are left to the reader).

Therefore, without loss of generality suppose $f(x) = g(x)$ for all $x \in (a, b)$. The proof follows by ϵ and is left as an exercise.

Exercises

Let f be in $\mathcal{R}[a, b]$. Prove that $-f$ is in $\mathcal{R}[a, b]$ and $\int_a^b -f(x) dx = -\int_a^b f(x) dx$.

Let f and g be in $\mathcal{R}[a, b]$. Prove that $f+g$ is in $\mathcal{R}[a, b]$ and $\int_a^b (f+g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$. Hint: Use to find a single partition P such that $U(P, f) - L(P, f) < \epsilon/2$ and $U(P, g) - L(P, g) < \epsilon/2$.

Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable. Let $g: [a, b] \rightarrow \mathbb{R}$ be a function such that $f(x) = g(x)$ for all $x \in (a, b)$. Prove that g is Riemann integrable and that $\int_a^b g = \int_a^b f$.

Prove the *mean value theorem for integrals*. That is, prove that if $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then there exists a $c \in [a, b]$ such that $\int_a^b f = f(c)(b-a)$.

If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all x .

If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function for all $x \in [a, b]$ and $\int_a^b f = 0$. Prove that there exists a $c \in [a, b]$ such that $f(c) = 0$ (Compare with the previous exercise).

If $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ are continuous functions such that $\int_a^b f = \int_a^b g$. Then show that there exists a $c \in [a, b]$ such that $f(c) = g(c)$.

Let α, β, γ be arbitrary numbers in $[a, b]$ (not necessarily ordered in any way). Prove $\int_a^b x^\alpha x^\beta x^\gamma dx = \int_a^b x^{\alpha+\beta+\gamma} dx$. Recall what $\int_a^b f$ means if $b \leq a$.

Prove.

[exercise:easyabsint] Suppose $f: [a, b] \rightarrow \mathbb{R}$ is bounded and has finitely many discontinuities. Show that as a function of x the expression $\left| \int_a^x f(t) dt \right|$ is bounded with finitely many discontinuities and is thus Riemann integrable. Then show $\int_a^b \left| \int_a^x f(t) dt \right| dx \leq \int_a^b |f(x)| dx$.

Show that the Thomae or popcorn function (see) is Riemann integrable. Therefore, there exists a function discontinuous at all rational numbers (a dense set) that is Riemann integrable.

In particular, define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ Show $\int_0^1 f = 0$.

If $I \subset \mathbb{R}$ is a bounded interval, then the function $\varphi_I(x) := \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$ is called an *elementary step function*.

Let I be an arbitrary bounded interval (you should consider all types of intervals: closed, open, half-open) and $a < b$, then using only the definition of the integral show that the elementary step function φ_I is integrable on $[a, b]$, and find the integral in terms of a, b , and the endpoints of I .

When a function f can be written as $f(x) = \sum_{k=1}^n \alpha_k \varphi_{I_k}(x)$ for some real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and some bounded intervals $I_1, I_2, \dots, I_n$, then f is called a *step function*.

Using the previous exercise, show that a step function is integrable on any interval $[a, b]$. Furthermore, find the integral in terms of a, b , the endpoints of I_k and the α_k .

[exercise:boundedvariationintegrable] Let $f: [a, b] \rightarrow \mathbb{R}$ be increasing. a) Show that f is Riemann integrable. Hint: Use a uniform partition; each subinterval of same length. b) Use part a to show that a decreasing function is Riemann integrable. c) Suppose $h = f - g$ where f and g are increasing functions on $[a, b]$. Show that h is Riemann integrable.

[exercise:hardabsint] Suppose $f: [a, b] \rightarrow \mathbb{R}$, then the function that takes x to $\left| \int_a^x f(t) dt \right|$ is also Riemann integrable on $[a, b]$. Then show the same inequality as .

[exercise:upperlowerlineineq] Suppose $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ are bounded. a) Show $\overline{\int_a^b (f+g)} \leq \overline{\int_a^b f} + \overline{\int_a^b g}$ and $\underline{\int_a^b (f+g)} \geq \underline{\int_a^b f} + \underline{\int_a^b g}$. b) Find example f and g where the inequality is strict. Hint: f and g should not be Riemann integrable.

Fundamental theorem of calculus

Notes: 1.5 lectures

In this chapter we discuss and prove the *fundamental theorem of calculus*. The entirety of integral calculus is built upon this theorem, ergo the name. The theorem relates the seemingly unrelated concepts of integral and derivative. It tells us how to compute the antiderivative of a function using the integral and vice-versa.

First form of the theorem

Let $F: [a, b] \rightarrow \mathbb{R}$ be a continuous function, differentiable on (a, b) . Let $f: (a, b) \rightarrow \mathbb{R}$ be such that $f(x) = F'(x)$ for $x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.

It is not hard to generalize the theorem to allow a finite number of points in $[a, b]$ where F is not differentiable, as long as it is continuous. This generalization is left as an exercise.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. For each interval $[x_{i-1}, x_i]$, use the to find a $c_i \in (x_{i-1}, x_i)$ such that $f(c_i) \Delta x_i = F(c_i) - F(x_{i-1}) = F(x_i) - F(x_{i-1})$. Using the notation from the definition of the integral, we have $m_i \leq f(c_i) \leq M_i$ and so $m_i \Delta x_i \leq F(x_i) - F(x_{i-1}) \leq M_i \Delta x_i$. We sum over $i = 1, 2, \dots, n$ to get $\sum_{i=1}^n m_i \Delta x_i \leq F(x_n) - F(x_0) \leq \sum_{i=1}^n M_i \Delta x_i$. In the middle sum, all the terms except the first and last cancel and we end up with $F(x_n) - F(x_0) = F(b) - F(a)$. The sums on the left and on the right are the lower and the upper sum respectively. So $L(P, f) \leq F(b) - F(a) \leq U(P, f)$. We take the supremum of $L(P, f)$ over all P and the left inequality yields $\underline{\int_a^b f} \leq F(b) - F(a)$. Similarly, taking the infimum of $U(P, f)$ over all partitions P yields $F(b) - F(a) \leq \overline{\int_a^b f}$. As f is Riemann integrable, we have $\int_a^b f = \underline{\int_a^b f} = \overline{\int_a^b f} = F(b) - F(a)$. The inequalities must be equalities and we are done.

The theorem is used to compute integrals. Suppose we know that the function $f(x)$ is a derivative of some other function $F(x)$, then we can find an explicit expression for $\int_a^b f$.

Suppose we are trying to compute $\int_0^1 x^2 dx$. We notice x^2 is the derivative of $\frac{1}{3}x^3$. We use the fundamental theorem to write $\int_0^1 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^1 = \frac{1}{3}$.

Second form of the theorem

The second form of the fundamental theorem gives us a way to solve the differential equation $F'(x) = f(x)$, where f is a known function and we are trying to find an F that satisfies the equation.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Define $F(x) := \int_a^x f$. First, F is continuous on $[a, b]$. Second, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

As f is bounded, there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Suppose $x, y \in [a, b]$ with $x > y$. Then $|F(x) - F(y)| = \left| \int_y^x f \right| \leq \int_y^x M = M(x - y)$. By symmetry, the same also holds if $x < y$. So F is Lipschitz continuous and hence continuous.

Now suppose f is continuous at c . Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that for $x \in [a, b]$ $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. In particular, for such x we have $f(c) - \epsilon \leq f(x) \leq f(c) + \epsilon$. Thus if $x > c$, then $\int_c^x (f(c) - \epsilon) \leq \int_c^x f \leq \int_c^x (f(c) + \epsilon)$. When $c > x$, then the inequality is reversed. Therefore, assuming $c \neq x$ we get $f(c) - \epsilon \leq \frac{F(x) - F(c)}{x - c} \leq f(c) + \epsilon$. As $\frac{F(x) - F(c)}{x - c} = \frac{\int_c^x f}{x - c} = \frac{\int_c^x (f - f(c) + f(c))}{x - c} = \frac{\int_c^x (f - f(c))}{x - c} + f(c) = \frac{\int_c^x (f - f(c))}{x - c} + f(c)$.

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$\left| \int_c^x f(t) dt - F(x) + F(c) \right| \leq \epsilon$. The result follows. It is left to the reader to see why it is OK that we just have a non-strict inequality.

Of course, if f is continuous on $[a, b]$, then it is automatically Riemann integrable, F is differentiable on all of $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

The second form of the fundamental theorem of calculus still holds if we let $d \in [a, b]$ and define $F(x) := \int_d^x f$. That is, we can use any point of $[a, b]$ as our base point. The proof is left as an exercise.

Let us look at what a simple discontinuity can do. Take $f(x) := -1$ if $x < 0$, and $f(x) := 1$ if $x \geq 0$. Let $F(x) := \int_0^x f$. It is not difficult to see that $F(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$. Notice that f is discontinuous at 0 and F is not differentiable at 0. However, the converse does not hold. Let us do another quick example. Let $g(x) := 0$ if $x \neq 0$, and $g(0) = 1$. Letting $G(x) := \int_0^x g$, we find that $G(x) = 0$ for all x . So g is discontinuous at 0, but $G'(0)$ exists and is equal to 0.

A common misunderstanding of the integral for calculus students is to think of integrals whose solution cannot be given in closed-form as somehow deficient. This is not the case. Most integrals we write down are not computable in closed-form. Even some integrals that we consider in closed-form are not really such. For example, how does a computer find the value of $\int_0^1 x^x dx$? One way to do it is to note that we define the natural log as the antiderivative of $\frac{1}{x}$ such that $\ln 1 = 0$. Therefore, $\ln x := \int_1^x \frac{1}{s} ds$. Then we can numerically approximate the integral. Morally, we did not really “simplify” $\int_0^1 x^x dx$ by writing down $\ln x$. We simply gave the integral a name. If we require numerical answers, it is possible we end up doing the calculation by approximating an integral anyway.

Another common function defined by an integral that cannot be evaluated symbolically is the erf function, defined as $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$. This function comes up often in applied mathematics. It is simply the antiderivative of $\frac{2}{\sqrt{\pi}} e^{-x^2}$ that is zero at zero. The second form of the fundamental theorem tells us that we can write the function as an integral. If we wish to compute any particular value, we numerically approximate the integral.

Change of variables

A theorem often used in calculus to solve integrals is the change of variables theorem. Let us prove it now. Recall a function is continuously differentiable if it is differentiable and the derivative is continuous.

Let $g: [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable function. If $g([a, b]) \subset [c, d]$ and $f: [c, d] \rightarrow \mathbb{R}$ is continuous, then $\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(s) ds$.

As g , g' , and f are continuous, we know $f(g(x))g'(x)$ is a continuous function on $[a, b]$, therefore it is Riemann integrable.

Define $F(y) := \int_{g(a)}^y f(s) ds$. By the second form of the fundamental theorem of calculus (using below) F is a differentiable function and $F'(y) = f(y)$. We apply the chain rule and write $(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$. We note that $F(g(a)) = 0$ and we use the first form of the fundamental theorem to obtain $\int_a^b f(g(x))g'(x) dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(s) ds$.

The change of variables theorem is often used to solve integrals by changing them to integrals that we know or that we can solve using the fundamental theorem of calculus.

From an exercise, we know that the derivative of $\sin(x)$ is $\cos(x)$. Therefore we solve $\int_0^{\pi/2} \cos(s) ds = \sin(s) \Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1$.

However, beware that we must satisfy the hypotheses of the theorem. The following example demonstrates why we should not just move symbols around mindlessly. We must be careful that those symbols really make sense.

Suppose we write down $\int_{-1}^1 \frac{1}{x} dx$. It may be tempting to take $g(x) := \ln|x|$. Then take $g'(x) = \frac{1}{x}$ and try to write $\int_{-1}^1 \frac{1}{x} dx = \ln|x| \Big|_{-1}^1 = \ln(1) - \ln(1) = 0$. This “solution” is incorrect, and it does not say that we can solve the given integral. First problem is that $\frac{1}{x}$ is not continuous on $[-1, 1]$. Second, $\ln|x|$ is not even Riemann integrable on $[-1, 1]$ (it is unbounded). The integral we wrote down simply does not make sense. Finally, g is not continuous on $[-1, 1]$ either.

Exercises

Compute $\int_0^1 x^2 dx$.

Compute $\int_0^1 x^2 \sin(x^2) dx$.

Suppose $F: [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on $[a, b] \setminus S$, where S is a finite set. Suppose there exists an $f \in \mathcal{C}([a, b])$ such that $F'(x) = f(x)$ for $x \in [a, b] \setminus S$. Show that $\int_a^b f = F(b) - F(a)$.

[second ftc exercise] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Let $c \in [a, b]$ be arbitrary. Define $F(x) := \int_c^x f$. Prove that F is differentiable and that $F'(x) = f(x)$ for all $x \in [a, b]$.

Prove *integration by parts*. That is, suppose F and G are continuously differentiable functions on $[a, b]$. Then prove $\int_a^b F(x)G'(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x) dx$.

Suppose F and G are continuously²³ differentiable functions defined on $[a, b]$ such that $F'(x) = G'(x)$ for all $x \in [a, b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$.

The next exercise shows how we can use the integral to “smooth out” a non-differentiable function.

[exercise:smoothingout] Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Let $\epsilon > 0$ be a constant. For $x \in [a + \epsilon, b - \epsilon]$, define $g(x) := \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f$. a) Show that g is differentiable and find the derivative.

b) Let f be differentiable and fix $x \in (a, b)$ (let ϵ be small enough). What happens to $g'(x)$ as ϵ gets smaller?

c) Find g for $f(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, $\epsilon = 1$ (you can assume $[a, b]$ is large enough).

Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $\int_a^x f = \int_a^b f$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all rational $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

A function f is an *odd function* if $f(x) = -f(-x)$, and f is an *even function* if $f(x) = f(-x)$. Let $a > 0$. Assume f is continuous. Prove: a) If f is odd, then $\int_{-a}^a f = 0$. b) If f is even, then $\int_{-a}^a f = 2 \int_0^a f$.

a) Show that $f(x) := \sin(\frac{1}{x})$ is integrable on any interval (you can define $f(0)$ to be anything). b) Compute $\int_{-1}^1 \sin(\frac{1}{x}) dx$. (Mind the discontinuity)

§6 a) Suppose $f: [a, b] \rightarrow \mathbb{R}$ is increasing, by f is Riemann integrable. Suppose f has a discontinuity at $c \in (a, b)$, show that $F(x) := \int_a^x f$ is not differentiable at c .

b) In , you have constructed an increasing function $f: [0, 1] \rightarrow \mathbb{R}$ that is discontinuous at every $x \in [0, 1] \cap \mathbb{Q}$. Use this f to construct a function $F(x)$ that is continuous on $[0, 1]$, but not differentiable at every $x \in [0, 1] \cap \mathbb{Q}$.

The logarithm and the exponential

Note: 1 lecture (optional, requires the optional sections , ,)

We now have all that is required to finally properly define the exponential and the logarithm that you know from calculus so well. First recall that we have a good idea of what x^n means as long as n is a positive integer. Simply, $x^n := \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$. It makes sense to define $x^0 := 1$. For negative integers we define $x^{-n} := \frac{1}{x^n}$. If $x > 0$, we mentioned before that $x^{1/n}$ is defined as the unique positive n th root. Finally for any rational number $\frac{m}{n}$, we define $x^{m/n} := (\sqrt[n]{x^m})$. However, what do we mean by $\sqrt{2}$? Or x^y in general? In particular, what is e^x for all x ? And how do we solve $y = e^x$ for x ? This section answers these questions and more.

The logarithm

It is convenient to start with the logarithm. Let us show that a unique function with the right properties exists, and only then will we call it *the* logarithm.

Processing math: 52% We function $L: (0, \infty) \rightarrow \mathbb{R}$ such that

- i. $L(1) = 0$.
- ii. L is differentiable and $L'(x) = \frac{1}{x}$.
- iii. L is strictly increasing, bijective, and $\lim_{x \rightarrow 0^+} L(x) = -\infty$, $\lim_{x \rightarrow \infty} L(x) = \infty$.
- iv. $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$.
- v. If q is a rational number and $x > 0$, then $L(x^q) = q L(x)$.

To prove existence, let us define a candidate and show it satisfies all the properties. Define $L(x) := \int_1^x \frac{1}{t} dt$.

Obviously [\[it:log:i\]](#) holds. Property [\[it:log:ii\]](#) holds via the fundamental theorem of calculus.

To prove property [\[it:log:iv\]](#), we change variables $u=yt$ to obtain $L(x) = \int_1^x \frac{1}{t} dt = \int_{1/y}^{x/y} \frac{1}{u} du = \int_1^{x/y} \frac{1}{u} du - \int_1^{1/y} \frac{1}{u} du = L(x/y) - L(1/y)$.

Property [\[it:log:ii\]](#) together with the fact that $L'(x) = \frac{1}{x} > 0$ for $x > 0$, implies that L is strictly increasing and hence one-to-one. Let us show L is onto. As $\frac{1}{t} \geq \frac{1}{2}$ when $t \in [1, 2]$, $L(2) = \int_1^2 \frac{1}{t} dt \geq \frac{1}{2}$. By induction, [\[it:log:iv\]](#) implies that for $n \in \mathbb{N}$ $L(2^n) = L(2) + L(2) + \dots + L(2) = n L(2)$. Given any $y > 0$, by the of the real numbers (notice $L(2) > 0$), there is an $n \in \mathbb{N}$ such that $L(2^n) > y$. By the there is an $x_1 \in (1, 2^n)$ such that $L(x_1) = y$. We get $(0, \infty)$ is in the image of L . As L is increasing, $L(x) > y$ for all $x > x_1$, and so $\lim_{x \rightarrow \infty} L(x) = \infty$. Next $0 = L(\frac{1}{x}) = L(x) + L(\frac{1}{x})$, and so $L(x) = -L(\frac{1}{x})$. Using $x=2^{-n}$, we obtain as above that L achieves all negative numbers. And $\lim_{x \rightarrow 0^+} L(x) = \lim_{x \rightarrow 0^+} -L(\frac{1}{x}) = \lim_{x \rightarrow \infty} -L(x) = -\infty$. In the limits, note that only $x > 0$ are in the domain of L .

Let us now prove [\[it:log:iv\]](#). As above, [\[it:log:iv\]](#) implies for $n \in \mathbb{N}$ we have $L(x^n) = n L(x)$. We already saw that $L(x) = -L(\frac{1}{x})$ so $L(x^{-n}) = -L(\frac{1}{x^n}) = -n L(x)$. Then for $m \in \mathbb{N}$ $L(x) = L(\frac{1}{x^{-m}}) = -L(x^{-m}) = -m L(x^{-1/m})$. Putting everything together for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have $L(x^{n/m}) = n L(x^{1/m}) = (\frac{n}{m}) L(x)$.

Finally for uniqueness, let us use properties [\[it:log:i\]](#) and [\[it:log:ii\]](#). Via the fundamental theorem of calculus $L(x) = \int_1^x \frac{1}{t} dt$ is the unique function such that $L(1) = 0$ and $L'(x) = \frac{1}{x}$.

Having proved that there is a unique function with these properties we simply define the *logarithm* or sometimes called the *natural logarithm*: $\ln(x) := L(x)$. Often mathematicians write $\log(x)$ instead of $\ln(x)$, which is more familiar to calculus students.

The exponential

Just as with the logarithm we define the exponential via a list of properties.

There exists a unique function $E: (0, \infty) \rightarrow \mathbb{R}$ such that

- i. $E(1) = 1$.
- ii. E is differentiable and $E'(x) = E(x)$.
- iii. E is strictly increasing, bijective, and $\lim_{x \rightarrow 0^+} E(x) = 0$, $\lim_{x \rightarrow \infty} E(x) = \infty$.
- iv. $E(x+y) = E(x)E(y)$ for all $x, y \in (0, \infty)$.
- v. If $q \in \mathbb{Q}$, then $E(x^q) = (E(x))^q$.

Again, let us prove existence of such a function by defining a candidate, and prove that it satisfies all the properties. The L defined above is invertible. Let E be the inverse function of L . Property [\[it:exp:i\]](#) is immediate.

Property [\[it:exp:ii\]](#) follows via the inverse function theorem, in particular: L satisfies all the hypotheses of the lemma, and hence $E'(x) = \frac{1}{L'(E(x))} = E(x)$.

Let us look at property [\[it:exp:iii\]](#). The function E is strictly increasing since $E(x) > 0$ and $E'(x) = E(x) > 0$. As E is the inverse of L , it must also be bijective. To find the limits, we use that E is strictly increasing and onto $(0, \infty)$. For every $M > 0$, there is an x_0 such that $E(x_0) = M$ and $E(x) \geq M$ for all $x \geq x_0$. Similarly for every $\epsilon > 0$, there is an x_0 such that $E(x_0) = \epsilon$ and $E(x) < \epsilon$ for all $x < x_0$. Therefore, $\lim_{x \rightarrow 0^+} E(x) = 0$, $\lim_{x \rightarrow \infty} E(x) = \infty$.

To prove property [\[it:exp:iv\]](#) we use the corresponding property for the logarithm. Take $x, y \in (0, \infty)$. As L is bijective, find a and b such that $x = L(a)$ and $y = L(b)$. Then $E(x+y) = E(\frac{1}{L(a)} + \frac{1}{L(b)}) = E(\frac{L(a)+L(b)}{L(a)L(b)}) = E(\frac{L(ab)}{L(a)L(b)}) = ab = E(x)E(y)$.

Property [\[it:exp:v\]](#) also follows from the corresponding property of L . Given $x \in (0, \infty)$, let a be such that $x = L(a)$ and $E(qx) = E(\frac{1}{L(a)} + \frac{1}{L(a^q)}) = E(\frac{L(a)+L(a^q)}{L(a)L(a^q)}) = a^q = (E(x))^q$.

Finally, uniqueness follows from [\[it:exp:i\]](#) and [\[it:exp:ii\]](#). Let E and F be two functions satisfying [\[it:exp:i\]](#) and [\[it:exp:ii\]](#). $\frac{d}{dx} (F(x)E(-x)) = F'(x)E(-x) - F(x)E'(-x) = F(x)E(-x) - E(-x)F(x) = 0$. Therefore $F(x)E(-x) = F(0)E(-0) = 1$ for all $x \in (0, \infty)$. Doing the computation with $F = E$, we obtain $E(x)E(-x) = 1$. Then $0 = 1 - 1 = F(x)E(-x) - E(x)E(-x) = \frac{d}{dx} (F(x)E(-x))$. Since $E(x)E(-x) = 1$, then $E(-x) \neq 0$ for all x . So $F(x)E(-x) = 0$ for all x , and we are done.

Having proved E is unique, we define the *exponential* function as $\exp(x) := E(x)$.

We can now make sense of exponentiation x^y for arbitrary numbers when $x > 0$. First suppose $y \in \mathbb{Q}$. Then $x^y = \exp(\frac{1}{L(x)} \ln(x^y)) = \exp(\frac{y \ln(x)}{L(x)})$. Therefore when $x > 0$ and y is irrational let us define $x^y := \exp(\frac{y \ln(x)}{L(x)})$. As $\exp$ is continuous then x^y is a continuous function of y . Therefore, we would obtain the same result had we taken a sequence of rational numbers $\{y_n\}$ approaching y and defined $x^y = \lim x^{y_n}$.

Define the number e as $e := \exp(1)$. The number e is sometimes called *Euler's number* or the *base of the natural logarithm*. We notice $e^x = \exp(\frac{1}{L(e)} \ln(e^x)) = \exp(x)$. We have justified the notation e^x for $\exp(x)$.

Finally, let us extend properties of logarithm and exponential to irrational powers. The proof is immediate.

Let $x, y \in \mathbb{R}$.

- i. $\exp(xy) = (\exp(x))^y$.
- ii. If $x > 0$ then $\ln(x^y) = y \ln(x)$.

Exercises

Let y be any real number and $b > 0$. Define $f: (0, \infty) \rightarrow \mathbb{R}$ and $g: (0, \infty) \rightarrow \mathbb{R}$ as, $f(x) := x^y$ and $g(x) := b^x$. Show that f and g are differentiable and find their derivative.

Let $b > 0$ be given.

- a) Show that for every $y > 0$, there exists a unique number x such that $y = b^x$. Define the *logarithm base b*, $\log_b: (0, \infty) \rightarrow \mathbb{R}$, by $\log_b(y) := x$.
- b) Show that $\log_b(x) = \frac{\ln(x)}{\ln(b)}$.
- c) Prove that if $c > 0$, then $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$.
- d) Prove $\log_b(xy) = \log_b(x) + \log_b(y)$, and $\log_b(x^y) = y \log_b(x)$.

§3 Use to study the remainder term and show that for all $x \in \mathbb{R}$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Hint: Do not differentiate the series term by term (unless you would prove that it works).

Use the geometric sum formula to show (for $t \neq -1$) $\sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$.

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Using this fact show $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$

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$$\{n\} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \cdots = \ln 2.$$

Show $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$. Hint: Take the logarithm.

Note: The expression $\left(1 + \frac{x}{n}\right)^n$ arises in compound interest calculations. It is the amount of money in a bank account after 1 year if 1 dollar was deposited initially at interest x and the interest was compounded n times during the year. Therefore e^x is the result of continuous compounding.

a) Prove that for $n \in \mathbb{N}$ we have $\sum_{k=2}^n \frac{1}{k} \leq \ln(n) \leq \sum_{k=1}^{n-1} \frac{1}{k}$.

b) Prove that the limit $\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right)$ exists. This constant is known as the *Euler-Mascheroni constant*²⁴. It is not known if this constant is rational or not, it is approximately $\gamma \approx 0.5772$.

Show $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0$.

Show that e^x is *convex*, in other words, show that if $a \leq x \leq b$ then $e^x \leq e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a}$.

Using the logarithm find $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$, $\lim_{n \rightarrow \infty} n^{1/n} = 1$.

Show that $E(x) = e^x$ is the unique continuous function such that $E(x+y) = E(x)E(y)$ and $E(1) = e$. Similarly prove that $L(x) = \ln(x)$ is the unique continuous function defined on positive x such that $L(xy) = L(x) + L(y)$ and $L(e) = 1$.

Improper integrals

Note: 2–3 lectures (optional section, can safely be skipped, requires the optional)

Often it is necessary to integrate over the entire real line, or a infinite interval of the form $[a, \infty)$ or $(-\infty, b]$. Also, we may wish to integrate functions defined on a finite interval (a, b) but not bounded. Such functions are not Riemann integrable, but we may want to write down the integral anyway in the spirit of . These integrals are called *improper integrals*, and are limits of integrals rather than integrals themselves.

Suppose $f: [a, b) \rightarrow \mathbb{R}$ is a function (not necessarily bounded) that is Riemann integrable on $[a, c]$ for all $c < b$. We define $\int_a^b f := \lim_{c \rightarrow b^-} \int_a^c f$, if the limit exists.

Suppose $f: (a, \infty) \rightarrow \mathbb{R}$ is a function such that f is Riemann integrable on $[a, c]$ for all $c < \infty$. We define $\int_a^\infty f := \lim_{c \rightarrow \infty} \int_a^c f$, if the limit exists.

If the limit exists, we say the improper integral *converges*. If the limit does not exist, we say the improper integral *diverges*.

We similarly define improper integrals for the left hand endpoint, we leave this to the reader.

For a finite endpoint b , using we see that if f is bounded, then we have defined nothing new. What is new is that we can apply this definition to unbounded functions. The following set of examples is so useful that we state it as a proposition.

[impropiemann:ptest] The improper integral $\int_1^\infty x^{-p} dx$ converges to $\frac{1}{p-1}$ if $p > 1$ and diverges if $0 < p \leq 1$.

The improper integral $\int_0^1 x^{-p} dx$ converges to $\frac{1}{1-p}$ if $0 < p < 1$ and diverges if $p \geq 1$.

The proof follows by application of the fundamental theorem of calculus. Let us do the proof for $p > 1$ for the infinite right endpoint, and we leave the rest to the reader. Hint: You should handle $p=1$ separately.

Suppose $p > 1$. Then $\int_1^b x^{-p} dx = \frac{1}{1-p} x^{1-p} \Big|_1^b = \frac{1}{1-p} (b^{1-p} - 1)$. As $p > 1$, then $1-p < 0$. Taking the limit as $b \rightarrow \infty$ we obtain that $\frac{1}{b^{p-1}}$ goes to 0, and the result follows.

We state the following proposition for just one type of improper integral, though the proof is straight forward and the same for other types of improper integrals.

[impropiemann:tail] Let $f: [a, \infty) \rightarrow \mathbb{R}$ be a function that is Riemann integrable on $[a, b]$ for all $b > a$. Given any $b > a$, $\int_b^\infty f$ converges if and only if $\int_a^\infty f$ converges, in which case $\int_a^\infty f = \int_a^b f + \int_b^\infty f$.

Let $c > b$. Then $\int_a^c f = \int_a^b f + \int_b^c f$. Taking the limit $c \rightarrow \infty$ finishes the proof.

Nonnegative functions are easier to work with as the following proposition demonstrates. The exercises will show that this proposition holds only for nonnegative functions. Analogues of this proposition exist for all the other types of improper limits are left to the student.

[impropiemann:possimp] Suppose $f: [a, \infty) \rightarrow \mathbb{R}$ is nonnegative ($f(x) \geq 0$ for all x) and such that f is Riemann integrable on $[a, b]$ for all $b > a$.

- $\int_a^\infty f = \sup \left\{ \int_a^x f : x \geq a \right\}$.
- Suppose $\{x_n\}$ is a sequence with $\lim_{n \rightarrow \infty} x_n = \infty$. Then $\int_a^\infty f$ converges if and only if $\lim_{n \rightarrow \infty} \int_a^{x_n} f$ exists, in which case $\int_a^\infty f = \lim_{n \rightarrow \infty} \int_a^{x_n} f$.

In the first item we allow for the value of ∞ in the supremum indicating that the integral diverges to infinity.

Let us start with the first item. Notice that as f is nonnegative, then $\int_a^x f$ is increasing as a function of x . If the supremum is infinite, then for every $M \in \mathbb{R}$ we find N such that $\int_a^N f \geq M$. As $\int_a^x f$ is increasing then $\int_a^x f \geq M$ for all $x \geq N$. So $\int_a^\infty f$ diverges to infinity.

Next suppose the supremum is finite, say $A = \sup \left\{ \int_a^x f : x \geq a \right\}$. For every $\epsilon > 0$, we find an N such that $A - \int_a^N f < \epsilon$. As $\int_a^x f$ is increasing, then $A - \int_a^x f < \epsilon$ for all $x \geq N$ and hence $\int_a^\infty f$ converges to A .

Let us look at the second item. If $\int_a^\infty f$ converges then every sequence $\{x_n\}$ going to infinity works. The trick is proving the other direction. Suppose $\{x_n\}$ is such that $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} \int_a^{x_n} f = A$ converges. Given $\epsilon > 0$, pick N such that for all $n \geq N$ we have $A - \epsilon < \int_a^{x_n} f < A + \epsilon$. Because $\int_a^x f$ is increasing as a function of x , we have that for all $x \geq x_N$ $A - \epsilon < \int_a^x f \leq \int_a^{x_n} f < A + \epsilon$. As $\{x_n\}$ goes to ∞ , then for any given x , there is an x_m such that $m \geq N$ and $x \leq x_m$. Then $\int_a^x f \leq \int_a^{x_m} f < A + \epsilon$. In particular, for all $x \geq x_N$ we have $\left| \int_a^x f - A \right| < \epsilon$.

Let $f, g: [a, \infty) \rightarrow \mathbb{R}$ and $g: [a, \infty) \rightarrow \mathbb{R}$ be functions that are Riemann integrable on $[a, b]$ for all $b > a$. Suppose that for all $x \geq a$ we have $|f(x)| \leq g(x)$.

- If $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges, and in this case $\left| \int_a^\infty f \right| \leq \int_a^\infty g$.
- If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Let us start with the first item. For any b and c , such that $a \leq b \leq c$, we have $-g(x) \leq f(x) \leq g(x)$, and so $\int_b^c -g \leq \int_b^c f \leq \int_b^c g$. In other words, $\left| \int_b^c f \right| \leq \int_b^c g$.

Let $\epsilon > 0$ be given. Because of we have $\int_a^\infty g = \int_a^b g + \int_b^\infty g$. As $\int_a^b g$ goes to $\int_a^\infty g$ as b goes to infinity, then $\int_b^\infty g$ goes to 0 as b goes to infinity. Choose B such that $\int_B^\infty g < \epsilon$. As g is nonnegative, then if $B \leq b < c$, then $\int_b^c g < \epsilon$ as well. Let $\{x_n\}$ be a sequence going to infinity. Let M be such that $x_n \geq B$ for all $n \geq M$. Take $n, m \geq M$, with $x_n \leq x_m$, $\left| \int_a^{x_n} f - \int_a^{x_m} f \right| \leq \left| \int_{x_n}^{x_m} f \right| \leq \int_{x_n}^{x_m} g < \epsilon$. Therefore the sequence $\left\{ \int_a^{x_n} f \right\}_{n=1}^\infty$ is Cauchy and hence converges.

We need to show that the limit is unique. Suppose $\{x_n\}$ is a sequence converging to infinity such that $\left\{ \int_a^{x_n} f \right\}$ converges to L_1 , and $\{y_n\}$ is a sequence converging to infinity such that $\left\{ \int_a^{y_n} f \right\}$ converges to L_2 . Then there must be some n such that $\left| \int_a^{x_n} f - L_1 \right| < \epsilon$ and $\left| \int_a^{y_n} f - L_2 \right| < \epsilon$. We can also suppose $x_n \geq B$ and $y_n \geq B$. Then $\left| L_1 - L_2 \right| \leq \left| \int_a^{x_n} f - \int_a^{y_n} f \right| \leq \left| \int_{x_n}^{y_n} f \right| \leq \int_{x_n}^{y_n} g < \epsilon$. As $\epsilon > 0$ was arbitrary, $L_1 = L_2$, and hence $\int_a^\infty f$ converges. Above we have shown that $\left| \int_a^c f \right| \leq \int_a^c g$ for all $c > a$. By taking the limit $c \rightarrow \infty$, the first item is proved.

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$k-1$ we get $\frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{k^2} + \frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2}$. In other words, $\frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2}$ is an estimate for the sum to within $\frac{1}{k^2}$. Therefore, if we wish to find the sum to within 0.01, we note $\frac{1}{10^2} = 0.01$. We obtain $1.6397 \dots \approx \frac{1}{10} + \sum_{n=1}^9 \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{100} + \frac{1}{10} + \sum_{n=1}^9 \frac{1}{n^2} \approx 1.6497 \dots$. The actual sum is $\frac{\pi^2}{6} \approx 1.6449 \dots$.

Exercises

Finish the proof of .

Find out for which $a \in \mathbb{R}$ does $\sum_{n=1}^{\infty} e^{an}$ converge. When the series converges, find an upper bound for the sum.

a) Estimate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ correct to within 0.01 using the integral test. b) Compute the limit of the series exactly and compare. Hint: the sum telescopes.

Prove $\int_{-\infty}^{\infty} \left| \operatorname{sinc}(x) \right| dx = \infty$. Hint: again, it is enough to show this on just one side.

Can you interpret $\int_{-1}^1 \frac{1}{\sqrt{|x|}} dx$ as an improper integral? If so, compute its value.

Take $f: [0, \infty) \rightarrow \mathbb{R}$, Riemann integrable on every interval $[0, b]$, and such that there exist M, a , and T , such that $|f(t)| \leq M e^{at}$ for all $t \geq T$. Show that the *Laplace transform* of f exists. That is, for every $s > a$ the following integral converges: $F(s) := \int_0^{\infty} f(t) e^{-st} dt$.

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a Riemann integrable function on every interval $[a, b]$, and such that $\int_a^b f(x) dx < \infty$. Show that the *Fourier sine and cosine transforms* exist. That is, for every $\omega \geq 0$ the following integrals converge $F^s(\omega) := \frac{1}{\pi} \int_0^{\infty} f(t) \sin(\omega t) dt$, $F^c(\omega) := \frac{1}{\pi} \int_0^{\infty} f(t) \cos(\omega t) dt$. Furthermore, show that F^s and F^c are bounded functions.

Suppose $f: [0, \infty) \rightarrow \mathbb{R}$ is Riemann integrable on every interval $[0, b]$. Show that $\int_0^{\infty} f$ converges if and only if for every $\epsilon > 0$ there exists an M such that if $M \leq a < b$ then $|\int_a^b f| < \epsilon$.

Suppose $f: [0, \infty) \rightarrow \mathbb{R}$ is nonnegative and decreasing. a) Show that if $\int_0^{\infty} f < \infty$, then $\lim_{x \rightarrow \infty} f(x) = 0$. b) Show that the converse does not hold.

Find an example of an *unbounded* continuous function $f: [0, \infty) \rightarrow \mathbb{R}$ that is nonnegative and such that $\int_0^{\infty} f < \infty$. Note that this means that $\lim_{x \rightarrow \infty} f(x)$ does not exist; compare previous exercise. Hint: on each interval $[k, k+1]$, $k \in \mathbb{N}$, define a function whose integral over this interval is less than say 2^{-k} .

Find an example of a function $f: [0, \infty) \rightarrow \mathbb{R}$ integrable on all intervals such that $\lim_{n \rightarrow \infty} \int_0^n f$ converges as a limit of a sequence, but such that $\int_0^{\infty} f$ does not exist. Hint: for all $n \in \mathbb{N}$, divide $[n, n+1]$ into two halves. In one half make the function negative, on the other make the function positive.

Show that if $f: [1, \infty) \rightarrow \mathbb{R}$ is such that $g(x) := x^2 f(x)$ is a bounded function, then $\int_1^{\infty} f$ converges.

It is sometimes desirable to assign a value to integrals that normally cannot be interpreted as even improper integrals, e.g. $\int_{-1}^1 \frac{1}{x} dx$. Suppose $f: [a, b] \rightarrow \mathbb{R}$ is a function and $a < c < b$, where f is Riemann integrable on all intervals $[a, c-\epsilon]$ and $[c+\epsilon, b]$ for all $\epsilon > 0$. Define the *Cauchy principal value* of $\int_a^b f$ as $p.v. \int_a^b f := \lim_{\epsilon \rightarrow 0} \left(\int_a^{c-\epsilon} f + \int_{c+\epsilon}^b f \right)$, if the limit exists.

a) Compute $p.v. \int_{-1}^1 \frac{1}{x} dx$.

b) Compute $\lim_{\epsilon \rightarrow 0} \left(\int_{-1}^{c-\epsilon} \frac{1}{x} dx + \int_{c+\epsilon}^1 \frac{1}{x} dx \right)$ and show it is not equal to the principal value.

c) Show that if f is integrable on $[a, b]$, then $p.v. \int_a^b f = \int_a^b f$.

d) Find an example of an f with a singularity at c as above such that $p.v. \int_a^b f$ exists, but the improper integrals $\int_a^c f$ and $\int_c^b f$ diverge.

e) Suppose $f: [-1, 1] \rightarrow \mathbb{R}$ is continuous. Show that $p.v. \int_{-1}^1 \frac{1}{x} f(x) dx$ exists.

Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions, where $g(x) = 0$ for all $x \notin [a, b]$ for some interval $[a, b]$.

a) Show that the *convolution* $(g * f)(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt$ is well-defined for all $x \in \mathbb{R}$.

b) Suppose $\int_{-\infty}^{\infty} f(t) dt < \infty$. Prove that $\lim_{x \rightarrow -\infty} (g * f)(x) = 0$, $\lim_{x \rightarrow \infty} (g * f)(x) = 0$.

Sequences of Functions

Pointwise and uniform convergence

Note: 1–1.5 lecture

Up till now when we talked about sequences we always talked about sequences of numbers. However, a very useful concept in analysis is to use a sequence of functions. For example, a solution to some differential equation might be found by finding only approximate solutions. Then the real solution is some sort of limit of those approximate solutions.

When talking about sequences of functions, the tricky part is that there are multiple notions of a limit. Let us describe two common notions of a limit of a sequence of functions.

Pointwise convergence

For every $n \in \mathbb{N}$ let $f_n: S \rightarrow \mathbb{R}$ be a function. We say the sequence $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* to $f: S \rightarrow \mathbb{R}$, if for every $x \in S$ we have $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

It is common to say that $f_n: S \rightarrow \mathbb{R}$ *converges to f on $T \subset S$* for some $f: T \rightarrow \mathbb{R}$. In that case we, of course, mean $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for every $x \in T$. We simply mean that the restrictions of f_n to T converge pointwise to f .

The sequence of functions defined by $f_n(x) := x^{2n}$ converges to $f: [-1, 1] \rightarrow \mathbb{R}$ on $[-1, 1]$, where $f(x) = \begin{cases} 1 & \text{if } x = -1 \text{ or } x = 1 \\ 0 & \text{otherwise} \end{cases}$. See .

To see this is so, first take $x \in (-1, 1)$. Then $0 \leq x^{2n} < 1$. We have seen before that $\lim_{n \rightarrow \infty} x^{2n} = 0$ for $|x| < 1$. Therefore $\lim_{n \rightarrow \infty} f_n(x) = 0$.

When $x = 1$ or $x = -1$, then $x^{2n} = 1$ for all n and hence $\lim_{n \rightarrow \infty} f_n(x) = 1$. We also note that $\{f_n\}$ does not converge for all other x .

Often, functions are given as a series. In this case, we use the notion of pointwise convergence to find the values of the function.

We write $\sum_{k=0}^{\infty} x^k$ to denote the limit of the functions $f_n(x) := \sum_{k=0}^n x^k$. When studying series, we have seen that on $x \in (-1, 1)$ the f_n converge pointwise to $\frac{1}{1-x}$.

The subtle point here is that while $\frac{1}{1-x}$ is defined for all $x \neq 1$, and f_n are defined for all x (even at $x=1$), convergence only happens on $(-1, 1)$.

Therefore, when we write $f(x) := \sum_{k=0}^{\infty} x^k$ we mean that f is defined on $(-1, 1)$ and is the pointwise limit of the partial sums.

Let $f_n(x) := \sin(xn)$. Then f_n does not converge pointwise to any function on any interval. It may converge at certain points, such as when $x=0$ or $x=\pi$. It is left as an exercise that in any interval $[a, b]$, there exists an x such that $\sin(xn)$ does not have a limit as n goes to infinity.

Before we move to uniform convergence, let us reformulate pointwise convergence in a different way. We leave the proof to the reader, it is a simple application of the definition of convergence of a sequence of real numbers.

[ptsconv:prop] Let $f_n: S \rightarrow \mathbb{R}$ and $f: S \rightarrow \mathbb{R}$ be functions. Then $\{f_n\}$ converges pointwise to f if and only if for every $x \in S$, and every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

The key point here is that N can depend on x , not just on ϵ . That is, for each x we can pick a different N . If we can pick one N for all x , we have what is called uniform convergence.

Uniform convergence

Let $f_n: S \rightarrow \mathbb{R}$ be functions. We say the sequence $\{f_n\}$ *converges uniformly* to $f: S \rightarrow \mathbb{R}$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

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Note that N now cannot depend on x . Given $\epsilon > 0$ we must find an N that works for all $x \in S$. Because of , we see that uniform convergence implies pointwise convergence.

Let $\{f_n\}$ be a sequence of functions $f_n : S \rightarrow \mathbb{R}$. If $\{f_n\}$ converges uniformly to $f : S \rightarrow \mathbb{R}$, then $\{f_n\}$ converges pointwise to f .

The converse does not hold.

The functions $f_n(x) := x^{2n}$ do not converge uniformly on $[-1, 1]$, even though they converge pointwise. To see this, suppose for contradiction that the convergence is uniform. For $\epsilon = \frac{1}{2}$, there would have to exist an N such that $x^{2N} = |f_n(x) - 0| \leq \frac{1}{2}$ for all $x \in (-1, 1)$ (as $f_n(x)$ converges to 0 on $(-1, 1)$). But that means that for any sequence $\{x_k\}$ in $(-1, 1)$ such that $\lim_{k \rightarrow \infty} x_k = 1$ we have $x_k^{2N} < \frac{1}{2}$ for all k . On the other hand x^{2N} is a continuous function of x (it is a polynomial), therefore we obtain a contradiction $1 = \lim_{k \rightarrow \infty} x_k^{2N} \leq \frac{1}{2}$.

However, if we restrict our domain to $[-a, a]$ where $0 < a < 1$, then $\{f_n\}$ converges uniformly to 0 on $[-a, a]$. First note that $a^{2n} \rightarrow 0$ as $n \rightarrow \infty$. Thus given $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $a^{2N} < \epsilon$ for all $n \geq N$. Then for any $x \in [-a, a]$ we have $|f_n(x)| \leq a^{2n} \leq a^{2N} < \epsilon$. Therefore, for $n \geq N$ $|f_n(x) - 0| \leq a^{2N} < \epsilon$.

Convergence in uniform norm

For bounded functions there is another more abstract way to think of uniform convergence. To every bounded function we assign a certain nonnegative number (called the uniform norm). This number measures the “distance” of the function from 0. We can then “measure” how far two functions are from each other. We simply translate a statement about uniform convergence into a statement about a certain sequence of real numbers converging to zero.

[def:uniform] Let $f : S \rightarrow \mathbb{R}$ be a bounded function. Define $\|f\|_\infty := \sup_{x \in S} |f(x)|$. $\|f\|_\infty$ is called the *uniform norm*.

To use this notation ²⁶ and this concept, the domain S must be fixed. Some authors use the notation $\|f\|_{\infty, S}$ to emphasize the dependence on S .

A sequence of bounded functions $f_n : S \rightarrow \mathbb{R}$ converges uniformly to $f : S \rightarrow \mathbb{R}$, if and only if $\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0$.

First suppose $\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0$. Let $\epsilon > 0$ be given. Then there exists an N such that for $n \geq N$ we have $\|f_n - f\|_\infty < \epsilon$. As $\|f_n - f\|_\infty$ is the supremum of $|f_n(x) - f(x)|$, we see that for all x we have $|f_n(x) - f(x)| < \epsilon$.

On the other hand, suppose $\{f_n\}$ converges uniformly to f . Let $\epsilon > 0$ be given. Then find N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$. Taking the supremum we see that $\|f_n - f\|_\infty < \epsilon$. Hence $\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = 0$.

Sometimes it is said that $\{f_n\}$ *converges to f in uniform norm* instead of *converges uniformly*. The proposition says that the two notions are the same thing.

Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $f_n(x) := \frac{1}{n} \sin(nx^2)$. Then we claim $\{f_n\}$ converges uniformly to $f(x) := 0$. Let us compute: $\|f_n - f\|_\infty = \sup_{x \in [0, 1]} \left| \frac{1}{n} \sin(nx^2) \right| \leq \frac{1}{n} \sup_{x \in [0, 1]} |\sin(nx^2)| \leq \frac{1}{n}$.

Using uniform norm, we define Cauchy sequences in a similar way as we define Cauchy sequences of real numbers.

Let $f_n : S \rightarrow \mathbb{R}$ be bounded functions. The sequence is *Cauchy in the uniform norm* or *uniformly Cauchy* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, k \geq N$ we have $\|f_m - f_k\|_\infty < \epsilon$.

[prop:uniformcauchy] Let $f_n : S \rightarrow \mathbb{R}$ be bounded functions. Then $\{f_n\}$ is Cauchy in the uniform norm if and only if there exists an $f : S \rightarrow \mathbb{R}$ and $\{f_n\}$ converges uniformly to f .

Let us first suppose $\{f_n\}$ is Cauchy in the uniform norm. Let us define f . Fix x , then the sequence $\{f_n(x)\}$ is Cauchy because $|f_m(x) - f_k(x)| \leq \|f_m - f_k\|_\infty$. Thus $\{f_n(x)\}$ converges to some real number. Define $f : S \rightarrow \mathbb{R}$ by $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. The sequence $\{f_n\}$ converges pointwise to f . To show that the convergence is uniform, let $\epsilon > 0$ be given. Find an N such that for $m, k \geq N$ we have $\|f_m - f_k\|_\infty < \epsilon$. In other words for all x we have $|f_m(x) - f_k(x)| < \epsilon$. We take the limit as k goes to infinity. Then $|f_m(x) - f(x)| < \epsilon$. Consequently for all x we get $|f_m(x) - f(x)| < \epsilon$. And hence $\{f_n\}$ converges uniformly.

For the other direction, suppose $\{f_n\}$ converges uniformly to f . Given $\epsilon > 0$, find N such that for all $n \geq N$ we have $\|f_n - f\|_\infty < \frac{\epsilon}{4}$ for all $x \in S$. Therefore for all $m, k \geq N$ we have $|f_m(x) - f_k(x)| \leq \|f_m - f\|_\infty + \|f - f_k\|_\infty < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$. Take supremum over all x to obtain $\|f_m - f_k\|_\infty < \frac{\epsilon}{2}$.

Exercises

Let f and g be bounded functions on $[a, b]$. Prove $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

- Find the pointwise limit $\lim_{n \rightarrow \infty} e^{x/n}$ for $x \in \mathbb{R}$.
- Is the limit uniform on $\mathbb{R}$?
- Is the limit uniform on $[0, 1]$?

Suppose $f_n : S \rightarrow \mathbb{R}$ are functions that converge uniformly to $f : S \rightarrow \mathbb{R}$. Suppose $A \subset S$. Show that the sequence of restrictions $\{f_n|_A\}$ converges uniformly to $f|_A$.

Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively pointwise. Show that $\{f_n + g_n\}$ converges pointwise to $f + g$.

Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively uniformly on A . Show that $\{f_n + g_n\}$ converges uniformly to $f + g$ on A .

Find an example of a sequence of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A , but such that $\{f_n g_n\}$ (the multiple) does not converge uniformly to fg on A . Hint: Let $A := \mathbb{R}$, let $f(x) = g(x) = x$. You can even pick $f_n = g_n$.

Suppose there exists a sequence of functions $\{f_n\}$ uniformly converging to 0 on A . Now suppose we have a sequence of functions $\{f_n\}$ and a function f on A such that $|f_n(x) - f(x)| \leq g_n(x)$ for all $x \in A$. Show that $\{f_n\}$ converges uniformly to f on A .

Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of functions on $[a, b]$. Suppose $\{f_n\}$ and $\{h_n\}$ converge uniformly to some function $f : [a, b] \rightarrow \mathbb{R}$ and suppose $f_n(x) \leq g_n(x) \leq h_n(x)$ for all $x \in [a, b]$. Show that $\{g_n\}$ converges uniformly to f .

Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of increasing functions (that is, $f_n(x) \leq f_n(y)$ whenever $x \leq y$). Suppose $f_n(0) = 0$ and $\lim_{n \rightarrow \infty} f_n(1) = 0$. Show that $\{f_n\}$ converges uniformly to 0.

Let $\{f_n\}$ be a sequence of functions defined on $[0, 1]$. Suppose there exists a sequence of distinct numbers $x_n \in [0, 1]$ such that $f_n(x_n) = 1$. Prove or disprove the following statements:

- True or false: There exists $\{f_n\}$ as above that converges to 0 pointwise.
- True or false: There exists $\{f_n\}$ as above that converges to 0 uniformly on $[0, 1]$.

Fix a continuous $h : [a, b] \rightarrow \mathbb{R}$. Let $f(x) := h(x)$ for $x \in [a, b]$, $f(x) := h(a)$ for $x < a$ and $f(x) := h(b)$ for $x > b$. First show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Now let f_n be the function g from with $\epsilon = \frac{1}{n}$, defined on the interval $[a, b]$. Show that $\{f_n\}$ converges uniformly to h on $[a, b]$.

Interchange of limits

Note: 1–1.5 lectures

Large parts of modern analysis deal mainly with the question of the interchange of two limiting operations. When we have a chain of two limits, we cannot always just swap the limits. For example, $0 = \lim_{n \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \frac{1}{n+k} \right) \neq \lim_{k \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{1}{n+k} \right) = 1$.

Processing math: 52% sequences of functions, interchange of limits comes up quite often. We treat two cases. First we look at continuity of the limit, and second we look at the integral of the limit.

Suppose $\{f_n\}$ is a sequence of Riemann integrable functions on $[0,1]$ that converges uniformly to 0. Show that $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0$.

Find a sequence of Riemann integrable functions $\{f_n\}$ on $[0,1]$ that converges pointwise to 0, but $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ does not exist (is $\$infinity\$$).

Prove *Dini's theorem*: Let f_n on $[a,b]$ to $\mathbb{R}$ be a sequence of continuous functions such that $0 \leq f_{n+1}(x) \leq f_n(x) \leq \dots \leq f_1(x) \leq \dots$ for all $n \in \mathbb{N}$. Suppose $\{f_n\}$ converges pointwise to 0. Show that $\{f_n\}$ converges to zero uniformly.

Suppose f_n on $[a,b]$ to $\mathbb{R}$ is a sequence of continuous functions that converges pointwise to a continuous f on $[a,b]$ to $\mathbb{R}$. Suppose that for any $x \in [a,b]$ the sequence $\{f_n(x) - f(x)\}$ is monotone. Show that the sequence $\{f_n\}$ converges uniformly.

Find a sequence of Riemann integrable functions f_n on $[0,1]$ to $\mathbb{R}$ such that $\{f_n\}$ converges to zero pointwise, and such that a) $\int_0^1 f_n(x) dx$ increases without bound, b) $\int_0^1 f_n(x) dx$ is the sequence $-1, 1, -1, 1, -1, \dots$.

It is possible to define a *joint limit* of a double sequence $\{x_{n,m}\}$ of real numbers (that is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$). We say L is the joint limit of $\{x_{n,m}\}$ and write $\lim_{(n,m) \rightarrow \infty} x_{n,m} = L$, if for every $\epsilon > 0$, there exists an M such that if $n \geq M$ and $m \geq M$, then $|x_{n,m} - L| < \epsilon$.

Suppose the joint limit of $\{x_{n,m}\}$ is L , and suppose that for all n , $\lim_{m \rightarrow \infty} x_{n,m}$ exists, and for all m , $\lim_{n \rightarrow \infty} x_{n,m}$ exists. Then show $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_{n,m} = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_{n,m} = L$.

A joint limit does not mean the iterated limits even exist. Consider $x_{n,m} := \frac{1}{n+m}$

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$\{\min\{n,m\}\}$.

a) Show that for no n does $\lim_{m \rightarrow \infty} x_{n,m}$ exist, and for no m does $\lim_{n \rightarrow \infty} x_{n,m}$ exist. So neither $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x_{n,m}$ nor $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} x_{n,m}$ makes any sense at all.

b) Show that the joint limit of $\{x_{n,m}\}$ exists and is 0.

Picard's theorem

Note: 1–2 lectures (can be safely skipped)

A first semester course in analysis should have a *pièce de résistance* caliber theorem. We pick a theorem whose proof combines everything we have learned. It is more sophisticated than the fundamental theorem of calculus, the first highlight theorem of this course. The theorem we are talking about is Picard's theorem²⁷ on existence and uniqueness of a solution to an ordinary differential equation. Both the statement and the proof are beautiful examples of what one can do with all we have learned. It is also a good example of how analysis is applied as differential equations are indispensable in science.

First order ordinary differential equation

Modern science is described in the language of *differential equations*. That is, equations involving not only the unknown, but also its derivatives. The simplest nontrivial form of a differential equation is the so-called *first order ordinary differential equation* $y' = F(x,y)$. Generally we also specify $y(x_0) = y_0$. The solution of the equation is a function $y(x)$ such that $y(x_0) = y_0$ and $y'(x) = F(x,y(x))$.

When F involves only the x variable, the solution is given by the fundamental theorem of calculus. On the other hand, when F depends on both x and y we need far more firepower. It is not always true that a solution exists, and if it does, that it is the unique solution. Picard's theorem gives us certain sufficient conditions for existence and uniqueness.

The theorem

We need a definition of continuity in two variables. First, a point in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is denoted by an ordered pair (x,y) . To make matters simple, let us give the following sequential definition of continuity.

Let $U \subset \mathbb{R}^2$ be a set and $F: U \rightarrow \mathbb{R}$ be a function. Let $(x,y) \in U$ be a point. The function F is *continuous* at (x,y) if for every sequence $\{(x_n, y_n)\}_{n=1}^\infty$ of points in U such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, we have $\lim_{n \rightarrow \infty} F(x_n, y_n) = F(x,y)$. We say F is continuous if it is continuous at all points in U .

Let $I, J \subset \mathbb{R}$ be closed bounded intervals, let I_0 and J_0 be their interiors, and let $(x_0, y_0) \in I_0 \times J_0$. Suppose $F: I \times J \rightarrow \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, there exists a number L such that $|F(x,y) - F(x,z)| \leq L|y - z|$ for all $(x,y), (x,z) \in I \times J$. Then there exists an $h > 0$ and a unique differentiable function $f: [x_0 - h, x_0 + h] \rightarrow J$ such that $f(x_0) = y_0$ and $f'(x) = F(x, f(x))$.

Suppose we could find a solution f . Using the fundamental theorem of calculus we integrate the equation $f'(x) = F(x, f(x))$, $f(x_0) = y_0$, and write [\[picard:diffq\]](#) as the integral equation $f(x) = y_0 + \int_{x_0}^x F(t, f(t)) dt$. The idea of our proof is that we try to plug in approximations to a solution to the right-hand side of [\[picard:inteq\]](#) to get better approximations on the left hand side of [\[picard:inteq\]](#). We hope that in the end the sequence converges and solves [\[picard:inteq\]](#) and hence [\[picard:diffq\]](#). The technique below is called *Picard iteration*, and the individual functions f_k are called the *Picard iterates*.

Without loss of generality, suppose $x_0 = 0$ (exercise below). Another exercise tells us that F is bounded as it is continuous. Therefore pick some $M > 0$ so that $|F(x,y)| \leq M$ for all $(x,y) \in I \times J$. Pick $\alpha > 0$ such that $[-\alpha, \alpha] \subset I$ and $[y_0 - \alpha, y_0 + \alpha] \subset J$. Define $h := \min\{\alpha, \frac{1}{M+L}\}$. Observe $[-h, h] \subset I$.

Set $f_0(x) := y_0$. We define f_k inductively. Assuming $f_{k-1}: [-h, h] \rightarrow [y_0 - \alpha, y_0 + \alpha]$, we see $F(t, f_{k-1}(t))$ is a well defined function of t for $t \in [-h, h]$. Further if f_{k-1} is continuous on $[-h, h]$, then $F(t, f_{k-1}(t))$ is continuous as a function of t on $[-h, h]$ (left as an exercise). Define $f_k(x) := y_0 + \int_0^x F(t, f_{k-1}(t)) dt$, and f_k is continuous on $[-h, h]$ by the fundamental theorem of calculus. To see that f_k maps $[-h, h]$ to $[y_0 - \alpha, y_0 + \alpha]$, we compute for $x \in [-h, h]$ $|f_k(x) - y_0| \leq \int_0^x |F(t, f_{k-1}(t))| dt \leq M|x| \leq Mh \leq M \frac{1}{M+L} \leq \alpha$. We now define f_{k+1} and so on, and we have defined a sequence $\{f_k\}$ of functions. We need to show that it converges to a function f that solves the equation [\[picard:inteq\]](#) and therefore [\[picard:diffq\]](#).

We wish to show that the sequence $\{f_k\}$ converges uniformly to some function on $[-h, h]$. First, for $t \in [-h, h]$ we have the following useful bound $|F(t, f_n(t)) - F(t, f_k(t))| \leq L|f_n(t) - f_k(t)|$, where $|f_n(t) - f_k(t)|$ is the uniform norm, that is the supremum of $|f_n(t) - f_k(t)|$ for $t \in [-h, h]$. Now note that $|f_n(t) - f_k(t)| \leq \int_0^t |F(s, f_{n-1}(s)) - F(s, f_{k-1}(s))| ds \leq L \int_0^t |f_{n-1}(s) - f_{k-1}(s)| ds$. Therefore $|f_n(t) - f_k(t)| \leq L \int_0^t |f_{n-1}(s) - f_{k-1}(s)| ds$. We now define $C := \frac{1}{1-L}$ and note that $C < 1$. Taking supremum on the left-hand side we get $|f_n - f_k| \leq C|f_{n-1} - f_{k-1}|$. Without loss of generality, suppose $n \geq k$. Then by we can show $|f_n - f_k| \leq C^k |f_0 - f_{k-1}|$. For $x \in [-h, h]$ we have $|f_n(x) - f_k(x)| \leq C^k |f_0(x) - y_0| \leq C^k \alpha$. Therefore, $|f_n - f_k| \leq C^k \alpha$. As $C < 1$, $\{f_n\}$ is uniformly Cauchy and by we obtain that $\{f_n\}$ converges uniformly on $[-h, h]$ to some function $f: [-h, h] \rightarrow \mathbb{R}$. The function f is the uniform limit of continuous functions and therefore continuous. Furthermore since all the f_n map $[-h, h]$ to $[y_0 - \alpha, y_0 + \alpha]$, then f maps $[-h, h]$ to $[y_0 - \alpha, y_0 + \alpha]$ (why?).

We now need to show that f solves [\[picard:inteq\]](#). First, as before we notice $|F(t, f_n(t)) - F(t, f(t))| \leq L|f_n(t) - f(t)|$. As $|f_n - f| \leq C^k \alpha$, we have $|F(t, f_n(t)) - F(t, f(t))| \leq LC^k \alpha$. Therefore, $|f_n(x) - f(x)| \leq \int_0^x |F(t, f_n(t)) - F(t, f(t))| dt \leq LC^k \alpha x \leq LC^k \alpha h \leq LC^k \alpha \frac{1}{M+L} \leq C^k \alpha$. We apply the fundamental theorem of calculus to show that f is differentiable and its derivative is $F(x, f(x))$. It is obvious that $f(0) = y_0$.

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Finally, what is left to do is to show uniqueness. Suppose $g: [-h, h] \rightarrow \mathbb{R}$ is another solution. As before we use the fact that $\|F(g(t)) - F(f(t))\| \leq L \|g(t) - f(t)\|$ for $t \in [-h, h]$. Then $\|f - g\| \leq \int_0^h \|f'(t) - g'(t)\| dt = \int_0^h \|F(f(t)) - F(g(t))\| dt \leq \int_0^h L \|f(t) - g(t)\| dt \leq L \int_0^h \|f - g\| dt \leq L \|f - g\| h$. As before, $C = \frac{L}{1 - Lh} < 1$. By taking supremum over $x \in [-h, h]$ on the left hand side we obtain $\|f - g\| \leq C \|f - g\|$. This is only possible if $\|f - g\| = 0$. Therefore, $f = g$, and the solution is unique.

Examples

Let us look at some examples. The proof of the theorem gives us an explicit way to find an h that works. It does not, however, give us the best h . It is often possible to find a much larger h for which the conclusion of the theorem holds.

The proof also gives us the Picard iterates as approximations to the solution. So the proof actually tells us how to obtain the solution, not just that the solution exists.

Consider $f'(x) = f(x)$, $f(0) = 1$. That is, we let $F(x, y) = y$, and we are looking for a function f such that $f'(x) = f(x)$. We pick any I that contains 0 in the interior. We pick an arbitrary J that contains 1 in its interior. We can use $L = 1$. The theorem guarantees an $h > 0$ such that there exists a unique solution $f: [-h, h] \rightarrow \mathbb{R}$. This solution is usually denoted by $e^x := f(x)$. We leave it to the reader to verify that by picking I and J large enough the proof of the theorem guarantees that we are able to pick α such that we get any h we want as long as $h < \frac{1}{2}$. We omit the calculation.

Of course, we know this function exists as a function for all x , so an arbitrary h ought to work. By same reasoning as above, no matter what x_0 and y_0 are, the proof guarantees an arbitrary h as long as $h < \frac{1}{2}$. Fix such an h . We get a unique function defined on $[x_0 - h, x_0 + h]$. After defining the function on $[-h, h]$ we find a solution on the interval $[0, 2h]$ and notice that the two functions must coincide on $[0, h]$ by uniqueness. We thus iteratively construct the exponential for all $x \in \mathbb{R}$. Therefore Picard's theorem could be used to prove the existence and uniqueness of the exponential.

Let us compute the Picard iterates. We start with the constant function $f_0(x) := 1$. Then $f_1(x) = 1 + \int_0^x f_0(s) ds = 1 + x$, $f_2(x) = 1 + \int_0^x f_1(s) ds = 1 + \int_0^x (1+s) ds = 1 + x + \frac{x^2}{2}$, $f_3(x) = 1 + \int_0^x f_2(s) ds = 1 + \int_0^x (1+s+\frac{s^2}{2}) ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. We recognize the beginning of the Taylor series for the exponential.

Suppose we have the equation $f'(x) = \sqrt{f(x)}$ $f(0) = 1$. From elementary differential equations we know $f(x) = \frac{1}{4}(1-x)^2$ is the solution. The solution is only defined on $(-\infty, 1)$. That is, we are able to use $h < 1$, but never a larger h . The function that takes y to y^2 is not Lipschitz as a function on all of $\mathbb{R}$. As we approach $x=1$ from the left, the solution becomes larger and larger. The derivative of the solution grows as y^2 , and therefore the L required will have to be larger and larger as y_0 grows. Thus if we apply the theorem with x_0 close to 1 and $y_0 = \frac{1}{4}(1-x_0)^2$ we find that the h that the proof guarantees will be smaller and smaller as x_0 approaches 1.

By picking α correctly, the proof of the theorem guarantees $h = \frac{1}{\sqrt{3}} \approx 0.577$ (we omit the calculation) for $x_0=0$ and $y_0=1$, even though we saw above that any $h < 1$ should work.

Consider the equation $f'(x) = 2\sqrt{f(x)}$ $f(0) = 0$. The function $F(x, y) = 2\sqrt{y}$ is continuous, but not Lipschitz in y (why?). The equation does not satisfy the hypotheses of the theorem. The function $f(x) = x^2$ is a solution, but $f(x) = 0$ is also a solution. A solution exists, but is not unique.

Consider $y' = \varphi(x)$ where $\varphi(x) := 0$ if $x \in \mathbb{Q}$ and $\varphi(x) = 1$ if $x \notin \mathbb{Q}$. The equation has no solution regardless of the initial conditions. A solution would have derivative φ , but φ does not have the intermediate value property at any point (why?). No solution exists by . Therefore to obtain existence of a solution, some continuity hypothesis on F is necessary.

Exercises

Let $I, J \subset \mathbb{R}$ be intervals. Let $F: I \times J \rightarrow \mathbb{R}$ be a continuous function of two variables and suppose $f: I \rightarrow J$ be a continuous function. Show that $F(x, f(x))$ is a continuous function on I .

Let $I, J \subset \mathbb{R}$ be closed bounded intervals. Show that if $F: I \times J \rightarrow \mathbb{R}$ is continuous, then F is bounded.

We proved Picard's theorem under the assumption that $x_0 = 0$. Prove the full statement of Picard's theorem for an arbitrary x_0 .

Let $f'(x) = x f(x)$ be our equation. Start with the initial condition $f(0) = 2$ and find the Picard iterates f_0, f_1, f_2, f_3, f_4 .

Suppose $F: I \times J \rightarrow \mathbb{R}$ is a function that is continuous in the first variable, that is, for any fixed y the function that takes x to $F(x, y)$ is continuous. Further, suppose F is Lipschitz in the second variable, that is, there exists a number L such that $\|F(x, y) - F(x, z)\| \leq L \|y - z\|$ for all $y, z \in J, x \in I$. Show that F is continuous as a function of two variables. Therefore, the hypotheses in the theorem could be made even weaker.

A common type of equation one encounters are *linear first order differential equations*, that is equations of the form $y' + p(x)y = q(x)$, $y(x_0) = y_0$. Prove Picard's theorem for linear equations. Suppose I is an interval, $x_0 \in I$, and $p: I \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$ are continuous. Show that there exists a unique differentiable $f: I \rightarrow \mathbb{R}$, such that $y = f(x)$ satisfies the equation and the initial condition. Hint: Assume existence of the exponential function and use the integrating factor formula for existence of f (prove that it works): $f(x) := e^{\int_{x_0}^x p(s) ds} \left(\int_{x_0}^x e^{-\int_{x_0}^s p(s) ds} q(t) dt + y_0 \right)$.

Metric Spaces

Metric spaces

Note: 1.5 lectures

As mentioned in the introduction, the main idea in analysis is to take limits. In we learned to take limits of sequences of real numbers. And in we learned to take limits of functions as a real number approached some other real number.

We want to take limits in more complicated contexts. For example, we want to have sequences of points in 3-dimensional space. We wish to define continuous functions of several variables. We even want to define functions on spaces that are a little harder to describe, such as the surface of the earth. We still want to talk about limits there.

Finally, we have seen the limit of a sequence of functions in . We wish to unify all these notions so that we do not have to reprove theorems over and over again in each context. The concept of a metric space is an elementary yet powerful tool in analysis. And while it is not sufficient to describe every type of limit we find in modern analysis, it gets us very far indeed.

Let X be a set, and let $d: X \times X \rightarrow \mathbb{R}$ be a function such that

- $d(x, y) \geq 0$ for all $x, y \in X$,
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Then the pair (X, d) is called a *metric space*. The function d is called the *metric* or sometimes the *distance function*. Sometimes we just say X is a metric space if the metric is clear from context.

The geometric idea is that d is the distance between two points. Items [\[metric:pos\]](#)–[\[metric:com\]](#) have obvious geometric interpretation: distance is always nonnegative, the only point that is distance 0 away from x is x itself, and finally that the distance from x to y is the same as the distance from y to x . The triangle inequality [\[metric:triang\]](#) has the interpretation given in .

For the purposes of drawing, it is convenient to draw figures and diagrams in the plane and have the metric be the standard distance. However, that is only one particular metric space. Just because a certain fact seems to be clear from drawing a picture does not mean it is true. You might be getting sidetracked by intuition from euclidean geometry, whereas the concept of a metric space is a lot more general.

Let us give some examples of metric spaces.

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The set of real numbers $\{\mathbb{R}\}$ is a metric space with the metric $d(x,y) := \left| \frac{x-y}{1+y} \right|$. Items [metric:pos]–[metric:com] of the definition are easy to verify. The triangle inequality [metric:triang] follows immediately from the standard triangle inequality for real numbers: $d(x,z) = \left| \frac{x-z}{1+z} \right| = \left| \frac{x-y+y-z}{1+z} \right| \leq \left| \frac{x-y}{1+z} \right| + \left| \frac{y-z}{1+z} \right| = d(x,y) + d(y,z)$. This metric is the *standard metric on $\{\mathbb{R}\}$* . If we talk about $\{\mathbb{R}\}$ as a metric space without mentioning a specific metric, we mean this particular metric.

We can also put a different metric on the set of real numbers. For example, take the set of real numbers $\{\mathbb{R}\}$ together with the metric $d(x,y) := \frac{|x-y|}{1+|x-y|}$. Items [metric:pos]–[metric:com] are again easy to verify. The triangle inequality [metric:triang] is a little bit more difficult. Note that $d(x,y) = \varphi(|x-y|)$ where $\varphi(t) = \frac{t}{1+t}$ and φ is an increasing function (positive derivative). Hence $d(x,z) = \varphi(|x-z|) = \varphi(|x-y+y-z|) \leq \varphi(|x-y| + |y-z|) \leq \varphi(|x-y|) + \varphi(|y-z|) = d(x,y) + d(y,z)$. Here we have an example of a nonstandard metric on $\{\mathbb{R}\}$. With this metric we see for example that $d(x,y) < 1$ for all $x,y \in \{\mathbb{R}\}$. That is, any two points are less than 1 unit apart.

An important metric space is the n -dimensional *euclidean space* $\{\mathbb{R}^n\}$. We use the following notation for points: $x = (x_1, x_2, \dots, x_n) \in \{\mathbb{R}^n\}$. We also simply write $0 \in \{\mathbb{R}^n\}$ to mean the vector $(0, 0, \dots, 0)$. Before making $\{\mathbb{R}^n\}$ a metric space, let us prove an important inequality, the so-called Cauchy-Schwarz inequality.

Take $x = (x_1, x_2, \dots, x_n) \in \{\mathbb{R}^n\}$ and $y = (y_1, y_2, \dots, y_n) \in \{\mathbb{R}^n\}$. Then $\left(\sum_{j=1}^n x_j y_j \right)^2 \leq \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{j=1}^n y_j^2 \right)$.

Any square of a real number is nonnegative. Hence any sum of squares is nonnegative: $0 \leq \sum_{j=1}^n \sum_{k=1}^n (x_j y_k - x_k y_j)^2 = \sum_{j=1}^n \sum_{k=1}^n (x_j^2 y_k^2 - 2 x_j x_k y_j y_k + x_k^2 y_j^2) = \sum_{j=1}^n \sum_{k=1}^n x_j^2 y_k^2 - 2 \sum_{j=1}^n \sum_{k=1}^n x_j x_k y_j y_k + \sum_{j=1}^n \sum_{k=1}^n x_k^2 y_j^2 = 2 \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{k=1}^n y_k^2 \right) - 2 \left(\sum_{j=1}^n x_j y_j \right)^2$. We relabel and divide by 2 to obtain $0 \leq \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{j=1}^n y_j^2 \right) - \left(\sum_{j=1}^n x_j y_j \right)^2$, which is precisely what we wanted.

Let us construct the standard metric for $\{\mathbb{R}^n\}$. Define $d(x,y) := \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$. For $n=1$, the real line, this metric agrees with what we did above. Again, the only tricky part of the definition to check is the triangle inequality. It is less messy to work with the square of the metric. In the following, note the use of the Cauchy-Schwarz inequality. $d(x,z)^2 = \sum_{j=1}^n (x_j - z_j)^2 = \sum_{j=1}^n (x_j - y_j + y_j - z_j)^2 \leq \sum_{j=1}^n (x_j - y_j)^2 + \sum_{j=1}^n (y_j - z_j)^2 + 2 \sum_{j=1}^n (x_j - y_j)(y_j - z_j) \leq \sum_{j=1}^n (x_j - y_j)^2 + \sum_{j=1}^n (y_j - z_j)^2 + 2 \sqrt{\sum_{j=1}^n (x_j - y_j)^2} \sqrt{\sum_{j=1}^n (y_j - z_j)^2} = d(x,y)^2 + d(y,z)^2 + 2 d(x,y) d(y,z)$. Taking the square root of both sides we obtain the correct inequality, because the square root is an increasing function.

An example to keep in mind is the so-called *discrete metric*. Let X be any set and define $d(x,y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. That is, all points are equally distant from each other. When X is a finite set, we can draw a diagram, see for example . Things become subtle when X is an infinite set such as the real numbers.

While this particular example seldom comes up in practice, it gives a useful “smell test.” If you make a statement about metric spaces, try it with the discrete metric. To show that (X,d) is indeed a metric space is left as an exercise.

[example:msC01] Let $C([a,b], \{\mathbb{R}\})$ be the set of continuous real-valued functions on the interval $[a,b]$. Define the metric on $C([a,b], \{\mathbb{R}\})$ as $d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$. Let us check the properties. First, $d(f,g)$ is finite as $f(x) - g(x)$ is a continuous function on a closed bounded interval $[a,b]$, and so is bounded. It is clear that $d(f,g) \geq 0$, it is the supremum of nonnegative numbers. If $f = g$ then $d(f,g) = 0$ for all x and hence $d(f,g) = 0$. Conversely if $d(f,g) = 0$, then for any x we have $|f(x) - g(x)| \leq d(f,g) = 0$ and hence $f(x) = g(x)$ for all x and $f = g$. That $d(f,g) = d(g,f)$ is equally trivial. To show the triangle inequality we use the standard triangle inequality. $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| \leq \sup_{x \in [a,b]} |f(x) - h(x)| + \sup_{x \in [a,b]} |h(x) - g(x)| = d(f,h) + d(h,g)$. When treating $C([a,b], \{\mathbb{R}\})$ as a metric space without mentioning a metric, we mean this particular metric. Notice that $d(f,g) = \|f - g\|_\infty$, the uniform norm of .

This example may seem esoteric at first, but it turns out that working with spaces such as $C([a,b], \{\mathbb{R}\})$ is really the meat of a large part of modern analysis. Treating sets of functions as metric spaces allows us to abstract away a lot of the grubby detail and prove powerful results such as Picard’s theorem with less work.

Oftentimes it is useful to consider a subset of a larger metric space as a metric space itself. We obtain the following proposition, which has a trivial proof.

Let (X,d) be a metric space and $Y \subset X$, then the restriction $d|_Y$ is a metric on Y .

If (X,d) is a metric space, $Y \subset X$, and $d' := d|_Y$, then (Y,d') is said to be a *subspace* of (X,d) .

It is common to simply write d for the metric on Y , as it is the restriction of the metric on X . Sometimes we say d' is the *subspace metric* and Y has the *subspace topology*.

A subset of the real numbers is bounded whenever all its elements are at most some fixed distance from 0. We also define bounded sets in a metric space. When dealing with an arbitrary metric space there may not be some natural fixed point 0. For the purposes of boundedness it does not matter.

Let (X,d) be a metric space. A subset $S \subset X$ is said to be *bounded* if there exists a $p \in X$ and a $B \in \{\mathbb{R}\}$ such that $d(p,x) \leq B$ for all $x \in S$. We say (X,d) is bounded if X itself is a bounded subset.

For example, the set of real numbers with the standard metric is not a bounded metric space. It is not hard to see that a subset of the real numbers is bounded in the sense of if and only if it is bounded as a subset of the metric space of real numbers with the standard metric.

On the other hand, if we take the real numbers with the discrete metric, then we obtain a bounded metric space. In fact, any set with the discrete metric is bounded.

Exercises

Show that for any set X , the discrete metric ($d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$) does give a metric space (X,d) .

Let $X := \{0\}$ be a set. Can you make it into a metric space?

Let $X := \{a, b\}$ be a set. Can you make it into two distinct metric spaces? (define two distinct metrics on it)

Let the set $X := \{A, B, C\}$ represent 3 buildings on campus. Suppose we wish our distance to be the time it takes to walk from one building to the other. It takes 5 minutes either way between buildings A and B. However, building C is on a hill and it takes 10 minutes from A and 15 minutes from B to get to C. On the other hand it takes 5 minutes to go from C to A and 7 minutes to go from C to B, as we are going downhill. Do these distances define a metric? If so, prove it, if not, say why not.

Suppose (X,d) is a metric space and $\varphi: [0, \infty) \rightarrow \{\mathbb{R}\}$ is a function such that $\varphi(t) \geq 0$ for all t and $\varphi(t) = 0$ if and only if $t = 0$. Also suppose φ is *subadditive*, that is, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$. Show that with $d'(x,y) := \varphi(d(x,y))$, we obtain a new metric space (X,d') .

[exercise:mscross] Let (X,d_X) and (Y,d_Y) be metric spaces.

a) Show that $(X \times Y, d)$ with $d((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2)$ is a metric space.

b) Show that $(X \times Y, d)$ with $d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ is a metric space.

Let X be the set of continuous functions on $[0,1]$. Let $\varphi: [0,1] \rightarrow (0, \infty)$ be continuous. Define $d(f,g) := \int_0^1 |f(x) - g(x)| \varphi(x) dx$. Show that (X,d) is a metric space.

[exercise:mshausdorffpseudo] Let (X,d) be a metric space. For nonempty bounded subsets A and B let $d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$. Note: d_H can be defined for arbitrary nonempty subsets if we allow the extended reals.

a) Let $Y \subset \mathcal{P}(X)$ be the set of bounded nonempty subsets. Prove that (Y, d_H) is a so-called *pseudometric space*: d_H satisfies the metric properties [metric:pos], [metric:com], [metric:triang], and further $d_H(A,A) = 0$ for all $A \in Y$.

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- b) Show by example that d itself is not symmetric, that is $d(A,B) \neq d(B,A)$.
 c) Find a metric space X and two different nonempty bounded subsets A and B such that $d_H(A,B) = 0$.

Open and closed sets

Note: 2 lectures

Topology

It is useful to define a so-called *topology*. That is we define closed and open sets in a metric space. Before doing so, let us define two special sets.

Let (X,d) be a metric space, $x \in X$ and $\delta > 0$. Then define the *open ball* or simply *ball* of radius δ around x as $B(x,\delta) := \{y \in X : d(x,y) < \delta\}$. Similarly we define the *closed ball* as $C(x,\delta) := \{y \in X : d(x,y) \leq \delta\}$.

When we are dealing with different metric spaces, it is sometimes convenient to emphasize which metric space the ball is in. We do this by writing $B_X(x,\delta) := B(x,\delta)$ or $C_X(x,\delta) := C(x,\delta)$.

Take the metric space $(\mathbb{R})$ with the standard metric. For $x \in \mathbb{R}$, and $\delta > 0$ we get $B(x,\delta) = (x-\delta, x+\delta) \sqcup \{x\} \sqcup C(x,\delta) = [x-\delta, x+\delta]$.

Be careful when working on a subspace. Suppose we take the metric space $[0,1]$ as a subspace of $(\mathbb{R})$. Then in $[0,1]$ we get $B(0, \frac{1}{2}) = B_{[0,1]}(0, \frac{1}{2}) = [0, \frac{1}{2})$. This is different from $B_{\mathbb{R}}(0, \frac{1}{2})$.

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$B(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap [0,1]$. The important thing to keep in mind is which metric space we are working in.

Let (X,d) be a metric space. A set $V \subset X$ is *open* if for every $x \in V$, there exists a $\delta > 0$ such that $B(x,\delta) \subset V$. See . A set $E \subset X$ is *closed* if the complement $E^c = X \setminus E$ is open. When the ambient space X is not clear from context we say V is open in X and E is closed in X .

If $x \in V$ and V is open, then we say V is an *open neighborhood* of x (or sometimes just *neighborhood*).

Intuitively, an open set is a set that does not include its “boundary,” wherever we are at in the set, we are allowed to “wiggle” a little bit and stay in the set. Note that not every set is either open or closed, in fact generally most subsets are neither.

The set $(0,1) \subset \mathbb{R}$ is neither open nor closed. First, every ball in $\mathbb{R}$ around $0, (-\delta, \delta)$, contains negative numbers and hence is not contained in $[0,1]$ and so $[0,1]$ is not open. Second, every ball in $\mathbb{R}$ around $1, (1-\delta, 1+\delta)$ contains numbers strictly less than 1 and greater than 1 (e.g. $1-\frac{\delta}{2}$ as long as $\delta < 2$). Thus $\mathbb{R} \setminus [0,1]$ is not open, and so $[0,1]$ is not closed.

[prop:topology:open] Let (X,d) be a metric space.

- $\emptyset$ and X are open in X .
- [topology:openii] If $V_1, V_2, \dots, V_k$ are open then $\bigcap_{j=1}^k V_j$ is also open. That is, finite intersection of open sets is open.
- [topology:openiii] If $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of open sets, then $\bigcup_{\lambda \in I} V_\lambda$ is also open. That is, union of open sets is open.

Note that the index set in [topology:openiii] is arbitrarily large. By $\bigcup_{\lambda \in I} V_\lambda$ we simply mean the set of all x such that $x \in V_\lambda$ for at least one $\lambda \in I$.

The sets X and $\emptyset$ are obviously open in X .

Let us prove [topology:openii]. If $x \in \bigcap_{j=1}^k V_j$, then $x \in V_j$ for all j . As V_j are all open, for every j there exists a $\delta_j > 0$ such that $B(x, \delta_j) \subset V_j$. Take $\delta := \min\{\delta_1, \delta_2, \dots, \delta_k\}$ and notice $\delta > 0$. We have $B(x, \delta) \subset B(x, \delta_j) \subset V_j$ for every j and so $B(x, \delta) \subset \bigcap_{j=1}^k V_j$. Consequently the intersection is open.

Let us prove [topology:openiii]. If $x \in \bigcup_{\lambda \in I} V_\lambda$, then $x \in V_\lambda$ for some $\lambda \in I$. As V_λ is open, there exists a $\delta > 0$ such that $B(x, \delta) \subset V_\lambda$. But then $B(x, \delta) \subset \bigcup_{\lambda \in I} V_\lambda$ and so the union is open.

The main thing to notice is the difference between items [topology:openii] and [topology:openiii]. Item [topology:openii] is not true for an arbitrary intersection, for example $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open.

The proof of the following analogous proposition for closed sets is left as an exercise.

[prop:topology:closed] Let (X,d) be a metric space.

- $\emptyset$ and X are closed in X .
- [topology:closedii] If $\{E_\lambda\}_{\lambda \in I}$ is an arbitrary collection of closed sets, then $\bigcap_{\lambda \in I} E_\lambda$ is also closed. That is, intersection of closed sets is closed.
- [topology:closediii] If $E_1, E_2, \dots, E_k$ are closed then $\bigcup_{j=1}^k E_j$ is also closed. That is, finite union of closed sets is closed.

We have not yet shown that the open ball is open and the closed ball is closed. Let us show this fact now to justify the terminology.

[prop:topology:ballsopenclosed] Let (X,d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x,\delta)$ is open and $C(x,\delta)$ is closed.

Let $y \in B(x,\delta)$. Let $\alpha := \delta - d(x,y)$. Of course $\alpha > 0$. Now let $z \in B(y,\alpha)$. Then $d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \alpha = d(x,y) + \delta - d(x,y) = \delta$. Therefore $z \in B(x,\delta)$ for every $z \in B(y,\alpha)$. So $B(y,\alpha) \subset B(x,\delta)$ and $B(x,\delta)$ is open.

The proof that $C(x,\delta)$ is closed is left as an exercise.

Again be careful about what is the ambient metric space. As $[0, \frac{1}{2})$ is an open ball in $[0,1]$, this means that $[0, \frac{1}{2})$ is an open set in $[0,1]$. On the other hand $[0, \frac{1}{2})$ is neither open nor closed in $\mathbb{R}$.

A useful way to think about an open set is as a union of open balls. If U is open, then for each $x \in U$, there is a $\delta_x > 0$ (depending on x) such that $B(x, \delta_x) \subset U$. Then $U = \bigcup_{x \in U} B(x, \delta_x)$.

The proof of the following proposition is left as an exercise. Note that there are many other open and closed sets in $(\mathbb{R})$.

[prop:topology:intervals:openclosed] Let $a < b$ be two real numbers. Then (a,b) , (a, ∞) , and $(-\infty, b)$ are open in $(\mathbb{R})$. Also $[a,b]$, $[a, \infty)$, and $(-\infty, b]$ are closed in $(\mathbb{R})$.

Connected sets

A nonempty metric space (X,d) is *connected* if the only subsets of X that are both open and closed are $\emptyset$ and X itself. If (X,d) is not connected we say it is *disconnected*.

When we apply the term *connected* to a nonempty subset $A \subset X$, we simply mean that A with the subspace topology is connected.

In other words, a nonempty X is connected if whenever we write $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are open, then either $X_1 = \emptyset$ or $X_2 = \emptyset$. So to show X is disconnected, we need to find nonempty disjoint open sets X_1 and X_2 whose union is X . For subsets, we state this idea as a proposition.

Let (X,d) be a metric space. A nonempty set $S \subset X$ is not connected if and only if there exist open sets U_1 and U_2 in X , such that $U_1 \cap U_2 \cap S = \emptyset$, $U_1 \cap S \neq \emptyset$, $U_2 \cap S \neq \emptyset$, and $S = \bigcup_{i=1}^2 (U_i \cap S)$.

If U_j is open in X , then $U_j \cap S$ is open in S in the subspace topology (with subspace metric). To see this, note that if $B_X(x,\delta) \subset U_j$, then $B_S(x,\delta) = S \cap B_X(x,\delta)$, we have $B_S(x,\delta) \subset U_j \cap S$. So if U_1 and U_2 as above exist, then X is disconnected based on the discussion above.

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The proof of the other direction follows by using to find U_1 and U_2 from two open disjoint subsets of S .

Let $S \subseteq \mathbb{R}$ be such that $x < z < y$ with $x, y \in S$ and $z \notin S$. Claim: S is not connected. Proof: Notice $\bigl((-\infty, z) \cap S \bigr) \cup \bigl((z, \infty) \cap S \bigr) = S$.

A nonempty set $S \subseteq \mathbb{R}$ is connected if and only if it is an interval or a single point.

Suppose S is connected. If S is a single point then we are done. So suppose $x < y$ and $x, y \in S$. If z is such that $x < z < y$, then $(-\infty, z) \cap S$ is nonempty and $(z, \infty) \cap S$ is nonempty. The two sets are disjoint. As S is connected, we must have their union is not S , so $z \notin S$.

Suppose S is bounded, connected, but not a single point. Let $\alpha := \inf S$ and $\beta := \sup S$ and note that $\alpha < \beta$. Suppose $\alpha < z < \beta$. As α is the infimum, then there is an $x \in S$ such that $\alpha \leq x < z$. Similarly there is a $y \in S$ such that $\beta \geq y > z$. We have shown above that $z \notin S$, so $(\alpha, \beta) \subseteq S$. If $w < \alpha$, then $w \notin S$ as α was the infimum, similarly if $w > \beta$ then $w \notin S$. Therefore the only possibilities for S are (α, β) , $[\alpha, \beta)$, $(\alpha, \beta]$, $[\alpha, \beta]$.

The proof that an unbounded connected S is an interval is left as an exercise.

On the other hand suppose S is an interval. Suppose U_1 and U_2 are open subsets of $\mathbb{R}$, $U_1 \cap S$ and $U_2 \cap S$ are nonempty, and $S = \bigl(U_1 \cap S \bigr) \cup \bigl(U_2 \cap S \bigr)$. We will show that $U_1 \cap S$ and $U_2 \cap S$ contain a common point, so they are not disjoint, and hence S must be connected. Suppose there is $x \in U_1 \cap S$ and $y \in U_2 \cap S$. Without loss of generality, assume $x < y$. As S is an interval $[x, y] \subseteq S$. Let $z := \inf(U_2 \cap [x, y])$. If $z = x$, then $x \in U_2$. If $z > x$, then for any $\delta > 0$ the ball $B(z, \delta) = (z - \delta, z + \delta)$ contains points of S that are not in U_2 , and so $z \notin U_2$ as U_2 is open. Therefore, $z \in U_1$. As U_1 is open, $B(z, \delta) \subseteq U_1$ for a small enough $\delta > 0$. As z is the infimum of $U_2 \cap [x, y]$, there must exist some $w \in U_2 \cap [x, y]$ such that $w \in (z, z + \delta) \subseteq B(z, \delta) \subseteq U_1$. Therefore $w \in U_1 \cap U_2 \cap [x, y]$. So $U_1 \cap S$ and $U_2 \cap S$ are not disjoint and hence S is connected.

In many cases a ball $B(x, \delta)$ is connected. But this is not necessarily true in every metric space. For a simplest example, take a two point space $\{a, b\}$ with the discrete metric. Then $B(a, 2) = \{a, b\}$, which is not connected as $B(a, 1) = \{a\}$ and $B(b, 1) = \{b\}$ are open and disjoint.

Closure and boundary

Sometimes we wish to take a set and throw in everything that we can approach from the set. This concept is called the closure.

Let (X, d) be a metric space and $A \subseteq X$. Then the *closure* of A is the set $\overline{A} := \bigcap \{E \subseteq X : \text{\textit{E is closed and }} A \subseteq E\}$. That is, $\overline{A}$ is the intersection of all closed sets that contain A .

Let (X, d) be a metric space and $A \subseteq X$. The closure $\overline{A}$ is closed. Furthermore if A is closed then $\overline{A} = A$.

First, the closure is the intersection of closed sets, so it is closed. Second, if A is closed, then take $E = A$, hence the intersection of all closed sets E containing A must be equal to A .

The closure of $(0, 1)$ in $\mathbb{R}$ is $[0, 1]$. Proof: Simply notice that if E is closed and contains $(0, 1)$, then E must contain 0 and 1 (why?). Thus $[0, 1] \subseteq E$. But $[0, 1]$ is also closed. Therefore the closure $\overline{(0, 1)} = [0, 1]$.

Be careful to notice what ambient metric space you are working with. If $X = (0, \infty)$, then the closure of $(0, 1)$ in $(0, \infty)$ is $(0, 1]$. Proof: Similarly as above $(0, 1]$ is closed in $(0, \infty)$ (why?). Any closed set E that contains $(0, 1)$ must contain 1 (why?). Therefore $(0, 1] \subseteq E$, and hence $\overline{(0, 1)} = (0, 1]$ when working in $(0, \infty)$.

Let us justify the statement that the closure is everything that we can “approach” from the set.

[prop:msclosureappr] Let (X, d) be a metric space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if for every $\delta > 0$, $B(x, \delta) \cap A \neq \emptyset$.

Let us prove the two contrapositives. Let us show that $x \notin \overline{A}$ if and only if there exists a $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$.

First suppose $x \notin \overline{A}$. We know $\overline{A}$ is closed. Thus there is a $\delta > 0$ such that $B(x, \delta) \subseteq \overline{A}^c$. As $A \subseteq \overline{A}$ we see that $B(x, \delta) \subseteq A^c$ and hence $B(x, \delta) \cap A = \emptyset$.

On the other hand suppose there is a $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$. Then $\{B(x, \delta)\}^c$ is a closed set and we have that $A \subseteq \{B(x, \delta)\}^c$, but $x \notin \{B(x, \delta)\}^c$. Thus as $\overline{A}$ is the intersection of closed sets containing A , we have $x \notin \overline{A}$.

We can also talk about what is in the interior of a set and what is on the boundary.

Let (X, d) be a metric space and $A \subseteq X$, then the *interior* of A is the set $A^\circ := \{x \in A : \text{\textit{there exists a }} \delta > 0 \text{\textit{ such that }} B(x, \delta) \subseteq A\}$. The *boundary* of A is the set $\partial A = \overline{A} \setminus A^\circ$.

Suppose $A = (0, 1)$ and $X = \mathbb{R}$. Then it is not hard to see that $\overline{A} = [0, 1]$, $A^\circ = (0, 1)$, and $\partial A = \{0, 1\}$.

Suppose $X = \{a, b\}$ with the discrete metric. Let $A = \{a\}$, then $\overline{A} = A^\circ = A$ and $\partial A = \emptyset$.

Let (X, d) be a metric space and $A \subseteq X$. Then A° is open and ∂A is closed.

Given $x \in A^\circ$ we have $\delta > 0$ such that $B(x, \delta) \subseteq A$. If $z \in B(x, \delta)$, then as open balls are open, there is an $\epsilon > 0$ such that $B(z, \epsilon) \subseteq B(x, \delta) \subseteq A$, so z is in A° . Therefore $B(x, \delta) \subseteq A^\circ$ and so A° is open.

As A° is open, then $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (A^\circ)^c$ is closed.

The boundary is the set of points that are close to both the set and its complement.

Let (X, d) be a metric space and $A \subseteq X$. Then $x \in \partial A$ if and only if for every $\delta > 0$, $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ are both nonempty.

Suppose $x \in \partial A = \overline{A} \setminus A^\circ$ and let $\delta > 0$ be arbitrary. By $\overline{A}$, $B(x, \delta)$ contains a point from A . If $B(x, \delta)$ contained no points of A^c , then x would be in A° . Hence $B(x, \delta)$ contains a point of A^c as well.

Let us prove the other direction by contrapositive. If $x \notin \partial A$, then there is some $\delta > 0$ such that $B(x, \delta) \subseteq \overline{A}^c$ as $\overline{A}$ is closed. So $B(x, \delta)$ contains no points of A , because $\overline{A}^c \subseteq A^c$.

Now suppose $x \in A^\circ$, then there exists a $\delta > 0$ such that $B(x, \delta) \subseteq A$, but that means $B(x, \delta)$ contains no points of A^c .

We obtain the following immediate corollary about closures of A and A^c . We simply apply .

Let (X, d) be a metric space and $A \subseteq X$. Then $\partial A = \overline{A} \cap \overline{A^c}$.

Exercises

Prove . Hint: consider the complements of the sets and apply .

Finish the proof of by proving that $C(x, \delta)$ is closed.

Prove .

Suppose (X, d) is a nonempty metric space with the discrete topology. Show that X is connected if and only if it contains exactly one element.

Show that if $S \subseteq \mathbb{R}$ is a connected unbounded set, then it is an (unbounded) interval.

Show that every open set can be written as a union of closed sets.

a) Show that E is closed if and only if $\partial E \subseteq E$. b) Show that U is open if and only if $\partial U \cap U = \emptyset$.

a) Show that A is open if and only if $A^\circ = A$. b) Suppose that U is an open set and $U \subseteq A$. Show that $U \subseteq A^\circ$.

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Let X be a set and d, d' be two metrics on X . Suppose there exists an $\alpha > 0$ and $\beta > 0$ such that $\alpha d(x,y) \leq d'(x,y) \leq \beta d(x,y)$ for all $x,y \in X$. Show that U is open in (X,d) if and only if U is open in (X,d') . That is, the topologies of (X,d) and (X,d') are the same.

Suppose $\{S_i\}_{i \in \mathbb{N}}$ is a collection of connected subsets of a metric space (X,d) . Suppose there exists an $x \in X$ such that $x \in S_i$ for all $i \in \mathbb{N}$. Show that $\bigcup_{i=1}^{\infty} S_i$ is connected.

Let A be a connected set. a) Is $\overline{A}$ connected? Prove or find a counterexample. b) Is A° connected? Prove or find a counterexample. Hint: Think of sets in $\mathbb{R}^2$.

The definition of open sets in the following exercise is usually called the *subspace topology*. You are asked to show that we obtain the same topology by considering the subspace metric.

[exercise:mssubspace] Suppose (X,d) is a metric space and $Y \subset X$. Show that with the subspace metric on Y , a set $U \subset Y$ is open (in Y) whenever there exists an open set $V \subset X$ such that $U = V \cap Y$.

Let (X,d) be a metric space. a) For any $x \in X$ and $\delta > 0$, show $\overline{B(x,\delta)} \subset C(x,\delta)$. b) Is it always true that $\overline{B(x,\delta)} = C(x,\delta)$? Prove or find a counterexample.

Let (X,d) be a metric space and $A \subset X$. Show that $A^\circ = \bigcup \{V : V \subset A \text{ is open}\}$.

Sequences and convergence

Note: 1 lecture

Sequences

The notion of a sequence in a metric space is very similar to a sequence of real numbers. The definitions are essentially the same as those for real numbers in the sense of where $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$ is replaced by an arbitrary metric space (X,d) .

A sequence in a metric space (X,d) is a function $x: \mathbb{N} \rightarrow X$. As before we write x_n for the n th element in the sequence and use the notation $\{x_n\}$, or more precisely $\{x_n\}_{n=1}^{\infty}$.

A sequence $\{x_n\}$ is *bounded* if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that $d(p, x_n) \leq B$ for all $n \in \mathbb{N}$. In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

If $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_{j+1} > n_j$ for all j , then the sequence $\{x_{n_j}\}_{j=1}^{\infty}$ is said to be a *subsequence* of $\{x_n\}$.

Similarly we also define convergence. Again, we will be cheating a little bit and we will use the definite article in front of the word *limit* before we prove that the limit is unique.

A sequence $\{x_n\}$ in a metric space (X,d) is said to *converge* to a point $p \in X$, if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq M$. The point p is said to be the *limit* of $\{x_n\}$. We write $\lim_{n \rightarrow \infty} x_n = p$.

A sequence that converges is said to be *convergent*. Otherwise, the sequence is said to be *divergent*.

Let us prove that the limit is unique. Note that the proof is almost identical to the proof of the same fact for sequences of real numbers. Many results we know for sequences of real numbers can be proved in the more general settings of metric spaces. We must replace $|x-y|$ with $d(x,y)$ in the proofs and apply the triangle inequality correctly.

[prop:mslimisunique] A convergent sequence in a metric space has a unique limit.

Suppose the sequence $\{x_n\}$ has the limit x and the limit y . Take an arbitrary $\epsilon > 0$. From the definition find an M_1 such that for all $n \geq M_1$, $d(x_n, x) < \frac{\epsilon}{2}$. Similarly find an M_2 such that for all $n \geq M_2$ we have $d(x_n, y) < \frac{\epsilon}{2}$. Now take an n such that $n \geq M_1$ and also $n \geq M_2$. $d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. As $d(x, y) < \epsilon$ for all $\epsilon > 0$, then $d(x, y) = 0$ and $y = x$. Hence the limit (if it exists) is unique.

The proofs of the following propositions are left as exercises.

[prop:msconvbound] A convergent sequence in a metric space is bounded.

[prop:msconvifa] A sequence $\{x_n\}$ in a metric space (X,d) converges to $p \in X$ if and only if there exists a sequence $\{a_n\}$ of real numbers such that $d(x_n, p) \leq a_n$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} a_n = 0$.

[prop:mssubseq] Let $\{x_n\}$ be a sequence in a metric space (X,d) .

- If $\{x_n\}$ converges to $p \in X$, then every subsequence $\{x_{n_k}\}$ converges to p .
- If for some $K \in \mathbb{N}$ the K -tail $\{x_n\}_{n=K+1}^{\infty}$ converges to $p \in X$, then $\{x_n\}$ converges to p .

Convergence in euclidean space

It is useful to note what convergence means in the euclidean space $\mathbb{R}^n$.

[prop:msconvecu] Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, where we write $x_j = (x_{j,1}, x_{j,2}, \dots, x_{j,n})$. Then $\{x_j\}_{j=1}^{\infty}$ converges if and only if $\{x_{j,k}\}_{j=1}^{\infty}$ converges for every k , in which case $\lim_{j \rightarrow \infty} x_j = (\lim_{j \rightarrow \infty} x_{j,1}, \lim_{j \rightarrow \infty} x_{j,2}, \dots, \lim_{j \rightarrow \infty} x_{j,n})$.

Let $\{x_j\}_{j=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$, where we write $x_j = (x_{j,1}, x_{j,2}, \dots, x_{j,n})$. Let $y = (y_1, y_2, \dots, y_n)$ be the limit. Given $\epsilon > 0$, there exists an M such that for all $j \geq M$ we have $d(y, x_j) < \epsilon$. Fix some $k=1, 2, \dots, n$. For $j \geq M$ we have $|y_k - x_{j,k}| \leq \sqrt{\sum_{\ell=1}^n |y_\ell - x_{j,\ell}|^2} = d(y, x_j) < \epsilon$. Hence the sequence $\{x_{j,k}\}_{j=1}^{\infty}$ converges to y_k .

For the other direction suppose $\{x_{j,k}\}_{j=1}^{\infty}$ converges to y_k for every $k=1, 2, \dots, n$. Hence, given $\epsilon > 0$, pick an M , such that if $j \geq M$ then $|y_k - x_{j,k}| \leq \sqrt{\sum_{k=1}^n |y_k - x_{j,k}|^2} < \epsilon$. Then $d(y, x_j) = \sqrt{\sum_{k=1}^n |y_k - x_{j,k}|^2} < \epsilon$. Hence the sequence $\{x_j\}_{j=1}^{\infty}$ converges to y .

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$\{n\} = \epsilon$. The sequence $\{x_j\}$ converges to $y \in \mathbb{R}^n$ and we are done.

Convergence and topology

The topology, that is, the set of open sets of a space encodes which sequences converge.

[prop:msconvtopo] Let (X,d) be a metric space and $\{x_n\}$ a sequence in X . Then $\{x_n\}$ converges to $x \in X$ if and only if for every open neighborhood U of x , there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $x_n \in U$.

First suppose $\{x_n\}$ converges. Let U be an open neighborhood of x , then there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. As the sequence converges, find an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $d(x, x_n) < \epsilon$ or in other words $x_n \in B(x, \epsilon) \subset U$.

Let us prove the other direction. Given $\epsilon > 0$ let $U := B(x, \epsilon)$ be the neighborhood of x . Then there is an $M \in \mathbb{N}$ such that for $n \geq M$ we have $x_n \in U = B(x, \epsilon)$ or in other words, $d(x, x_n) < \epsilon$.

A set is closed when it contains the limits of its convergent sequences.

[prop:msclosedlim] Let (X,d) be a metric space, $E \subset X$ a closed set and $\{x_n\}$ a sequence in E that converges to some $x \in X$. Then $x \in E$.

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Let us prove the contrapositive. Suppose $\{x_n\}$ is a sequence in X that converges to $x \in E^c$. As E^c is open, says there is an M such that for all $n \geq M$, $x_n \in E^c$. So $\{x_n\}$ is not a sequence in E .

When we take a closure of a set A , we really throw in precisely those points that are limits of sequences in A .

[prop:msclosureapprseq] Let (X,d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ of elements in A such that $\lim x_n = x$.

Let $x \in \overline{A}$. We know by that given $\frac{1}{n}$, there exists a point $x_n \in B(x, \frac{1}{n}) \cap A$. As $d(x, x_n) < \frac{1}{n}$, we have $\lim x_n = x$.

For the other direction, see .

Exercises

[exercise:reverseclosedseq] Let (X,d) be a metric space and let $A \subset X$. Let E be the set of all $x \in X$ such that there exists a sequence $\{x_n\}$ in A that converges to x . Show $E = \overline{A}$.

a) Show that $d(x,y) := \min \{1, \left| \frac{x-y}{2} \right| \}$ defines a metric on $(\mathbb{R}, d)$. b) Show that a sequence converges in $(\mathbb{R}, d)$ if and only if it converges in the standard metric. c) Find a bounded sequence in $(\mathbb{R}, d)$ that contains no convergent subsequence.

Prove .

Prove .

Suppose $\{x_n\}_{n=1}^\infty$ converges to x . Suppose $f: \mathbb{N} \rightarrow \mathbb{N}$ is a one-to-one function. Show that $\{x_{f(n)}\}_{n=1}^\infty$ converges to x .

If (X,d) is a metric space where d is the discrete metric. Suppose $\{x_n\}$ is a convergent sequence in X . Show that there exists a $K \in \mathbb{N}$ such that for all $n \geq K$ we have $x_n = x_K$.

A set $S \subset X$ is said to be dense in X if for every $x \in X$, there exists a sequence $\{x_n\}$ in S that converges to x . Prove that $(\mathbb{R})^n$ contains a countable dense subset.

Suppose $\{U_n\}_{n=1}^\infty$ be a decreasing ($U_{n+1} \subset U_n$ for all n) sequence of open sets in a metric space (X,d) such that $\bigcap_{n=1}^\infty U_n = \{p\}$ for some $p \in X$. Suppose $\{x_n\}$ is a sequence of points in X such that $x_n \in U_n$. Does $\{x_n\}$ necessarily converge to p ? Prove or construct a counterexample.

Let $E \subset X$ be closed and let $\{x_n\}$ be a sequence in X converging to $p \in X$. Suppose $x_n \in E$ for infinitely many $n \in \mathbb{N}$. Show $p \in E$.

Take $(\mathbb{R})^+ = \{x \in \mathbb{R} : x \geq 0\}$ be the extended reals. Define $d(x,y) := \left| \frac{x-y}{1+\sqrt{x}+\sqrt{y}} \right|$ if $x, y \in \mathbb{R}$, define $d(\infty, x) := \left| \frac{1}{1+\sqrt{x}} \right|$ if $x \in \mathbb{R}$, and let $d(\infty, \infty) := 0$. a) Show that $(\mathbb{R}^+, d)$ is a metric space. b) Suppose $\{x_n\}$ is a sequence of real numbers such that for every $M \in \mathbb{R}$, there exists an N such that $x_n \geq M$ for all $n \geq N$. Show that $\lim x_n = \infty$ in $(\mathbb{R}^+, d)$. c) Show that a sequence of real numbers converges to a real number in $(\mathbb{R}^+, d)$ if and only if it converges in $(\mathbb{R})^+$ with the standard metric.

Suppose $\{V_n\}_{n=1}^\infty$ is a collection of open sets in (X,d) such that $V_{n+1} \subset V_n$. Let $\{x_n\}$ be a sequence such that $x_n \in V_{n+1} \setminus V_n$ and suppose $\{x_n\}$ converges to $p \in X$. Show that $p \in \bigcap_{n=1}^\infty V_n$.

Prove .

Completeness and compactness

Note: 2 lectures

Cauchy sequences and completeness

Just like with sequences of real numbers we define Cauchy sequences.

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$ we have $d(x_n, x_k) < \epsilon$.

The definition is again simply a translation of the concept from the real numbers to metric spaces. So a sequence of real numbers is Cauchy in the sense of if and only if it is Cauchy in the sense above, provided we equip the real numbers with the standard metric $d(x,y) = |x-y|$.

A convergent sequence in a metric space is Cauchy.

Suppose $\{x_n\}$ converges to x . Given $\epsilon > 0$ there is an M such that for $n \geq M$ we have $d(x, x_n) < \frac{\epsilon}{2}$. Hence for all $n, k \geq M$ we have $d(x_n, x_k) \leq d(x_n, x) + d(x, x_k) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Let (X,d) be a metric space. We say X is *complete* or *Cauchy-complete* if every Cauchy sequence $\{x_n\}$ in X converges to an $x \in X$.

The space $(\mathbb{R})^n$ with the standard metric is a complete metric space.

For $(\mathbb{R})^1 = \mathbb{R}$ completeness was proved in . The proof of the above proposition is a reduction to the one dimensional case.

Let $\{x_j\}_{j=1}^\infty$ be a Cauchy sequence in $(\mathbb{R})^n$, where we write $x_j = (x_{j,1}, x_{j,2}, \dots, x_{j,n})$ in $(\mathbb{R})^n$. As the sequence is Cauchy, given $\epsilon > 0$, there exists an M such that for all $i, j \geq M$ we have $d(x_i, x_j) < \epsilon$.

Fix some $k=1, 2, \dots, n$, for $i, j \geq M$ we have $|x_{i,k} - x_{j,k}| \leq d(x_i, x_j) < \epsilon$

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$\{x_{j,k}\}$ is complete the sequence converges; there exists an $y_k \in \mathbb{R}$ such that $y_k = \lim_{j \rightarrow \infty} x_{j,k}$.

Write $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. By we have that $\{x_j\}$ converges to $y \in \mathbb{R}^n$ and hence $(\mathbb{R})^n$ is complete.

Note that a subset of $(\mathbb{R})^n$ with the subspace metric need not be complete. For example, $(0,1)$ with the subspace metric is not complete as $\{\frac{1}{n}\}$ is a Cauchy sequence in $(0,1)$ with no limit in $(0,1)$. But see also .

Compactness

Let (X,d) be a metric space and $K \subset X$. The set K is said to be *compact* if for any collection of open sets $\{U_\lambda\}_{\lambda \in I}$ such that $K \subset \bigcup_{\lambda \in I} U_\lambda$, there exists a finite subset $\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset I$ such that $K \subset \bigcup_{j=1}^k U_{\lambda_j}$.

A collection of open sets $\{U_\lambda\}_{\lambda \in I}$ as above is said to be an *open cover* of K . So a way to say that K is compact is to say that *every open cover of K has a finite subcover*.

Let (X,d) be a metric space. A compact set $K \subset X$ is closed and bounded.

First, we prove that a compact set is bounded. Fix $p \in X$. We have the open cover $K \subset \bigcup_{n=1}^\infty B(p, n) = X$. If K is compact, then there exists some set of indices $n_1 < n_2 < \dots < n_k$ such that $K \subset \bigcup_{j=1}^k B(p, n_j) = B(p, n_k)$. As K is contained in a ball, K is bounded.

Next, we show a set that is not closed is not compact. Suppose $\overline{K} \neq K$, that is, there is a point $x \in \overline{K} \setminus K$. If $y \neq x$, then for n with $\frac{1}{n} < d(x,y)$ we have $y \notin C(x, \frac{1}{n})$. Furthermore $x \notin K$, so $K \subset \bigcup_{n=1}^\infty C(x, \frac{1}{n})^c$. As a closed ball is closed, $C(x, \frac{1}{n})^c$ is open, and so we have an open cover. If we take any finite collection of indices $n_1 < n_2 < \dots < n_k$, then $\bigcup_{j=1}^k C(x, \frac{1}{n_j})^c = C(x, \frac{1}{n_k})^c$. As x is in the closure, $C(x, \frac{1}{n_k}) \cap K \neq \emptyset$. So there is no finite subcover and K is not compact.

We prove below that in finite dimensional euclidean space every closed bounded set is compact. So closed bounded sets of $(\mathbb{R})^n$ are examples of compact sets. It is not true that in every

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and bounded is equivalent to compact. A simple example would be an incomplete metric space such as $(0,1)$ with the subspace metric. But there are many complete and very

useful metric spaces where closed and bounded is not enough to give compactness, see : $C([a,b], \mathbb{R})$ is a complete metric space, but the closed unit ball $C(0,1)$ is not compact. However, see .

A useful property of compact sets in a metric space is that every sequence has a convergent subsequence. Such sets are sometimes called *sequentially compact*. Let us prove that in the context of metric spaces, a set is compact if and only if it is sequentially compact. First we prove a lemma.

[ms:lebesgue] Let (X,d) be a metric space and $K \subset X$. Suppose every sequence in K has a subsequence convergent in K . Given an open cover $\{U_\lambda\}_{\lambda \in I}$ of K , there exists a $\delta > 0$ such that for every $x \in K$, there exists a $\lambda \in I$ with $B(x,\delta) \subset U_\lambda$.

It is important to recognize what the lemma says. It says that given any cover there is a single $\delta > 0$. The δ can depend on the cover, but of course it does not depend on x .

Let us prove the lemma by contrapositive. If the conclusion is not true, then there is an open cover $\{U_\lambda\}_{\lambda \in I}$ of K with the following property. For every $n \in \mathbb{N}$ there exists an $x_n \in K$ such that $B(x_n, \frac{1}{n})$ is not a subset of any U_λ . Given any $x \in K$, there is a $\lambda \in I$ such that $x \in U_\lambda$. Hence there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset U_\lambda$. Take M such that $\frac{1}{M} < \frac{\epsilon}{2}$. If $y \in B(x, \frac{\epsilon}{2})$ and $n \geq M$, then by triangle inequality $B(y, \frac{1}{n}) \subset B(x, \frac{\epsilon}{2}) \subset U_\lambda$. In other words, for all $n \geq M$, $x_n \notin B(x, \frac{\epsilon}{2})$. Hence the sequence cannot have a subsequence converging to x . As $x \in K$ was arbitrary we are done.

[thm:mscompactiseqcpt] Let (X,d) be a metric space. Then $K \subset X$ is a compact set if and only if every sequence in K has a subsequence converging to a point in K .

Let $K \subset X$ be a set and $\{x_n\}$ a sequence in K . Suppose that for each $x \in K$, there is a ball $B(x, \alpha_x)$ for some $\alpha_x > 0$ such that $x_n \in B(x, \alpha_x)$ for only finitely many $n \in \mathbb{N}$. Then $K \subset \bigcup_{x \in K} B(x, \alpha_x)$. Any finite collection of these balls is going to contain only finitely many x_n . Thus for any finite collection of such balls there is an $x_n \in K$ that is not in the union. Therefore, K is not compact.

So if K is compact, then there exists an $x \in K$ such that for any $\delta > 0$, $B(x, \delta)$ contains x_k for infinitely many $k \in \mathbb{N}$. The ball $B(x, 1)$ contains some x_k so let $n_1 := k$. If n_{j-1} is defined, then there must exist a $k > n_{j-1}$ such that $x_k \in B(x, \frac{1}{j})$, so define $n_j := k$. Notice that $d(x, x_{n_j}) < \frac{1}{j}$. By $\lim_{j \rightarrow \infty} x_{n_j} = x$.

For the other direction, suppose every sequence in K has a subsequence converging in K . Take an open cover $\{U_\lambda\}_{\lambda \in I}$ of K . Using the Lebesgue covering lemma above, we find a $\delta > 0$ such that for every x , there is a $\lambda \in I$ with $B(x, \delta) \subset U_\lambda$.

Pick $x_1 \in K$ and find $\lambda_1 \in I$ such that $B(x_1, \delta) \subset U_{\lambda_1}$. If $K \setminus U_{\lambda_1} \neq \emptyset$, we stop as we have found a finite subcover. Otherwise, there must be a point $x_2 \in K \setminus U_{\lambda_1}$. Note that $d(x_2, x_1) \geq \delta$. There must exist some $\lambda_2 \in I$ such that $B(x_2, \delta) \subset U_{\lambda_2}$. We work inductively. Suppose λ_{n-1} is defined. Either $U_{\lambda_{n-1}} \cup \dots \cup U_{\lambda_n}$ is a finite cover of K , in which case we stop, or there must be a point $x_n \in K \setminus \bigcup_{i=1}^n U_{\lambda_i}$. Note that $d(x_n, x_j) \geq \delta$ for all $j = 1, 2, \dots, n-1$. Next, there must be some $\lambda_n \in I$ such that $B(x_n, \delta) \subset U_{\lambda_n}$.

Either at some point we obtain a finite subcover of K or we obtain an infinite sequence $\{x_n\}$ as above. For contradiction suppose that there is no finite subcover and we have the sequence $\{x_n\}$. For all n and k , $n \neq k$, we have $d(x_n, x_k) \geq \delta$, so no subsequence of $\{x_n\}$ can be Cauchy. Hence no subsequence of $\{x_n\}$ can be convergent, which is a contradiction.

The Bolzano-Weierstrass theorem for sequences of real numbers () says that any bounded sequence in $\mathbb{R}$ has a convergent subsequence. Therefore any sequence in a closed interval $[a,b] \subset \mathbb{R}$ has a convergent subsequence. The limit must also be in $[a,b]$ as limits preserve non-strict inequalities. Hence a closed bounded interval $[a,b] \subset \mathbb{R}$ is compact.

Let (X,d) be a metric space and let $K \subset X$ be compact. If $E \subset K$ is a closed set, then E is compact.

Let $\{x_n\}$ be a sequence in E . It is also a sequence in K . Therefore it has a convergent subsequence $\{x_{n_j}\}$ that converges to some $x \in K$. As E is closed the limit of a sequence in E is also in E and so $x \in E$. Thus E must be compact.

[thm:msbw] A closed bounded subset $K \subset \mathbb{R}^n$ is compact.

So subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded, a condition that is much easier to check. Let us reiterate that the Heine-Borel theorem only holds for $\mathbb{R}^n$ and not for metric spaces in general. In general, compact implies closed and bounded, but not vice versa.

For $\mathbb{R}$ the Bolzano-Weierstrass theorem for sequences of real numbers () says that any bounded sequence in $\mathbb{R}$ is closed and bounded, then any sequence $\{x_k\}$ in K is bounded, so it has a convergent subsequence by Bolzano-Weierstrass theorem (). As K is closed, the limit of the subsequence must be an element of K . So K is compact.

Let us carry out the proof for $n=2$ and leave arbitrary n as an exercise. As $K \subset \mathbb{R}^2$ is bounded, there exists a set $B=[a,b] \times [c,d] \subset \mathbb{R}^2$ such that $K \subset B$. We will show that B is compact. Then K , being a closed subset of a compact B , is also compact.

Let $\{x_k, y_k\}_{k=1}^\infty$ be a sequence in B . That is, $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k . A bounded sequence of real numbers has a convergent subsequence so there is a subsequence $\{x_{k_j}\}_{j=1}^\infty$ that is convergent. The subsequence $\{y_{k_j}\}_{j=1}^\infty$ is also a bounded sequence so there exists a subsequence $\{y_{k_{j_i}}\}_{i=1}^\infty$ that is convergent. A subsequence of a convergent sequence is still convergent, so $\{x_{k_{j_i}}\}_{i=1}^\infty$ is convergent. Let $x := \lim_{i \rightarrow \infty} x_{k_{j_i}}$ and $y := \lim_{i \rightarrow \infty} y_{k_{j_i}}$. By $\bigwedge_{i=1}^\infty (x_{k_{j_i}} \in B \wedge y_{k_{j_i}} \in B)$ $\bigwedge_{i=1}^\infty$ converges to (x,y) . Furthermore, as $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k , we know that $(x,y) \in B$.

The discrete metric provides interesting counterexamples again. Let (X,d) be a metric space with the discrete metric, that is $d(x,y) = 1$ if $x \neq y$. Suppose X is an infinite set. Then:

- (X,d) is a complete metric space.
- Any subset $K \subset X$ is closed and bounded.
- A subset $K \subset X$ is compact if and only if it is a finite set.
- The conclusion of the Lebesgue covering lemma is always satisfied with e.g. $\delta = \frac{1}{2}$, even for noncompact $K \subset X$.

The proofs of these statements are either trivial or are relegated to the exercises below.

Exercises

Let (X,d) be a metric space and A a finite subset of X . Show that A is compact.

Let $A = \{ \frac{1}{n} : n \in \mathbb{N} \} \subset \mathbb{R}$. a) Show that A is not compact directly using the definition. b) Show that $A \cup \{0\}$ is compact directly using the definition.

Let (X,d) be a metric space with the discrete metric. a) Prove that X is complete. b) Prove that X is compact if and only if X is a finite set.

a) Show that the union of finitely many compact sets is a compact set. b) Find an example where the union of infinitely many compact sets is not compact.

Prove for arbitrary dimension. Hint: The trick is to use the correct notation.

Show that a compact set K is a complete metric space (using the subspace metric).

[exercise:CabRcomplete] Let $C([a,b], \mathbb{R})$ be the metric space as in . Show that $C([a,b], \mathbb{R})$ is a complete metric space.

[exercise:msclbounnotcompt] Let $C([0,1], \mathbb{R})$ be the metric space of . Let 0 denote the zero function. Then show that the closed ball $C(0,1)$ is not compact (even though it is closed and bounded). Hints: Construct a sequence of distinct continuous functions $\{f_n\}$ such that $d(f_n, 0) = 1$ and $d(f_n, f_k) = 1$ for all $n \neq k$. Show that the set $\{f_n : n \in \mathbb{N}\} \subset C(0,1)$ is closed but not compact. See for inspiration.

Show that there exists a metric on $\mathbb{R}$ that makes $\mathbb{R}$ into a compact set.

Suppose (X,d) is complete and suppose we have a countably infinite collection of nonempty compact sets $E_1 \supset E_2 \supset E_3 \supset \dots$ then prove $\bigcap_{j=1}^\infty E_j \neq \emptyset$.

Let $C([0,1], \mathbb{R})$ be the metric space of . Let K be the set of $f \in C([0,1], \mathbb{R})$ such that f is equal to a quadratic polynomial, i.e. $f(x) = a+bx+cx^2$, and such that $\|f\| \leq 1$ for all $x \in [0,1]$, that is $f \in C(0,1)$. Show that K is compact.

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[exercise:mstotbound] Let (X, d) be a complete metric space. Show that $K \subset X$ is compact if and only if K is closed and such that for every $\epsilon > 0$ there exists a finite set of points $x_1, x_2, \dots, x_n$ with $K \subset \bigcup_{j=1}^n B(x_j, \epsilon)$. Note: Such a set K is said to be *totally bounded*, so in a complete metric space a set is compact if and only if it is closed and totally bounded.

Take $\mathbb{N} \subset \mathbb{R}$ using the standard metric. Find an open cover of $\mathbb{N}$ such that the conclusion of the Lebesgue covering lemma does not hold.

Prove the general Bolzano-Weierstrass theorem: Any bounded sequence $\{x_k\}$ in $\mathbb{R}$ has a convergent subsequence.

Let X be a metric space and $C \subset \mathcal{P}(X)$ the set of nonempty compact subsets of X . Using the Hausdorff metric from , show that (C, d_H) is a metric space. That is, show that if L and K are nonempty compact subsets then $d_H(L, K) = 0$ if and only if $L = K$.

[exercise:closedcomplete] Let (X, d) be a complete metric space and $E \subset X$ a closed set. Show that E with the subspace metric is a complete metric space.

Continuous functions

Note: 1 lecture

Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces and $c \in X$. Then $f: X \rightarrow Y$ is *continuous at c* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in X$ and $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$.

When $f: X \rightarrow Y$ is continuous at all $c \in X$, then we simply say that f is a *continuous function*.

The definition agrees with the definition from when f is a real-valued function on the real line, if we take the standard metric on $\mathbb{R}$.

[prop:contiscont] Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f: X \rightarrow Y$ is continuous at $c \in X$ if and only if for every sequence $\{x_n\}$ in X converging to c , the sequence $\{f(x_n)\}$ converges to $f(c)$.

Suppose f is continuous at c . Let $\{x_n\}$ be a sequence in X converging to c . Given $\epsilon > 0$, there is a $\delta > 0$ such that $d_X(x, c) < \delta$ implies $d_Y(f(x), f(c)) < \epsilon$. So take M such that for all $n \geq M$, we have $d_X(x_n, c) < \delta$, then $d_Y(f(x_n), f(c)) < \epsilon$. Hence $\{f(x_n)\}$ converges to $f(c)$.

On the other hand suppose f is not continuous at c . Then there exists an $\epsilon > 0$, such that for every $n \in \mathbb{N}$ there exists an $x_n \in X$, with $d_X(x_n, c) < \frac{1}{n}$ such that $d_Y(f(x_n), f(c)) \geq \epsilon$. Then $\{x_n\}$ converges to c , but $\{f(x_n)\}$ does not converge to $f(c)$.

Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial. That is, $f(x, y) = \sum_{j=0}^d \sum_{k=0}^{d-j} a_{jk} x^j y^k = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \dots + a_{dd}$, for some $d \in \mathbb{N}$ (the degree) and $a_{jk} \in \mathbb{R}$. Then we claim f is continuous. Let $\{(x_n, y_n)\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^2$ that converges to $(x, y) \in \mathbb{R}^2$. We have proved that this means that $\lim x_n = x$ and $\lim y_n = y$. So by we have $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} \sum_{j=0}^d \sum_{k=0}^{d-j} a_{jk} x_n^j y_n^k = \sum_{j=0}^d \sum_{k=0}^{d-j} a_{jk} x^j y^k = f(x, y)$. So f is continuous at (x, y) , and as (x, y) was arbitrary f is continuous everywhere. Similarly, a polynomial in n variables is continuous.

Compactness and continuity

Continuous maps do not map closed sets to closed sets. For example, $f: (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) := x$ takes the set $(0, 1)$, which is closed in $(0, 1)$, to the set $(0, 1)$, which is not closed in $\mathbb{R}$. On the other hand continuous maps do preserve compact sets.

[lemma:continuouscompact] Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \rightarrow Y$ a continuous function. If $K \subset X$ is a compact set, then $f(K)$ is a compact set.

A sequence in $f(K)$ can be written as $\{f(x_n)\}_{n=1}^\infty$, where $\{x_n\}_{n=1}^\infty$ is a sequence in K . The set K is compact and therefore there is a subsequence $\{x_{n_i}\}_{i=1}^\infty$ that converges to some $x \in K$. By continuity, $\lim_{i \rightarrow \infty} f(x_{n_i}) = f(x) \in f(K)$. So every sequence in $f(K)$ has a subsequence convergent to a point in $f(K)$, and $f(K)$ is compact by .

As before, $f: X \rightarrow \mathbb{R}$ achieves an *absolute minimum* at $c \in X$ if $f(x) \geq f(c)$ for all $x \in X$. On the other hand, f achieves an *absolute maximum* at $c \in X$ if $f(x) \leq f(c)$ for all $x \in X$.

Let (X, d) be a compact metric space and $f: X \rightarrow \mathbb{R}$ a continuous function. Then f is bounded and in fact f achieves an absolute minimum and an absolute maximum on X .

As X is compact and f is continuous, we have that $f(X) \subset \mathbb{R}$ is compact. Hence $f(X)$ is closed and bounded. In particular, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$, because both the sup and inf can be achieved by sequences in $f(X)$ and $f(X)$ is closed. Therefore there is some $x \in X$ such that $f(x) = \sup f(X)$ and some $y \in X$ such that $f(y) = \inf f(X)$.

Continuity and topology

Let us see how to define continuity in terms of the topology, that is, the open sets. We have already seen that topology determines which sequences converge, and so it is no wonder that the topology also determines continuity of functions.

[lemma:mstopocontloc] Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \rightarrow Y$ is continuous at $c \in X$ if and only if for every open neighborhood U of $f(c)$ in Y , the set $f^{-1}(U)$ contains an open neighborhood of c in X .

First suppose that f is continuous at c . Let U be an open neighborhood of $f(c)$ in Y , then $B_Y(f(c), \epsilon) \subset U$ for some $\epsilon > 0$. By continuity of f , there exists a $\delta > 0$ such that whenever x is such that $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$. In other words, $B_X(c, \delta) \subset f^{-1}(B_Y(f(c), \epsilon)) \subset f^{-1}(U)$, and $B_X(c, \delta)$ is an open neighborhood of c .

For the other direction, let $\epsilon > 0$ be given. If $f^{-1}(B_Y(f(c), \epsilon))$ contains an open neighborhood W of c , it contains a ball. That is, there is some $\delta > 0$ such that $B_X(c, \delta) \subset W \subset f^{-1}(B_Y(f(c), \epsilon))$. That means precisely that if $d_X(x, c) < \delta$ then $d_Y(f(x), f(c)) < \epsilon$, and so f is continuous at c .

[thm:mstopocont] Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \rightarrow Y$ is continuous if and only if for every open $U \subset Y$, $f^{-1}(U)$ is open in X .

The proof follows from and is left as an exercise.

Let $f: X \rightarrow Y$ be a continuous function. tells us that if $E \subset Y$ is closed, then $f^{-1}(E) = X \setminus f^{-1}(Y \setminus E)$ is also closed. Therefore if we have a continuous function $f: X \rightarrow \mathbb{R}$, then the *zero set* of f , that is, $f^{-1}(0) = \{x \in X : f(x) = 0\}$, is closed. We have just proved the most basic result in *algebraic geometry*, the study of zero sets of polynomials.

Similarly the set where f is nonnegative, that is, $f^{-1}([0, \infty)) = \{x \in X : f(x) \geq 0\}$ is closed. On the other hand the set where f is positive, $f^{-1}((0, \infty)) = \{x \in X : f(x) > 0\}$ is open.

Uniform continuity

As for continuous functions on the real line, in the definition of continuity it is sometimes convenient to be able to pick one δ for all points.

Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f: X \rightarrow Y$ is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x, c \in X$ and $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$.

A uniformly continuous function is continuous, but not necessarily vice-versa as we have seen.

[thm:Xcompactfuniformcont] Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f: X \rightarrow Y$ is continuous and X compact. Then f is uniformly continuous.

Let $\epsilon > 0$ be given. For each $c \in X$, pick $\delta_c > 0$ such that $d_Y(f(x), f(c)) < \frac{\epsilon}{2}$ whenever $d_X(x, c) < \delta_c$. The balls $B(c, \delta_c)$ cover X , and the space X is compact. Apply the to obtain a $\delta > 0$ such that for every $x \in X$, there is a $c \in X$ for which $B(x, \delta) \subset B(c, \delta_c)$.

If $x_1, x_2 \in X$ where $d_X(x_1, x_2) < \delta$, find a $c \in X$ such that $B(x_1, \delta) \subset B(c, \delta_c)$. Then $x_2 \in B(c, \delta_c)$. By the triangle inequality and the definition of δ_c we have $d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(c)) + d_Y(f(c), f(x_2)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Useful examples of uniformly continuous functions are again the so-called *Lipschitz continuous* functions. That is if (X, d_X) and (Y, d_Y) are metric spaces, then $f: X \rightarrow Y$ is called Lipschitz or *Lipschitz continuous* if there exists a $K \in \mathbb{R}$ such that $d_Y(f(x), f(c)) \leq K d_X(x, c)$ for all $x, c \in X$. It is not difficult to prove that Lipschitz implies uniformly continuous,

just take $\delta = \frac{1}{K} \epsilon$. And we already saw in the case of functions on the real line, a function can be uniformly continuous but not Lipschitz.

It is worth mentioning that, if a function is Lipschitz, it tends to be easiest to simply show it is Lipschitz even if we are only interested in knowing continuity.

Exercises

Consider $\mathbb{N} \subset \mathbb{R}$ with the standard metric. Let (X, d) be a metric space and $f: X \rightarrow \mathbb{N}$ a continuous function. a) Prove that if X is connected, then f is constant (the range of f is a single value). b) Find an example where X is disconnected and f is not constant.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(0,0) := 0$, and $f(x,y) := \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$. a) Show that for any fixed x , the function that takes y to $f(x,y)$ is continuous. Similarly for any fixed y , the function that takes x to $f(x,y)$ is continuous. b) Show that f is not continuous.

Suppose that $f: X \rightarrow Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y) . Let $A \subset X$. a) Show that $f(\overline{A}) \subset \overline{f(A)}$. b) Show that the subset can be proper.

Prove . Hint: Use .

[exercise:msconnconn] Suppose $f: X \rightarrow Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y) . Show that if X is connected, then $f(X)$ is connected.

Prove the following version of the . Let (X, d) be a connected metric space and $f: X \rightarrow \mathbb{R}$ a continuous function. Suppose that there exist $x_0, x_1 \in X$ and $y \in \mathbb{R}$ such that $f(x_0) < y < f(x_1)$. Then prove that there exists a $z \in X$ such that $f(z) = y$. Hint: See .

A continuous function $f: X \rightarrow Y$ for metric spaces (X, d_X) and (Y, d_Y) is said to be *proper* if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is compact. Suppose a continuous $f: (0,1) \rightarrow (0,1)$ is proper and $\{x_n\}$ is a sequence in $(0,1)$ that converges to 0. Show that $\{f(x_n)\}$ has no subsequence that converges in $(0,1)$.

Let (X, d_X) and (Y, d_Y) be metric space and $f: X \rightarrow Y$ be a one-to-one and onto continuous function. Suppose X is compact. Prove that the inverse $f^{-1}: Y \rightarrow X$ is continuous.

Take the metric space of continuous functions $C([0,1], \mathbb{R})$. Let $k: [0,1] \times [0,1] \rightarrow \mathbb{R}$ be a continuous function. Given $f \in C([0,1], \mathbb{R})$ define $\varphi_f(x) := \int_0^1 k(x,y) f(y) dy$. a) Show that $T(f) := \varphi_f$ defines a function $T: C([0,1], \mathbb{R}) \rightarrow C([0,1], \mathbb{R})$. b) Show that T is continuous.

Let (X, d) be a metric space.

a) If $p \in X$, show that $f: X \rightarrow \mathbb{R}$ defined by $f(x) := d(x,p)$ is continuous.

b) Define a metric on $X \times X$ as in part b, and show that $g: X \times X \rightarrow \mathbb{R}$ defined by $g(x,y) := d(x,y)$ is continuous.

c) Show that if K_1 and K_2 are compact subsets of X , then there exists a $p \in K_1$ and $q \in K_2$ such that $d(p,q)$ is minimal, that is, $d(p,q) = \inf \{d(x,y) : x \in K_1, y \in K_2\}$.

Fixed point theorem and Picard's theorem again

Note: 1 lecture (optional, does not require)

In this section we prove the fixed point theorem for contraction mappings. As an application we prove Picard's theorem, which we proved without metric spaces in . The proof we present here is similar, but the proof goes a lot smoother with metric spaces and the fixed point theorem.

Fixed point theorem

Let (X, d) and (X', d') be metric spaces. $f: X \rightarrow X'$ is said to be a *contraction* (or a contractive map) if it is a k -Lipschitz map for some $k < 1$, i.e. if there exists a $k < 1$ such that $d'(f(x), f(y)) \leq k d(x, y)$ for all $x, y \in X$.

If $f: X \rightarrow X$ is a map, $x \in X$ is called a *fixed point* if $f(x) = x$.

[Contraction mapping principle or Fixed point theorem] [thm:contr] Let (X, d) be a nonempty complete metric space and $f: X \rightarrow X$ a contraction. Then f has a unique fixed point.

The words *complete* and *contraction* are necessary. See .

Pick any $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{n+1} := f(x_n)$. $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq k d(x_n, x_{n-1}) \leq \dots \leq k^n d(x_1, x_0)$. Suppose $m > n$, then $d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) \leq k^n d(x_1, x_0) \sum_{i=0}^{m-n-1} k^i \leq k^n d(x_1, x_0) \frac{1-k}{1-k^{m-n}}$. In particular the sequence is Cauchy (why?). Since X is complete we let $x := \lim x_n$, and we claim that x is our unique fixed point.

Fixed point? The function f is continuous as it is a contraction, so Lipschitz continuous. Hence $f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x$.

Unique? Let x and y both be fixed points. $d(x, y) = d(f(x), f(y)) \leq k d(x, y)$. As $k < 1$ this means that $d(x, y) = 0$ and hence $x = y$. The theorem is proved.

The proof is constructive. Not only do we know a unique fixed point exists. We also know how to find it. We start with any point $x_0 \in X$ and simply iterate $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, etc... In fact, you can even find how far away from the fixed point you are, see the exercises. The idea of the proof is therefore used in real world applications.

Picard's theorem

Before we get to Picard, let us mention what metric space we will be applying the fixed point theorem to. We will use the metric space $C([a,b], \mathbb{R})$ of . That is, $C([a,b], \mathbb{R})$ is the space of continuous functions $f: [a,b] \rightarrow \mathbb{R}$ with the metric $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. Convergence in this metric is convergence in uniform norm, or in other words, uniform convergence. Therefore, see , $C([a,b], \mathbb{R})$ is a complete metric space.

Let us use the fixed point theorem to prove the classical Picard theorem on the existence and uniqueness of ordinary differential equations. Consider the equation $\frac{dy}{dx} = F(x,y)$. Given some x_0, y_0 we are looking for a function $y = f(x)$ such that $f(x_0) = y_0$ and such that $F(x) = F(x, f(x))$. To avoid having to come up with many names we often simply write $y' = F(x,y)$ and $y(x)$ for the solution.

The simplest example is for example the equation $y' = y$, $y(0) = 1$. The solution is the exponential $y(x) = e^x$. A somewhat more complicated example is $y' = -2xy$, $y(0) = 1$, whose solution is the Gaussian $y(x) = e^{-x^2}$.

There are some subtle issues, for example how long does the solution exist. Look at the equation $y' = y^2$, $y(0) = 1$. Then $y(x) = \frac{1}{1-x}$ is a solution. While F is a reasonably "nice" function and in particular exists for all x and y , the solution "blows up" at $x=1$. For more examples related to Picard's theorem see .

Let $I, J \subset \mathbb{R}$ be compact intervals, let I_0 and J_0 be their interiors, and let $(x_0, y_0) \in I_0 \times J_0$. Suppose $F: I \times J \rightarrow \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, there exists an $L \in \mathbb{R}$ such that $|F(x,y) - F(x,z)| \leq L |y - z|$ for all $(x,y), (x,z) \in I \times J$. Then there exists an $h > 0$ and a unique differentiable function $f: [x_0 - h, x_0 + h] \rightarrow J$ such that $F(x) = F(x, f(x))$ and $f(x_0) = y_0$.

Without loss of generality assume $x_0 = 0$. As $I \times J$ is compact and $F(x,y)$ is continuous, it is bounded. So find an $M > 0$, such that $|F(x,y)| \leq M$ for all $(x,y) \in I \times J$. Pick $\alpha > 0$ such that $[-\alpha, \alpha] \subset I$ and $[y_0 - \alpha, y_0 + \alpha] \subset J$. Let $h = \min\{\alpha, \frac{\alpha}{M+L}\}$. Note $[-h, h] \subset I$. Define the set $Y := \{f \in C([-h, h], \mathbb{R}) : f(-h) \in [y_0 - \alpha, y_0 + \alpha]\}$. That is, Y is the space of continuous functions on $[-h, h]$ with values in J , in other words, exactly those functions where $F(x, f(x))$ makes sense. The metric used is the standard metric given above.

Show that $Y \subset C([-h, h], \mathbb{R})$ is closed. Hint: J is closed.

The space $C([-h, h], \mathbb{R})$ is complete, and a closed subset of a complete metric space is a complete metric space with the subspace metric, see . So Y with the subspace metric is complete.

Define a mapping $T: Y \rightarrow C([-h, h], \mathbb{R})$ by $T(f)(x) := y_0 + \int_0^x F(t, f(t)) dt$.

Show that if $f: [-h, h] \rightarrow J$ is continuous then $F(t, f(t))$ is continuous on $[-h, h]$ as a function of t . Use this to show that T is well defined and that $T(f) \in C([-h, h], \mathbb{R})$.

Let $f \in Y$ and $\|f\| \leq h$. As F is bounded by M we have $\|T(f) - y_0\| \leq \int_0^h |F(t, f(t))| dt \leq hM \leq \frac{\alpha}{2}$. So $T(f) \in Y$. In other words, $T(Y) \subset Y$.

Processing math: 52% T as a mapping of Y to Y .

We claim $T: \text{Colon } Y \rightarrow Y$ is a contraction. First, for $x \in [-h, h]$ and $f, g \in Y$ we have $\left| F(x, f(x)) - F(x, g(x)) \right| \leq L \left| f(x) - g(x) \right|$. Therefore, $\left| T(f)(x) - T(g)(x) \right| \leq \left| \int_0^x F(t, f(t)) - F(t, g(t)) dt \right| \leq \int_0^x L \left| f(t) - g(t) \right| dt \leq L \int_0^x \frac{1}{M+L} dt \leq \frac{1}{M+L} \int_0^x 1 dt \leq \frac{1}{M+L} x \leq \frac{1}{M+L} h$. We chose $M > 0$ and so $\frac{1}{M+L} h < 1$. The claim is proved by taking supremum over $x \in [-h, h]$ of the left hand side above to obtain $\|T(f) - T(g)\| \leq \frac{1}{M+L} \|f - g\|$.

We apply the fixed point theorem () to find a unique $f \in Y$ such that $T(f) = f$, that is, $f(x) = y_0 + \int_0^x F(t, f(t)) dt$. By the fundamental theorem of calculus, $T(f)$ is the unique differentiable function whose derivative is $F(x, f(x))$ and $T(f)(0) = y_0$. Therefore f is the unique solution of $f'(x) = F(x, f(x))$ and $f(0) = y_0$.

Prove that the statement “Without loss of generality assume $x_0 = 0$ ” is justified. That is, prove that if we know the theorem with $x_0 = 0$, the theorem is true as stated.

Exercises

For more exercises related to Picard’s theorem see .

Let $F: \text{Colon } \mathbb{R} \rightarrow \mathbb{R}$ be defined by $F(x) := kx + b$ where $0 < k < 1$, $b \in \mathbb{R}$.

- Show that F is a contraction.
- Find the fixed point and show directly that it is unique.

Let $f: [0, \frac{1}{4}] \rightarrow [0, \frac{1}{4}]$ be defined by $f(x) := x^2$ is a contraction.

- Show that f is a contraction, and find the best (smallest) k from the definition that works.
- Find the fixed point and show directly that it is unique.

[exercise:nofixedpoint] a) Find an example of a contraction $f: X \rightarrow X$ of non-complete metric space X with no fixed point. b) Find a 1-Lipschitz map $f: X \rightarrow X$ of a complete metric space X with no fixed point.

Consider $y' = y^2$, $y(0) = 1$. Use the iteration scheme from the proof of the contraction mapping principle. Start with $f_0(x) = 1$. Find a few iterates (at least up to f_2). Prove that the pointwise limit of f_n is $\frac{1}{1-x}$, that is for every x with $|x| < 1$ for some $h > 0$, prove that $\lim_{n \rightarrow \infty} f_n(x) = \frac{1}{1-x}$.

Suppose $f: X \rightarrow X$ is a contraction for $k < 1$. Suppose you use the iteration procedure with $x_{n+1} := f(x_n)$ as in the proof of the fixed point theorem. Suppose x is the fixed point of f .

- Show that $d(x, x_n) \leq k^n d(x, x_0)$ for all $n \in \mathbb{N}$.
- Suppose $d(y_1, y_2) \leq 16$ for all $y_1, y_2 \in X$, and $k = \frac{1}{2}$. Find an N such that starting at any point $x_0 \in X$, $d(x, x_n) \leq 2^{-n}$ for all $n \geq N$.

Let $f(x) := x - \frac{x^2-2}{2x}$. (You may recognize Newton’s method for $\sqrt{2}$)

- Prove $f(\mathbb{R} \setminus \{0\}) \subset \mathbb{R} \setminus \{0\}$.
- Prove that $f: [1, \infty) \rightarrow [1, \infty)$ is a contraction.
- Apply the fixed point theorem to find an $x \geq 1$ such that $f(x) = x$, and show that $x = \sqrt{2}$.

Suppose $f: X \rightarrow X$ is a contraction, and (X, d) is a metric space with the discrete metric, that is $d(x, y) = 1$ whenever $x \neq y$. Show that f is constant, that is, there exists a $c \in X$ such that $f(x) = c$ for all $x \in X$.

- The term “modern” refers to late 19th century up to the present.
- For the fans of the TV show *Futurama*, there is a movie theater in one episode called an $\aleph_0$ -plex.
- An algebraist would say that $\mathbb{Z}$ is an ordered ring, or perhaps more precisely a commutative ordered ring.
- Uniqueness is up to isomorphism, but we wish to avoid excessive use of algebra. For us, it is simply enough to assume that a set of real numbers exists. See Rudin for the construction and more details.
- Named after the Swiss mathematician **Jacob Bernoulli** (1655 – 1705).
- The boundedness hypothesis is for simplicity, it can be dropped if we allow for the extended real numbers.
- use the notation (x_n) to denote a sequence instead of $\{x_n\}$, which is what uses. Both are common.
- Named after the English physicist and mathematician **Isaac Newton** (1642 – 1726/7).
- Named after the Czech mathematician **Bernhard Placidor Johann Nepomuk Bolzano** (1781 – 1848), and the German mathematician **Karl Theodor Wilhelm Weierstrass** (1815 – 1897).
- Sometimes it is said that (x_n) converges to infinity.
- Named after the French mathematician **Augustin-Louis Cauchy** (1789–1857).
- The divergence of the harmonic series was known before the theory of series was made rigorous. In fact the proof we give is the earliest proof and was given by **Nicole Oresme** (1323?–1382).
- Demonstration of this fact is what made the Swiss mathematician Leonhard Paul Euler (1707 – 1783) famous.
- Named for the Italian mathematician **Ernesto Cesàro** (1859 – 1906).
- Named after the German mathematician **Johannes Karl Thoma** (1840 – 1921).
- Named after the German mathematician **Rudolf Otto Sigismund Lipschitz** (1832–1903).
- Named for the German mathematician Gottfried Wilhelm Leibniz (1646–1716).
- Named after the French mathematician Michel Rolle (1652–1719).
- Named for the English mathematician Brook Taylor (1685–1731). It was first found by the Scottish mathematician James Gregory (1638 – 1675). The statement we give was proved by Joseph-Louis Lagrange (1736 – 1813).
- Named after the German mathematician Georg Friedrich Bernhard Riemann (1826–1866).
- Named after the French mathematician Jean-Gaston Darboux (1842–1917).
- Such an h is said to be of *bounded variation*.
- Compare this hypothesis to .
- Named for the Swiss mathematician **Leonhard Paul Euler** (1707 – 1783) and the Italian mathematician Lorenzo Mascheroni (1750 – 1800).
- Shortened from Latin: *sinus cardinalis*.
- The notation nor terminology is not completely standardized. The norm is also called the *sup norm* or *infinity norm*, and in addition to $\|f\|_\infty$ and $\|f\|_{\infty, S}$ it is sometimes written as $\|f\|_{\infty}$ or $\|f\|_{\infty, S}$.
- Named for the French mathematician Charles Émile Picard (1856–1941).

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2.2: The set of real numbers

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2.3: Absolute value

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2.4: Intervals and the size of $\mathbb{R}$

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2.5: Decimal representation of the reals

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CHAPTER OVERVIEW

3: Sequences and Series

Topic hierarchy

- [3.1: Sequences and Limits](#)
- [3.2: Facts about limits of sequences](#)
- [3.3: Limit superior, limit inferior, and Bolzano-Weierstrass](#)
- [3.4: Cauchy sequences](#)
- [3.5: Series](#)
- [3.6: More on Series](#)

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3.1: Sequences and Limits

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3.2: Facts about limits of sequences

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3.3: Limit superior, limit inferior, and Bolzano-Weierstrass

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3.4: Cauchy sequences

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3.5: Series

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3.6: More on Series

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CHAPTER OVERVIEW

4: Continuous Functions

Topic hierarchy

- [4.1: Limits of functions](#)
- [4.2: Continuous Functions](#)
- [4.3: Min-max and Intermediate Value Theorems](#)
- [4.4: Uniform Continuity](#)
- [4.5: Limits at Infinity](#)
- [4.6: Monotone Functions and Continuity](#)

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4.1: Limits of functions

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4.2: Continuous Functions

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4.3: Min-max and Intermediate Value Theorems

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4.4: Uniform Continuity

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4.5: Limits at Infinity

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4.6: Monotone Functions and Continuity

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CHAPTER OVERVIEW

5: The Derivative

Topic hierarchy

[5.1: The Derivative](#)

[5.2: Mean Value Theorem](#)

[5.3: Taylor's Theorem](#)

[5.4: Inverse Function Theorem](#)

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5.1: The Derivative

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5.2: Mean Value Theorem

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5.3: Taylor's Theorem

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5.4: Inverse Function Theorem

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CHAPTER OVERVIEW

6: The Riemann Integral

Topic hierarchy

- [6.1: The Riemann integral](#)
- [6.2: Properties of the Integral](#)
- [6.3: Fundamental Theorem of Calculus](#)
- [6.4: The Logarithm and the Exponential](#)
- [6.5: Improper Integrals](#)
- [6.6: temp](#)

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6.1: The Riemann integral

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6.2: Properties of the Integral

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6.3: Fundamental Theorem of Calculus

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6.4: The Logarithm and the Exponential

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6.5: Improper Integrals

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Riemann-Stieltjes integral

FIXME: we'd need to redo a bunch of things from Riemann integral. Perhaps useful, but those are missing below and sort of make this more and more out of scope of the book.

A common useful generalization of the Riemann integral is the Riemann-Stieltjes integral¹. If we think of the Riemann integral as a sum where all terms are weighted equally, it is natural that we may want to do a weighed sum. That is, we may wish to give some points "more weight" than to other points. A particular simple example of what we might want to accomplish is an integral which evaluates a function at a point. You may have seen this concept in your calculus class as the delta function.

We will again define this integral using the Darboux approach for simplicity.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function and let $\alpha: [a, b] \rightarrow \mathbb{R}$ be a monotone increasing function. Let P be a partition of $[a, b]$, then define

$$\begin{aligned} m_i &:= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &:= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ L(P, f, \alpha) &:= \sum_{i=1}^n m_i (\alpha(x_i) - \alpha(x_{i-1})), \\ U(P, f, \alpha) &:= \sum_{i=1}^n M_i (\alpha(x_i) - \alpha(x_{i-1})). \end{aligned}$$

We call $L(P, f, \alpha)$ the and $U(P, f, \alpha)$ the . Then define

$$\begin{aligned} \int_a^b f d\alpha &:= \sup\{L(P, f, \alpha) : P \text{ a partition of } [a, b]\}, \\ \overline{\int_a^b f d\alpha} &:= \inf\{U(P, f, \alpha) : P \text{ a partition of } [a, b]\}. \end{aligned}$$

And we call $\underline{\int}$ the and $\overline{\int}$ the . Finally, if

$$\int_a^b f d\alpha = \overline{\int_a^b f d\alpha}. \quad (6.6.1)$$

Then we say that f is with respect to α .

When we need to specify the variable of integration we may write

$$\int_a^b f(x) d\alpha(x). \quad (6.6.2)$$

When we set $\alpha(x) := x$ we recover the Riemann integral. The notation $d\alpha$ suggests derivative, in this case $\alpha'(x) = 1$ and as we said, the Riemann integral is when all points are weighted equally.

If $\alpha(x) := x$, then a bounded function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if it is Riemann-Stieltjes integrable with respect to α . In this case

$$\int_a^b f = \int_a^b f d\alpha. \quad (6.6.3)$$

Simply plug in $\alpha(x) = x$ into the definition and note that the definition is now precisely the same as for the Riemann integral.

Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is continuous. Given $c \in (a, b)$, let

$$\alpha(x) := \begin{cases} 1 & \text{if } x \geq c, \\ 0 & \text{if } x < c. \end{cases} \quad (6.6.4)$$

We claim that f is Riemann-Stieltjes differentiable with respect to α and that

$$\int_a^b f d\alpha = f(c). \quad (6.6.5)$$

Proof: Given $\epsilon > 0$ take $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in [a, b]$ with $|x - c| < \delta$. Take the partition $P = \{a, c - \delta, c + \delta, b\}$. Then

$$\begin{aligned} L(P, f, \alpha) &= m_1(\alpha(c - \delta) - \alpha(a)) + m_2(\alpha(c + \delta) - \alpha(c - \delta)) + m_3(\alpha(b) - \alpha(c + \delta)) \\ &= m_2(1 - 0) = m_2 = \inf\{f(x) : x \in [c - \delta, c + \delta]\} \\ &> f(c) - \epsilon. \end{aligned}$$

Similarly $U(P, f, \alpha) < f(c) + \epsilon$. Therefore

$$U(P, f, \alpha) - L(P, f, \alpha) < 2\epsilon. \quad (6.6.6)$$

The notion of integrability really does depend on α . For a very trivial example, it is not difficult to see that if $\alpha(x) = 0$, then all bounded functions f on $[a, b]$ are integrable with respect to this α and

$$\int_a^b f d\alpha = 0. \quad (6.6.7)$$

If α is very nice, we can recover the Riemann-Stieltjes integral using the Riemann integral.

Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable and $\alpha: [a, b] \rightarrow \mathbb{R}$ is a continuously differentiable increasing function. Then f is Riemann-Stieltjes integrable with respect to α and

$$\int_a^b f(x) d\alpha(x) = \int_a^b f(x) \alpha'(x) dx. \quad (6.6.8)$$

FIXME

Exercises

Directly from the definition of the Riemann-Stieltjes integral prove that if $\alpha(x) = px$ for some $p \geq 0$, then If f is Riemann integrable, then it is Riemann-Stieltjes integrable with respect to α and $p \int_a^b f = \int_a^b f d\alpha$.

Let $\alpha: [a, b] \rightarrow \mathbb{R}$ and $\beta: [a, b] \rightarrow \mathbb{R}$ be increasing functions and suppose that $\alpha(x) = \beta(x) + C$ for some constant C . If $f: [a, b] \rightarrow \mathbb{R}$ is integrable with respect to α , show that it is integrable with respect to β and $\int_a^b f d\alpha = \int_a^b f d\beta$.

1. Named for ...↩

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CHAPTER OVERVIEW

7: Sequences of Functions

Topic hierarchy

[7.1: Pointwise and Uniform Convergence](#)

[7.2: Interchange of Limits](#)

[7.3: Picard's theorem](#)

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7.1: Pointwise and Uniform Convergence

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7.2: Interchange of Limits

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7.3: Picard's theorem

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CHAPTER OVERVIEW

8: Metric Spaces

Topic hierarchy

- 8.1: Metric Spaces
- 8.2: Open and Closed Sets
- 8.3: Sequences and Convergence
- 8.4: Completeness and Compactness
- 8.5: Continuous Functions
- 8.6: Fixed point theorem and Picard's theorem again

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8.1: Metric Spaces

As mentioned in the introduction, the main idea in analysis is to take limits. In we learned to take limits of sequences of real numbers. And in we learned to take limits of functions as a real number approached some other real number.

We want to take limits in more complicated contexts. For example, we might want to have sequences of points in 3-dimensional space. Or perhaps we wish to define continuous functions of several variables. We might even want to define functions on spaces that are a little harder to describe, such as the surface of the earth. We still want to talk about limits there.

Finally, we have seen the limit of a sequence of functions in . We wish to unify all these notions so that we do not have to reprove theorems over and over again in each context. The concept of a metric space is an elementary yet powerful tool in analysis. And while it is not sufficient to describe every type of limit we can find in modern analysis, it gets us very far indeed.

Definition: Metric Space

Let X be a set and let $d: X \times X \rightarrow \mathbb{R}$ be a function such that

- i. [metric:pos] $d(x, y) \geq 0$ for all x, y in X ,
- ii. [metric:zero] $d(x, y) = 0$ if and only if $x = y$,
- iii. [metric:com] $d(x, y) = d(y, x)$,
- iv. [metric:triang] $d(x, z) = d(x, y) + d(y, z)$ (*triangle inequality*).

Then the pair (X, d) is called a *metric space*. The function d is called the *metric* or sometimes the *distance function*. Sometimes we just say X is a metric space if the metric is clear from context.

The geometric idea is that d is the distance between two points. Items [metric:pos]–[metric:com] have obvious geometric interpretation: distance is always nonnegative, the only point that is distance 0 away from x is x itself, and finally that the distance from x to y is the same as the distance from y to x . The triangle inequality [metric:triang] has the interpretation given in

For the purposes of drawing, it is convenient to draw figures and diagrams in the plane and have the metric be the standard distance. However, that is only one particular metric space. Just because a certain fact seems to be clear from drawing a picture does not mean it is true. You might be getting sidetracked by intuition from euclidean geometry, whereas the concept of a metric space is a lot more general.

Let us give some examples of metric spaces.

The set of real numbers $\mathbb{R}$ is a metric space with the metric

$$d(x, y) := |x - y|. \quad (8.1.1)$$

Items [metric:pos]–[metric:com] of the definition are easy to verify. The triangle inequality [metric:triang] follows immediately from the standard triangle inequality for real numbers:

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z). \quad (8.1.2)$$

This metric is the *standard metric on $\mathbb{R}$* . If we talk about $\mathbb{R}$ as a metric space without mentioning a specific metric, we mean this particular metric.

We can also put a different metric on the set of real numbers. For example take the set of real numbers $\mathbb{R}$ together with the metric

$$d(x, y) := \frac{|x - y|}{|x - y| + 1}. \quad (8.1.3)$$

Items [metric:pos]–[metric:com] are again easy to verify. The triangle inequality [metric:triang] is a little bit more difficult. Note that $d(x, y) = \varphi(|x - y|)$ where $\varphi(t) = \frac{t}{t+1}$ and note that φ is an increasing function (positive derivative) hence

$$\begin{aligned} d(x, z) &= \varphi(|x - z|) = \varphi(|x - y + y - z|) \leq \varphi(|x - y| + |y - z|) \\ &= \frac{|x - y| + |y - z|}{|x - y| + |y - z| + 1} = \frac{|x - y|}{|x - y| + |y - z| + 1} + \frac{|y - z|}{|x - y| + |y - z| + 1} \\ &\leq \frac{|x - y|}{|x - y| + 1} + \frac{|y - z|}{|y - z| + 1} = d(x, y) + d(y, z). \end{aligned}$$

Here we have an example of a nonstandard metric on $\mathbb{R}$. With this metric we can see for example that $d(x, y) < 1$ for all $x, y \in \mathbb{R}$. That is, any two points are less than 1 unit apart.

An important metric space is the n -dimensional *euclidean space* $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. We use the following notation for points: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We also simply write $0 \in \mathbb{R}^n$ to mean the vector $(0, 0, \dots, 0)$. Before making $\mathbb{R}^n$ a metric space, let us prove an important inequality, the so-called Cauchy-Schwarz inequality.

Take $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Then

$$\left(\sum_{j=1}^n x_j y_j \right)^2 \leq \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{j=1}^n y_j^2 \right). \quad (8.1.4)$$

Any square of a real number is nonnegative. Hence any sum of squares is nonnegative:

$$\begin{aligned} 0 &\leq \sum_{j=1}^n \sum_{k=1}^n (x_j y_k - x_k y_j)^2 \\ &= \sum_{j=1}^n \sum_{k=1}^n (x_j^2 y_k^2 + x_k^2 y_j^2 - 2x_j x_k y_j y_k) \\ &= \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{k=1}^n y_k^2 \right) + \left(\sum_{j=1}^n y_j^2 \right) \left(\sum_{k=1}^n x_k^2 \right) - 2 \left(\sum_{j=1}^n x_j y_j \right) \left(\sum_{k=1}^n x_k y_k \right) \end{aligned}$$

We relabel and divide by 2 to obtain

$$0 \leq \left(\sum_{j=1}^n x_j^2 \right) \left(\sum_{j=1}^n y_j^2 \right) - \left(\sum_{j=1}^n x_j y_j \right)^2, \quad (8.1.5)$$

which is precisely what we wanted.

Let us construct standard metric for $\mathbb{R}^n$. Define

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}. \quad (8.1.6)$$

For $n = 1$, the real line, this metric agrees with what we did above. Again, the only tricky part of the definition to check is the triangle inequality. It is less messy to work with the square of the metric. In the following, note the use of the Cauchy-Schwarz inequality.

$$\begin{aligned} d(x, z)^2 &= \sum_{j=1}^n (x_j - z_j)^2 \\ &= \sum_{j=1}^n (x_j - y_j + y_j - z_j)^2 \\ &= \sum_{j=1}^n \left((x_j - y_j)^2 + (y_j - z_j)^2 + 2(x_j - y_j)(y_j - z_j) \right) \\ &= \sum_{j=1}^n (x_j - y_j)^2 + \sum_{j=1}^n (y_j - z_j)^2 + \sum_{j=1}^n 2(x_j - y_j)(y_j - z_j) \\ &\leq \sum_{j=1}^n (x_j - y_j)^2 + \sum_{j=1}^n (y_j - z_j)^2 + 2 \sqrt{\sum_{j=1}^n (x_j - y_j)^2} \sqrt{\sum_{j=1}^n (y_j - z_j)^2} \\ &= \left(\sqrt{\sum_{j=1}^n (x_j - y_j)^2} + \sqrt{\sum_{j=1}^n (y_j - z_j)^2} \right)^2 = (d(x, y) + d(y, z))^2. \end{aligned}$$

Taking the square root of both sides we obtain the correct inequality.

An example to keep in mind is the so-called *discrete metric*. Let X be any set and define

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases} \quad (8.1.7)$$

That is, all points are equally distant from each other. When X is a finite set, we can draw a diagram, see for example . Things become subtle when X is an infinite set such as the real numbers.

While this particular example seldom comes up in practice, it gives a useful “smell test.” If you make a statement about metric spaces, try it with the discrete metric. To show that (X, d) is indeed a metric space is left as an exercise.

[example:msC01] Let $C([a, b])$ be the set of continuous real-valued functions on the interval $[a, b]$. Define the metric on $C([a, b])$ as

$$d(f, g) := \sup_{x \in [a, b]} |f(x) - g(x)|. \quad (8.1.8)$$

Let us check the properties. First, $d(f, g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a, b]$, and so is bounded. It is clear that $d(f, g) \geq 0$, it is the supremum of nonnegative numbers. If $f = g$ then $|f(x) - g(x)| = 0$ for all x and hence $d(f, g) = 0$. Conversely if $d(f, g) = 0$, then for any x we have $|f(x) - g(x)| \leq d(f, g) = 0$ and hence $f(x) = g(x)$ for all x and $f = g$. That $d(f, g) = d(g, f)$ is equally trivial. To show the triangle inequality we use the standard triangle inequality.

$$\begin{aligned} d(f, h) &= \sup_{x \in [a, b]} |f(x) - g(x)| = \sup_{x \in [a, b]} |f(x) - h(x) + h(x) - g(x)| \\ &\leq \sup_{x \in [a, b]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sup_{x \in [a, b]} |f(x) - h(x)| + \sup_{x \in [a, b]} |h(x) - g(x)| = d(f, h) + d(h, g). \end{aligned}$$

When treat $C([a, b])$ as a metric space without mentioning a metric, we mean this particular metric.

This example may seem esoteric at first, but it turns out that working with spaces such as $C([a, b])$ is really the meat of a large part of modern analysis. Treating sets of functions as metric spaces allows us to abstract away a lot of the grubby detail and prove powerful results such as Picard’s theorem with less work.

Oftentimes it is useful to consider a subset of a larger metric space as a metric space. We obtain the following proposition, which has a trivial proof.

Let (X, d) be a metric space and $Y \subset X$, then the restriction $d|_{Y \times Y}$ is a metric on Y .

If (X, d) is a metric space, $Y \subset X$, and $d' := d|_{Y \times Y}$, then (Y, d') is said to be a *subspace* of (X, d) .

It is common to simply write d for the metric on Y , as it is the restriction of the metric on X . Sometimes we will say that d' is the *subspace metric* and that Y has the *subspace topology*.

A subset of the real numbers is bounded whenever all its elements are at most some fixed distance from 0. We can also define bounded sets in a metric space. When dealing with an arbitrary metric space there may not be some natural fixed point 0. For the purposes of boundedness it does not matter.

Let (X, d) be a metric space. A subset $S \subset X$ is said to be *bounded* if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S. \quad (8.1.9)$$

We say that (X, d) is bounded if X itself is a bounded subset.

For example, the set of real numbers with the standard metric is not a bounded metric space. It is not hard to see that a subset of the real numbers is bounded in the sense of if and only if it is bounded as a subset of the metric space of real numbers with the standard metric.

On the other hand, if we take the real numbers with the discrete metric, then we obtain a bounded metric space. In fact, any set with the discrete metric is bounded.

Exercises

Show that for any set X , the discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$) does give a metric space (X, d) .

Let $X := \{0\}$ be a set. Can you make it into a metric space?

Let $X := \{a, b\}$ be a set. Can you make it into two distinct metric spaces? (define two distinct metrics on it)

Let the set $X := \{A, B, C\}$ represent 3 buildings on campus. Suppose we wish to our distance to be the time it takes to walk from one building to the other. It takes 5 minutes either way between buildings A and B . However, building C is on a hill and it takes 10 minutes from A and 15 minutes from B to get to C . On the other hand it takes 5 minutes to go from C to A and 7 minutes to go from C to B , as we are going downhill. Do these distances define a metric? If so, prove it, if not say why not.

Suppose that (X, d) is a metric space and $\varphi: [0, \infty] \rightarrow \mathbb{R}$ is an increasing function such that $\varphi(t) \geq 0$ for all t and $\varphi(t) = 0$ if and only if $t = 0$. Also suppose that φ is *subadditive*, that is $\varphi(s+t) \leq \varphi(s) + \varphi(t)$. Show that with $d'(x, y) := \varphi(d(x, y))$, we obtain a new metric space (X, d') .

Let (X, d_X) and (Y, d_Y) be metric spaces.

a) Show that $(X \times Y, d)$ with $d((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2)$ is a metric space.

b) Show that $(X \times Y, d)$ with $d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ is a metric space.

Let X be the set of continuous functions on $[0, 1]$. Let $\varphi: [0, 1] \rightarrow (0, \infty)$ be continuous. Define

$$d(f, g) := \int_0^1 |f(x) - g(x)| \varphi(x) dx. \quad (8.1.10)$$

Show that (X, d) is a metric space.

Let (X, d) be a metric space. For nonempty bounded subsets A and B let

$$d(x, B) := \inf\{d(x, b) : b \in B\} \quad \text{and} \quad d(A, B) := \sup\{d(a, B) : a \in A\}. \quad (8.1.11)$$

Now define the *Hausdorff metric* as

$$d_H(A, B) := \max\{d(A, B), d(B, A)\}. \quad (8.1.12)$$

Note: d_H can be defined for arbitrary nonempty subsets if we allow the extended reals.

a) Let $Y \subset \mathcal{P}(X)$ be the set of bounded nonempty subsets. Show that (Y, d_H) is a metric space. b) Show by example that d itself is not a metric. That is, d is not always symmetric.

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8.2: Open and Closed Sets

Topology

It is useful to define a so-called *topology*. That is we define closed and open sets in a metric space. Before doing so, let us define two special sets.

Let (X, d) be a metric space, $x \in X$ and $\delta > 0$. Then define the *open ball* or simply *ball* of radius δ around x as

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}. \quad (8.2.1)$$

Similarly we define the *closed ball* as

$$C(x, \delta) := \{y \in X : d(x, y) \leq \delta\}. \quad (8.2.2)$$

When we are dealing with different metric spaces, it is sometimes convenient to emphasize which metric space the ball is in. We do this by writing $B_X(x, \delta) := B(x, \delta)$ or $C_X(x, \delta) := C(x, \delta)$.

Take the metric space $\mathbb{R}$ with the standard metric. For $x \in \mathbb{R}$, and $\delta > 0$ we get

$$B(x, \delta) = (x - \delta, x + \delta) \quad \text{and} \quad C(x, \delta) = [x - \delta, x + \delta]. \quad (8.2.3)$$

Be careful when working on a subspace. Suppose we take the metric space $[0, 1]$ as a subspace of $\mathbb{R}$. Then in $[0, 1]$ we get

$$B(0, \frac{1}{2}) = B_{[0,1]}(0, \frac{1}{2}) = [0, \frac{1}{2}). \quad (8.2.4)$$

This is of course different from $B_{\mathbb{R}}(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2})$.

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$B(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2})$. The important thing to keep in mind is which metric space we are working in.

Let (X, d) be a metric space. A set $V \subset X$ is *open* if for every $x \in V$, there exists a $\delta > 0$ such that $B(x, \delta) \subset V$. See . A set $E \subset X$ is *closed* if the complement $E^c = X \setminus E$ is open. When the ambient space X is not clear from context we say V is open in X and E is closed in X .

If $x \in V$ and V is open, then we say that V is an *open neighborhood* of x (or sometimes just *neighborhood*).

Intuitively, an open set is a set that does not include its “boundary.” Note that not every set is either open or closed, in fact generally most subsets are neither.

The set $[0, 1) \subset \mathbb{R}$ is neither open nor closed. First, every ball in $\mathbb{R}$ around 0, $(-\delta, \delta)$ contains negative numbers and hence is not contained in $[0, 1)$ and so $[0, 1)$ is not open. Second, every ball in $\mathbb{R}$ around 1, $(1 - \delta, 1 + \delta)$ contains numbers strictly less than 1 and greater than 0 (e.g. $1 - \frac{\delta}{2}$ as long as $\delta < 2$). Thus $\mathbb{R} \setminus [0, 1)$ is not open, and so $[0, 1)$ is not closed.

[prop:topology:open] Let (X, d) be a metric space.

- i. [topology:openi] $\emptyset$ and X are open in X .
- ii. [topology:openii] If $V_1, V_2, \dots, V_k$ are open then

$$\bigcap_{j=1}^k V_j \quad (8.2.5)$$

is also open. That is, finite intersection of open sets is open.

- iii. [topology:openiii] If $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of open sets, then

$$\bigcup_{\lambda \in I} V_\lambda \quad (8.2.6)$$

is also open. That is, union of open sets is open.

Note that the index set in [topology:openiii] is arbitrarily large. By $\bigcup_{\lambda \in I} V_\lambda$ we simply mean the set of all x such that $x \in V_\lambda$ for at least one $\lambda \in I$.

The set X and $\emptyset$ are obviously open in X .

Let us prove [topology:openii]. If $x \in \bigcap_{j=1}^k V_j$, then $x \in V_j$ for all j . As V_j are all open, there exists a $\delta_j > 0$ for every j such that $B(x, \delta_j) \subset V_j$. Take $\delta := \min\{\delta_1, \dots, \delta_k\}$ and note that $\delta > 0$. We have $B(x, \delta) \subset B(x, \delta_j) \subset V_j$ for every j and thus $B(x, \delta) \subset \bigcap_{j=1}^k V_j$. Thus the intersection is open.

Let us prove [topology:openiii]. If $x \in \bigcup_{\lambda \in I} V_\lambda$, then $x \in V_\lambda$ for some $\lambda \in I$. As V_λ is open then there exists a $\delta > 0$ such that $B(x, \delta) \subset V_\lambda$. But then $B(x, \delta) \subset \bigcup_{\lambda \in I} V_\lambda$ and so the union is open.

The main thing to notice is the difference between items [topology:openii] and [topology:openiii]. Item [topology:openii] is not true for an arbitrary intersection, for example $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open.

The proof of the following analogous proposition for closed sets is left as an exercise.

[prop:topology:closed] Let (X, d) be a metric space.

- i. [topology:closedi] $\emptyset$ and X are closed in X .
- ii. [topology:closedii] If $\{E_\lambda\}_{\lambda \in I}$ is an arbitrary collection of closed sets, then

$$\bigcap_{\lambda \in I} E_\lambda \quad (8.2.7)$$

is also closed. That is, intersection of closed sets is closed.

- iii. [topology:closediii] If $E_1, E_2, \dots, E_k$ are closed then

$$\bigcup_{j=1}^k E_j \quad (8.2.8)$$

is also closed. That is, finite union of closed sets is closed.

We have not yet shown that the open ball is open and the closed ball is closed. Let us show this fact now to justify the terminology.

[prop:topology:ballsopenclosed] Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x, \delta)$ is open and $C(x, \delta)$ is closed.

Let $y \in B(x, \delta)$. Let $\alpha := \delta - d(x, y)$. Of course $\alpha > 0$. Now let $z \in B(y, \alpha)$. Then

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \alpha = d(x, y) + \delta - d(x, y) = \delta. \quad (8.2.9)$$

Therefore $z \in B(x, \delta)$ for every $z \in B(y, \alpha)$. So $B(y, \alpha) \subset B(x, \delta)$ and $B(x, \delta)$ is open.

The proof that $C(x, \delta)$ is closed is left as an exercise.

Again be careful about what is the ambient metric space. As $[0, \frac{1}{2})$ is an open ball in $[0, 1]$, this means that $[0, \frac{1}{2})$ is an open set in $[0, 1]$. On the other hand $[0, \frac{1}{2})$ is neither open nor closed in $\mathbb{R}$.

A useful way to think about an open set is a union of open balls. If U is open, then for each $x \in U$, there is a $\delta_x > 0$ (depending on x of course) such that $B(x, \delta_x) \subset U$. Then $U = \bigcup_{x \in U} B(x, \delta_x)$.

The proof of the following proposition is left as an exercise. Note that there are other open and closed sets in $\mathbb{R}$.

[prop:topology:intervals:openclosed] Let $a < b$ be two real numbers. Then (a, b) , (a, ∞) , and $(-\infty, b)$ are open in $\mathbb{R}$. Also $[a, b]$, $[a, \infty)$, and $(-\infty, b]$ are closed in $\mathbb{R}$.

Connected sets

A nonempty metric space (X, d) is *connected* if the only subsets that are both open and closed are $\emptyset$ and X itself.

When we apply the term *connected* to a nonempty subset $A \subset X$, we simply mean that A with the subspace topology is connected.

In other words, a nonempty X is connected if whenever we write $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are open, then either $X_1 = \emptyset$ or $X_2 = \emptyset$. So to test for disconnectedness, we need to find nonempty disjoint open sets X_1 and X_2 whose union is X . For subsets, we state this idea as a proposition.

Let (X, d) be a metric space. A nonempty set $S \subset X$ is not connected if and only if there exist open sets U_1 and U_2 in X , such that $U_1 \cap U_2 \cap S = \emptyset$, $U_1 \cap S \neq \emptyset$, $U_2 \cap S \neq \emptyset$, and

$$S = (U_1 \cap S) \cup (U_2 \cap S). \quad (8.2.10)$$

If U_j is open in X , then $U_j \cap S$ is open in S in the subspace topology (with subspace metric). To see this, note that if $B_X(x, \delta) \subset U_j$, then as $B_S(x, \delta) = S \cap B_X(x, \delta)$, we have $B_S(x, \delta) \subset U_j \cap S$. The proof follows by the above discussion.

The proof of the other direction follows by using to find U_1 and U_2 from two open disjoint subsets of S .

Let $S \subset \mathbb{R}$ be such that $x < z < y$ with $x, y \in S$ and $z \notin S$. Claim: S is not connected. Proof: Notice

$$((-\infty, z) \cap S) \cup ((z, \infty) \cap S) = S. \quad (8.2.11)$$

A set $S \subset \mathbb{R}$ is connected if and only if it is an interval or a single point.

Suppose that S is connected (so also nonempty). If S is a single point then we are done. So suppose that $x < y$ and $x, y \in S$. If z is such that $x < z < y$, then $(-\infty, z) \cap S$ is nonempty and $(z, \infty) \cap S$ is nonempty. The two sets are disjoint. As S is connected, we must have their union is not S , so $z \in S$.

Suppose that S is bounded, connected, but not a single point. Let $\alpha := \inf S$ and $\beta := \sup S$ and note that $\alpha < \beta$. Suppose $\alpha < z < \beta$. As α is the infimum, then there is an $x \in S$ such that $\alpha \leq x < z$. Similarly there is a $y \in S$ such that $\beta \geq y > z$. We have shown above that $z \in S$, so $(\alpha, \beta) \subset S$. If $w < \alpha$, then $w \notin S$ as α was the infimum, similarly if $w > \beta$ then $w \notin S$. Therefore the only possibilities for S are (α, β) , $[\alpha, \beta)$, $(\alpha, \beta]$, $[\alpha, \beta]$.

The proof that an unbounded connected S is an interval is left as an exercise.

On the other hand suppose that S is an interval. Suppose that U_1 and U_2 are open subsets of $\mathbb{R}$, $U_1 \cap S$ and $U_2 \cap S$ are nonempty, and $S = (U_1 \cap S) \cup (U_2 \cap S)$. We will show that $U_1 \cap S$ and $U_2 \cap S$ contain a common point, so they are not disjoint, and hence S must be connected. Suppose that there is $x \in U_1 \cap S$ and $y \in U_2 \cap S$. We can assume that $x < y$. As S is an interval $[x, y] \subset S$. Let $z := \inf(U_2 \cap [x, y])$. If $z = x$, then $z \in U_1$. If $z > x$, then for any $\delta > 0$ the ball $B(z, \delta) = (z - \delta, z + \delta)$ contains points that are not in U_2 , and so $z \notin U_2$ as U_2 is open. Therefore, $z \in U_1$. As U_1 is open, $B(z, \delta) \subset U_1$ for a small enough $\delta > 0$. As z is the infimum of $U_2 \cap [x, y]$, there must exist some $w \in U_2 \cap [x, y]$ such that $w \in [z, z + \delta) \subset B(z, \delta) \subset U_1$. Therefore $w \in U_1 \cap U_2 \cap [x, y]$. So $U_1 \cap S$ and $U_2 \cap S$ are not disjoint and hence S is connected.

In many cases a ball $B(x, \delta)$ is connected. But this is not necessarily true in every metric space. For a simplest example, take a two point space $\{a, b\}$ with the discrete metric. Then $B(a, 2) = \{a, b\}$, which is not connected as $B(a, 1) = \{a\}$ and $B(b, 1) = \{b\}$ are open and disjoint.

Closure and boundary

Sometime we wish to take a set and throw in everything that we can approach from the set. This concept is called the closure.

Let (X, d) be a metric space and $A \subset X$. Then the *closure* of A is the set

$$\overline{A} := \bigcap \{E \subset X : E \text{ is closed and } A \subset E\}. \quad (8.2.12)$$

That is, $\overline{A}$ is the intersection of all closed sets that contain A .

Let (X, d) be a metric space and $A \subset X$. The closure $\overline{A}$ is closed. Furthermore if A is closed then $\overline{A} = A$.

First, the closure is the intersection of closed sets, so it is closed. Second, if A is closed, then take $E = A$, hence the intersection of all closed sets E containing A must be equal to A .

The closure of $(0, 1)$ in $\mathbb{R}$ is $[0, 1]$. Proof: Simply notice that if E is closed and contains $(0, 1)$, then E must contain 0 and 1 (why?). Thus $[0, 1] \subset E$. But $[0, 1]$ is also closed. Therefore the closure $\overline{(0, 1)} = [0, 1]$.

Be careful to notice what ambient metric space you are working with. If $X = (0, \infty)$, then the closure of $(0, 1)$ in $(0, \infty)$ is $(0, 1]$. Proof: Similarly as above $[0, 1]$ is closed in $(0, \infty)$ (why?). Any closed set E that contains $(0, 1)$ must contain 1 (why?). Therefore $(0, 1] \subset E$, and hence $\overline{(0, 1)} = (0, 1]$ when working in $(0, \infty)$.

Let us justify the statement that the closure is everything that we can “approach” from the set.

[prop:msclosureappr] Let (X, d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ if and only if for every $\delta > 0$, $B(x, \delta) \cap A \neq \emptyset$.

Let us prove the two contrapositives. Let us show that $x \notin \overline{A}$ if and only if there exists a $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$.

First suppose that $x \notin \overline{A}$. We know $\overline{A}$ is closed. Thus there is a $\delta > 0$ such that $B(x, \delta) \subset \overline{A}^c$. As $A \subset \overline{A}$ we see that $B(x, \delta) \subset A^c$ and hence $B(x, \delta) \cap A = \emptyset$.

On the other hand suppose that there is a $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$. Then $B(x, \delta)^c$ is a closed set and we have that $A \subset B(x, \delta)^c$, but $x \notin B(x, \delta)^c$. Thus as $\overline{A}$ is the intersection of closed sets containing A , we have $x \notin \overline{A}$.

We can also talk about what is in the interior of a set and what is on the boundary.

Let (X, d) be a metric space and $A \subset X$, then the *interior* of A is the set

$$A^\circ := \{x \in A : \text{there exists a } \delta > 0 \text{ such that } B(x, \delta) \subset A\}. \quad (8.2.13)$$

The *boundary* of A is the set

$$\partial A := \overline{A} \setminus A^\circ. \quad (8.2.14)$$

Suppose $A = (0, 1]$ and $X = \mathbb{R}$. Then it is not hard to see that $\overline{A} = [0, 1]$, $A^\circ = (0, 1)$, and $\partial A = \{0, 1\}$.

Suppose $X = \{a, b\}$ with the discrete metric. Let $A = \{a\}$, then $\overline{A} = A^\circ$ and $\partial A = \emptyset$.

Let (X, d) be a metric space and $A \subset X$. Then A° is open and ∂A is closed.

Given $x \in A^\circ$ we have $\delta > 0$ such that $B(x, \delta) \subset A$. If $z \in B(x, \delta)$, then as open balls are open, there is an $\epsilon > 0$ such that $B(z, \epsilon) \subset B(x, \delta) \subset A$, so z is in A° . Therefore $B(x, \delta) \subset A^\circ$ and so A° is open.

As A° is open, then $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (A^\circ)^c$ is closed.

The boundary is the set of points that are close to both the set and its complement.

Let (X, d) be a metric space and $A \subset X$. Then $x \in \partial A$ if and only if for every $\delta > 0$, $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ are both nonempty.

If $x \notin \overline{A}$, then there is some $\delta > 0$ such that $B(x, \delta) \subset \overline{A}^c$ as $\overline{A}$ is closed. So $B(x, \delta)$ contains no points of A .

Now suppose that $x \in A^\circ$, then there exists a $\delta > 0$ such that $B(x, \delta) \subset A$, but that means that $B(x, \delta)$ contains no points of A^c .

Finally suppose that $x \in \overline{A} \setminus A^\circ$. Let $\delta > 0$ be arbitrary. By $B(x, \delta)$ contains a point from A . Also, if $B(x, \delta)$ contained no points of A^c , then x would be in A° . Hence $B(x, \delta)$ contains a points of A^c as well.

We obtain the following immediate corollary about closures of A and A^c . We simply apply .

Let (X, d) be a metric space and $A \subset X$. Then $\partial A = \overline{A} \cap \overline{A^c}$.

Exercises

Prove . Hint: consider the complements of the sets and apply .

Finish the proof of by proving that $C(x, \delta)$ is closed.

Prove .

Suppose that (X, d) is a nonempty metric space with the discrete topology. Show that X is connected if and only if it contains exactly one element.

Show that if $S \subset \mathbb{R}$ is a connected unbounded set, then it is an (unbounded) interval.

Show that every open set can be written as a union of closed sets.

a) Show that E is closed if and only if $\partial E \subset E$. b) Show that U is open if and only if $\partial U \cap U = \emptyset$.

a) Show that A is open if and only if $A^\circ = A$. b) Suppose that U is an open set and $U \subset A$. Show that $U \subset A^\circ$.

Let X be a set and d, d' be two metrics on X . Suppose that there exists an $\alpha > 0$ and $\beta > 0$ such that $\alpha d(x, y) \leq d'(x, y) \leq \beta d(x, y)$ for all $x, y \in X$. Show that U is open in (X, d) if and only if U is open in (X, d') . That is, the topologies of (X, d) and (X, d') are the same.

Suppose that $\{S_i\}$, $i \in \mathbb{N}$ is a collection of connected subsets of a metric space (X, d) . Suppose that there exists an $x \in X$ such that $x \in S_i$ for all $i \in \mathbb{N}$. Show that $\bigcup_{i=1}^{\infty} S_i$ is connected.

Let A be a connected set. a) Is $\overline{A}$ connected? Prove or find a counterexample. b) Is A° connected? Prove or find a counterexample. Hint: Think of sets in $\mathbb{R}^2$.

The definition of open sets in the following exercise is usually called the *subspace topology*. You are asked to show that we obtain the same topology by considering the subspace metric.

[exercise:mssubspace] Suppose (X, d) is a metric space and $Y \subset X$. Show that with the subspace metric on Y , a set $U \subset Y$ is open (in Y) whenever there exists an open set $V \subset X$ such that $U = V \cap Y$.

Let (X, d) be a metric space. a) For any $x \in X$ and $\delta > 0$, show $\overline{B(x, \delta)} \subset C(x, \delta)$. b) Is it always true that $\overline{B(x, \delta)} = C(x, \delta)$? Prove or find a counterexample.

Let (X, d) be a metric space and $A \subset X$. Show that $A^\circ = \bigcup \{V : V \subset A \text{ is open}\}$.

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8.3: Sequences and Convergence

Sequences

The notion of a sequence in a metric space is very similar to a sequence of real numbers.

A *sequence* in a metric space (X, d) is a function $x: \mathbb{N} \rightarrow X$. As before we write x_n for the n th element in the sequence and use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty}. \quad (8.3.1)$$

A sequence $\{x_n\}$ is *bounded* if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x_n) \leq B \quad \text{for all } n \in \mathbb{N}. \quad (8.3.2)$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

If $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_{j+1} > n_j$ for all j then the sequence $\{x_{n_j}\}_{j=1}^{\infty}$ is said to be a *subsequence* of $\{x_n\}$.

Similarly we also define convergence. Again, we will be cheating a little bit and we will use the definite article in front of the word *limit* before we prove that the limit is unique.

A sequence $\{x_n\}$ in a metric space (X, d) is said to *converge* to a point $p \in X$, if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq M$. The point p is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n \rightarrow \infty} x_n := p. \quad (8.3.3)$$

A sequence that converges is said to be *convergent*. Otherwise, the sequence is said to be *divergent*.

Let us prove that the limit is unique. Note that the proof is almost identical to the proof of the same fact for sequences of real numbers. In fact many results we know for sequences of real numbers can be proved in the more general settings of metric spaces. We must replace $|x - y|$ with $d(x, y)$ in the proofs and apply the triangle inequality correctly.

[prop:mslimisunique] A convergent sequence in a metric space has a unique limit.

Suppose that the sequence $\{x_n\}$ has the limit x and the limit y . Take an arbitrary $\epsilon > 0$. From the definition we find an M_1 such that for all $n \geq M_1$, $d(x_n, x) < \frac{\epsilon}{2}$. Similarly we find an M_2 such that for all $n \geq M_2$ we have $d(x_n, y) < \frac{\epsilon}{2}$. Now take an n such that $n \geq M_1$ and also $n \geq M_2$

$$\begin{aligned} d(y, x) &\leq d(y, x_n) + d(x_n, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As $d(y, x) < \epsilon$ for all $\epsilon > 0$, then $d(x, y) = 0$ and $y = x$. Hence the limit (if it exists) is unique.

The proofs of the following propositions are left as exercises.

[prop:msconvbound] A convergent sequence in a metric space is bounded.

[prop:msconvifa] A sequence $\{x_n\}$ in a metric space (X, d) converges to $p \in X$ if and only if there exists a sequence $\{a_n\}$ of real numbers such that

$$d(x_n, p) \leq a_n \quad \text{for all } n \in \mathbb{N}, \quad (8.3.4)$$

and

$$\lim_{n \rightarrow \infty} a_n = 0. \quad (8.3.5)$$

Convergence in euclidean space

It is useful to note what convergence means in the euclidean space $\mathbb{R}^n$.

[prop:msconveuc] Let $\{x^j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, where we write $x^j = (x_1^j, x_2^j, \dots, x_n^j) \in \mathbb{R}^n$. Then $\{x^j\}_{j=1}^{\infty}$ converges if and only if $\{x_k^j\}_{j=1}^{\infty}$ converges for every k , in which case

$$\lim_{j \rightarrow \infty} x^j = \left(\lim_{j \rightarrow \infty} x_1^j, \lim_{j \rightarrow \infty} x_2^j, \dots, \lim_{j \rightarrow \infty} x_n^j \right). \quad (8.3.6)$$

For $\mathbb{R} = \mathbb{R}^1$ the result is immediate. So let $n > 1$.

Let $\{x^j\}_{j=1}^\infty$ be a convergent sequence in $\mathbb{R}^n$, where we write $x^j = (x_1^j, x_2^j, \dots, x_n^j) \in \mathbb{R}^n$. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be the limit. Given $\epsilon > 0$, there exists an M such that for all $j \geq M$ we have

$$d(x, x^j) < \epsilon. \quad (8.3.7)$$

Fix some $k = 1, 2, \dots, n$. For $j \geq M$ we have

$$|x_k - x_k^j| = \sqrt{(x_k - x_k^j)^2} \leq \sqrt{\sum_{\ell=1}^n (x_\ell - x_\ell^j)^2} = d(x, x^j) < \epsilon. \quad (8.3.8)$$

Hence the sequence $\{x_k^j\}_{j=1}^\infty$ converges to x_k .

For the other direction suppose that $\{x_k^j\}_{j=1}^\infty$ converges to x_k for every $k = 1, 2, \dots, n$. Hence, given $\epsilon > 0$, pick an M , such that if $j \geq M$ then $|x_k - x_k^j| < \frac{\epsilon}{\sqrt{n}}$ for all $k = 1, 2, \dots, n$. Then $d(x, x^j) = \sqrt{\sum_{k=1}^n |x_k - x_k^j|^2} < \sqrt{\sum_{k=1}^n \left(\frac{\epsilon}{\sqrt{n}}\right)^2} = \sqrt{\sum_{k=1}^n \frac{\epsilon^2}{n}} = \sqrt{\epsilon^2} = \epsilon$

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$\{n\} = \epsilon$.] The sequence $\{x^j\}$ converges to $x \in \mathbb{R}^n$ and we are done.

Convergence and topology

The topology, that is, the set of open sets of a space encodes which sequences converge.

[prop:msconvtopo] Let (X, d) be a metric space and $\{x_n\}$ a sequence in X . Then $\{x_n\}$ converges to $x \in X$ if and only if for every open neighborhood U of x , there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $x_n \in U$.

First suppose that $\{x_n\}$ converges. Let U be an open neighborhood of x , then there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. As the sequence converges, find an $M \in \mathbb{N}$ such that for all $n \geq M$ we have $d(x, x_n) < \epsilon$ or in other words $x_n \in B(x, \epsilon) \subset U$.

Let us prove the other direction. Given $\epsilon > 0$ let $U := B(x, \epsilon)$ be the neighborhood of x . Then there is an $M \in \mathbb{N}$ such that for $n \geq M$ we have $x_n \in U = B(x, \epsilon)$ or in other words, $d(x, x_n) < \epsilon$.

A set is closed when it contains the limits of its convergent sequences.

[prop:msclosedlim] Let (X, d) be a metric space, $E \subset X$ a closed set and $\{x_n\}$ a sequence in E that converges to some $x \in X$. Then $x \in E$.

Let us prove the contrapositive. Suppose $\{x_n\}$ is a sequence in X that converges to $x \in E^c$. As E^c is open, says there is an M such that for all $n \geq M$, $x_n \in E^c$. So $\{x_n\}$ is not a sequence in E .

When we take a closure of a set A , we really throw in precisely those points that are limits of sequences in A .

[prop:msclosureapprseq] Let (X, d) be a metric space and $A \subset X$. If $x \in \overline{A}$, then there exists a sequence $\{x_n\}$ of elements in A such that $\lim x_n = x$.

Let $x \in \overline{A}$. We know by that given $\frac{1}{n}$, there exists a point $x_n \in B(x, \frac{1}{n}) \cap A$. As $d(x, x_n) < \frac{1}{n}$, we have that $\lim x_n = x$.

Exercises

Let (X, d) be a metric space and let $A \subset X$. Let E be the set of all $x \in X$ such that there exists a sequence $\{x_n\}$ in A that converges to x . Show that $E = \overline{A}$.

a) Show that $d(x, y) := \min\{1, |x - y|\}$ defines a metric on $\mathbb{R}$. b) Show that a sequence converges in $(\mathbb{R}, d)$ if and only if it converges in the standard metric. c) Find a bounded sequence in $(\mathbb{R}, d)$ that contains no convergent subsequence.

Prove

Prove

Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x . Suppose that $f: \mathbb{N} \rightarrow \mathbb{N}$ is a one-to-one and onto function. Show that $\{x_{f(n)}\}_{n=1}^{\infty}$ converges to x .

If (X, d) is a metric space where d is the discrete metric. Suppose that $\{x_n\}$ is a convergent sequence in X . Show that there exists a $K \in \mathbb{N}$ such that for all $n \geq K$ we have $x_n = x_K$.

A set $S \subset X$ is said to be dense in X if for every $x \in X$, there exists a sequence $\{x_n\}$ in S that converges to x . Prove that $\mathbb{R}^n$ contains a countable dense subset.

Suppose $\{U_n\}_{n=1}^{\infty}$ be a decreasing ($U_{n+1} \subset U_n$ for all n) sequence of open sets in a metric space (X, d) such that $\bigcap_{n=1}^{\infty} U_n = \{p\}$ for some $p \in X$. Suppose that $\{x_n\}$ is a sequence of points in X such that $x_n \in U_n$. Does $\{x_n\}$ necessarily converge to p ? Prove or construct a counterexample.

Let $E \subset X$ be closed and let $\{x_n\}$ be a sequence in X converging to $p \in X$. Suppose $x_n \in E$ for infinitely many $n \in \mathbb{N}$. Show $p \in E$.

Take $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ be the extended reals. Define $d(x, y) := \frac{|x-y|}{1+|x-y|}$ if $x, y \in \mathbb{R}$, define $d(\infty, x) := d(-\infty, x) = 1$ for all $x \in \mathbb{R}$, and let $d(\infty, -\infty) := 2$. a) Show that $(\mathbb{R}^*, d)$ is a metric space. b) Suppose that $\{x_n\}$ is a sequence of real numbers such that for $x_n \geq n$ for all n . Show that $\lim x_n = \infty$ in $(\mathbb{R}^*, d)$.

Suppose that $\{V_n\}_{n=1}^{\infty}$ is a collection of open sets in (X, d) such that $V_{n+1} \supset V_n$. Let $\{x_n\}$ be a sequence such that $x_n \in V_{n+1} \setminus V_n$ and suppose that $\{x_n\}$ converges to $p \in X$. Show that $p \in \partial V$ where $V = \bigcup_{n=1}^{\infty} V_n$.

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8.4: Completeness and Compactness

Cauchy sequences and completeness

Just like with sequences of real numbers we can define Cauchy sequences.

Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$ we have

$$d(x_n, x_k) < \epsilon. \quad (8.4.1)$$

The definition is again simply a translation of the concept from the real numbers to metric spaces. So a sequence of real numbers is Cauchy in the sense of if and only if it is Cauchy in the sense above, provided we equip the real numbers with the standard metric $d(x, y) = |x - y|$.

Let (X, d) be a metric space. We say that X is *complete* or *Cauchy-complete* if every Cauchy sequence $\{x_n\}$ in X converges to an $x \in X$.

The space $\mathbb{R}^n$ with the standard metric is a complete metric space.

For $\mathbb{R} = \mathbb{R}^1$ this was proved in .

Take $n > 1$. Let $\{x^j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^n$, where we write $x^j = (x_1^j, x_2^j, \dots, x_n^j) \in \mathbb{R}^n$. As the sequence is Cauchy, given $\epsilon > 0$, there exists an M such that for all $i, j \geq M$ we have

$$d(x^i, x^j) < \epsilon. \quad (8.4.2)$$

Fix some $k = 1, 2, \dots, n$, for $i, j \geq M$ we have $\sqrt[n]{|x_k^i - x_k^j|} < \sqrt[n]{\epsilon}$

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$\sqrt[n]{\epsilon}$ is complete the sequence converges; there exists an $x_k \in \mathbb{R}$ such that $x_k = \lim_{j \rightarrow \infty} x_k^j$.

Write $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. By we have that $\{x^j\}$ converges to $x \in \mathbb{R}^n$ and hence $\mathbb{R}^n$ is complete.

Compactness

Let (X, d) be a metric space and $K \subset X$. The set K is set to be *compact* if for any collection of open sets $\{U_\lambda\}_{\lambda \in I}$ such that

$$K \subset \bigcup_{\lambda \in I} U_\lambda, \quad (8.4.3)$$

there exists a finite subset $\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset I$ such that

$$K \subset \bigcup_{j=1}^k U_{\lambda_j}. \quad (8.4.4)$$

A collection of open sets $\{U_\lambda\}_{\lambda \in I}$ as above is said to be a *open cover* of K . So a way to say that K is compact is to say that *every open cover of K has a finite subcover*.

Let (X, d) be a metric space. A compact set $K \subset X$ is closed and bounded.

First, we prove that a compact set is bounded. Fix $p \in X$. We have the open cover

$$K \subset \bigcup_{n=1}^\infty B(p, n) = X. \quad (8.4.5)$$

If K is compact, then there exists some set of indices $n_1 < n_2 < \dots < n_k$ such that

$$K \subset \bigcup_{j=1}^k B(p, n_j) = B(p, n_k). \quad (8.4.6)$$

As K is contained in a ball, K is bounded.

Next, we show a set that is not closed is not compact. Suppose that $\overline{K} \neq K$, that is, there is a point $x \in \overline{K} \setminus K$. If $y \neq x$, then for n with $\frac{1}{n} < d(x, y)$ we have $y \notin C(x, \frac{1}{n})$. Furthermore $x \notin K$, so

$$K \subset \bigcup_{n=1}^{\infty} C(x, \frac{1}{n})^c. \quad (8.4.7)$$

As a closed ball is closed, $C(x, \frac{1}{n})^c$ is open, and so we have an open cover. If we take any finite collection of indices $n_1 < n_2 < \dots < n_k$, then

$$\bigcup_{j=1}^k C(x, \frac{1}{n_j})^c = C(x, \frac{1}{n_k})^c \quad (8.4.8)$$

As x is in the closure, we have $C(x, \frac{1}{n_k}) \cap K \neq \emptyset$, so there is no finite subcover and K is not compact.

We prove below that in finite dimensional euclidean space every closed bounded set is compact. So closed bounded sets of $\mathbb{R}^n$ are examples of compact sets. It is not true that in every metric space, closed and bounded is equivalent to compact. There are many metric spaces where closed and bounded is not enough to give compactness, see for example .

A useful property of compact sets in a metric space is that every sequence has a convergent subsequence. Such sets are sometimes called *sequentially compact*. Let us prove that in the context of metric spaces, a set is compact if and only if it is sequentially compact.

[thm:mscompactisseqcpt] Let (X, d) be a metric space. Then $K \subset X$ is a compact set if and only if every sequence in K has a subsequence converging to a point in K .

Let $K \subset X$ be a set and $\{x_n\}$ a sequence in K . Suppose that for each $x \in K$, there is a ball $B(x, \alpha_x)$ for some $\alpha_x > 0$ such that $x_n \in B(x, \alpha_x)$ for only finitely many $n \in \mathbb{N}$. Then

$$K \subset \bigcup_{x \in K} B(x, \alpha_x). \quad (8.4.9)$$

Any finite collection of these balls is going to contain only finitely many x_n . Thus for any finite collection of such balls there is an $x_n \in K$ that is not in the union. Therefore, K is not compact.

So if K is compact, then there exists an $x \in K$ such that for any $\delta > 0$, $B(x, \delta)$ contains x_k for infinitely many $k \in \mathbb{N}$. $B(x, 1)$ contains some x_k so let $n_1 := k$. If n_{j-1} is defined, then there must exist a $k > n_{j-1}$ such that $x_k \in B(x, \frac{1}{j})$, so define $n_j := k$. Notice that $d(x, x_{n_j}) < \frac{1}{j}$. By, $\lim x_{n_j} = x$.

For the other direction, suppose that every sequence in K has a subsequence converging in K . Take an open cover $\{U_\lambda\}_{\lambda \in I}$ of K . For every $x \in K$, define

$$\delta(x) := \sup\{\delta \in (0, 1) : B(x, \delta) \subset U_\lambda \text{ for some } \lambda \in I\}. \quad (8.4.10)$$

As $\{U_\lambda\}$ is an open cover of K , $\delta(x) > 0$ for each $x \in K$. By construction, for any positive $\epsilon < \delta(x)$ there must exist a $\lambda \in I$ such that $B(x, \epsilon) \subset U_\lambda$.

Pick a $\lambda_0 \in I$ and look at U_{λ_0} . If $K \subset U_{\lambda_0}$, we stop as we have found a finite subcover. Otherwise, there must be a point $x_1 \in K \setminus U_{\lambda_0}$. There must exist some $\lambda_1 \in I$ such that $x_1 \in U_{\lambda_1}$ and in fact $B(x_1, \frac{1}{2}\delta(x_1)) \subset U_{\lambda_1}$. We work inductively. Suppose that λ_{n-1} is defined. Either $U_{\lambda_0} \cup U_{\lambda_1} \cup \dots \cup U_{\lambda_{n-1}}$ is a finite cover of K , in which case we stop, or there must be a point $x_n \in K \setminus (U_{\lambda_0} \cup U_{\lambda_1} \cup \dots \cup U_{\lambda_{n-1}})$. In this case, there must be some $\lambda_n \in I$ such that $x_n \in U_{\lambda_n}$, and in fact

$$B(x_n, \frac{1}{2}\delta(x_n)) \subset U_{\lambda_n}. \quad (8.4.11)$$

So either we obtained a finite subcover or we obtained an infinite sequence $\{x_n\}$ as above. For contradiction suppose that there was no finite subcover and we have the sequence $\{x_n\}$. Then there is a subsequence $\{x_{n_k}\}$ that converges, that is, $x = \lim x_{n_k} \in K$. We take $\lambda \in I$ such that $B(x, \frac{1}{2}\delta(x)) \subset U_\lambda$. As the subsequence converges, there is a k such that $d(x_{n_k}, x) < \frac{1}{8}\delta(x)$. By the triangle inequality, $B(x_{n_k}, \frac{3}{8}\delta(x)) \subset B(x, \frac{1}{2}\delta(x)) \subset U_\lambda$. So $\frac{3}{8}\delta(x) < \delta(x_{n_k})$, which implies

$$B(x_{n_k}, \frac{3}{16}\delta(x)) \subset B(x_{n_k}, \frac{1}{2}\delta(x_{n_k})) \subset U_{\lambda_{n_k}}. \quad (8.4.12)$$

As $\frac{1}{16} < \frac{3}{16}\delta(x)$ we have $x \in B(x_{n_k}, \frac{3}{16}\delta(x))$, or $x \in U_{\lambda_{n_k}}$. As $\lim x_{n_j} = x$, for all j large enough we have $x_{n_j} \in U_{\lambda_{n_k}}$ by . Let us fix one of those j such that $j > k$. But by construction $x_{n_j} \notin U_{\lambda_{n_k}}$ if $j > k$, which is a contradiction.

By the Bolzano-Weierstrass theorem for sequences () we have that any bounded sequence has a convergent subsequence. Therefore any sequence in a closed interval $[a, b] \subset \mathbb{R}$ has a convergent subsequence. The limit must also be in $[a, b]$ as limits preserve non-strict inequalities. Hence a closed bounded interval $[a, b] \subset \mathbb{R}$ is compact.

Let (X, d) be a metric space and let $K \subset X$ be compact. Suppose that $E \subset K$ is a closed set, then E is compact.

Let $\{x_n\}$ be a sequence in E . It is also a sequence in K . Therefore it has a convergent subsequence $\{x_{n_j}\}$ that converges to $x \in K$. As E is closed the limit of a sequence in E is also in E and so $x \in E$. Thus E must be compact.

[thm:msbw] A closed bounded subset $K \subset \mathbb{R}^n$ is compact.

For $\mathbb{R} = \mathbb{R}^1$ if $K \subset \mathbb{R}$ is closed and bounded, then any sequence $\{x_n\}$ in K is bounded, so it has a convergent subsequence by Bolzano-Weierstrass theorem for sequences (). As K is closed, the limit of the subsequence must be an element of K . So K is compact.

Let us carry out the proof for $n = 2$ and leave arbitrary n as an exercise.

As K is bounded, there exists a set $B = [a, b] \times [c, d] \subset \mathbb{R}^2$ such that $K \subset B$. If we can show that B is compact, then K , being a closed subset of a compact B , is also compact.

Let $\{(x_k, y_k)\}_{k=1}^\infty$ be a sequence in B . That is, $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k . A bounded sequence has a convergent subsequence so there is a subsequence $\{x_{k_j}\}_{j=1}^\infty$ that is convergent. The subsequence $\{y_{k_j}\}_{j=1}^\infty$ is also a bounded sequence so there exists a subsequence $\{y_{k_{j_i}}\}_{i=1}^\infty$ that is convergent. A subsequence of a convergent sequence is still convergent, so $\{x_{k_{j_i}}\}_{i=1}^\infty$ is convergent. Let

$$x := \lim_{i \rightarrow \infty} x_{k_{j_i}} \quad \text{and} \quad y := \lim_{i \rightarrow \infty} y_{k_{j_i}}. \quad (8.4.13)$$

By , $\{(x_{k_{j_i}}, y_{k_{j_i}})\}_{i=1}^\infty$ converges to (x, y) as i goes to ∞ . Furthermore, as $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k , we know that $(x, y) \in B$.

Exercises

Let (X, d) be a metric space and A a finite subset of X . Show that A is compact.

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. a) Show that A is not compact directly using the definition. b) Show that $A \cup \{0\}$ is compact directly using the definition.

Let (X, d) be a metric space with the discrete metric. a) Prove that X is complete. b) Prove that X is compact if and only if X is a finite set.

a) Show that the union of finitely many compact sets is a compact set. b) Find an example where the union of infinitely many compact sets is not compact.

Prove for arbitrary dimension. Hint: The trick is to use the correct notation.

Show that a compact set K is a complete metric space.

Let $C([a, b])$ be the metric space as in . Show that $C([a, b])$ is a complete metric space.

[exercise:msclbounnotcompt] Let $C([0, 1])$ be the metric space of . Let 0 denote the zero function. Then show that the closed ball $C(0, 1)$ is not compact (even though it is closed and bounded). Hints: Construct a sequence of distinct continuous functions $\{f_n\}$ such that $d(f_n, 0) = 1$ and $d(f_n, f_k) = 1$ for all $n \neq k$. Show that the set $\{f_n : n \in \mathbb{N}\} \subset C(0, 1)$ is closed but not compact. See for inspiration.

Show that there exists a metric on $\mathbb{R}$ that makes $\mathbb{R}$ into a compact set.

Suppose that (X, d) is complete and suppose we have a countably infinite collection of nonempty compact sets $E_1 \supset E_2 \supset E_3 \supset \dots$ then prove $\bigcap_{j=1}^\infty E_j \neq \emptyset$.

Let $C([0, 1])$ be the metric space of . Let K be the set of $f \in C([0, 1])$ such that f is equal to a quadratic polynomial, i.e. $f(x) = a + bx + cx^2$, and such that $|f(x)| \leq 1$ for all $x \in [0, 1]$, that is $f \in C(0, 1)$. Show that K is compact.

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8.5: Continuous Functions

Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces and $c \in X$. Then $f: X \rightarrow Y$ is *continuous at c* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in X$ and $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$.

When $f: X \rightarrow Y$ is continuous at all $c \in X$, then we simply say that f is a *continuous function*.

The definition agrees with the definition from when f is a real-valued function on the real line, when we take the standard metric on $\mathbb{R}$.

[prop:contiscont] Let (X, d_X) and (Y, d_Y) be metric spaces and $c \in X$. Then $f: X \rightarrow Y$ is continuous at c if and only if for every sequence $\{x_n\}$ in X converging to c , the sequence $\{f(x_n)\}$ converges to $f(c)$.

Suppose that f is continuous at c . Let $\{x_n\}$ be a sequence in X converging to c . Given $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, c) < \delta$ implies $d(f(x), f(c)) < \epsilon$. So take M such that for all $n \geq M$, we have $d(x_n, c) < \delta$, then $d(f(x_n), f(c)) < \epsilon$. Hence $\{f(x_n)\}$ converges to $f(c)$.

On the other hand suppose that f is not continuous at c . Then there exists an $\epsilon > 0$, such that for every $\frac{1}{n}$ there exists an $x_n \in X$, $d(x_n, c) < \frac{1}{n}$ such that $d(f(x_n), f(c)) \geq \epsilon$. Therefore $\{f(x_n)\}$ does not converge to $f(c)$.

Compactness and continuity

Continuous maps do not map closed sets to closed sets. For example, $f: (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) := x$ takes the set $(0, 1)$, which is closed in $(0, 1)$, to the set $(0, 1)$, which is not closed in $\mathbb{R}$. On the other hand continuous maps do preserve compact sets.

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f: X \rightarrow Y$ is a continuous function. If $K \subset X$ is a compact set, then $f(K)$ is a compact set.

Let $\{f(x_n)\}_{n=1}^\infty$ be a sequence in $f(K)$, then $\{x_n\}_{n=1}^\infty$ is a sequence in K . The set K is compact and therefore has a subsequence $\{x_{n_i}\}_{i=1}^\infty$ that converges to some $x \in K$. By continuity,

$$\lim_{i \rightarrow \infty} f(x_{n_i}) = f(x) \in f(K). \quad (8.5.1)$$

Therefore every sequence in $f(K)$ has a subsequence convergent to a point in $f(K)$, so $f(K)$ is compact by .

As before, $f: X \rightarrow \mathbb{R}$ achieves an *absolute minimum* at $c \in X$ if

$$f(x) \geq f(c) \quad \text{for all } x \in X. \quad (8.5.2)$$

On the other hand, f achieves an *absolute maximum* at $c \in X$ if

$$f(x) \leq f(c) \quad \text{for all } x \in X. \quad (8.5.3)$$

Let (X, d) be a compact metric space, and $f: X \rightarrow \mathbb{R}$ is a continuous function. Then $f(X)$ is compact and in fact f achieves an absolute minimum and an absolute maximum on X .

As X is compact and f is continuous, we have that $f(X) \subset \mathbb{R}$ is compact. Hence $f(X)$ is closed and bounded. In particular, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$. That is because both the sup and inf can be achieved by sequences in $f(X)$ and $f(X)$ is closed. Therefore there is some $x \in X$ such that $f(x) = \sup f(X)$ and some $y \in X$ such that $f(y) = \inf f(X)$.

Continuity and topology

Let us see how to define continuity just in the terms of topology, that is, the open sets. We have already seen that topology determines which sequences converge, and so it is no wonder that the topology also determines continuity of functions.

[lemma:mstopocontloc] Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \rightarrow Y$ is continuous at $c \in X$ if and only if for every open neighbourhood U of $f(c)$ in Y , the set $f^{-1}(U)$ contains an open neighbourhood of c in X .

Suppose that f is continuous at c . Let U be an open neighbourhood of $f(c)$ in Y , then $B_Y(f(c), \epsilon) \subset U$ for some $\epsilon > 0$. As f is continuous, then there exists a $\delta > 0$ such that whenever x is such that $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$. In other words,

$$B_X(c, \delta) \subset f^{-1}(B_Y(f(c), \epsilon)). \quad (8.5.4)$$

and $B_X(c, \delta)$ is an open neighbourhood of c .

For the other direction, let $\epsilon > 0$ be given. If $f^{-1}(B_Y(f(c), \epsilon))$ contains an open neighbourhood, it contains a ball, that is there is some $\delta > 0$ such that

$$B_X(c, \delta) \subset f^{-1}(B_Y(f(c), \epsilon)). \quad (8.5.5)$$

That means precisely that if $d_X(x, c) < \delta$ then $d_Y(f(x), f(c)) < \epsilon$ and so f is continuous at c .

[thm:mstopocont] Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \rightarrow Y$ is continuous if and only if for every open $U \subset Y$, $f^{-1}(U)$ is open in X .

The proof follows from and is left as an exercise.

Exercises

Consider $\mathbb{N} \subset \mathbb{R}$ with the standard metric. Let (X, d) be a metric space and $f: X \rightarrow \mathbb{N}$ a continuous function. a) Prove that if X is connected, then f is constant (the range of f is a single value). b) Find an example where X is disconnected and f is not constant.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(0, 0) := 0$, and $f(x, y) := \frac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$. a) Show that for any fixed x , the function that takes y to $f(x, y)$ is continuous. Similarly for any fixed y , the function that takes x to $f(x, y)$ is continuous. b) Show that f is not continuous.

Suppose that $f: X \rightarrow Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y) . Let $A \subset X$. a) Show that $f(\overline{A}) \subset \overline{f(A)}$. b) Show that the subset can be proper.

Prove . Hint: Use .

[exercise:msconnconn] Suppose that $f: X \rightarrow Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y) . Show that if X is connected, then $f(X)$ is connected.

Prove the following version of the intermediate value theorem. Let (X, d) be a connected metric space and $f: X \rightarrow \mathbb{R}$ a continuous function. Suppose that there exist $x_0, x_1 \in X$ and $y \in \mathbb{R}$ such that $f(x_0) < y < f(x_1)$. Then prove that there exists a $z \in X$ such that $f(z) = y$. Hint: see .

A continuous function $f: X \rightarrow Y$ for metric spaces (X, d_X) and (Y, d_Y) is said to be *proper* if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is compact. Suppose that a continuous $f: (0, 1) \rightarrow (0, 1)$ is proper and $\{x_n\}$ is a sequence in $(0, 1)$ that converges to 0. Show that $\{f(x_n)\}$ has no subsequence that converges in $(0, 1)$.

Let (X, d_X) and (Y, d_Y) be metric space and $f: X \rightarrow Y$ be a one to one and onto continuous function. Suppose that X is compact. Prove that the inverse $f^{-1}: Y \rightarrow X$ is continuous.

Take the metric space of continuous functions $C([0, 1])$. Let $k: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Given $f \in C([0, 1])$ define

$$\varphi_f(x) := \int_0^1 k(x, y) f(y) dy. \quad (8.5.6)$$

a) Show that $T(f) := \varphi_f$ defines a function $T: C([0, 1]) \rightarrow C([0, 1])$ b) Show that T is continuous.

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8.6: Fixed point theorem and Picard's theorem again

In this section we prove a fixed point theorem for contraction mappings. As an application we prove Picard's theorem. We have proved Picard's theorem without metric spaces in . The proof we present here is similar, but the proof goes a lot smoother by using metric space concepts and the fixed point theorem. For more examples on using Picard's theorem see .

Let (X, d) and (X', d') be metric spaces. $F: X \rightarrow X'$ is said to be a *contraction* (or a contractive map) if it is a k -Lipschitz map for some $k < 1$, i.e. if there exists a $k < 1$ such that

$$d'(F(x), F(y)) \leq kd(x, y) \quad \text{for all } x, y \in X. \quad (8.6.1)$$

If $T: X \rightarrow X$ is a map, $x \in X$ is called a *fixed point* if $T(x) = x$.

[Contraction mapping principle or Fixed point theorem] [thm:contr] Let (X, d) be a nonempty complete metric space and $T: X \rightarrow X$ is a contraction. Then T has a fixed point.

Note that the words *complete* and *contraction* are necessary. See .

Pick any $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{n+1} := T(x_n)$.

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq kd(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0). \quad (8.6.2)$$

So let $m \geq n$

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) \\ &= k^n d(x_1, x_0) \sum_{i=0}^{m-n-1} k^i \\ &\leq k^n d(x_1, x_0) \sum_{i=0}^{\infty} k^i = k^n d(x_1, x_0) \frac{1}{1-k}. \end{aligned}$$

In particular the sequence is Cauchy. Since X is complete we let $x := \lim_{n \rightarrow \infty} x_n$ and claim that x is our unique fixed point.

Fixed point? Note that T is continuous because it is a contraction. Hence

$$T(x) = \lim T(x_n) = \lim x_{n+1} = x. \quad (8.6.3)$$

Unique? Let y be a fixed point.

$$d(x, y) = d(T(x), T(y)) = kd(x, y). \quad (8.6.4)$$

As $k < 1$ this means that $d(x, y) = 0$ and hence $x = y$. The theorem is proved.

Note that the proof is constructive. Not only do we know that a unique fixed point exists. We also know how to find it. Let us use the theorem to prove the classical Picard theorem on the existence and uniqueness of ordinary differential equations.

Consider the equation

$$\frac{dx}{dt} = F(t, x). \quad (8.6.5)$$

Given some t_0, x_0 we are looking for a function $f(t)$ such that $f'(t_0) = x_0$ and such that

$$f'(t) = F(t, f(t)). \quad (8.6.6)$$

There are some subtle issues. Look at the equation $x' = x^2$, $x(0) = 1$. Then $x(t) = \frac{1}{1-t}$ is a solution. While F is a reasonably "nice" function and in particular exists for all x and t , the solution "blows up" at $t = 1$.

Let $I, J \subset \mathbb{R}$ be compact intervals and let I_0 and J_0 be their interiors. Suppose $F: I \times J \rightarrow \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, there exists $L \in \mathbb{R}$ such that

$$|F(t, x) - F(t, y)| \leq L|x - y| \quad \text{for all } x, y \in J, t \in I. \quad (8.6.7)$$

Let $(t_0, x_0) \in I_0 \times J_0$. Then there exists $h > 0$ and a unique differentiable $f: [t_0 - h, t_0 + h] \rightarrow \mathbb{R}$, such that $f'(t) = F(t, f(t))$ and $f(t_0) = x_0$.

Without loss of generality assume $t_0 = 0$. Let $M := \sup\{|F(t, x)| : (t, x) \in I \times J\}$. As $I \times J$ is compact, $M < \infty$. Pick $\alpha > 0$ such that $[-\alpha, \alpha] \subset I$ and $[x_0 - \alpha, x_0 + \alpha] \subset J$. Let

$$h := \min \left\{ \alpha, \frac{\alpha}{M + L\alpha} \right\}. \quad (8.6.8)$$

Note $[-h, h] \subset I$. Define the set

$$Y := \{f \in C([-h, h]) : f([-h, h]) \subset [x_0 - \alpha, x_0 + \alpha]\}. \quad (8.6.9)$$

Here $C([-h, h])$ is equipped with the standard metric $d(f, g) := \sup\{|f(x) - g(x)| : x \in [-h, h]\}$. With this metric we have shown in an exercise that $C([-h, h])$ is a complete metric space.

Show that $Y \subset C([-h, h])$ is closed.

Define a mapping $T: Y \rightarrow C([-h, h])$ by

$$T(f)(t) := x_0 + \int_0^t F(s, f(s)) \, ds. \quad (8.6.10)$$

Show that T really maps into $C([-h, h])$.

Let $f \in Y$ and $|t| \leq h$. As F is bounded by M we have

$$\begin{aligned} |T(f)(t) - x_0| &= \left| \int_0^t F(s, f(s)) \, ds \right| \\ &\leq |t| M \leq hM \leq \alpha. \end{aligned}$$

Therefore, $T(Y) \subset Y$. We can thus consider T as a mapping of Y to Y .

We claim T is a contraction. First, for $t \in [-h, h]$ and $f, g \in Y$ we have

$$|F(t, f(t)) - F(t, g(t))| \leq L|f(t) - g(t)| \leq L d(f, g). \quad (8.6.11)$$

Therefore,

$$\begin{aligned} |T(f)(t) - T(g)(t)| &= \left| \int_0^t F(s, f(s)) - F(s, g(s)) \, ds \right| \\ &\leq |t| L d(f, g) \\ &\leq hL d(f, g) \\ &\leq \frac{L\alpha}{M + L\alpha} d(f, g). \end{aligned}$$

We can assume $M > 0$ (why?). Then $\frac{L\alpha}{M + L\alpha} < 1$ and the claim is proved.

Now apply the fixed point theorem () to find a unique $f \in Y$ such that $T(f) = f$, that is,

$$f(t) = x_0 + \int_0^t F(s, f(s)) \, ds. \quad (8.6.12)$$

By the fundamental theorem of calculus, f is differentiable and $f'(t) = F(t, f(t))$.

We have shown that f is the unique function in Y . Why is it the unique continuous function $f: [-h, h] \rightarrow J$ that solves $T(f) = f$?
Hint: Look at the last estimate in the proof.

Exercises

Suppose $X = X' = \mathbb{R}$ with the standard metric. Let $0 < k < 1$, $b \in \mathbb{R}$. a) Show that the map $F(x) = kx + b$ is a contraction. b) Find the fixed point and show directly that it is unique.

Suppose $X = X' = [0, \frac{1}{4}]$ with the standard metric. a) Show that the map $F(x) = x^2$ is a contraction, and find the best (largest) k that works. b) Find the fixed point and show directly that it is unique.

[exercise:nofixedpoint] a) Find an example of a contraction of non-complete metric space with no fixed point. b) Find a 1-Lipschitz map of a complete metric space with no fixed point.

Consider $x' = x^2$, $x(0) = 1$. Start with $f_0(t) = 1$. Find a few iterates (at least up to f_2). Prove that the limit of f_n is $\frac{1}{1-t}$.

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CHAPTER OVERVIEW

9: Several Variables and Partial Derivatives

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9.1: Vector Spaces, linear Mappings, and Convexity

The euclidean space $\mathbb{R}^n$ has already made an appearance in the metric space chapter. In this chapter, we will extend the differential calculus we created for one variable to several variables. The key idea in differential calculus is to approximate functions by lines and linear functions. In several variables we must introduce a little bit of linear algebra before we can move on. So let us start with vector spaces and linear functions on vector spaces. While it is common to use $\vec{x}$ or the bold $\mathbf{x}$ for elements of $\mathbb{R}^n$, especially in the applied sciences, we use just plain x , which is common in mathematics. That is $x \in \mathbb{R}^n$ is a vector, which means that $x = (x^1, x^2, \dots, x^n)$ is an n -tuple of real numbers. We use upper indices for identifying components, leaving us the lower index for sequences of vectors. For example, we can have vectors x_1 and x_2 in $\mathbb{R}^n$ and then $x_1 = (x_1^1, x_1^2, \dots, x_1^n)$ and $x_2 = (x_2^1, x_2^2, \dots, x_2^n)$. It is common to write vectors as column vectors, that is, $n \times 1$ matrices:

$$x = (x^1, x^2, \dots, x^n) = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}. \quad (9.1.1)$$

We will do so when convenient. We call real numbers scalars to distinguish them from vectors. Let X be a set together with operations of addition, $+: X \times X \rightarrow X$, and multiplication, $\cdot: \mathbb{R} \times X \rightarrow X$, (we write ax instead of $a \cdot x$). X is called a vector space (or a real vector space) if the following conditions are satisfied: (Addition is associative) If $u, v, w \in X$, then $u + (v + w) = (u + v) + w$. (Addition is commutative) If $u, v \in X$, then $u + v = v + u$. (Additive identity) There is a $0 \in X$ such that $v + 0 = v$ for all $v \in X$. (Additive inverse) For every $v \in X$, there is a $-v \in X$, such that $v + (-v) = 0$. (Distributive law) If $a \in \mathbb{R}$, $u, v \in X$, then $a(u + v) = au + av$. (Distributive law) If $a, b \in \mathbb{R}$, $v \in X$, then $(a + b)v = av + bv$. (Multiplication is associative) If $a, b \in \mathbb{R}$, $v \in X$, then $(ab)v = a(bv)$. (Multiplicative identity) $1v = v$ for all $v \in X$. Elements of a vector space are usually called vectors, even if they are not elements of $\mathbb{R}^n$ (vectors in the “traditional” sense). An example vector space is $\mathbb{R}^n$, where addition and multiplication by a constant is done componentwise: if $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} x + y &:= (x^1, x^2, \dots, x^n) + (y^1, y^2, \dots, y^n) = (x^1 + y^1, x^2 + y^2, \dots, x^n + y^n), \\ \alpha x &:= \alpha(x^1, x^2, \dots, x^n) = (\alpha x^1, \alpha x^2, \dots, \alpha x^n). \end{aligned}$$

In this book we mostly deal with vector spaces that can be regarded as subsets of $\mathbb{R}^n$, but there are other vector spaces that are useful in analysis. For example, the space $C([0, 1], \mathbb{R})$ of continuous functions on the interval $[0, 1]$ is a vector space. A trivial example of a vector space (the smallest one in fact) is just $X = \{0\}$. The operations are defined in the obvious way. You always need a zero vector to exist, so all vector spaces are nonempty sets. It is also possible to use other fields than $\mathbb{R}$ in the definition (for example it is common to use the complex numbers $\mathbb{C}$), but let us stick with the real numbers¹. A function $f: X \rightarrow Y$, when Y is not $\mathbb{R}$ is often called a mapping or a map rather than a function. Linear combinations and dimension If we have vectors $x_1, \dots, x_k \in \mathbb{R}^n$ and scalars $a^1, \dots, a^k \in \mathbb{R}$, then

$$a^1 x_1 + a^2 x_2 + \dots + a^k x_k \quad (9.1.2)$$

is called a linear combination of the vectors $x_1, \dots, x_k$. If $Y \subset \mathbb{R}^n$ is a set then the span of Y , or in notation $\text{span}(Y)$, is the set of all linear combinations of some finite number of elements of Y . We also say Y spans $\text{span}(Y)$. Let $Y := \{(1, 1)\} \subset \mathbb{R}^2$. Then

$$\text{span}(Y) = \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}. \quad (9.1.3)$$

That is, $\text{span}(Y)$ is the line through the origin and the point $(1, 1)$. [example:vecspr2span] Let $Y := \{(1, 1), (0, 1)\} \subset \mathbb{R}^2$. Then

$$\text{span}(Y) = \mathbb{R}^2, \quad (9.1.4)$$

as any point $(x, y) \in \mathbb{R}^2$ can be written as a linear combination

$$(x, y) = x(1, 1) + (y - x)(0, 1). \quad (9.1.5)$$

A sum of two linear combinations is again a linear combination, and a scalar multiple of a linear combination is a linear combination, which proves the following proposition. Let X be a vector space. For any $Y \subset X$, the set $\text{span}(Y)$ is a vector space itself. If Y is already a vector space then $\text{span}(Y) = Y$. A set of vectors $\{x_1, x_2, \dots, x_k\}$ is linearly independent, if the only solution to

$$a^1 x_1 + a^2 x_2 + \dots + a^k x_k = 0 \quad (9.1.6)$$

is the trivial solution $a^1 = a^2 = \dots = a^k = 0$. A set that is not linearly independent, is linearly dependent. A linearly independent set B of vectors such that $\text{span}(B) = X$ is called a basis of X . For example the set Y of the two vectors in is a basis of $\mathbb{R}^2$. If a vector space X contains a linearly independent set of d vectors, but no linearly independent set of $d+1$ vectors then we say the dimension or $\dim X := d$. If for all $d \in \mathbb{N}$ the vector space X contains a set of d linearly independent vectors, we say X is infinite dimensional and write $\dim X := \infty$. Clearly for the trivial vector space, $\dim \{0\} = 0$. We will see in a moment that any vector space that is a subset of $\mathbb{R}^n$ has a finite dimension, and that dimension is less than or equal to n . If a set is linearly dependent, then one of the vectors is a linear combination of the others. In other words, if $a^j \neq 0$, then we can solve for x_j

$$x_j = \frac{a^1}{a^j} x_1 + \dots + \frac{a^{j-1}}{a^j} x_{j-1} + \frac{a^{j+1}}{a^j} x_{j+1} + \dots + \frac{a^k}{a^j} x_k. \quad (9.1.7)$$

Clearly then the vector x_j has at least two different representations as linear combinations of $\{x_1, x_2, \dots, x_k\}$. If $B = \{x_1, x_2, \dots, x_k\}$ is a basis of a vector space X , then every point $y \in X$ has a unique representation of the form

$$y = \sum_{j=1}^k \alpha^j x_j \quad (9.1.8)$$

for some numbers $\alpha^1, \alpha^2, \dots, \alpha^k$. Every $y \in X$ is a linear combination of elements of B since X is the span of B . For uniqueness suppose

$$y = \sum_{j=1}^k \alpha^j x_j = \sum_{j=1}^k \beta^j x_j, \quad (9.1.9)$$

then

$$\sum_{j=1}^k (\alpha^j - \beta^j) x_j = 0. \quad (9.1.10)$$

By linear independence of the basis $\alpha^j = \beta^j$ for all j . For $\mathbb{R}^n$ we define

$$e_1 := (1, 0, 0, \dots, 0), \quad e_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad e_n := (0, 0, 0, \dots, 1), \quad (9.1.11)$$

and call this the standard basis of $\mathbb{R}^n$. We use the same letters e_j for any $\mathbb{R}^n$, and which space $\mathbb{R}^n$ we are working in is understood from context. A direct computation shows that $\{e_1, e_2, \dots, e_n\}$ is really a basis of $\mathbb{R}^n$; it is easy to show that it spans $\mathbb{R}^n$ and is linearly independent. In fact,

$$x = (x^1, x^2, \dots, x^n) = \sum_{j=1}^n x^j e_j. \quad (9.1.12)$$

Let X be a vector space. If X is spanned by d vectors, then $\dim X \leq d$. $\dim X = d$ if and only if X has a basis of d vectors (and so every basis has d vectors). In particular, $\dim \mathbb{R}^n = n$. If $Y \subset X$ is a vector space and $\dim X = d$, then $\dim Y \leq d$. If $\dim X = d$ and a set T of d vectors spans X , then T is linearly independent. If $\dim X = d$ and a set T of m vectors is linearly independent, then there is a set S of $d-m$ vectors such that $T \cup S$ is a basis of X . Let us start with (i). Suppose $S = \{x_1, x_2, \dots, x_d\}$ spans X , and $T = \{y_1, y_2, \dots, y_m\}$ is a set of linearly independent vectors of X . We wish to show that $m \leq d$. Write

$$y_1 = \sum_{k=1}^d \alpha_1^k x_k, \quad (9.1.13)$$

which we can do as S spans X . One of the α_1^k is nonzero (otherwise y_1 would be zero), so suppose without loss of generality that this is α_1^1 . Then we can solve

$$x_1 = \frac{1}{\alpha_1^1} y_1 - \sum_{k=2}^d \frac{\alpha_1^k}{\alpha_1^1} x_k. \quad (9.1.14)$$

In particular $\{y_1, x_2, \dots, x_d\}$ span X , since x_1 can be obtained from $\{y_1, x_2, \dots, x_d\}$. Next,

$$y_2 = \alpha_2^1 y_1 + \sum_{k=2}^d \alpha_2^k x_k. \quad (9.1.15)$$

As T is linearly independent, we must have that one of the α_2^k for $k \geq 2$ must be nonzero. Without loss of generality suppose $\alpha_2^2 \neq 0$. Proceed to solve for

$$x_2 = \frac{1}{\alpha_2^2} y_2 - \frac{\alpha_2^1}{\alpha_2^2} y_1 - \sum_{k=3}^d \frac{\alpha_2^k}{\alpha_2^2} x_k. \quad (9.1.16)$$

In particular $\{y_1, y_2, x_3, \dots, x_d\}$ spans X . The astute reader will think back to linear algebra and notice that we are row-reducing a matrix. We continue this procedure. If $m < d$, then we are done. So suppose $m \geq d$. After d steps we obtain that $\{y_1, y_2, \dots, y_d\}$ spans X . Any other vector v in X is a linear combination of $\{y_1, y_2, \dots, y_d\}$, and hence cannot be in T as T is linearly independent. So $m = d$. Let us look at (ii). First notice that if we have a set T of k linearly independent vectors that do not span X , then we can always choose a vector $v \in X \setminus \text{span}(T)$. The set $T \cup \{v\}$ is linearly independent (exercise). If $\dim X = d$, then there must exist some linearly independent set of d vectors T , and it must span X , otherwise we could choose a larger set of linearly independent vectors. So we have a basis of d vectors. On the other hand if we have a basis of d vectors, it is linearly independent and spans X . By (i) we know there is no set of $d+1$ linearly independent vectors, so dimension must be d . For (iii) notice that $\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$. To see (iv), suppose Y is a vector space and $Y \subset X$, where $\dim X = d$. As X cannot contain $d+1$ linearly independent vectors, neither can Y . For (v) suppose T is a set of m vectors that is linearly dependent and spans X . Then one of the vectors is a linear combination of the others. Therefore if we remove it from T we obtain a set of $m-1$ vectors that still span X and hence $\dim X \leq m-1$. For (vi) suppose $T = \{x_1, \dots, x_m\}$ is a linearly independent set. We follow the procedure above in the proof of (ii) to keep adding vectors while keeping the set linearly independent. As the dimension is d we can add a vector exactly $d-m$ times. Linear mappings A mapping $A: X \rightarrow Y$ of vector spaces X and Y is linear (or a linear transformation) if for every $a \in \mathbb{R}$ and $x, y \in X$ we have

$$A(ax) = aA(x) \quad A(x+y) = A(x) + A(y). \quad (9.1.17)$$

We usually write Ax instead of $A(x)$ if A is linear. If A is one-to-one and onto then we say A is invertible and we denote the inverse by A^{-1} . If $A: X \rightarrow X$ is linear then we say A is a linear operator on X . We write $L(X, Y)$ for the set of all linear transformations from X to Y , and just $L(X)$ for the set of linear operators on X . If $a, b \in \mathbb{R}$ and $A, B \in L(X, Y)$, define the transformation $aA + bB$

$$(aA + bB)(x) = aAx + bBx. \quad (9.1.18)$$

If $A \in L(Y, Z)$ and $B \in L(X, Y)$, define the transformation AB as

$$ABx := A(Bx). \quad (9.1.19)$$

Finally denote by $I \in L(X)$ the identity: the linear operator such that $Ix = x$ for all x . It is not hard to see that $aA + bB \in L(X, Y)$, and that $AB \in L(X, Z)$. In particular, $L(X, Y)$ is a vector space. It is obvious that if A is linear then $A0 = 0$. If $A: X \rightarrow Y$ is invertible, then A^{-1} is linear. Let $a \in \mathbb{R}$ and $y \in Y$. As A is onto, then there is an x such that $y = Ax$, and further as it is also one-to-one $A^{-1}(Az) = z$ for all $z \in X$. So

$$A^{-1}(ay) = A^{-1}(aAx) = A^{-1}(A(ax)) = ax = aA^{-1}(y). \quad (9.1.20)$$

Similarly let $y_1, y_2 \in Y$, and $x_1, x_2 \in X$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$, then

$$A^{-1}(y_1 + y_2) = A^{-1}(Ax_1 + Ax_2) = A^{-1}(A(x_1 + x_2)) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2). \quad (9.1.21)$$

If $A: X \rightarrow Y$ is linear then it is completely determined by its values on a basis of X . Furthermore, if B is a basis, then any function $\tilde{A}: B \rightarrow Y$ extends to a linear function on X . For infinite dimensional spaces, the proof is essentially the same, but a little trickier to write, so let us stick with finitely many dimensions. We leave the infinite dimensional case to the reader. Let $\{x_1, x_2, \dots, x_n\}$ be a basis and suppose $A(x_j) = y_j$. Then every $x \in X$ has a unique representation

$$x = \sum_{j=1}^n b^j x_j \quad (9.1.22)$$

for some numbers $b^1, b^2, \dots, b^n$. Then by linearity

$$Ax = A \sum_{j=1}^n b^j x_j = \sum_{j=1}^n b^j Ax_j = \sum_{j=1}^n b^j y_j. \quad (9.1.23)$$

The “furthermore” follows by defining the extension $Ax = \sum_{j=1}^n b^j y_j$, and noting that this is well defined by uniqueness of the representation of x . If X is a finite dimensional vector space and $A: X \rightarrow X$ is linear, then A is one-to-one if and only if it is onto. Let $\{x_1, x_2, \dots, x_n\}$ be a basis for X . Suppose A is one-to-one. Now suppose

$$\sum_{j=1}^n c^j Ax_j = A \sum_{j=1}^n c^j x_j = 0. \quad (9.1.24)$$

As A is one-to-one, the only vector that is taken to 0 is 0 itself. Hence,

$$0 = \sum_{j=1}^n c^j x_j \quad (9.1.25)$$

and so $c^j = 0$ for all j . Therefore, $\{Ax_1, Ax_2, \dots, Ax_n\}$ is linearly independent. By an above proposition and the fact that the dimension is n , we have that $\{Ax_1, Ax_2, \dots, Ax_n\}$ span X . As any point $x \in X$ can be written as

$$x = \sum_{j=1}^n a^j Ax_j = A \sum_{j=1}^n a^j x_j, \quad (9.1.26)$$

so A is onto. Now suppose A is onto. As A is determined by the action on the basis we see that every element of X has to be in the span of $\{Ax_1, \dots, Ax_n\}$. Suppose

$$A \sum_{j=1}^n c^j x_j = \sum_{j=1}^n c^j Ax_j = 0. \quad (9.1.27)$$

By the same proposition as $\{Ax_1, Ax_2, \dots, Ax_n\}$ span X , the set is independent, and hence $c^j = 0$ for all j . This means that A is one-to-one. If $Ax = Ay$, then $A(x - y) = 0$ and so $x = y$. Convexity A subset U of a vector space is convex if whenever $x, y \in U$, the line segment from x to y lies in U . That is, if the convex combination $(1 - t)x + ty$ is in U for all $t \in [0, 1]$. See . Note that in $\mathbb{R}$, every connected interval is convex. In $\mathbb{R}^2$ (or higher dimensions) there are lots of nonconvex connected sets. For example the set $\mathbb{R}^2 \setminus \{0\}$ is not convex but it is connected. To see this simply take any $x \in \mathbb{R}^2 \setminus \{0\}$ and let $y := -x$. Then $(\frac{1}{2})x + (\frac{1}{2})y = 0$, which is not in the set. On the other hand, the ball $B(x, r) \subset \mathbb{R}^n$ (using the standard metric on $\mathbb{R}^n$) is always convex by the triangle inequality. Show that in $\mathbb{R}^n$ any ball $B(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$ is convex. Any subspace V of a vector space X is convex. A somewhat more complicated example is given by the following. Let $C([0, 1], \mathbb{R})$ be the vector space of continuous real valued functions on $\mathbb{R}$. Let $X \subset C([0, 1], \mathbb{R})$ be the set of those f such

$$\int_0^1 f(x) dx \leq 1 \quad \text{and} \quad f(x) \geq 0 \text{ for all } x \in [0, 1]. \quad (9.1.28)$$

Then X is convex. Take $t \in [0, 1]$ and note that if $f, g \in X$ then $tf(x) + (1 - t)g(x) \geq 0$ for all x . Furthermore

$$\int_0^1 (tf(x) + (1 - t)g(x)) dx = t \int_0^1 f(x) dx + (1 - t) \int_0^1 g(x) dx \leq 1. \quad (9.1.29)$$

Note that X is not a subspace of $C([0, 1], \mathbb{R})$. The intersection two closed sets is convex. In fact, If $\{C_\lambda\}_{\lambda \in I}$ is an arbitrary collection of convex sets, then

$$C := \bigcap_{\lambda \in I} C_\lambda \quad (9.1.30)$$

is convex. The proof is easy. If $x, y \in C$, then $x, y \in C_\lambda$ for all $\lambda \in I$, and hence if $t \in [0, 1]$, then $tx + (1 - t)y \in C_\lambda$ for all $\lambda \in I$. Therefore $tx + (1 - t)y \in C$ and C is convex. Let $T: V \rightarrow W$ be a linear mapping between two vector spaces and let $C \subset V$ be a convex set. Then $T(C)$ is convex. Take any two points $p, q \in T(C)$. Then pick $x, y \in C$ such that $T(x) = p$ and $T(y) = q$. As C is convex then for all $t \in [0, 1]$ we have $tx + (1 - t)y \in C$, so

$$T(tx + (1 - t)y) = tT(x) + (1 - t)T(y) = tp + (1 - t)q \quad (9.1.31)$$

is in $T(C)$. For completeness, let us A very useful construction is the convex hull. Given any set $S \subset V$ of a vector space, define the convex hull of S , by

$$\text{co}(S) := \bigcap \{C \subset V : S \subset C, \text{ and } C \text{ is convex}\}. \quad (9.1.32)$$

That is, the convex hull is the smallest convex set containing S . Note that by a proposition above, the intersection of convex sets is convex and hence, the convex hull is convex. The convex hull of 0 and 1 in $\mathbb{R}$ is $[0, 1]$. Proof: Any convex set containing 0 and 1 must contain $[0, 1]$. The set $[0, 1]$ is convex, therefore it must be the convex hull. Exercises Verify that $\mathbb{R}^n$ is a vector space. Let X be a vector space. Prove that a finite set of vectors $\{x_1, \dots, x_n\} \subset X$ is linearly independent if and only if for every $j = 1, 2, \dots, n$

$$\text{span}(\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}) \subsetneq \text{span}(\{x_1, \dots, x_n\}). \quad (9.1.33)$$

That is, the span of the set with one vector removed is strictly smaller. Prove that $C([0, 1], \mathbb{R})$ is an infinite dimensional vector space where the operations are defined in the obvious way: $s = f + g$ and $m = fg$ are defined as $s(x) := f(x) + g(x)$ and $m(x) := f(x)g(x)$. Hint: for the dimension, think of functions that are only nonzero on the interval $(\frac{1}{n+1}, \frac{1}{n})$. Let $k: [0, 1]^2 \rightarrow \mathbb{R}$ be continuous. Show that $L: C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ defined by

$$Lf(y) := \int_0^1 k(x, y)f(x) dx \quad (9.1.34)$$

is a linear operator. That is, show that L is well defined (that Lf is continuous), and that L is linear.

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9.2: Analysis with Vector spaces

Norms Let us start measuring distance. If X is a vector space, then we say a real valued function $\|\cdot\|$ is a norm if: $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$. $\|cx\| = |c| \|x\|$ for all $c \in \mathbb{R}$ and $x \in X$. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (Triangle inequality). Before defining the standard norm on $\mathbb{R}^n$, let us define the standard scalar dot product on $\mathbb{R}^n$. For two vectors if $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$ and $y = (y^1, y^2, \dots, y^n) \in \mathbb{R}^n$, define

$$x \cdot y := \sum_{j=1}^n x^j y^j. \quad (9.2.1)$$

It is easy to see that the dot product is linear in each variable separately, that is, it is a linear mapping when you keep one of the variables constant. The Euclidean norm is then defined as

$$\|x\| := \sqrt{x \cdot x} = \sqrt{(x^1)^2 + (x^2)^2 + \dots + (x^n)^2}. \quad (9.2.2)$$

It is easy to see that the Euclidean norm satisfies (i) and (ii). To prove that (iii) holds, the key key inequality in the so-called Cauchy-Schwarz inequality that we have seen before. As this inequality is so important let us restate and reprove it using the notation of this chapter. Let $x, y \in \mathbb{R}^n$, then

$$|x \cdot y| \leq \|x\| \|y\| = \sqrt{x \cdot x} \sqrt{y \cdot y}, \quad (9.2.3)$$

with equality if and only if the vectors are scalar multiples of each other. If $x = 0$ or $y = 0$, then the theorem holds trivially. So assume $x \neq 0$ and $y \neq 0$. If x is a scalar multiple of y , that is $x = \lambda y$ for some $\lambda \in \mathbb{R}$, then the theorem holds with equality:

$$|\lambda y \cdot y| = |\lambda| |y \cdot y| = |\lambda| \|y\|^2 = \|\lambda y\| \|y\|. \quad (9.2.4)$$

Next take $x + ty$,

$$\|x + ty\|^2 = (x + ty) \cdot (x + ty) = x \cdot x + x \cdot ty + ty \cdot x + ty \cdot ty = \|x\|^2 + 2t(x \cdot y) + t^2 \|y\|^2. \quad (9.2.5)$$

If x is not a scalar multiple of y , then $\|x + ty\|^2 > 0$ for all t . So the above polynomial in t is never zero. From elementary algebra it follows that the discriminant must be negative:

$$4(x \cdot y)^2 - 4\|x\|^2 \|y\|^2 < 0, \quad (9.2.6)$$

or in other words $(x \cdot y)^2 < \|x\|^2 \|y\|^2$. Item (iii), the triangle inequality, follows via a simple computation:

$$\|x + y\|^2 = x \cdot x + y \cdot y + 2(x \cdot y) \leq \|x\|^2 + \|y\|^2 + 2(\|x\| \|y\|) = (\|x\| + \|y\|)^2. \quad (9.2.7)$$

The distance $d(x, y) := \|x - y\|$ is the standard distance function on $\mathbb{R}^n$ that we used when we talked about metric spaces. In fact, on any vector space X , once we have a norm (any norm), we define a distance $d(x, y) := \|x - y\|$ that makes X into a metric space (an easy exercise). Let $A \in L(X, Y)$. Define

$$\|A\| := \sup\{\|Ax\| : x \in X \text{ with } \|x\| = 1\}. \quad (9.2.8)$$

The number $\|A\|$ is called the operator norm. We will see below that indeed it is a norm (at least for finite dimensional spaces). By linearity we get

$$\|A\| = \sup\{\|Ax\| : x \in X \text{ with } \|x\| = 1\} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}. \quad (9.2.9)$$

This implies that

$$\|Ax\| \leq \|A\| \|x\|. \quad (9.2.10)$$

It is not hard to see from the definition that $\|A\| = 0$ if and only if $A = 0$, that is, if A takes every vector to the zero vector. For finite dimensional spaces $\|A\|$ is always finite as we prove below. This also implies that A is continuous. For infinite dimensional spaces neither statement needs to be true. For a simple example, take the vector space of continuously differentiable functions on $[0, 1]$ and as the norm use the uniform norm. The functions $\sin(nx)$ have norm 1, but the derivatives have norm n . So differentiation (which is a linear operator) has unbounded norm on this space. But let us stick to finite dimensional spaces now. If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is uniformly continuous (Lipschitz with constant $\|A\|$). If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $c \in \mathbb{R}$, then

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|cA\| = |c| \|A\|. \quad (9.2.11)$$

In particular $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with distance $\|A - B\|$. If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then

$$\|BA\| \leq \|B\| \|A\|. \quad (9.2.12)$$

For (i), let $x \in \mathbb{R}^n$. We know that A is defined by its action on a basis. Write

$$x = \sum_{j=1}^n c^j e_j. \quad (9.2.13)$$

Then

$$\|Ax\| = \left\| \sum_{j=1}^n c^j A e_j \right\| \leq \sum_{j=1}^n |c^j| \|A e_j\|. \quad (9.2.14)$$

If $\|x\| = 1$, then it is easy to see that $|c^j| \leq 1$ for all j , so

$$\|Ax\| \leq \sum_{j=1}^n |c^j| \|A e_j\| \leq \sum_{j=1}^n \|A e_j\|. \quad (9.2.15)$$

The right hand side does not depend on x and so we are done, we have found a finite upper bound. Next,

$$\|A(x - y)\| \leq \|A\| \|x - y\| \quad (9.2.16)$$

as we mentioned above. So if $\|A\| < \infty$, then this says that A is Lipschitz with constant $\|A\|$. For (ii), let us note that

$$\|(A + B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| \|x\| + \|B\| \|x\| = (\|A\| + \|B\|) \|x\|. \quad (9.2.17)$$

So $\|A + B\| \leq \|A\| + \|B\|$. Similarly

$$\|(cA)x\| = |c| \|Ax\| \leq (|c| \|A\|) \|x\|. \quad (9.2.18)$$

Thus $\|cA\| \leq |c| \|A\|$. Next note

$$|c| \|Ax\| = \|cAx\| \leq \|cA\| \|x\|. \quad (9.2.19)$$

Hence $|c| \|A\| \leq \|cA\|$. That we have a metric space follows pretty easily, and is left to student. For (iii) write

$$\|BAx\| \leq \|B\| \|Ax\| \leq \|B\| \|A\| \|x\|. \quad (9.2.20)$$

As a norm defines a metric, we have defined a metric space topology on $L(\mathbb{R}^n, \mathbb{R}^m)$ so we can talk about open/closed sets, continuity, and convergence. Note that we have defined a norm only on $\mathbb{R}^n$ and not on an arbitrary finite dimensional vector space. However, after picking bases, we can define a norm on any vector space in the same way. So we really have a topology on any $L(X, Y)$, although the precise metric would depend on the basis picked. Let $U \subset L(\mathbb{R}^n)$ be the set of invertible linear operators. If $A \in U$ and $B \in L(\mathbb{R}^n)$, and

$$\|A - B\| < \frac{1}{\|A^{-1}\|}, \quad (9.2.21)$$

then B is invertible. U is open and $A \mapsto A^{-1}$ is a continuous function on U . The proposition says that U is an open set and $A \mapsto A^{-1}$ is continuous on U . You should always think back to $\mathbb{R}^1$, where linear operators are just numbers a . The operator a is invertible ($a^{-1} = \frac{1}{a}$) whenever $a \neq 0$. Of course $a \mapsto \frac{1}{a}$ is continuous. When $n > 1$, then there are other noninvertible operators, and in general things are a bit more difficult. Let us prove (i). First a straight forward computation

$$\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\| \|Ax\| \leq \|A^{-1}\| (\|(A - B)x\| + \|Bx\|) \leq \|A^{-1}\| \|A - B\| \|x\| + \|A^{-1}\| \|Bx\|. \quad (9.2.22)$$

Now assume $x \neq 0$ and so $\|x\| \neq 0$. Using [eqcontineq] we obtain

$$\|x\| < \|x\| + \|A^{-1}\| \|Bx\|, \quad (9.2.23)$$

or in other words $\|Bx\| \neq 0$ for all nonzero x , and hence $Bx \neq 0$ for all nonzero x . This is enough to see that B is one-to-one (if $Bx = By$, then $B(x - y) = 0$, so $x = y$). As B is one-to-one operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ it is onto and hence invertible. Let us look at (ii). Let B be invertible and near A^{-1} , that is [eqcontineq] is satisfied. In fact, suppose $\|A - B\| \|A^{-1}\| < \frac{1}{2}$. Then we have shown above (using $B^{-1}y$ instead of x)

$$\|B^{-1}y\| \leq \|A^{-1}\| \|A - B\| \|B^{-1}y\| + \|A^{-1}\| \|y\| \leq \frac{1}{2} \|B^{-1}y\| + \|A^{-1}\| \|y\|, \quad (9.2.24)$$

or $\frac{\|B^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}\|}{\|A^{-1}\|} \frac{\|A-B\|}{\|A^{-1}\|} \frac{\|A^{-1}\|}{\|A^{-1}\|} \frac{\|y\|}{\|y\|} \leq 2 \frac{\|A^{-1}\|}{\|A^{-1}\|} \frac{\|A-B\|}{\|A^{-1}\|} \frac{\|A^{-1}\|}{\|A^{-1}\|} \frac{\|y\|}{\|y\|}$. So $\|B^{-1}\| \leq 2 \|A^{-1}\|$. Now note that

$$A^{-1}(A-B)B^{-1} = A^{-1}(AB^{-1} - I) = B^{-1} - A^{-1}, \quad (9.2.25)$$

and

$$\|B^{-1} - A^{-1}\| = \|A^{-1}(A-B)B^{-1}\| \leq \|A^{-1}\| \|A-B\| \|B^{-1}\| \leq \quad (9.2.26)$$

FIXME: continuity of vector space Matrices Finally let us get to matrices, which are a convenient way to represent finite-dimensional operators. If we have bases $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$ for vector spaces X and Y , then we know that a linear operator is determined by its values on the basis. Given $A \in L(X, Y)$, define the numbers $\{a_i^j\}$ as follows

$$Ax_j = \sum_{i=1}^m a_i^j y_i, \quad (9.2.27)$$

and write them as a matrix

$$A = \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{bmatrix}. \quad (9.2.28)$$

Note that the columns of the matrix are precisely the coefficients that represent Ax_j . Let us derive the familiar rule for matrix multiplication. When

$$x = \sum_{j=1}^n \gamma^j x_j, \quad (9.2.29)$$

then

$$Ax = \sum_{j=1}^n \sum_{i=1}^m \gamma^j a_i^j y_i = \sum_{i=1}^m \left(\sum_{j=1}^n \gamma^j a_i^j \right) y_i, \quad (9.2.30)$$

which gives rise to the familiar rule for matrix multiplication. There is a one-to-one correspondence between matrices and linear operators in $L(X, Y)$. That is, once we fix a basis in X and in Y . If we would choose a different basis, we would get different matrices. This is important, the operator A acts on elements of X , the matrix is something that works with n -tuples of numbers. If B is an r -by- m matrix with entries b_k^j , then the matrix for BA has the i, k th entry c_k^i being

$$c_k^i = \sum_{j=1}^m b_k^j a_j^i. \quad (9.2.31)$$

Note how upper and lower indices line up. A linear mapping changing one basis to another is then just a square matrix in which the columns represent basis elements of the second basis in terms of the first basis. We call such a linear mapping an change of basis. Now suppose all the bases are just the standard bases and $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. If we recall the Cauchy-Schwarz inequality we note that

$$\|Ax\|^2 = \sum_{i=1}^m \left(\sum_{j=1}^n \gamma^j a_j^i \right)^2 \leq \sum_{i=1}^m \left(\sum_{j=1}^n (\gamma^j)^2 \right) \left(\sum_{j=1}^n (a_j^i)^2 \right) = \sum_{i=1}^m \left(\sum_{j=1}^n (a_j^i)^2 \right) \|x\|^2. \quad (9.2.32)$$

In other words, we have a bound on the operator norm

$$\|A\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n (a_j^i)^2}. \quad (9.2.33)$$

If the entries go to zero, then $\|A\|$ goes to zero. In particular, if A is fixed and B is changing such that the entries of $A - B$ go to zero then B goes to A in operator norm. That is B goes to A in the metric space topology induced by the operator norm. We have proved the first part of: If $f: S \rightarrow \mathbb{R}^{nm}$ is a continuous function for a metric space S , then taking the components of f as the entries of a matrix, f is a continuous mapping from S to $L(\mathbb{R}^n, \mathbb{R}^m)$. Conversely if $f: S \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function then the

entries of the matrix are continuous functions. The proof of the second part is rather easy. Take $f(x)e_j$ and note that is a continuous function to $\mathbb{R}^m$ with standard Euclidean norm (Note $\|(A - B)e_j\| \leq \|A - B\|$). Such a function recall from last semester that such a function is continuous if and only if its components are continuous and these are the components of the j th column of the matrix $f(x)$. Determinants It would be nice to have an easy test for when is a matrix invertible. This is where determinants come in. First define the symbol $\text{sgn}(x)$ for a number is defined by

$$\text{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (9.2.34)$$

Suppose $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of the integers $(1, \dots, n)$. It is not hard to see that any permutation can be obtained by a sequence of transpositions (switchings of two elements). Call a permutation even (resp. odd) if it takes an even (resp. odd) number of transpositions to get from σ to $(1, \dots, n)$. It can be shown that this is well defined, in fact it is not hard to show that

$$\text{sgn}(\sigma) := \text{sgn}(\sigma_1, \dots, \sigma_n) = \prod_{p < q} \text{sgn}(\sigma_q - \sigma_p) \quad (9.2.35)$$

is 1 if σ is even and -1 if σ is odd. This fact can be proved by noting that applying a transposition changes the sign, which is not hard to prove by induction on n . Then note that the sign of $(1, 2, \dots, n)$ is 1. Let S_n be the set of all permutations on n elements (the symmetric group). Let $A = [a_j^i]$ be a matrix. Define the determinant of A

$$\det(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma_i}^i. \quad (9.2.36)$$

$\det(I) = 1$. $\det([x_1 x_2 \dots x_n])$ as a function of column vectors x_j is linear in each variable x_j separately. If two columns of a matrix are interchanged, then the determinant changes sign. If two columns of A are equal, then $\det(A) = 0$. If a column is zero, then $\det(A) = 0$. $A \mapsto \det(A)$ is a continuous function. $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and $\det[a] = a$. In fact, the determinant is the unique function that satisfies (i), (ii), and (iii). But we digress. We go through the proof quickly, as you have likely seen this before. (i) is trivial. For (ii) Notice that each term in the definition of the determinant contains exactly one factor from each column. Part (iii) follows by noting that switching two columns is like switching the two corresponding numbers in every element in S_n . Hence all the signs are changed. Part (iv) follows because if two columns are equal and we switch them we get the same matrix back and so part (iii) says the determinant must have been 0. Part (v) follows because the product in each term in the definition includes one element from the zero column. Part (vi) follows as $\det$ is a polynomial in the entries of the matrix and hence continuous. We have seen that a function defined on matrices is continuous in the operator norm if it is continuous in the entries. Finally, part (vii) is a direct computation. If A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \det(B)$. In particular, A is invertible if and only if $\det(A) \neq 0$ and in this case, $\det(A^{-1}) = \frac{1}{\det(A)}$. Let $b_1, b_2, \dots, b_n$ be the columns of B . Then

$$AB = [Ab_1 \quad Ab_2 \quad \dots \quad Ab_n]. \quad (9.2.37)$$

That is, the columns of AB are $Ab_1, Ab_2, \dots, Ab_n$. Let b_j^i denote the elements of B and a_j the columns of A . Note that $Ae_j = a_j$. By linearity of the determinant as proved above we have

$$\begin{aligned} \det(AB) &= \det([Ab_1 \quad Ab_2 \quad \dots \quad Ab_n]) = \det\left(\left[\sum_{j=1}^n b_1^j a_j \quad Ab_2 \quad \dots \quad Ab_n\right]\right) \\ &= \sum_{j=1}^n b_1^j \det([a_j \quad Ab_2 \quad \dots \quad Ab_n]) \\ &= \sum_{1 \leq j_1, j_2, \dots, j_n \leq n} b_1^{j_1} b_2^{j_2} \dots b_n^{j_n} \det([a_{j_1} \quad a_{j_2} \quad \dots \quad a_{j_n}]) \\ &= \left(\sum_{(j_1, j_2, \dots, j_n) \in S_n} b_1^{j_1} b_2^{j_2} \dots b_n^{j_n} \text{sgn}(j_1, j_2, \dots, j_n) \right) \det([a_1 \quad a_2 \quad \dots \quad a_n]). \end{aligned}$$

In the above, go from all integers between 1 and n , to just elements of S_n by noting that when two columns in the determinant are the same then the determinant is zero. We then reorder the columns to the original ordering and obtain the sgn . The conclusion follows by recognizing the determinant of B . The rows and columns are swapped, but a moment's reflection reveals it does not matter. We could also just plug in $A = I$ above. For the second part of the theorem note that if A is invertible, then $A^{-1}A = I$ and so $\det(A^{-1}) \det(A) = 1$. If A is not invertible, then the columns are linearly dependent. That is, suppose

$$\sum_{j=1}^n c^j a_j = 0. \quad (9.2.38)$$

Without loss of generality suppose $c^1 \neq 1$. Take

$$B := \begin{bmatrix} c^1 & 0 & 0 & \cdots & 0 \\ c^2 & 1 & 0 & \cdots & 0 \\ c^3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c^n & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (9.2.39)$$

It is not hard to see from the definition that $\det(B) = c^1 \neq 0$. Then $\det(AB) = \det(A) \det(B) = c^1 \det(A)$. Note that the first column of AB is zero, and hence $\det(AB) = 0$. Thus $\det(A) = 0$. There are three types of so-called elementary matrices. First for some $j = 1, 2, \dots, n$ and some $\lambda \in \mathbb{R}, \lambda \neq 0$, an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ \lambda e_i & \text{if } i = j. \end{cases} \quad (9.2.40)$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with the k th row multiplied by λ . It is an easy computation (exercise) that $\det(E) = \lambda$. Second, for some j and k with $j \neq k$, and $\lambda \in \mathbb{R}$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ e_i + \lambda e_k & \text{if } i = j. \end{cases} \quad (9.2.41)$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with λ times the k th row added to the j th row. It is an easy computation (exercise) that $\det(E) = 1$. Finally for some j and k with $j \neq k$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j \text{ and } i \neq k, \\ e_k & \text{if } i = j, \\ e_j & \text{if } i = k. \end{cases} \quad (9.2.42)$$

Given any $n \times m$ matrix M the matrix EM is the same matrix with j th and k th rows swapped. It is an easy computation (exercise) that $\det(E) = -1$. Elementary matrices are useful for computing the determinant. The proof of the following proposition is left as an exercise. [prop:elematrixdecomp] Let T be an $n \times n$ invertible matrix. Then there exists a finite sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that

$$T = E_1 E_2 \cdots E_k, \quad (9.2.43)$$

and

$$\det(T) = \det(E_1) \det(E_2) \cdots \det(E_k). \quad (9.2.44)$$

Determinant is independent of the basis. In other words, if B is invertible then,

$$\det(A) = \det(B^{-1}AB). \quad (9.2.45)$$

The proof is immediate. If in one basis A is the matrix representing a linear operator, then for another basis we can find a matrix B such that the matrix $B^{-1}AB$ takes us to the first basis, applies A in the first basis, and takes us back to the basis we started with. Therefore, the determinant can be defined as a function on the space $L(X)$ for some finite dimensional metric space X , not just on matrices. We choose a basis on X , and we can represent a linear mapping using a matrix with respect to this basis. We obtain the same determinant as if we had used any other basis. It follows from the two propositions that

$$\det: L(X) \rightarrow \mathbb{R} \quad (9.2.46)$$

is a well-defined and continuous function. Exercises If X is a vector space with a norm $\|\cdot\|$, then show that $d(x, y) := \|x - y\|$ makes X a metric space. Verify the computation of the determinant for the three types of elementary matrices.

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9.3: The Derivative

Recall that when we had a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we defined the derivative at x as

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

In other words, there was a number a (the derivative of f at x) such that

$$\lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} - a \right| = \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x) - ah}{h} \right| = \lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - ah|}{|h|} = 0.$$

Multiplying by a is a linear map in one dimension. That is, we think of $a \in L(\mathbb{R}^1, \mathbb{R}^1)$. We use this definition to extend differentiation to more variables. Let $U \subset \mathbb{R}^n$ be an open subset and $f: U \rightarrow \mathbb{R}^m$. We say f is differentiable at $x \in U$ if there exists an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}^n}} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

We define $Df(x) := A$, or $f'(x) := A$, and we say A is the derivative of f at x . When f is differentiable at all $x \in U$, we say simply that f is differentiable. For a differentiable function, the derivative of f is a function from U to $L(\mathbb{R}^n, \mathbb{R}^m)$. Compare to the one dimensional case, where the derivative is a function from U to $\mathbb{R}$, but we really want to think of $\mathbb{R}$ here as $L(\mathbb{R}^1, \mathbb{R}^1)$. The norms above must be in the right spaces of course. The norm in the numerator is in $\mathbb{R}^m$, and the norm in the denominator is $\mathbb{R}^n$ where h lives. Normally it is understood that $h \in \mathbb{R}^n$ from context. We will not explicitly say so from now on. We have again cheated somewhat and said that A is the derivative. We have not shown yet that there is only one, let us do that now. Let $U \subset \mathbb{R}^n$ be an open subset and $f: U \rightarrow \mathbb{R}^m$. Suppose $x \in U$ and there exist $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|} = 0.$$

Then $A = B$.

$$\begin{aligned} \frac{\|(A-B)h\|}{\|h\|} &= \frac{\|f(x+h) - f(x) - Ah - (f(x+h) - f(x) - Bh)\|}{\|h\|} \\ &\leq \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} + \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|}. \end{aligned}$$

So $\frac{\|(A-B)h\|}{\|h\|} \rightarrow 0$ as $h \rightarrow 0$. That is, given $\epsilon > 0$, then for all h in some δ -ball around the origin

$$\epsilon > \frac{\|(A-B)h\|}{\|h\|} = \|(A-B)\frac{h}{\|h\|}\|.$$

For any x with $\|x\| = 1$ let $h = (\frac{\epsilon}{\|A-B\|})x$, then $\|h\| < \delta$ and $\frac{h}{\|h\|} = x$ and so $\|A-B\| \leq \epsilon$. So $A = B$. If $f(x) = Ax$ for a linear mapping A , then $f'(x) = A$. This is easily seen:

$$\frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = \frac{\|A(x+h) - Ax - Ah\|}{\|h\|} = \frac{0}{\|h\|} = 0.$$

Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^m$ be differentiable at x_0 . Then f is continuous at x_0 . Another way to write the differentiability is to write

$$r(h) := f(x_0+h) - f(x_0) - f'(x_0)h.$$

As $\frac{\|r(h)\|}{\|h\|}$ must go to zero as $h \rightarrow 0$, then $r(h)$ itself must go to zero. The mapping $h \mapsto f'(x_0)h$ is linear mapping between finite dimensional spaces. Therefore it is continuous and goes to zero.

Therefore $f(x_0+h)$ must go to $f(x_0)$ as $h \rightarrow 0$. That is, f is continuous at x_0 . Let $U \subset \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^m$ be differentiable at $x_0 \in U$. Let $V \subset \mathbb{R}^m$ be open, $f(U) \subset V$ and let $g: V \rightarrow \mathbb{R}^l$ be differentiable at $f(x_0)$. Then

$$F(x) = g(f(x))$$

is differentiable at x_0 and

$$F'(x_0) = g'(f(x_0))f'(x_0).$$

Without the points this is sometimes written as $F' = (f \circ g)' = g' f'$. The way to understand it is that the derivative of the composition $g \circ f$ is the composition of the derivatives of g and f . That is, if $A := f'(x_0)$ and $B := g'(f(x_0))$, then $F'(x_0) = BA$. Let $A := f'(x_0)$ and $B := g'(f(x_0))$. Take $h \in \mathbb{R}^n$ and write $y_0 = f(x_0)$, $k = f(x_0+h) - f(x_0)$. Let

$$r(h) := f(x_0+h) - f(x_0) - Ah = k - Ah.$$

Then

$$\left| \frac{F(x_0+h) - F(x_0) - BAh}{\|h\|} \right| = \left| \frac{g(f(x_0+h)) - g(f(x_0)) - BAh}{\|h\|} \right| = \left| \frac{g(y_0+k) - g(y_0) - Bk}{\|k\|} \right| \cdot \left| \frac{\|k\|}{\|h\|} \right|$$

First, $\|B\|$ is constant and f is differentiable at x_0 , so the term $\|B\| \frac{\|r(h)\|}{\|h\|}$ goes to 0. Next as f is continuous at x_0 , we have that as h goes to 0, then k goes to 0. Therefore $\frac{\|g(y_0+k) - g(y_0) - Bk\|}{\|k\|}$ goes to 0 because g is differentiable at y_0 . Finally

$$\frac{\|f(x_0+h) - f(x_0)\|}{\|h\|} \leq \frac{\|f(x_0+h) - f(x_0) - Ah\|}{\|h\|} + \frac{\|Ah\|}{\|h\|} \leq \frac{\|f(x_0+h) - f(x_0) - Ah\|}{\|h\|} + \|A\|.$$

As f is differentiable at x_0 , the term $\frac{\|f(x_0+h) - f(x_0)\|}{\|h\|}$ stays bounded as h goes to 0. Therefore, $\frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|}$ goes to zero, and $F'(x_0) = BA$, which is what was claimed. Partial derivatives

There is another way to generalize the derivative from one dimension. We can hold all but one variables constant and take the regular derivative. Let $f: U \rightarrow \mathbb{R}$ be a function on an open set $U \subset \mathbb{R}^n$. If the following limit exists we write

$$\frac{\partial f}{\partial x^j}(x) := \lim_{h \rightarrow 0} \frac{f(x^1, \dots, x^{j-1}, x^j + h, x^{j+1}, \dots, x^n) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x + he_j) - f(x)}{h}.$$

We call $\frac{\partial f}{\partial x^j}(x)$ the partial derivative of f with respect to x^j . Sometimes we write $D_j f$ instead. For a mapping $f: U \rightarrow \mathbb{R}^m$ we write $f = (f^1, f^2, \dots, f^m)$, where f^k are real-valued functions. Then we define $\frac{\partial f^k}{\partial x^j}$ (or write it as $D_j f^k$). Partial derivatives are easier to compute with all the machinery of calculus, and they provide a way to compute the total derivative of a function. Let $U \subset \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^m$ be differentiable at $x_0 \in U$. Then all the partial derivatives at x_0 exist and in terms of the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, $f'(x_0)$ is represented by the matrix

$$\begin{bmatrix} \frac{\partial f^1}{\partial x^1}(x_0) & \frac{\partial f^1}{\partial x^2}(x_0) & \dots & \frac{\partial f^1}{\partial x^n}(x_0) \\ \frac{\partial f^2}{\partial x^1}(x_0) & \frac{\partial f^2}{\partial x^2}(x_0) & \dots & \frac{\partial f^2}{\partial x^n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x_0) & \frac{\partial f^m}{\partial x^2}(x_0) & \dots & \frac{\partial f^m}{\partial x^n}(x_0) \end{bmatrix}.$$

In other words

$$f'(x_0)e_j = \sum_{k=1}^m \frac{\partial f^k}{\partial x^j}(x_0)e_k.$$

If $h = \sum_{j=1}^n c^j e_j$, then

$$f'(x_0)h = \sum_{j=1}^n \sum_{k=1}^m c^j \frac{\partial f^k}{\partial x^j}(x_0)e_k.$$

Again note the up-down pattern with the indices being summed over. That is on purpose. Fix a j and note that

$$\begin{aligned} \left\| \frac{f(x_0 + he_j) - f(x_0)}{h} - f'(x_0)e_j \right\| &= \left\| \frac{f(x_0 + he_j) - f(x_0) - f'(x_0)he_j}{h} \right\| \\ &= \frac{\|f(x_0 + he_j) - f(x_0) - f'(x_0)he_j\|}{\|he_j\|}. \end{aligned}$$

As h goes to 0, the right hand side goes to zero by differentiability of f , and hence

$$\lim_{h \rightarrow 0} \frac{f(x_0 + he_j) - f(x_0)}{h} = f'(x_0)e_j.$$

Note that f is vector valued. So represent f by components $f = (f^1, f^2, \dots, f^m)$, and note that taking a limit in $\mathbb{R}^m$ is the same as taking the limit in each component separately. Therefore for any k the partial derivative

$$\frac{\partial f^k}{\partial x^j}(x_0) = \lim_{h \rightarrow 0} \frac{f^k(x_0 + he_j) - f^k(x_0)}{h}$$

exists and is equal to the k th component of $f'(x_0)e_j$, and we are done. One of the consequences of the theorem is that if f is differentiable on U , then $f': U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function if and only if all the $\frac{\partial f^k}{\partial x^j}$ are continuous functions. Gradient and directional derivatives Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}$ is a differentiable function. We define the gradient as

$$\nabla f(x) := \sum_{j=1}^n \frac{\partial f}{\partial x^j}(x)e_j.$$

Here the upper-lower indices do not really match up. Suppose $\gamma: (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is a differentiable function and the image $\gamma((a, b)) \subset U$. Write $\gamma = (\gamma^1, \gamma^2, \dots, \gamma^n)$. Let

$$g(t) := f(\gamma(t)).$$

The function g is differentiable and the derivative is

$$g'(t) = \sum_{j=1}^n \frac{\partial f}{\partial x^j}(\gamma(t)) \frac{d\gamma^j}{dt}(t) = \sum_{j=1}^n \frac{\partial f}{\partial x^j} \frac{d\gamma^j}{dt}.$$

For convenience, we sometimes leave out the points where we are evaluating as on the right hand side above. Notice

$$g'(t) = (\nabla f)(\gamma(t)) \cdot \gamma'(t) = \nabla f \cdot \gamma',$$

where the dot is the standard scalar dot product. We use this idea to define derivatives in a specific direction. A direction is simply a vector pointing in that direction. So pick a vector $u \in \mathbb{R}^n$ such that $\|u\| = 1$. Fix $x \in U$. Then define

$$\gamma(t) := x + tu.$$

It is easy to compute that $\gamma'(t) = u$ for all t . By chain rule

$$\frac{d}{dt} \Big|_{t=0} [f(x + tu)] = (\nabla f)(x) \cdot u,$$

where the notation $\frac{d}{dt} \Big|_{t=0}$ represents the derivative evaluated at $t = 0$. We also compute directly

$$\frac{d}{dt} \Big|_{t=0} [f(x + tu)] = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}.$$

We obtain the directional derivative, denoted by

$$D_u f(x) := \frac{d}{dt} \Big|_{t=0} [f(x + tu)],$$

which can be computed by one of the methods above. Let us suppose $(\nabla f)(x) \neq 0$. By Cauchy-Schwarz inequality we have

$$|D_u f(x)| \leq \|(\nabla f)(x)\|.$$

Equality is achieved when u is a scalar multiple of $(\nabla f)(x)$. That is, when

$$u = \frac{(\nabla f)(x)}{\|(\nabla f)(x)\|}.$$

We get $D_u f(x) = \|(\nabla f)(x)\|$. The gradient points in the direction in which the function grows fastest, in other words, in the direction in which $D_u f(x)$ is maximal. Bounding the derivative Let us prove a "mean value theorem" for vector valued functions. If $\varphi: [a, b] \rightarrow \mathbb{R}^n$ is differentiable on (a, b) and continuous on $[a, b]$, then there exists a t such that

$$\|\varphi(b) - \varphi(a)\| \leq (b - a) \|\varphi'(t)\|.$$

By mean value theorem on the function $(\varphi(b) - \varphi(a)) \cdot \varphi'(t)$ (the dot is the scalar dot product again) we obtain there is a t such that

$$(\varphi(b) - \varphi(a)) \cdot \varphi(b) - (\varphi(b) - \varphi(a)) \cdot \varphi(a) = \|\varphi(b) - \varphi(a)\|^2 = (\varphi(b) - \varphi(a)) \cdot \varphi'(t)$$

where we treat φ' as a simply a column vector of numbers by abuse of notation. Note that in this case, it is not hard to see that $\|\varphi'(t)\| = \sqrt{\sum_{i=1}^n (\varphi'_i(t))^2}$.

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U be a convex open set, $f: U \rightarrow \mathbb{R}^m$ a differentiable function, and an M such that

$$\|f'(x)\| \leq M$$

for all $x \in U$. Then f is Lipschitz with constant M , that is

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

for all $x, y \in U$. Fix x and y in U and note that $(1 - t)x + ty \in U$ for all $t \in [0, 1]$ by convexity. Next

$$\frac{d}{dt} [f((1 - t)x + ty)] = f'((1 - t)x + ty)(y - x).$$

By mean value theorem above we get

$$\|f(x) - f(y)\| \leq \left\| \frac{d}{dt} [f((1 - t)x + ty)] \right\| \leq \|f'((1 - t)x + ty)\| \|y - x\| \leq M\|y - x\|. \quad \text{qedhere}$$

If U is not convex the proposition is not true. To see this fact, take the set

$$U = \{(x, y) : 0.9 < x^2 + y^2 < 1.1\} \setminus \{(x, 0) : x < 0\}.$$

Let $f(x, y)$ be the angle that the line from the origin to (x, y) makes with the positive x axis. You can even write the formula for f :

$$f(x, y) = 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right).$$

Think spiral staircase with room in the middle. See . The function is differentiable, and the derivative is bounded on U , which is not hard to see. Thinking of what happens near where the negative x -axis cuts the annulus in half, we see that the conclusion cannot hold. Let us solve the differential equation $f' = 0$. If $U \subset \mathbb{R}^n$ is connected and $f: U \rightarrow \mathbb{R}^m$ is differentiable and $f'(x) = 0$, for all $x \in U$, then f is constant. For any $x \in U$, there is a ball $B(x, \delta) \subset U$. The ball $B(x, \delta)$ is convex. Since $\|f'(y)\| \leq 0$ for all $y \in B(x, \delta)$ then by the theorem, $\|f(x) - f(y)\| \leq 0\|x - y\| = 0$, so $f(x) = f(y)$ for all $y \in B(x, \delta)$. This means that $f^{-1}(c)$ is open for any $c \in \mathbb{R}^m$. Suppose $f^{-1}(c)$ is nonempty. The two sets

$$U' = f^{-1}(c), \quad U'' = f^{-1}(\mathbb{R}^m \setminus \{c\}) = \bigcup_{\substack{a \in \mathbb{R}^m \\ a \neq c}} f^{-1}(a)$$

are open disjoint, and further $U = U' \cup U''$. So as U' is nonempty, and U is connected, we have that $U'' = \emptyset$. So $f(x) = c$ for all $x \in U$. Continuously differentiable functions We say $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable, or $C^1(U)$ if f is differentiable and $f': U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous. Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^m$. The function f is continuously differentiable if and only if all the partial derivatives exist and are continuous. Without continuity the theorem does not hold. Just because partial derivatives exist does not mean that f is differentiable, in fact, f may not even be continuous. See the exercises FIXME. We have seen that if f is differentiable, then the partial derivatives exist. Furthermore, the partial derivatives are the entries of the matrix of $f'(x)$. So if $f': U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then the entries are continuous, hence the partial derivatives are continuous. To prove the opposite direction, suppose the partial derivatives exist and are continuous. Fix $x \in U$. If we can show that $f'(x)$ exists we are done, because the entries of the matrix $f'(x)$ are then the partial derivatives and if the entries are continuous functions, the matrix valued function f' is continuous. Let us do induction on dimension. First let us note that the conclusion is true when $n = 1$. In this case the derivative is just the regular derivative (exercise: you should check that the fact that the function is vector valued is not a problem). Suppose the conclusion is true for $\mathbb{R}^{n-1}$, that is, if we restrict to the first $n - 1$ variables, the conclusion is true. It is easy to see that the

first $n - 1$ partial derivatives of f restricted to the set where the last coordinate is fixed are the same as those for f . In the following we think of $\mathbb{R}^{n-1}$ as a subset of $\mathbb{R}^n$, that is the set in $\mathbb{R}^n$ where $x^n = 0$. Let

$$A = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x) & \cdots & \frac{\partial f^m}{\partial x^n}(x) \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x) & \cdots & \frac{\partial f^m}{\partial x^{n-1}}(x) \end{bmatrix}, \quad v =$$

Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that for any $k \in \mathbb{R}^{n-1}$ with $\|k\| < \delta$ we have

$$\frac{\|f(x+k) - f(x) - A_1 k\|}{\|k\|} < \epsilon.$$

By continuity of the partial derivatives, suppose δ is small enough so that

$$\left| \frac{\partial f^j}{\partial x^n}(x+h) - \frac{\partial f^j}{\partial x^n}(x) \right| < \epsilon,$$

for all j and all h with $\|h\| < \delta$. Let $h = h_1 + te_n$ be a vector in $\mathbb{R}^n$ where $h_1 \in \mathbb{R}^{n-1}$ such that $\|h\| < \delta$. Then $\|h_1\| \leq \|h\| < \delta$. Note that $Ah = A_1 h_1 + tv$.

$$\begin{aligned} \|f(x+h) - f(x) - Ah\| &= \|f(x+h_1+te_n) - f(x+h_1) - tv + f(x+h_1) - f(x) - A_1 h_1\| \\ &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \|f(x+h_1) - f(x) - A_1 h_1\| \\ &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \epsilon \|h_1\|. \end{aligned}$$

As all the partial derivatives exist then by the mean value theorem for each j there is some $\theta_j \in [0, t]$ (or $[t, 0]$ if $t < 0$), such that

$$\dot{f}(x+h_1+te_n) - \dot{f}(x+h_1) = t \frac{\partial \dot{f}^j}{\partial x^n}(x+h_1+\theta_j e_n).$$

Note that if $\|h\| < \delta$ then $\|h_1 + \theta_j e_n\| \leq \|h\| < \delta$. So to finish the estimate

$$\begin{aligned} \|f(x+h) - f(x) - Ah\| &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \epsilon \|h_1\| \\ &\leq \sqrt{\sum_{j=1}^m \left(t \frac{\partial f^j}{\partial x^n}(x+h_1+\theta_j e_n) - t \frac{\partial f^j}{\partial x^n}(x) \right)^2} + \epsilon \|h_1\| \\ &\leq \sqrt{m} \epsilon |t| + \epsilon \|h_1\| \\ &\leq (\sqrt{m} + 1) \epsilon \|h\|. \end{aligned}$$

The Jacobian Let $U \subset \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^m$ be a differentiable mapping. Then define the Jacobian of f at x as

$$J_f(x) := \det \left(f'(x) \right).$$

Sometimes this is written as

$$\frac{\partial(f^1, \dots, f^m)}{\partial(x^1, \dots, x^n)}.$$

This last piece of notation may seem somewhat confusing, but it is useful when you need to specify the exact variables and function components used. The Jacobian J_f is a real valued function, and when $n = 1$ it is simply the derivative. When f is C^1 , then J_f is a continuous function. From the chain rule it follows that:

$$J_{f \circ g}(x) = J_f(g(x)) J_g(x).$$

It can be computed directly that the determinant tells us what happens to area/volume. Suppose we are in $\mathbb{R}^2$. Then if A is a linear transformation, it follows by direct computation that the direct image of the unit square $A([0, 1]^2)$ has area $|\det(A)|$. Note that the sign of the determinant determines "orientation". If the determinant is negative, then the two sides of the unit square will be flipped in the image. We claim without proof that this follows for arbitrary figures, not just the square. Similarly, the Jacobian measures how much a differentiable mapping stretches things locally, and if it flips orientation. Exercises Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $f(x, y) = \sqrt{x^2 + y^2}$. Show that f is not differentiable at the origin. Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

a) Show that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points (including the origin). b) Show that f is not continuous at the origin (and hence not differentiable). Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

a) Show that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points. b) Show that f is continuous at the origin. c) Show that f is not differentiable at the origin.

9.4: Continuity and the Derivative

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9.5: Inverse and implicit function Theorem

Inverse and implicit function theorem Note: FIXME lectures To prove the inverse function theorem we use the contraction mapping principle we have seen in FIXME and that we have used to prove Picard's theorem. Recall that a mapping $f: X \rightarrow X'$ between two metric spaces (X, d) and (X', d') is called a contraction if there exists a $k < 1$ such that

$$d'(f(x), f(y)) \leq kd(x, y) \quad \text{for all } x, y \in X.$$

The contraction mapping principle says that if $f: X \rightarrow X$ is a contraction and X is a complete metric space, then there exists a fixed point, that is, there exists an $x \in X$ such that $f(x) = x$. Intuitively if a function is differentiable, then it locally "behaves like" the derivative (which is a linear function). The idea of the inverse function theorem is that if a function is differentiable and the derivative is invertible, the function is (locally) invertible. Let $U \subset \mathbb{R}^n$ be a set and let $f: U \rightarrow \mathbb{R}^n$ be a continuously differentiable function. Also suppose $x_0 \in U$, $f(x_0) = y_0$, and $f'(x_0)$ is invertible (that is, $J_f(x_0) \neq 0$). Then there exist open sets $V, W \subset \mathbb{R}^n$ such that $x_0 \in V \subset U$, $f(V) = W$ and $f|_V$ is one-to-one and onto. Furthermore, the inverse $g(y) = (f|_V)^{-1}(y)$ is continuously differentiable and

$$g'(y) = (f'(x))^{-1}, \quad \text{for all } x \in V, y = f(x).$$

Write $A = f'(x_0)$. As f' is continuous, there exists an open ball V around x_0 such that

$$\|A - f'(x)\| < \frac{1}{2\|A^{-1}\|} \quad \text{for all } x \in V.$$

Note that $f'(x)$ is invertible for all $x \in V$. Given $y \in \mathbb{R}^n$ we define $\varphi_y: V \rightarrow \mathbb{R}^n$

$$\varphi_y(x) = x + A^{-1}(y - f(x)).$$

As A^{-1} is one-to-one, then $\varphi_y(x) = x$ (x is a fixed point) if only if $y - f(x) = 0$, or in other words $f(x) = y$. Using chain rule we obtain

$$\varphi_y'(x) = I - A^{-1}f'(x) = A^{-1}(A - f'(x)).$$

So for $x \in V$ we have

$$\|\varphi_y'(x)\| \leq \|A^{-1}\| \|A - f'(x)\| < \frac{1}{2}.$$

As V is a ball it is convex, and hence

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in V.$$

In other words φ_y is a contraction defined on V , though we so far do not know what is the range of φ_y . We cannot apply the fixed point theorem, but we can say that φ_y has at most one fixed point (note proof of uniqueness in the contraction mapping principle). That is, there exists at most one $x \in V$ such that $f(x) = y$, and so $f|_V$ is one-to-one. Let $W = f(V)$. We need to show that W is open. Take a $y_1 \in W$, then there is a unique $x_1 \in V$ such that $f(x_1) = y_1$. Let $r > 0$ be small enough such that the closed ball $C(x_1, r) \subset V$ (such $r > 0$ exists as V is open). Suppose y is such that

$$\|y - y_1\| < \frac{r}{2\|A^{-1}\|}.$$

If we can show that $y \in W$, then we have shown that W is open. Define $\varphi_y(x) = x + A^{-1}(y - f(x))$ as before. If $x \in C(x_1, r)$, then

$$\begin{aligned}
 \|\varphi_y(x) - x_1\| &\leq \|\varphi_y(x) - \varphi_y(x_1)\| + \|\varphi_y(x_1) - x_1\| \\
 &\leq \frac{1}{2}\|x - x_1\| + \|A^{-1}(y - y_1)\| \\
 &\leq \frac{1}{2}r + \|A^{-1}\|\|y - y_1\| \\
 &< \frac{1}{2}r + \|A^{-1}\|\frac{r}{2\|A^{-1}\|} = r.
 \end{aligned}$$

So φ_y takes $C(x_1, r)$ into $B(x_1, r) \subset C(x_1, r)$. It is a contraction on $C(x_1, r)$ and $C(x_1, r)$ is complete (closed subset of $\mathbb{R}^n$ is complete). Apply the contraction mapping principle to obtain a fixed point x , i.e. $\varphi_y(x) = x$. That is $f(x) = y$. So

$y \in f(C(x_1, r)) \subset f(V) = W$. Therefore W is open. Next we need to show that g is continuously differentiable and compute its derivative. First let us show that it is differentiable. Let $y \in W$ and $k \in \mathbb{R}^n$, $k \neq 0$, such that $y + k \in W$. Then there are unique $x \in V$ and $h \in \mathbb{R}^n$, $h \neq 0$ and $x + h \in V$, such that $f(x) = y$ and $f(x + h) = y + k$ as $f|_V$ is a one-to-one and onto mapping of V onto W . In other words, $g(y) = x$ and $g(y + k) = x + h$. We can still squeeze some information from the fact that φ_y is a contraction.

$$\varphi_y(x + h) - \varphi_y(x) = h + A^{-1}(f(x) - f(x + h)) = h - A^{-1}k.$$

So

$$\|h - A^{-1}k\| = \|\varphi_y(x + h) - \varphi_y(x)\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2}.$$

By the inverse triangle inequality $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\|$ so

$$\|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\|\|k\|.$$

In particular as k goes to 0, so does h . As $x \in V$, then $f'(x)$ is invertible. Let $B = (f'(x))^{-1}$, which is what we think the derivative of g at y is. Then

$$\begin{aligned}
 \frac{\|g(y + k) - g(y) - Bk\|}{\|k\|} &= \frac{\|h - Bk\|}{\|k\|} \\
 &= \frac{\|h - B(f(x + h) - f(x))\|}{\|k\|} \\
 &= \frac{\|B(f(x + h) - f(x) - f'(x)h)\|}{\|k\|} \\
 &\leq \|B\| \frac{\|h\|}{\|k\|} \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|} \\
 &\leq 2\|B\|\|A^{-1}\| \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|}.
 \end{aligned}$$

As k goes to 0, so does h . So the right hand side goes to 0 as f is differentiable, and hence the left hand side also goes to 0. And B is precisely what we wanted $g'(y)$ to be. We have g is differentiable, let us show it is $C^1(W)$. Now, $g: W \rightarrow V$ is continuous (it is differentiable), f' is a continuous function from V to $L(\mathbb{R}^n)$, and $X \rightarrow X^{-1}$ is a continuous function. $g'(y) = (f'(g(y)))^{-1}$ is the composition of these three continuous functions and hence is continuous. Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}^n$ is a continuously differentiable mapping such that $f'(x)$ is invertible for all $x \in U$. Then given any open set $V \subset U$, $f(V)$ is open. (f is an open mapping). Without loss of generality, suppose $U = V$. For each point $y \in f(V)$, we pick $x \in f^{-1}(y)$ (there could be more than one such point), then by the inverse function theorem there is a neighbourhood of x in V that maps onto a neighbourhood of y . Hence $f(V)$ is open. The theorem, and the corollary, is not true if $f'(x)$ is not invertible for some x . For example, the map $f(x, y) = (x, xy)$, maps $\mathbb{R}^2$ onto the set $\mathbb{R}^2 \setminus \{(0, y) : y \neq 0\}$, which is neither open nor closed. In fact $f^{-1}(0, 0) = \{(0, y) : y \in \mathbb{R}\}$. Note that this bad behaviour only occurs on the y -axis, everywhere else the function is locally invertible. In fact if we avoid the y -

axis it is even one to one. Also note that just because $f'(x)$ is invertible everywhere does not mean that f is one-to-one globally. It is “locally” one-to-one but perhaps not “globally.” For an example, take the map $f: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (x^2 - y^2, 2xy)$. It is left to student to show that f is differentiable and the derivative is invertible. On the other hand, the mapping is 2-to-1 globally. For every (a, b) that is not the origin, there are exactly two solutions to $x^2 - y^2 = a$ and $2xy = b$. We leave it to the student to show that there is at least one solution, and then notice that replacing x and y with $-x$ and $-y$ we obtain another solution. Also note that the invertibility of the derivative is not a necessary condition, just sufficient for having a continuous inverse and being an open mapping. For example the function $f(x) = x^3$ is an open mapping from $\mathbb{R}$ to $\mathbb{R}$ and is globally one-to-one with a continuous inverse. Implicit function theorem The inverse function theorem is really a special case of the implicit function theorem which we prove next. Although somewhat ironically we prove the implicit function theorem using the inverse function theorem. What we were showing in the inverse function theorem was that the equation $x - f(y) = 0$ was solvable for y in terms of x if the derivative in terms of y was invertible, that is if $f'(y)$ was invertible. That is there was locally a function g such that $x - f(g(x)) = 0$. OK, so how about we look at the equation $f(x, y) = 0$. Obviously this is not solvable for y in terms of x in every case. For example, when $f(x, y)$ does not actually depend on y . For a slightly more complicated example, notice that $x^2 + y^2 - 1 = 0$ defines the unit circle, and we can locally solve for y in terms of x when 1) we are near a point which lies on the unit circle and 2) when we are not at a point where the circle has a vertical tangency, or in other words where $\frac{\partial f}{\partial y} = 0$. To make things simple we fix some notation. We let $(x, y) \in \mathbb{R}^{n+m}$ denote the coordinates $(x^1, \dots, x^n, y^1, \dots, y^m)$. A linear transformation $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^m)$ can then always be written as $A = [A_x \ A_y]$ so that $A(x, y) = A_x x + A_y y$, where $A_x \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $A_y \in L(\mathbb{R}^m, \mathbb{R}^m)$. Let $A = [A_x \ A_y] \in L(\mathbb{R}^{n+m}, \mathbb{R}^m)$ and suppose A_y is invertible, then let $B = -(A_y)^{-1}A_x$ and note that

$$0 = A(x, Bx) = A_x x + A_y Bx.$$

The proof is obvious. We simply solve and obtain $y = Bx$. Let us therefore show that the same can be done for C^1 functions. [thm:implicit] Let $U \subset \mathbb{R}^{n+m}$ be an open set and let $f: U \rightarrow \mathbb{R}^m$ be a $C^1(U)$ mapping. Let $(x_0, y_0) \in U$ be a point such that $f(x_0, y_0) = 0$ and such that

$$\frac{\partial(f^1, \dots, f^m)}{\partial(y^1, \dots, y^m)}(x_0, y_0) \neq 0.$$

Then there exists an open set $W \subset \mathbb{R}^n$ with $x_0 \in W$, an open set $W' \subset \mathbb{R}^m$ with $y_0 \in W'$, with $W \times W' \subset U$, and a $C^1(W)$ mapping $g: W \rightarrow W'$, with $g(x_0) = y_0$, and for all $x \in W$, the point $g(x)$ is the unique point in W' such that

$$f(x, g(x)) = 0.$$

Furthermore, if $[A_x \ A_y] = f'(x_0, y_0)$, then

$$g'(x_0) = -(A_y)^{-1}A_x.$$

FIXME: and these are ALL the points where f vanishes near x_0, y_0 . The condition $\frac{\partial(f^1, \dots, f^m)}{\partial(y^1, \dots, y^m)}(x_0, y_0) = \det(A_y) \neq 0$ simply means that A_y is invertible. Define $F: U \rightarrow \mathbb{R}^{n+m}$ by $F(x, y) := (x, f(x, y))$. It is clear that F is C^1 , and we want to show that the derivative at (x_0, y_0) is invertible. Let us compute the derivative. We know that

$$\frac{\|f(x_0 + h, y_0 + k) - f(x_0, y_0) - A_x h - A_y k\|}{\|(h, k)\|}$$

goes to zero as $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ goes to zero. But then so does

$$\frac{\|(h, f(x_0 + h, y_0 + k) - f(x_0, y_0)) - (h, A_x h + A_y k)\|}{\|(h, k)\|} = \frac{\|f(x_0 + h, y_0 + k) - f(x_0, y_0) - A_x h - A_y k\|}{\|(h, k)\|}.$$

So the derivative of F at (x_0, y_0) takes (h, k) to $(h, A_x h + A_y k)$. If $(h, A_x h + A_y k) = (0, 0)$, then $h = 0$, and so $A_y k = 0$. As A_y is one-to-one, then $k = 0$. Therefore $F'(x_0, y_0)$ is one-to-one or in other words invertible and we can apply the inverse function theorem. That is, there exists some open set $V \subset \mathbb{R}^{n+m}$ with $(x_0, 0) \in V$, and an inverse mapping $G: V \rightarrow \mathbb{R}^{n+m}$, that is $F(G(x, s)) = (x, s)$ for all $(x, s) \in V$ (where $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$). Write $G = (G_1, G_2)$ (the first n and the second m components of G). Then

$$F(G_1(x, s), G_2(x, s)) = (G_1(x, s), f(G_1(x, s), G_2(x, s))) = (x, s).$$

So $x = G_1(x, s)$ and $f(G_1(x, s), G_2(x, s)) = f(x, G_2(x, s)) = s$. Plugging in $s = 0$ we obtain

$$f(x, G_2(x, 0)) = 0.$$

The set $G(V)$ contains a whole neighbourhood of the point (x_0, y_0) and therefore there are some open The set V is open and hence there exist some open sets $\tilde{W}$ and W' such that $\tilde{W} \times W' \subset G(V)$ with $x_0 \in \tilde{W}$ and $y_0 \in W'$. Then take

$W = \{x \in \tilde{W} : G_2(x, 0) \in W'\}$. The function that takes x to $G_2(x, 0)$ is continuous and therefore W is open. We define $g: W \rightarrow \mathbb{R}^m$ by $g(x) := G_2(x, 0)$ which is the g in the theorem. The fact that $g(x)$ is the unique point in W' follows because $W \times W' \subset G(V)$ and G is one-to-one and onto $G(V)$. Next differentiate

$$x \mapsto f(x, g(x)),$$

at x_0 , which should be the zero map. The derivative is done in the same way as above. We get that for all $h \in \mathbb{R}^n$

$$0 = A(h, g'(x_0)h) = A_x h + A_y g'(x_0)h,$$

and we obtain the desired derivative for g as well. In other words, in the context of the theorem we have m equations in $n + m$ unknowns.

$$\begin{aligned} f^1(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \\ &\vdots \\ f^m(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \end{aligned}$$

And the condition guaranteeing a solution is that this is a C^1 mapping (that all the components are C^1 , or in other words all the partial derivatives exist and are continuous), and the matrix

$$\begin{bmatrix} \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m} \end{bmatrix}$$

is invertible at (x_0, y_0) . Consider the set $x^2 + y^2 - (z + 1)^3 = -1$, $e^x + e^y + e^z = 3$ near the point $(0, 0, 0)$. The function we are looking at is

$$f(x, y, z) = (x^2 + y^2 - (z + 1)^3 + 1, e^x + e^y + e^z - 3).$$

We find that

$$Df = \begin{bmatrix} 2x & 2y & -3(z+1)^2 \\ e^x & e^y & e^z \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2(0) & -3(0+1)^2 \\ e^0 & e^0 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

is invertible. Hence near $(0, 0, 0)$ we can find y and z as C^1 functions of x such that for x near 0 we have

$$x^2 + y(x)^2 - (z(x) + 1)^3 = -1, \quad e^x + e^{y(x)} + e^{z(x)} = 3.$$

The theorem does not tell us how to find $y(x)$ and $z(x)$ explicitly, it just tells us they exist. In other words, near the origin the set of solutions is a smooth curve in $\mathbb{R}^3$ that goes through the origin. Note that there are versions of the theorem for arbitrarily many derivatives. If f has k continuous derivatives, then the solution also has k derivatives. Exercises

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9.6: Higher Order Derivatives

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CHAPTER OVERVIEW

10: One dimensional integrals in several variables

Topic hierarchy

[10.1: Differentiation under the Integral](#)

[10.2: Path Integrals](#)

[10.3: Path Independence](#)

[10.4: temp](#)

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10.1: Differentiation under the Integral

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10.3: Path Independence

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10.4: temp

[rn:chapter] [seq:chapter] [lim:chapter] [der:chapter] [int:chapter] [fs:chapter] [ms:chapter]

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Introduction

About this book

This book is the continuation of “Basic Analysis”. The book is meant to be a seamless continuation, so the chapters are numbered to start where the first volume left off. The book started with my notes for a second semester undergraduate analysis at University of Wisconsin—Madison in 2012, where I used my notes together with Rudin’s book. In 2016, I taught a second semester undergraduate analysis at Oklahoma State University and heavily modified and cleaned up the notes, this time using them as the main text.

I plan on eventually adding more topics especially at the end. I will try to preserve the current numbering in subsequent editions as always. The new topics I have planned would add sections and chapters onto the end of the book rather than be inserted in the middle.

For the most part, this second volume depends on the non-optional parts of volume I, however, the optional bits such as higher order derivatives are sometimes used, for example in 6, 3, 6. This book is not necessarily the entire second semester course. What I had in mind for a two semester course is that some bits of the first volume, such as metric spaces, are covered in the second semester, while some of the optional topics of volume I are covered in the first semester. Leaving metric spaces for second semester makes more sense as then the second semester is the “multivariable” part of the course.

Several possibilities for the material in this book are:

- 1) 1–5, (perhaps 1), 1 and 2.
- 2) 1–6, 1–3, 1 and 2.
- 3) Everything.

When I ran the course at OSU, I covered the first book minus metric spaces and a couple of optional sections in the first semester. Then, in the second semester, I covered most of what I skipped from volume I, including metric spaces, and took option 2) above.

Several variables and partial derivatives

Vector spaces, linear mappings, and convexity

Note: 2–3 lectures

Vector spaces

The euclidean space $\mathbb{R}^n$ has already made an appearance in the metric space chapter. In this chapter, we will extend the differential calculus we created for one variable to several variables. The key idea in differential calculus is to approximate functions by lines and linear functions. In several variables we must introduce a little bit of linear algebra before we can move on. So let us start with vector spaces and linear functions on vector spaces.

While it is common to use $\bar{x}$ or the bold $\mathbf{x}$ for elements of $\mathbb{R}^n$, especially in the applied sciences, we use just plain x , which is common in mathematics. That is, $v \in \mathbb{R}^n$ is a *vector*, which means $v = (v_1, v_2, \dots, v_n)$ is an n -tuple of real numbers.¹

It is common to write and treat vectors as *column vectors*, that is, $n \times 1$ matrices:

$$v = (v_1, v_2, \dots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

We will do so when convenient. We call real numbers *scalars* to distinguish them from vectors.

The set $\mathbb{R}^n$ has a so-called *vector space* structure defined on it. However, even though we will be looking at functions defined on $\mathbb{R}^n$, not all spaces we wish to deal with are equal to $\mathbb{R}^n$. Therefore, let us define the abstract notion of the vector space.

Let X be a set together with operations of addition, $+: X \times X \rightarrow X$, and multiplication, $\cdot: \mathbb{R} \times X \rightarrow X$, (we usually write ax instead of $a \cdot x$). X is called a *vector space* (or a *real vector space*) if the following conditions are satisfied:

1. (Addition is associative) If $u, v, w \in X$, then $u + (v + w) = (u + v) + w$.
2. (Addition is commutative) If $u, v \in X$, then $u + v = v + u$.
3. (Additive identity) There is a $0 \in X$ such that $v + 0 = v$ for all $v \in X$.
4. (Additive inverse) For every $v \in X$, there is a $-v \in X$, such that $v + (-v) = 0$.
5. (Distributive law) If $a \in \mathbb{R}$, $u, v \in X$, then $a(u + v) = au + av$.
6. (Distributive law) If $a, b \in \mathbb{R}$, $v \in X$, then $(a + b)v = av + bv$.
7. (Multiplication is associative) If $a, b \in \mathbb{R}$, $v \in X$, then $(ab)v = a(bv)$.
8. (Multiplicative identity) $1v = v$ for all $v \in X$.

Elements of a vector space are usually called *vectors*, even if they are not elements of $\mathbb{R}^n$ (vectors in the “traditional” sense).

If $Y \subset X$ is a subset that is a vector space itself with the same operations, then Y is called a *subspace* or *vector subspace* of X .

An example vector space is $\mathbb{R}^n$, where addition and multiplication by a scalar is done componentwise: if $a \in \mathbb{R}$, $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, and $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, then

$$\begin{aligned} v + w &:= (v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n), \\ av &:= a(v_1, v_2, \dots, v_n) = (av_1, av_2, \dots, av_n). \end{aligned}$$

In this book we mostly deal with vector spaces that can be often regarded as subsets of $\mathbb{R}^n$, but there are other vector spaces useful in analysis. Let us give a couple of examples.

A trivial example of a vector space (the smallest one in fact) is just $X = \{0\}$. The operations are defined in the obvious way. You always need a zero vector to exist, so all vector spaces are nonempty sets.

The space $C([0, 1], \mathbb{R})$ of continuous functions on the interval $[0, 1]$ is a vector space. For two functions f and g in $C([0, 1], \mathbb{R})$ and $a \in \mathbb{R}$, we make the obvious definitions of $f + g$ and af :

$$(f + g)(x) := f(x) + g(x), \quad (af)(x) := a(f(x)).$$

The 0 is the function that is identically zero. We leave it as an exercise to check that all the vector space conditions are satisfied.

The space of polynomials $c_0 + c_1t + c_2t^2 + \cdots + c_mt^m$ is a vector space, let us denote it by $R[t]$ (coefficients are real and the variable is t). The operations are defined in the same way as for functions above. Suppose there are two polynomials, one of degree m and one of degree n . Assume $n \geq m$ for simplicity. Then

$$(c_0 + c_1t + c_2t^2 + \cdots + c_mt^m) + (d_0 + d_1t + d_2t^2 + \cdots + d_nt^n) = \\ (c_0 + d_0) + (c_1 + d_1)t + (c_2 + d_2)t^2 + \cdots + (c_m + d_m)t^m + d_{m+1}t^{m+1} + \cdots + d_nt^n$$

and

$$a(c_0 + c_1t + c_2t^2 + \cdots + c_mt^m) = (ac_0) + (ac_1)t + (ac_2)t^2 + \cdots + (ac_m)t^m.$$

Despite what it looks like, $R[t]$ is not equivalent to R^n for any n . In particular, it is not “finite dimensional”, we will make this notion precise in just a little bit. One can make a finite dimensional vector subspace by restricting the degree. For example, if we say $\mathcal{P}_n$ is the set of polynomials of degree n or less, then $\mathcal{P}_n$ is a finite dimensional vector space.

The space $R[t]$ can be thought of as a subspace of $C(R, R)$. If we restrict the range of t to $[0, 1]$, $R[t]$ can be identified with a subspace of $C([0, 1], R)$.

It is often better to think of even simpler “finite dimensional” vector spaces using the abstract notion rather than always R^n . It is possible to use other fields than R in the definition (for example it is common to use the complex numbers C), but let us stick with the real numbers².

Linear combinations and dimension

Suppose X is a vector space, $x_1, x_2, \dots, x_k \in X$ are vectors, and $a_1, a_2, \dots, a_k \in R$ are scalars. Then

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k$$

is called a *linear combination* of the vectors $x_1, x_2, \dots, x_k$.

If $Y \subset X$ is a set, then the *span* of Y , or in notation $\text{span}(Y)$, is the set of all linear combinations of all finite subsets of Y . We also say Y *spans* $\text{span}(Y)$.

Let $Y := \{(1, 1)\} \subset R^2$. Then

$$\text{span}(Y) = \{(x, x) \in R^2 : x \in R\}.$$

That is, $\text{span}(Y)$ is the line through the origin and the point $(1, 1)$.

[example:vecspr2span] Let $Y := \{(1, 1), (0, 1)\} \subset R^2$. Then

$$\text{span}(Y) = R^2,$$

as any point $(x, y) \in R^2$ can be written as a linear combination

$$(x, y) = x(1, 1) + (y - x)(0, 1).$$

A sum of two linear combinations is again a linear combination, and a scalar multiple of a linear combination is a linear combination, which proves the following proposition.

Let X be a vector space. For any $Y \subset X$, the set $\text{span}(Y)$ is a vector space itself. That is, $\text{span}(Y)$ is a subspace of X .

If Y is already a vector space, then $\text{span}(Y) = Y$.

A set of vectors $\{x_1, x_2, \dots, x_k\} \subset X$ is *linearly independent*, if the only solution to

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$$

is the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. A set that is not linearly independent, is *linearly dependent*.

A linearly independent set B of vectors such that $\text{span}(B) = X$ is called a *basis* of X . For example the set Y of the two vectors in is a basis of $\mathbb{R}^2$.

If a vector space X contains a linearly independent set of d vectors, but no linearly independent set of $d + 1$ vectors, then we say the *dimension* or $\dim X := d$. If for all $d \in \mathbb{N}$ the vector space X contains a set of d linearly independent vectors, we say X is infinite dimensional and write $\dim X := \infty$.

Clearly for the trivial vector space, $\dim \{0\} = 0$. We will see in a moment that any vector subspace of $\mathbb{R}^n$ has a finite dimension, and that dimension is less than or equal to n .

If a set is linearly dependent, then one of the vectors is a linear combination of the others. In other words, in [eq:lincomb] if $a_j \neq 0$, then we solve for x_j

$$x_j = \frac{a_1}{a_j}x_1 + \cdots + \frac{a_{j-1}}{a_j}x_{j-1} + \frac{a_{j+1}}{a_j}x_{j+1} + \cdots + \frac{a_k}{a_j}x_k.$$

The vector x_j has at least two different representations as linear combinations of $\{x_1, x_2, \dots, x_k\}$. The one above and x_j itself.

If $B = \{x_1, x_2, \dots, x_k\}$ is a basis of a vector space X , then every point $y \in X$ has a unique representation of the form

$$y = \sum_{j=1}^k a_j x_j$$

for some scalars $a_1, a_2, \dots, a_k$.

Every $y \in X$ is a linear combination of elements of B since X is the span of B . For uniqueness suppose

$$y = \sum_{j=1}^k a_j x_j = \sum_{j=1}^k b_j x_j,$$

then

$$\sum_{j=1}^k (a_j - b_j) x_j = 0.$$

By linear independence of the basis $a_j = b_j$ for all j .

For $\mathbb{R}^n$ we define

$$e_1 := (1, 0, 0, \dots, 0), \quad e_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad e_n := (0, 0, 0, \dots, 1),$$

and call this the *standard basis* of $\mathbb{R}^n$. We use the same letters e_j for any $\mathbb{R}^n$, and which space $\mathbb{R}^n$ we are working in is understood from context. A direct computation shows that $\{e_1, e_2, \dots, e_n\}$ is really a basis of $\mathbb{R}^n$; it spans $\mathbb{R}^n$ and is linearly independent. In fact,

$$x = (x_1, x_2, \dots, x_n) = \sum_{j=1}^n x_j e_j.$$

[mv:dimprop] Let X be a vector space and d a nonnegative integer.

1. [mv:dimprop:i] If X is spanned by d vectors, then $\dim X \leq d$.
2. [mv:dimprop:ii] $\dim X = d$ if and only if X has a basis of d vectors (and so every basis has d vectors).
3. [mv:dimprop:iii] In particular, $\dim \mathbb{R}^n = n$.
4. [mv:dimprop:iv] If $Y \subset X$ is a vector subspace and $\dim X = d$, then $\dim Y \leq d$.

5. [mv:dimprop:v] If $\dim X = d$ and a set T of d vectors spans X , then T is linearly independent.
6. [mv:dimprop:vi] If $\dim X = d$ and a set T of m vectors is linearly independent, then there is a set S of $d - m$ vectors such that $T \cup S$ is a basis of X .

Let us start with [mv:dimprop:i]. Suppose $S = \{x_1, x_2, \dots, x_d\}$ spans X , and $T = \{y_1, y_2, \dots, y_m\}$ is a set of linearly independent vectors of X . We wish to show that $m \leq d$. Write

$$y_1 = \sum_{k=1}^d a_{k,1} x_k,$$

for some numbers $a_{1,1}, a_{2,1}, \dots, a_{d,1}$, which we can do as S spans X . One of the $a_{k,1}$ is nonzero (otherwise y_1 would be zero), so suppose without loss of generality that this is $a_{1,1}$. Then we solve

$$x_1 = \frac{1}{a_{1,1}} y_1 - \sum_{k=2}^d \frac{a_{k,1}}{a_{1,1}} x_k.$$

In particular, $\{y_1, x_2, \dots, x_d\}$ span X , since x_1 can be obtained from $\{y_1, x_2, \dots, x_d\}$. Therefore, there are some numbers for some numbers $a_{1,2}, a_{2,2}, \dots, a_{d,2}$, such that

$$y_2 = a_{1,2} y_1 + \sum_{k=2}^d a_{k,2} x_k.$$

As T is linearly independent, one of the $a_{k,2}$ for $k \geq 2$ must be nonzero. Without loss of generality suppose $a_{2,2} \neq 0$. Proceed to solve for

$$x_2 = \frac{1}{a_{2,2}} y_2 - \frac{a_{1,2}}{a_{2,2}} y_1 - \sum_{k=3}^d \frac{a_{k,2}}{a_{2,2}} x_k.$$

In particular, $\{y_1, y_2, x_3, \dots, x_d\}$ spans X .

We continue this procedure. If $m < d$, then we are done. So suppose $m \geq d$. After d steps we obtain that $\{y_1, y_2, \dots, y_d\}$ spans X . Any other vector v in X is a linear combination of $\{y_1, y_2, \dots, y_d\}$, and hence cannot be in T as T is linearly independent. So $m = d$.

Let us look at [mv:dimprop:ii]. First, if T is a set of k linearly independent vectors that do not span X , that is $X \setminus \text{span}(T) \neq \emptyset$, then choose a vector $v \in X \setminus \text{span}(T)$. The set $T \cup \{v\}$ is linearly independent (exercise). If $\dim X = d$, then there must exist some linearly independent set of d vectors T , and it must span X , otherwise we could choose a larger set of linearly independent vectors. So we have a basis of d vectors. On the other hand if we have a basis of d vectors, it is linearly independent and spans X by definition. By [mv:dimprop:i] we know there is no set of $d + 1$ linearly independent vectors, so dimension must be d .

For [mv:dimprop:iii] notice that $\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

To see [mv:dimprop:iv], suppose Y is a vector space and $Y \subset X$, where $\dim X = d$. As X cannot contain $d + 1$ linearly independent vectors, neither can Y .

For [mv:dimprop:v] suppose T is a set of m vectors that is linearly dependent and spans X . Then one of the vectors is a linear combination of the others. Therefore if we remove it from T we obtain a set of $m - 1$ vectors that still span X and hence $\dim X \leq m - 1$ by [mv:dimprop:i].

For [mv:dimprop:vi] suppose $T = \{x_1, x_2, \dots, x_m\}$ is a linearly independent set. We follow the procedure above in the proof of [mv:dimprop:ii] to keep adding vectors while keeping the set linearly independent. As the dimension is d we can add a vector exactly $d - m$ times.

Linear mappings

A function $f: X \rightarrow Y$, when Y is not $\mathbb{R}$, is often called a *mapping* or a *map* rather than a *function*.

A mapping $A: X \rightarrow Y$ of vector spaces X and Y is *linear* (or a *linear transformation*) if for every $a \in \mathbb{R}$ and every $x, y \in X$,

$$A(ax) = aA(x), \quad \text{and} \quad A(x + y) = A(x) + A(y).$$

We usually write Ax instead of $A(x)$ if A is linear. If A is one-to-one and onto, then we say A is *invertible*, and we denote the inverse by A^{-1} . If $A: X \rightarrow X$ is linear, then we say A is a *linear operator* on X .

We write $L(X, Y)$ for the set of all linear transformations from X to Y , and just $L(X)$ for the set of linear operators on X . If $a \in \mathbb{R}$ and $A, B \in L(X, Y)$, define the transformations aA and $A + B$ by

$$(aA)(x) := aAx, \quad (A + B)(x) := Ax + Bx.$$

If $A \in L(Y, Z)$ and $B \in L(X, Y)$, define the transformation AB as the composition $A \circ B$, that is,

$$ABx := A(Bx).$$

Finally denote by $I \in L(X)$ the *identity*: the linear operator such that $Ix = x$ for all x .

It is not hard to see that $aA \in L(X, Y)$ and $A + B \in L(X, Y)$, and that $AB \in L(X, Z)$. In particular, $L(X, Y)$ is a vector space. As the set $L(X)$ is not only a vector space, but also admits a product, it is often called an *algebra*.

An immediate consequence of the definition of a linear mapping is: if A is linear, then $A0 = 0$.

If $A \in L(X, Y)$ is invertible, then A^{-1} is linear.

Let $a \in \mathbb{R}$ and $y \in Y$. As A is onto, then there is an x such that $y = Ax$, and further as it is also one-to-one $A^{-1}(Az) = z$ for all $z \in X$. So

$$A^{-1}(ay) = A^{-1}(aAx) = A^{-1}(A(ax)) = ax = aA^{-1}(y).$$

Similarly let $y_1, y_2 \in Y$, and $x_1, x_2 \in X$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$, then

$$A^{-1}(y_1 + y_2) = A^{-1}(Ax_1 + Ax_2) = A^{-1}(A(x_1 + x_2)) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2). \quad \text{\textcolor{red}{qedhere}}$$

[mv:lindefonbasis] If $A \in L(X, Y)$ is linear, then it is completely determined by its values on a basis of X . Furthermore, if B is a basis of X , then any function $\tilde{A}: B \rightarrow Y$ extends to a linear function on X .

We will only prove this proposition for finite dimensional spaces, as we do not need infinite dimensional spaces. For infinite dimensional spaces, the proof is essentially the same, but a little trickier to write, so let us stick with finitely many dimensions.

Let $\{x_1, x_2, \dots, x_n\}$ be a basis and suppose $Ax_j = y_j$. Every $x \in X$ has a unique representation

$$x = \sum_{j=1}^n b_j x_j$$

for some numbers $b_1, b_2, \dots, b_n$. By linearity

$$Ax = A \sum_{j=1}^n b_j x_j = \sum_{j=1}^n b_j Ax_j = \sum_{j=1}^n b_j y_j.$$

The “furthermore” follows by setting $y_j := \tilde{A}(x_j)$, and defining the extension as $Ax := \sum_{j=1}^n b_j y_j$. The function is well defined by uniqueness of the representation of x . We leave it to the reader to check that A is linear.

The next proposition only works for finite dimensional vector spaces. It is a special case of the so-called rank-nullity theorem from linear algebra.

[mv:prop:lin1onto] If X is a finite dimensional vector space and $A \in L(X)$, then A is one-to-one if and only if it is onto.

Let $\{x_1, x_2, \dots, x_n\}$ be a basis for X . Suppose A is one-to-one. Now suppose

$$\sum_{j=1}^n c_j Ax_j = A \sum_{j=1}^n c_j x_j = 0.$$

As A is one-to-one, the only vector that is taken to 0 is 0 itself. Hence,

$$0 = \sum_{j=1}^n c_j x_j$$

and $c_j = 0$ for all j . So $\{Ax_1, Ax_2, \dots, Ax_n\}$ is a linearly independent set. By and the fact that the dimension is n , we conclude $\{Ax_1, Ax_2, \dots, Ax_n\}$ span X . Any point $x \in X$ can be written as

$$x = \sum_{j=1}^n a_j Ax_j = A \sum_{j=1}^n a_j x_j,$$

so A is onto.

Now suppose A is onto. As A is determined by the action on the basis we see that every element of X has to be in the span of $\{Ax_1, Ax_2, \dots, Ax_n\}$. Suppose

$$A \sum_{j=1}^n c_j x_j = \sum_{j=1}^n c_j Ax_j = 0.$$

By as $\{Ax_1, Ax_2, \dots, Ax_n\}$ span X , the set is independent, and hence $c_j = 0$ for all j . In other words if $Ax = 0$, then $x = 0$. This means that A is one-to-one: If $Ax = Ay$, then $A(x - y) = 0$ and so $x = y$.

We leave the proof of the next proposition as an exercise.

[prop:LXYfinitedim] If X and Y are finite dimensional vector spaces, then $L(X, Y)$ is also finite dimensional.

Finally let us note that we often identify a finite dimensional vector space X of dimension n with $\mathbb{R}^n$, provided we fix a basis $\{x_1, x_2, \dots, x_n\}$ in X . That is, we define a bijective linear map $A \in L(X, \mathbb{R}^n)$ by $Ax_j = e_j$, where $\{e_1, e_2, \dots, e_n\}$. Then we have the correspondence

$$\sum_{j=1}^n c_j x_j \in X \xrightarrow{A} (c_1, c_2, \dots, c_n) \in \mathbb{R}^n.$$

Convexity

A subset U of a vector space is *convex* if whenever $x, y \in U$, the line segment from x to y lies in U . That is, if the *convex combination* $(1 - t)x + ty$ is in U for all $t \in [0, 1]$. See .

Note that in $\mathbb{R}$, every connected interval is convex. In $\mathbb{R}^2$ (or higher dimensions) there are lots of nonconvex connected sets. For example the set $\mathbb{R}^2 \setminus \{0\}$ is not convex but it is connected. To see this simply take any $x \in \mathbb{R}^2 \setminus \{0\}$ and let $y := -x$. Then $(\frac{1}{2})x + (\frac{1}{2})y = 0$, which is not in the set. On the other hand, the ball $B(x, r) \subset \mathbb{R}^n$ (using the standard metric on $\mathbb{R}^n$) is convex by the triangle inequality.

Show that in $\mathbb{R}^n$ any ball $B(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$ is convex.

Any subspace V of a vector space X is convex.

A somewhat more complicated example is given by the following. Let $C([0, 1], \mathbb{R})$ be the vector space of continuous real valued functions on $\mathbb{R}$. Let $X \subset C([0, 1], \mathbb{R})$ be the set of those f such that

$$\int_0^1 f(x) dx \leq 1 \quad \text{and} \quad f(x) \geq 0 \text{ for all } x \in [0, 1].$$

Then X is convex. Take $t \in [0, 1]$, and note that if $f, g \in X$, then $tf(x) + (1 - t)g(x) \geq 0$ for all x . Furthermore

$$\int_0^1 (tf(x) + (1 - t)g(x)) dx = t \int_0^1 f(x) dx + (1 - t) \int_0^1 g(x) dx \leq 1.$$

Note that X is not a subspace of $C([0, 1], \mathbb{R})$.

The intersection two convex sets is convex. In fact, if $\{C_\lambda\}_{\lambda \in I}$ is an arbitrary collection of convex sets, then

$$C := \bigcap_{\lambda \in I} C_\lambda$$

is convex.

If $x, y \in C$, then $x, y \in C_\lambda$ for all $\lambda \in I$, and hence if $t \in [0, 1]$, then $tx + (1 - t)y \in C_\lambda$ for all $\lambda \in I$. Therefore $tx + (1 - t)y \in C$ and C is convex.

Let $T: V \rightarrow W$ be a linear mapping between two vector spaces and let $C \subset V$ be a convex set. Then $T(C)$ is convex.

Take any two points $p, q \in T(C)$. Pick $x, y \in C$ such that $Tx = p$ and $Ty = q$. As C is convex, then $tx + (1 - t)y \in C$ for all $t \in [0, 1]$, so

$$tp + (1 - t)q = tTx + (1 - t)Ty = T(tx + (1 - t)y) \in T(C). \quad \text{\textcolor{red}{qedhere}}$$

For completeness, a very useful construction is the *convex hull*. Given any set $S \subset V$ of a vector space, define the convex hull of S , by

$$\text{co}(S) := \bigcap \{C \subset V : S \subset C, \text{ and } C \text{ is convex}\}.$$

That is, the convex hull is the smallest convex set containing S . By a proposition above, the intersection of convex sets is convex and hence, the convex hull is convex.

The convex hull of 0 and 1 in $\mathbb{R}$ is $[0, 1]$. Proof: Any convex set containing 0 and 1 must contain $[0, 1]$. The set $[0, 1]$ is convex, therefore it must be the convex hull.

Exercises

Verify that $\mathbb{R}^n$ is a vector space.

Let X be a vector space. Prove that a finite set of vectors $\{x_1, \dots, x_n\} \subset X$ is linearly independent if and only if for every $j = 1, 2, \dots, n$

$$\text{span}(\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}) \subsetneq \text{span}(\{x_1, \dots, x_n\}).$$

That is, the span of the set with one vector removed is strictly smaller.

Show that the set $X \subset C([0, 1], \mathbb{R})$ of those functions such that $\int_0^1 f = 0$ is a vector subspace.

Prove $C([0, 1], \mathbb{R})$ is an infinite dimensional vector space where the operations are defined in the obvious way: $s = f + g$ and $m = fg$ are defined as $s(x) := f(x) + g(x)$ and $m(x) := f(x)g(x)$. Hint: for the dimension, think of functions that are only nonzero on the interval $(\frac{1}{n+1}, \frac{1}{n})$.

Let $k: [0, 1]^2 \rightarrow \mathbb{R}$ be continuous. Show that $L: C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ defined by

$$Lf(y) := \int_0^1 k(x, y)f(x) dx$$

is a linear operator. That is, show that L is well defined (that Lf is continuous), and that L is linear.

Let $\mathcal{P}_n$ be the vector space of polynomials in one variable of degree n or less. Show that $\mathcal{P}_n$ is a vector space of dimension $n + 1$.

Let $R[t]$ be the vector space of polynomials in one variable t . Let $D: R[t] \rightarrow R[t]$ be the derivative operator (derivative in t). Show that D is a linear operator.

Let us show that only works in finite dimensions. Take $R[t]$ and define the operator $A: R[t] \rightarrow R[t]$ by $A(P(t)) = tP(t)$. Show that A is linear and one-to-one, but show that it is not onto.

Finish the proof of in the finite dimensional case. That is, suppose, $\{x_1, x_2, \dots, x_n\}$ is a basis of X , $\{y_1, y_2, \dots, y_n\} \subset Y$ and we define a function

$$Ax := \sum_{j=1}^n b_j y_j, \quad \text{if } x = \sum_{j=1}^n b_j x_j.$$

Then prove that $A: X \rightarrow Y$ is linear.

Prove . Hint: A linear operator is determined by its action on a basis. So given two bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ for X and Y respectively, consider the linear operators A_{jk} that send $A_{jk}x_j = y_k$, and $A_{jk}x_\ell = 0$ if $\ell \neq j$.

Suppose X and Y are vector spaces and $A \in L(X, Y)$ is a linear operator.

- Show that the nullspace $N := \{x \in X : Ax = 0\}$ is a vectorspace.
- Show that the range $R := \{y \in Y : Ax = y \text{ for some } x \in X\}$ is a vectorspace.

Show by example that a union of convex sets need not be convex.

Compute the convex hull of the set of 3 points $\{(0, 0), (0, 1), (1, 1)\}$ in $\mathbb{R}^2$.

Show that the set $\{(x, y) \in \mathbb{R}^2 : y > x^2\}$ is a convex set.

Show that the set $X \subset C([0, 1], \mathbb{R})$ of those functions such that $\int_0^1 f = 1$ is a convex set, but not a vector subspace.

Show that every convex set in $\mathbb{R}^n$ is connected using the standard topology on $\mathbb{R}^n$.

Suppose $K \subset \mathbb{R}^2$ is a convex set such that the only point of the form $(x, 0)$ in K is the point $(0, 0)$. Further suppose that there $(0, 1) \in K$ and $(1, 1) \in K$. Then show that if $(x, y) \in K$, then $y > 0$ unless $x = 0$.

Analysis with vector spaces

Note: 2-3 lectures

Norms

Let us start measuring distance.

If X is a vector space, then we say a function $\|\cdot\|: X \rightarrow \mathbb{R}$ is a *norm* if:

- $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{R}$ and $x \in X$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (Triangle inequality).

Before defining the standard norm on $\mathbb{R}^n$, let us define the standard scalar *dot product* on $\mathbb{R}^n$. For two vectors if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, define

$$x \cdot y := \sum_{j=1}^n x_j y_j.$$

It is easy to see that the dot product is linear in each variable separately, that is, it is a linear mapping when you keep one of the variables constant. The *Euclidean norm* is defined as

$$\|x\| := \|x\|_{\mathbb{R}^n} := \sqrt{x \cdot x} = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}.$$

We normally just use $\|x\|$, but sometimes it will be necessary to emphasize that we are talking about the euclidean norm and use $\|x\|_{\mathbb{R}^n}$. It is easy to see that the Euclidean norm satisfies [defn:norm:i] and [defn:norm:ii]. To prove that [defn:norm:iii] holds, the key inequality is the so-called Cauchy-Schwarz inequality we saw before. As this inequality is so important let us restate and reprove it using the notation of this chapter.

Let $x, y \in \mathbb{R}^n$, then

$$|x \cdot y| \leq \|x\| \|y\| = \sqrt{x \cdot x} \sqrt{y \cdot y},$$

with equality if and only if the vectors are scalar multiples of each other.

If $x = 0$ or $y = 0$, then the theorem holds trivially. So assume $x \neq 0$ and $y \neq 0$.

If x is a scalar multiple of y , that is $x = \lambda y$ for some $\lambda \in \mathbb{R}$, then the theorem holds with equality:

$$|\lambda y \cdot y| = |\lambda| |y \cdot y| = |\lambda| \|y\|^2 = \|\lambda y\| \|y\|.$$

Next take $x + ty$, we find that $\|x + ty\|^2$ is a quadratic polynomial in t :

$$\|x + ty\|^2 = (x + ty) \cdot (x + ty) = x \cdot x + x \cdot ty + ty \cdot x + ty \cdot ty = \|x\|^2 + 2t(x \cdot y) + t^2\|y\|^2.$$

If x is not a scalar multiple of y , then $\|x + ty\|^2 > 0$ for all t . So the polynomial $\|x + ty\|^2$ is never zero. Elementary algebra says that the discriminant must be negative:

$$4(x \cdot y)^2 - 4\|x\|^2\|y\|^2 < 0,$$

or in other words $(x \cdot y)^2 < \|x\|^2\|y\|^2$.

Item [defn:norm:iii], the triangle inequality, follows via a simple computation:

$$\|x + y\|^2 = x \cdot x + y \cdot y + 2(x \cdot y) \leq \|x\|^2 + \|y\|^2 + 2(\|x\| \|y\|) = (\|x\| + \|y\|)^2.$$

The distance $d(x, y) := \|x - y\|$ is the standard distance function on $\mathbb{R}^n$ that we used when we talked about metric spaces.

In fact, on any vector space X , once we have a norm (any norm), we define a distance $d(x, y) := \|x - y\|$ that makes X into a metric space (an easy exercise).

Let $A \in L(X, Y)$. Define

$$\|A\| := \sup \{ \|Ax\| : x \in X \text{ with } \|x\| = 1 \}.$$

The number $\|A\|$ is called the *operator norm*. We will see below that indeed it is a norm (at least for finite dimensional spaces). Again, when necessary to emphasize which norm we are talking about, we may write it as $\|A\|_{L(X, Y)}$.

By linearity, $\|A \frac{x}{\|x\|}\| = \frac{\|Ax\|}{\|x\|}$, for any nonzero $x \in X$. The vector $\frac{x}{\|x\|}$ is of norm 1. Therefore,

$$\|A\| = \sup \{ \|Ax\| : x \in X \text{ with } \|x\| = 1 \} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}.$$

This implies that

$$\|Ax\| \leq \|A\| \|x\|.$$

It is not hard to see from the definition that $\|A\| = 0$ if and only if $A = 0$, that is, if A takes every vector to the zero vector.

It is also not difficult to see the norm of the identity operator:

$$\|I\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ix\|}{\|x\|} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|x\|}{\|x\|} = 1.$$

For finite dimensional spaces, $\|A\|$ is always finite as we prove below. This also implies that A is continuous. For infinite dimensional spaces neither statement needs to be true. For a simple example, take the vector space of continuously differentiable functions on $[0, 1]$ and as the norm use the uniform norm. The functions $\sin(nx)$ have norm 1, but the derivatives have norm n . So differentiation (which is a linear operator) has unbounded norm on this space. But let us stick to finite dimensional spaces now.

When we talk about finite dimensional vector space, one often thinks of $\mathbb{R}^n$, although if we have a norm, the norm might perhaps not be the standard euclidean norm. In the exercises, you can prove that every norm is “equivalent” to the euclidean norm in that the topology it generates is the same. For simplicity, we only prove the following proposition for the euclidean space, and the proof for a general finite dimensional space is left as an exercise.

[prop:finitedimproprnormfin] Let X and Y be finite dimensional vector spaces with a norm. If $A \in L(X, Y)$, then $\|A\| < \infty$, and A is uniformly continuous (Lipschitz with constant $\|A\|$).

As we said we only prove the proposition for euclidean space so suppose that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and the norm is the standard euclidean norm. The general case is left as an exercise.

Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$, with $\|x\| = 1$, as

$$x = \sum_{j=1}^n c_j e_j.$$

Since $e_j \cdot e_\ell = 0$ whenever $j \neq \ell$ and $e_j \cdot e_j = 1$, then $c_j = x \cdot e_j$ and

$$|c_j| = |x \cdot e_j| \leq \|x\| \|e_j\| = 1.$$

Then

$$\|Ax\| = \left\| \sum_{j=1}^n c_j A e_j \right\| \leq \sum_{j=1}^n |c_j| \|A e_j\| \leq \sum_{j=1}^n \|A e_j\|.$$

The right hand side does not depend on x . We found a finite upper bound independent of x , so $\|A\| < \infty$.

Now for any vector spaces X and Y , and $A \in L(X, Y)$, suppose that $\|A\| < \infty$. For $v, w \in X$,

$$\|A(v - w)\| \leq \|A\| \|v - w\|.$$

As $\|A\| < \infty$, then this says A is Lipschitz with constant $\|A\|$.

[prop:finitedimproprnorm] Let X, Y , and Z be finite dimensional vector spaces with a norm.

1. [item:finitedimproprnorm:i] If $A, B \in L(X, Y)$ and $c \in \mathbb{R}$, then

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|cA\| = |c| \|A\|.$$

In particular, the operator norm is a norm on the vector space $L(X, Y)$.

2. [item:finitedimproprnorm:ii] If $A \in L(X, Y)$ and $B \in L(Y, Z)$, then

$$\|BA\| \leq \|B\| \|A\|.$$

For [item:finitedimproprnorm:i],

$$\|(A + B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| \|x\| + \|B\| \|x\| = (\|A\| + \|B\|) \|x\|.$$

So $\|A + B\| \leq \|A\| + \|B\|$.

Similarly,

$$\|(cA)x\| = |c|\|Ax\| \leq (|c|\|A\|)\|x\|.$$

Thus $\|cA\| \leq |c|\|A\|$. Next,

$$|c|\|Ax\| = \|cAx\| \leq \|cA\|\|x\|.$$

Hence $|c|\|A\| \leq \|cA\|$.

For [\[item:finitedimpropnorm:ii\]](#) write

$$\|BAx\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|. \quad \text{\textcolor{red}{qedhere}}$$

As a norm defines a metric, there is a metric space topology on $L(X, Y)$, so we can talk about open/closed sets, continuity, and convergence.

[\[prop:finitedimpropinv\]](#) Let X be a finite dimensional vector space with a norm. Let $U \subset L(X)$ be the set of invertible linear operators.

1. [\[finitedimpropinv:i\]](#) If $A \in U$ and $B \in L(X)$, and

$$\|A - B\| < \frac{1}{\|A^{-1}\|},$$

then B is invertible.

2. [\[finitedimpropinv:ii\]](#) U is open and $A \mapsto A^{-1}$ is a continuous function on U .

Let us make sense of this on a simple example. Think back to $\mathbb{R}^1$, where linear operators are just numbers a and the operator norm of a is simply $|a|$. The operator a is invertible ($a^{-1} = \text{\textcolor{red}{\frac{1}{a}}}$) whenever $a \neq 0$. The condition $|a - b| < \frac{1}{|a^{-1}|}$ does indeed

imply that b is not zero. And $a \mapsto \text{\textcolor{red}{\frac{1}{a}}}$ is a continuous map. When $n > 1$, then there are other noninvertible operators than just zero, and in general things are a bit more difficult.

Let us prove [\[finitedimpropinv:i\]](#). We know something about A^{-1} and something about $A - B$. These are linear operators so let us apply them to a vector.

$$A^{-1}(A - B)x = x - A^{-1}Bx.$$

Therefore,

$$\begin{aligned} \|x\| &= \|A^{-1}(A - B)x + A^{-1}Bx\| \\ &\leq \|A^{-1}\| \|A - B\| \|x\| + \|A^{-1}\| \|Bx\|. \end{aligned}$$

Now assume $x \neq 0$ and so $\|x\| \neq 0$. Using [\[eqcontineq\]](#) we obtain

$$\|x\| < \|x\| + \|A^{-1}\| \|Bx\|,$$

or in other words $\|Bx\| \neq 0$ for all nonzero x , and hence $Bx \neq 0$ for all nonzero x . This is enough to see that B is one-to-one (if $Bx = By$, then $B(x - y) = 0$, so $x = y$). As B is one-to-one operator from X to X which is finite dimensional and hence is invertible.

Let us look at [\[finitedimpropinv:ii\]](#). Fix some $A \in U$. Let B be invertible and near A , that is $\|A - B\| \|A^{-1}\| < \text{\textcolor{red}{\frac{1}{2}}}$. Then [\[eqcontineq\]](#) is satisfied. We have shown above (using $B^{-1}y$ instead of x)

$$\|B^{-1}y\| \leq \|A^{-1}\| \|A - B\| \|B^{-1}y\| + \|A^{-1}\| \|y\| \leq \text{\textcolor{red}{\frac{1}{2}}} \|B^{-1}y\| + \|A^{-1}\| \|y\|,$$

or $\|B^{-1}\| \leq \frac{1}{\|A - B\|} \|A\|$. So $\|B^{-1}\| \leq 2\|A^{-1}\|$.

Now

$$A^{-1}(A - B)B^{-1} = A^{-1}(AB^{-1} - I) = B^{-1} - A^{-1},$$

and

$$\|B^{-1} - A^{-1}\| = \|A^{-1}(A - B)B^{-1}\| \leq \|A^{-1}\| \|A - B\| \|B^{-1}\| \leq 2\|A^{-1}\|^2 \|A - B\|.$$

Therefore, if as B tends to A , $\|B^{-1} - A^{-1}\|$ tends to 0, and so the inverse operation is a continuous function at A .

Matrices

As we previously noted, once we fix a basis in a finite dimensional vector space X , we can represent a vector of X as an n -tuple of numbers, that is a vector in $\mathbb{R}^n$. The same thing can be done with $L(X, Y)$, which brings us to matrices, which are a convenient way to represent finite-dimensional linear transformations. Suppose $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$ are bases for vector spaces X and Y respectively. A linear operator is determined by its values on the basis. Given $A \in L(X, Y)$, Ax_j is an element of Y . Therefore, define the numbers $\{a_{i,j}\}$ as follows

$$Ax_j = \sum_{i=1}^m a_{i,j} y_i,$$

and write them as a *matrix*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

And we say A is an m -by- n matrix. The *columns* of the matrix are precisely the coefficients that represent Ax_j . Let us derive the familiar rule for matrix multiplication.

When

$$z = \sum_{j=1}^n c_j x_j,$$

then

$$Az = \sum_{j=1}^n c_j Ax_j = \sum_{j=1}^n c_j \left(\sum_{i=1}^m a_{i,j} y_i \right) = \sum_{i=1}^m \left(\sum_{j=1}^n a_{i,j} c_j \right) y_i,$$

which gives rise to the familiar rule for matrix multiplication.

There is a one-to-one correspondence between matrices and linear operators in $L(X, Y)$. That is, once we fix a basis in X and in Y . If we would choose a different basis, we would get different matrices. This is important, the operator A acts on elements of X , the matrix is something that works with n -tuples of numbers, that is, vectors of $\mathbb{R}^n$.

If B is an n -by- r matrix with entries $b_{j,k}$, then the matrix for $C = AB$ is an m -by- r matrix whose i, k th entry $c_{i,k}$ is

$$c_{i,k} = \sum_{j=1}^n a_{i,j} b_{j,k}.$$

A way to remember it is if you order the indices as we do, that is *row, column*, and put the elements in the same order as the matrices, then it is the “middle index” that is “summed-out.”

A linear mapping changing one basis to another is a square matrix in which the columns represent basis elements of the second basis in terms of the first basis. We call such a linear mapping a *change of basis*.

Suppose all the bases are just the standard bases and $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Recall the Cauchy-Schwarz inequality and compute

$$\|Az\|^2 = \sum_{i=1}^m \left(\sum_{j=1}^n a_{i,j} c_j \right)^2 \leq \sum_{i=1}^m \left(\sum_{j=1}^n (c_j)^2 \right) \left(\sum_{j=1}^n (a_{i,j})^2 \right) = \sum_{i=1}^m \left(\sum_{j=1}^n (a_{i,j})^2 \right) \|z\|^2.$$

In other words, we have a bound on the operator norm (note that equality rarely happens)

$$\|A\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n (a_{i,j})^2}.$$

If the entries go to zero, then $\|A\|$ goes to zero. In particular, if A is fixed and B is changing such that the entries of $A - B$ go to zero, then B goes to A in operator norm. That is, B goes to A in the metric space topology induced by the operator norm. We proved the first part of:

If $f: S \rightarrow \mathbb{R}^m$ is a continuous function for a metric space S , then taking the components of f as the entries of a matrix, f is a continuous mapping from S to $L(\mathbb{R}^n, \mathbb{R}^m)$. Conversely, if $f: S \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function, then the entries of the matrix are continuous functions.

The proof of the second part is rather easy. Take $f(x)e_j$ and note that is a continuous function to $\mathbb{R}^m$ with standard Euclidean norm: $\|f(x)e_j - f(y)e_j\| = \|(f(x) - f(y))e_j\| \leq \|f(x) - f(y)\|$, so as $x \rightarrow y$, then $\|f(x) - f(y)\| \rightarrow 0$ and so $\|f(x)e_j - f(y)e_j\| \rightarrow 0$. Such a function is continuous if and only if its components are continuous and these are the components of the j th column of the matrix $f(x)$.

Determinants

A certain number can be assigned to square matrices that measures how the corresponding linear mapping stretches space. In particular, this number, called the determinant, can be used to test for invertibility of a matrix.

First define the symbol $\text{sgn}(x)$ for a number is defined by

$$\text{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Suppose $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a *permutation* of the integers $(1, 2, \dots, n)$, that is, a reordering of $(1, 2, \dots, n)$. Any permutation can be obtained by a sequence of transpositions (switchings of two elements). Call a permutation *even* (resp. *odd*) if it takes an even (resp. odd) number of transpositions to get from σ to $(1, 2, \dots, n)$. It can be shown that this is well defined (exercise). In fact, define

$$\text{sgn}(\sigma) := \text{sgn}(\sigma_1, \dots, \sigma_n) = \prod_{p < q} \text{sgn}(\sigma_q - \sigma_p).$$

Then it can be shown that $\text{sgn}(\sigma)$ is 1 if σ is even and -1 if σ is odd. This fact can be proved by noting that applying a transposition changes the sign. Then note that the sign of $(1, 2, \dots, n)$ is 1.

Let S_n be the set of all permutations on n elements (the *symmetric group*). Let $A = [a_{i,j}]$ be a square $n \times n$ matrix. Define the *determinant* of A

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma_i}.$$

1. [prop:det:i] $\det(I) = 1$.
2. [prop:det:ii] $\det([x_1 \ x_2 \ \cdots \ x_n])$ as a function of column vectors x_j is linear in each variable x_j separately.
3. [prop:det:iii] If two columns of a matrix are interchanged, then the determinant changes sign.
4. [prop:det:iv] If two columns of A are equal, then $\det(A) = 0$.
5. [prop:det:v] If a column is zero, then $\det(A) = 0$.
6. [prop:det:vi] $A \mapsto \det(A)$ is a continuous function.
7. [prop:det:vii] $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and $\det[a] = a$.

In fact, the determinant is the unique function that satisfies [prop:det:i], [prop:det:ii], and [prop:det:iii]. But we digress. By [prop:det:ii], we mean that if we fix all the vectors $x_1, \dots, x_n$ except for x_j and think of the determinant as function of x_j , it is a linear function. That is, if $v, w \in \mathbb{R}^n$ are two vectors, and $a, b \in \mathbb{R}$ are scalars, then

$$\det([x_1 \ \cdots \ x_{j-1} \ (av + bw) \ x_{j+1} \ \cdots \ x_n]) = a \det([x_1 \ \cdots \ x_{j-1} \ v \ x_{j+1} \ \cdots \ x_n]) + b \det([x_1 \ \cdots \ x_{j-1} \ w \ x_{j+1} \ \cdots \ x_n]).$$

We go through the proof quickly, as you have likely seen this before.

[prop:det:i] is trivial. For [prop:det:ii], notice that each term in the definition of the determinant contains exactly one factor from each column.

Part [prop:det:iii] follows by noting that switching two columns is like switching the two corresponding numbers in every element in S_n . Hence all the signs are changed. Part [prop:det:iv] follows because if two columns are equal and we switch them we get the same matrix back and so part [prop:det:iii] says the determinant must have been 0.

Part [prop:det:v] follows because the product in each term in the definition includes one element from the zero column. Part [prop:det:vi] follows as $\det$ is a polynomial in the entries of the matrix and hence continuous. We have seen that a function defined on matrices is continuous in the operator norm if it is continuous in the entries. Finally, part [prop:det:vii] is a direct computation.

The determinant tells us about areas and volumes, and how they change. For example, in the 1×1 case, a matrix is just a number, and the determinant is exactly this number. It says how the linear mapping “stretches” the space. Similarly for $\mathbb{R}^2$ (and in fact for $\mathbb{R}^n$). Suppose $A \in L(\mathbb{R}^2)$ is a linear transformation. It can be checked directly that the area of the image of the unit square $A([0, 1]^2)$ is precisely $|\det(A)|$. The sign of the determinant tells us if the image is flipped or not. This works with arbitrary figures, not just the unit square. The determinant tells us the stretch in the area. In $\mathbb{R}^3$ it will tell us about the 3 dimensional volume, and in n -dimensions about the n -dimensional volume. We claim this without proof.

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \det(B)$. In particular, A is invertible if and only if $\det(A) \neq 0$ and in this case, $\det(A^{-1}) = \frac{1}{\det(A)}$.

Let $b_1, b_2, \dots, b_n$ be the columns of B . Then

$$AB = [Ab_1 \ Ab_2 \ \cdots \ Ab_n].$$

That is, the columns of AB are $Ab_1, Ab_2, \dots, Ab_n$.

Let $b_{j,k}$ denote the elements of B and a_j the columns of A . Note that $Ae_j = a_j$. By linearity of the determinant as proved above we have

$$\begin{aligned}
 \det(AB) &= \det([Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n]) = \det\left(\left[\sum_{j=1}^n b_{j,1}a_j \quad Ab_2 \quad \cdots \quad Ab_n\right]\right) \\
 &= \sum_{j=1}^n b_{j,1} \det([a_j \quad Ab_2 \quad \cdots \quad Ab_n]) \\
 &= \sum_{1 \leq j_1, j_2, \dots, j_n \leq n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \det([a_{j_1} \quad a_{j_2} \quad \cdots \quad a_{j_n}]) \\
 &= \left(\sum_{(j_1, j_2, \dots, j_n) \in S_n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \operatorname{sgn}(j_1, j_2, \dots, j_n) \right) \det([a_1 \quad a_2 \quad \cdots \quad a_n]).
 \end{aligned}$$

In the above, go from all integers between 1 and n , to just elements of S_n by noting that when two columns in the determinant are the same, then the determinant is zero. We then reorder the columns to the original ordering and obtain the sgn .

The conclusion that $\det(AB) = \det(A) \det(B)$ follows by recognizing the determinant of B . We obtain this by plugging in $A = I$. The expression we got for the determinant of B has rows and columns swapped, so as a side note, we have also just proved that the determinant of a matrix and its transpose are equal.

To prove the second part of the theorem, suppose A is invertible. Then $A^{-1}A = I$ and consequently $\det(A^{-1}) \det(A) = \det(A^{-1}A) = \det(I) = 1$. If A is not invertible, then the columns are linearly dependent. That is, suppose

$$\sum_{j=1}^n \gamma_j a_j = 0,$$

where not all γ_j are equal to 0. Without loss of generality suppose $\gamma_1 \neq 0$. Take

$$B := \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & 1 & 0 & \cdots & 0 \\ \gamma_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Applying the definition of the determinant we see $\det(B) = \gamma_1 \neq 0$. Then $\det(AB) = \det(A) \det(B) = \gamma_1 \det(A)$. The first column of AB is zero, and hence $\det(AB) = 0$. Thus $\det(A) = 0$.

Determinant is independent of the basis. In other words, if B is invertible, then

$$\det(A) = \det(B^{-1}AB).$$

Proof follows by noting $\det(B^{-1}AB) = \frac{1}{\det(B)} \det(A) \det(B) = \det(A)$. If in one basis A is the matrix representing a linear operator, then for another basis we can find a matrix B such that the matrix $B^{-1}AB$ takes us to the first basis, applies A in the first basis, and takes us back to the basis we started with. We choose a basis on X , and we represent a linear mapping using a matrix with respect to this basis. We obtain the same determinant as if we had used any other basis. It follows that

$$\det : L(X) \rightarrow \mathbb{R}$$

is a well-defined function (not just on matrices).

There are three types of so-called *elementary matrices*. Recall again that e_j are the standard basis of $\mathbb{R}^n$. First for some $j = 1, 2, \dots, n$ and some $\lambda \in \mathbb{R}, \lambda \neq 0$, an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ \lambda e_i & \text{if } i = j. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with the k th row multiplied by λ . It is an easy computation (exercise) that $\det(E) = \lambda$.

Second, for some j and k with $j \neq k$, and $\lambda \in \mathbb{R}$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j, \\ e_i + \lambda e_k & \text{if } i = j. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix as M except with λ times the k th row added to the j th row. It is an easy computation (exercise) that $\det(E) = 1$.

Finally, for some j and k with $j \neq k$ an $n \times n$ matrix E defined by

$$Ee_i = \begin{cases} e_i & \text{if } i \neq j \text{ and } i \neq k, \\ e_k & \text{if } i = j, \\ e_j & \text{if } i = k. \end{cases}$$

Given any $n \times m$ matrix M the matrix EM is the same matrix with j th and k th rows swapped. It is an easy computation (exercise) that $\det(E) = -1$.

Elementary matrices are useful for computing the determinant. The proof of the following proposition is left as an exercise.

[prop:elemmatrixdecomp] Let T be an $n \times n$ invertible matrix. Then there exists a finite sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that

$$T = E_1 E_2 \cdots E_k,$$

and

$$\det(T) = \det(E_1) \det(E_2) \cdots \det(E_k).$$

Exercises

If X is a vector space with a norm $\|\cdot\|$, then show that $d(x, y) := \|x - y\|$ makes X a metric space.

Show that for square matrices A and B , $\det(AB) = \det(BA)$.

For $\mathbb{R}^n$ define

$$\|x\|_\infty := \max \{|x_1|, |x_2|, \dots, |x_n|\},$$

sometimes called the sup or the max norm.

a) Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$ (defining a different distance).

b) What is the unit ball $B(0, 1)$ in this norm?

For $\mathbb{R}^n$ define

$$\|x\|_1 := \sum_{j=1}^n |x_j|,$$

sometimes called the 1-norm (or L^1 norm).

- Show that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$ (defining a different distance, sometimes called the taxicab distance).
- What is the unit ball $B(0, 1)$ in this norm?

Using the euclidean norm on $\mathbb{R}^2$. Compute the operator norm of the operators in $L(\mathbb{R}^2)$ given by the matrices:

$$\text{a) } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ b) } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ c) } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ d) } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

[exercise:normonedim] Using the standard euclidean norm $\mathbb{R}^n$. Show

- Suppose $A \in L(\mathbb{R}, \mathbb{R}^n)$ is defined for $x \in \mathbb{R}$ by $Ax = xa$ for a vector $a \in \mathbb{R}^n$. Then the operator norm $\|A\|_{L(\mathbb{R}, \mathbb{R}^n)} = \|a\|_{\mathbb{R}^n}$.

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$\mathbb{R}^n, \mathbb{R}^n)$ is defined for $x \in \mathbb{R}^n$ by $Bx = b \cdot x$ for a vector $b \in \mathbb{R}^n$. Then the operator norm $\|B\|_{L(\mathbb{R}^n, \mathbb{R}^n)} = \|b\|_{\mathbb{R}^n}$.

Suppose $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a permutation of $(1, 2, \dots, n)$.

- Show that we can make a finite number of transpositions (switching of two elements) to get to $(1, 2, \dots, n)$.
- Using the definition [eq:sgn] show that σ is even if $\text{sgn}(\sigma) = 1$ and σ is odd if $\text{sgn}(\sigma) = -1$. In particular, showing that being odd or even is well defined.

Verify the computation of the determinant for the three types of elementary matrices.

Prove .

- Suppose $D = [d_{i,j}]$ is an n -by- n diagonal matrix, that is, $d_{i,j} = 0$ whenever $i \neq j$. Show that $\det(D) = d_{1,1}d_{2,2}\cdots d_{n,n}$.
- Suppose A is a diagonalizable matrix. That is, there exists a matrix B such that $B^{-1}AB = D$ for a diagonal matrix $D = [d_{i,j}]$. Show that $\det(A) = d_{1,1}d_{2,2}\cdots d_{n,n}$.

Take the vectorspace of polynomials $\mathbb{R}[t]$ and the linear operator $D \in L(\mathbb{R}[t])$ that is the differentiation (we proved in an earlier exercise that D is a linear operator). Define the norm on $P(t) = c_0 + c_1t + \cdots + c_nt^n$ as $\|P\| := \sup \{|c_j| : j = 0, 1, \dots, n\}$.

- Show that $\|P\|$ is a norm on $\mathbb{R}[t]$.
- Show that D does not have bounded operator norm, that is $\|D\| = \infty$. Hint: consider the polynomials t^n as n tends to infinity.

In this exercise we finish the proof of . Let X be any finite dimensional vector space with a norm. Let $\{x_1, x_2, \dots, x_n\}$ be a basis for X .

- Show that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(c_1, c_2, \dots, c_n) = \|c_1x_1 + c_2x_2 + \cdots + c_nx_n\|$$

is continuous.

- Show that there exists numbers m and M such that if $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$ with $\|c\| = 1$ (standard euclidean norm), then $m \leq \|c_1x_1 + c_2x_2 + \cdots + c_nx_n\| \leq M$ (here the norm is on X).
- Show that there exists a number B such that if $\|c_1x_1 + c_2x_2 + \cdots + c_nx_n\| = 1$, then $|c_j| \leq B$.
- Use part (c) to show that if X and Y are finite dimensional vector spaces and $A \in L(X, Y)$, then $\|A\| < \infty$.

Let X be any finite dimensional vector space with a norm $\|\cdot\|$ and basis $\{x_1, x_2, \dots, x_n\}$. Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and $\|c\|$ be the standard euclidean norm on $\mathbb{R}^n$.

- Find that there exist positive numbers $m, M > 0$ such that for

$$m\|c\| \leq \|c_1x_1 + c_2x_2 + \cdots + c_nx_n\| \leq M\|c\|.$$

Hint: See previous exercise.

b) Use part (a) to show that of $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X , then there exist positive numbers $m, M > 0$ (perhaps different than above) such that for all $x \in X$ we have

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

c) Now show that $U \subset X$ is open in the metric defined by $\|x - y\|_1$ if and only if it is open in the metric defined by $\|x - y\|_2$. In other words, convergence of sequences, continuity of functions is the same in either norm.

The derivative

Note: 2–3 lectures

The derivative

Recall that for a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we defined the derivative at x as

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

In other words, there was a number a (the derivative of f at x) such that

$$\lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} - a \right| = \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x) - ah}{h} \right| = \lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - ah|}{|h|} = 0.$$

Multiplying by a is a linear map in one dimension. That is, we think of $a \in L(\mathbb{R}^1, \mathbb{R}^1)$ which is the best linear approximation of f near x . We use this definition to extend differentiation to more variables.

Let $U \subset \mathbb{R}^n$ be an open subset and $f: U \rightarrow \mathbb{R}^m$. We say f is *differentiable* at $x \in U$ if there exists an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}^n}} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

We write $Df(x) := A$, or $f'(x) := A$, and we say A is the *derivative* of f at x . When f is *differentiable* at all $x \in U$, we say simply that f is differentiable.

For a differentiable function, the derivative of f is a function from U to $L(\mathbb{R}^n, \mathbb{R}^m)$. Compare to the one dimensional case, where the derivative is a function from U to $\mathbb{R}$, but we really want to think of $\mathbb{R}$ here as $L(\mathbb{R}^1, \mathbb{R}^1)$.

The norms above must be on the right spaces of course. The norm in the numerator is on $\mathbb{R}^m$, and the norm in the denominator is on $\mathbb{R}^n$ where h lives. Normally it is understood that $h \in \mathbb{R}^n$ from context. We will not explicitly say so from now on.

We have again cheated somewhat and said that A is *the* derivative. We have not shown yet that there is only one, let us do that now.

Let $U \subset \mathbb{R}^n$ be an open subset and $f: U \rightarrow \mathbb{R}^m$. Suppose $x \in U$ and there exist $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|} = 0.$$

Then $A = B$.

$$\begin{aligned} \frac{\|(A-B)h\|}{\|h\|} &= \frac{\|f(x+h) - f(x) - Ah - (f(x+h) - f(x) - Bh)\|}{\|h\|} \\ &\leq \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} + \frac{\|f(x+h) - f(x) - Bh\|}{\|h\|}. \end{aligned}$$

So $\frac{\|(A-B)h\|}{\|h\|} \rightarrow 0$ as $h \rightarrow 0$. That is, given $\epsilon > 0$, then for all h in some δ -ball around the origin

$$\epsilon > \frac{\|(A-B)h\|}{\|h\|} = \|(A-B)\frac{h}{\|h\|}\|.$$

For any x with $\|x\| = 1$, let $h = (\text{nicefrac}{2}\delta)x$, then $\|h\| < \delta$ and $\frac{h}{\|h\|} = x$. So $\|(A-B)x\| < \epsilon$. Taking the supremum over all x with $\|x\| = 1$ we get the operator norm $\|A-B\| \leq \epsilon$. As $\epsilon > 0$ was arbitrary $\|A-B\| = 0$ or in other words $A = B$.

If $f(x) = Ax$ for a linear mapping A , then $f'(x) = A$. This is easily seen:

$$\frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = \frac{\|A(x+h) - Ax - Ah\|}{\|h\|} = \frac{0}{\|h\|} = 0.$$

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f(x, y) = (f_1(x, y), f_2(x, y)) := (1 + x + 2y + x^2, 2x + 3y + xy)$. Let us show that f is differentiable at the origin and let us compute the derivative, directly using the definition. The derivative is in $L(\mathbb{R}^2, \mathbb{R}^2)$ so it can be represented by

a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose $h = (h_1, h_2)$. We need the following expression to go to zero.

$$\frac{\|f(h_1, h_2) - f(0, 0) - (ah_1 + bh_2, ch_1 + dh_2)\|}{\|(h_1, h_2)\|} = \frac{\sqrt{((1-a)h_1 + (2-b)h_2 + h_1^2)^2 + ((2-c)h_1 + (3-d)h_2 + h_1h_2)^2}}{\sqrt{h_1^2 + h_2^2}}.$$

If we choose $a = 1, b = 2, c = 2, d = 3$, the expression becomes

$$\frac{\sqrt{h_1^4 + h_1^2h_2^2}}{\sqrt{h_1^2 + h_2^2}} = |h_1| \frac{\sqrt{h_1^2 + h_2^2}}{\sqrt{h_1^2 + h_2^2}} = |h_1|.$$

And this expression does indeed go to zero as $h \rightarrow 0$. Therefore the function is differentiable at the origin and the derivative can be represented by the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^m$ be differentiable at $p \in U$. Then f is continuous at p .

Another way to write the differentiability of f at p is to first write

$$r(h) := f(p+h) - f(p) - f'(p)h,$$

and $\frac{\|r(h)\|}{\|h\|}$ must go to zero as $h \rightarrow 0$. So $r(h)$ itself must go to zero. The mapping $h \mapsto f'(p)h$ is a linear mapping between finite dimensional spaces, it is therefore continuous and goes to zero as $h \rightarrow 0$. Therefore, $f(p+h)$ must go to $f(p)$ as $h \rightarrow 0$. That is, f is continuous at p .

Let $U \subset \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^m$ be differentiable at $p \in U$. Let $V \subset \mathbb{R}^m$ be open, $f(U) \subset V$ and let $g: V \rightarrow \mathbb{R}^\ell$ be differentiable at $f(p)$. Then

$$F(x) = g(f(x))$$

is differentiable at p and

$$F'(p) = g'(f(p))f'(p).$$

Without the points where things are evaluated, this is sometimes written as $F' = (f \circ g)' = g'f'$. The way to understand it is that the derivative of the composition $g \circ f$ is the composition of the derivatives of g and f . That is, if $f'(p) = A$ and $g'(f(p)) = B$, then $F'(p) = BA$.

Let $A := f'(p)$ and $B := g'(f(p))$. Take $h \in \mathbb{R}^n$ and write $q = f(p)$, $k = f(p+h) - f(p)$. Let

$$r(h) := f(p+h) - f(p) - Ah.$$

Then $r(h) = k - Ah$ or $Ah = k - r(h)$. We look at the quantity we need to go to zero:

$$\begin{aligned} \frac{\|F(p+h) - F(p) - BAh\|}{\|h\|} &= \frac{\|g(f(p+h)) - g(f(p)) - BAh\|}{\|h\|} \\ &= \frac{\|g(q+k) - g(q) - B(k - r(h))\|}{\|h\|} \\ &\leq \frac{\|g(q+k) - g(q) - Bk\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|} \\ &= \frac{\|g(q+k) - g(q) - Bk\|}{\|k\|} \frac{\|f(p+h) - f(p)\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|}. \end{aligned}$$

First, $\|B\|$ is constant and f is differentiable at p , so the term $\|B\| \frac{\|r(h)\|}{\|h\|}$ goes to 0. Next as f is continuous at p , we have that as h goes to 0, then k goes to 0. Therefore $\frac{\|g(q+k) - g(q) - Bk\|}{\|k\|}$ goes to 0 because g is differentiable at q . Finally

$$\frac{\|f(p+h) - f(p)\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \frac{\|Ah\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \|A\|.$$

As f is differentiable at p , for small enough h $\|f(p+h) - f(p) - Ah\|$ is bounded. Therefore the term $\frac{\|f(p+h) - f(p)\|}{\|h\|}$ stays bounded as h goes to 0. Therefore, $\frac{\|F(p+h) - F(p) - BAh\|}{\|h\|}$ goes to zero, and $F'(p) = BA$, which is what was claimed.

Partial derivatives

There is another way to generalize the derivative from one dimension. We hold all but one variable constant and take the regular derivative.

Let $f: U \rightarrow \mathbb{R}$ be a function on an open set $U \subset \mathbb{R}^n$. If the following limit exists we write

$$\frac{\partial f}{\partial x_j}(x) := \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x + he_j) - f(x)}{h}.$$

We call $\frac{\partial f}{\partial x_j}(x)$ the *partial derivative* of f with respect to x_j . Sometimes we write $D_j f$ instead.

For a mapping $f: U \rightarrow \mathbb{R}^m$ we write $f = (f_1, f_2, \dots, f_m)$, where f_k are real-valued functions. Then we define $\frac{\partial f_k}{\partial x_j}$ (or write it as $D_j f_k$).

Partial derivatives are easier to compute with all the machinery of calculus, and they provide a way to compute the derivative of a function.

[mv:prop:jacobianmatrix] Let $U \subset \mathbb{R}^n$ be open and let $f: U \rightarrow \mathbb{R}^m$ be differentiable at $p \in U$. Then all the partial derivatives at p exist and in terms of the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, $f'(p)$ is represented by the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \cdots & \frac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \frac{\partial f_m}{\partial x_2}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}.$$

In other words

$$f'(p)e_j = \sum_{k=1}^m \frac{\partial f_k}{\partial x_j}(p)e_k.$$

If $v = \sum_{j=1}^n c_j e_j = (c_1, c_2, \dots, c_n)$, then

$$f'(p)v = \sum_{j=1}^n \sum_{k=1}^m c_j \frac{\partial f_k}{\partial x_j}(p)e_k = \sum_{k=1}^m \left(\sum_{j=1}^n c_j \frac{\partial f_k}{\partial x_j}(p) \right) e_k.$$

Fix a j and note that

$$\begin{aligned} \left\| \frac{f(p + he_j) - f(p)}{h} - f'(p)e_j \right\| &= \left\| \frac{f(p + he_j) - f(p) - f'(p)he_j}{h} \right\| \\ &= \frac{\|f(p + he_j) - f(p) - f'(p)he_j\|}{\|he_j\|}. \end{aligned}$$

As h goes to 0, the right hand side goes to zero by differentiability of f , and hence

$$\lim_{h \rightarrow 0} \frac{f(p + he_j) - f(p)}{h} = f'(p)e_j.$$

Note that f is vector valued. So represent f by components $f = (f_1, f_2, \dots, f_m)$, and note that taking a limit in $\mathbb{R}^m$ is the same as taking the limit in each component separately. Therefore for any k the partial derivative

$$\frac{\partial f_k}{\partial x_j}(p) = \lim_{h \rightarrow 0} \frac{f_k(p + he_j) - f_k(p)}{h}$$

exists and is equal to the k th component of $f'(p)e_j$, and we are done.

The converse of the proposition is not true. Just because the partial derivatives exist, does not mean that the function is differentiable. See the exercises. However, when the partial derivatives are continuous, we will prove that the converse holds. One of the consequences of the proposition is that if f is differentiable on U , then $f' : U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function if and only if all the $\frac{\partial f_k}{\partial x_j}$ are continuous functions.

Gradient and directional derivatives

Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}$ is a differentiable function. We define the *gradient* as

$$\nabla f(x) := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x) e_j.$$

Notice that the gradient gives us a way to represent the action of the derivative as a dot product: $f'(x)v = \nabla f(x) \cdot v$.

Suppose $\gamma: (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is a differentiable function and the image $\gamma((a, b)) \subset U$. Such a function and its image is sometimes called a *curve*, or a *differentiable curve*. Write $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$. Let

$$g(t) := f(\gamma(t)).$$

The function g is differentiable. For purposes of computation we identify $L(\mathbb{R}^1)$ with $\mathbb{R}$, and hence $g'(t)$ can be computed as a number:

$$g'(t) = f'(\gamma(t))\gamma'(t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\gamma(t)) \frac{d\gamma_j}{dt}(t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{d\gamma_j}{dt}.$$

For convenience, we sometimes leave out the points where we are evaluating as on the right hand side above. Let us rewrite this with the notation of the gradient and the dot product:

$$g'(t) = (\nabla f)(\gamma(t)) \cdot \gamma'(t) = \nabla f \cdot \gamma'.$$

We use this idea to define derivatives in a specific direction. A direction is simply a vector pointing in that direction. So pick a vector $u \in \mathbb{R}^n$ such that $\|u\| = 1$. Fix $x \in U$. Then define a curve

$$\gamma(t) := x + tu.$$

It is easy to compute that $\gamma'(t) = u$ for all t . By chain rule

$$\left. \frac{d}{dt} \right|_{t=0} [f(x + tu)] = (\nabla f)(x) \cdot u,$$

where the notation $\left. \frac{d}{dt} \right|_{t=0}$ represents the derivative evaluated at $t = 0$. We also compute directly

$$\left. \frac{d}{dt} \right|_{t=0} [f(x + tu)] = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}.$$

We obtain the *directional derivative*, denoted by

$$D_u f(x) := \left. \frac{d}{dt} \right|_{t=0} [f(x + tu)],$$

which can be computed by one of the methods above.

Let us suppose $(\nabla f)(x) \neq 0$. By Cauchy-Schwarz inequality we have

$$|D_u f(x)| \leq \|(\nabla f)(x)\|.$$

Equality is achieved when u is a scalar multiple of $(\nabla f)(x)$. That is, when

$$u = \frac{(\nabla f)(x)}{\|(\nabla f)(x)\|},$$

we get $D_u f(x) = \|(\nabla f)(x)\|$. The gradient points in the direction in which the function grows fastest, in other words, in the direction in which $D_u f(x)$ is maximal.

The Jacobian

Let $U \subset \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^n$ be a differentiable mapping. Then define the *Jacobian*, or *Jacobian determinant* ³, of f at x as

$$J_f(x) := \det(f'(x)).$$

Sometimes this is written as

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}.$$

This last piece of notation may seem somewhat confusing, but it is useful when you need to specify the exact variables and function components used.

The Jacobian J_f is a real valued function, and when $n = 1$ it is simply the derivative. From the chain rule and the fact that $\det(AB) = \det(A) \det(B)$, it follows that:

$$J_{f \circ g}(x) = J_f(g(x))J_g(x).$$

As we mentioned the determinant tells us what happens to area/volume. Similarly, the Jacobian measures how much a differentiable mapping stretches things locally, and if it flips orientation. In particular, if the Jacobian is non-zero then we would assume that locally the mapping is invertible (and we would be correct as we will later see).

Exercises

Suppose $\gamma: (-1, 1) \rightarrow \mathbb{R}^n$ and $\alpha: (-1, 1) \rightarrow \mathbb{R}^n$ be two differentiable curves such that $\gamma(0) = \alpha(0)$ and $\gamma'(0) = \alpha'(0)$. Suppose $F: \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function. Show that

$$\left. \frac{d}{dt} \right|_{t=0} F(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} F(\alpha(t)).$$

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $f(x, y) = \sqrt{x^2 + y^2}$. Show that f is not differentiable at the origin.

Using only the definition of the derivative, show that the following $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are differentiable at the origin and find their derivative.

- a) $f(x, y) := (1 + x + xy, x)$,
- b) $f(x, y) := (y - y^{10}, x)$,
- c) $f(x, y) := ((x + y + 1)^2, (x - y + 2)^2)$.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are differentiable functions. Using only the definition of the derivative, show that $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $h(x, y) := (f(x), g(y))$ is a differentiable function and find the derivative at any point (x, y) .

[exercise:noncontpartialsexist] Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

- a) Show that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points (including the origin).
 b) Show that f is not continuous at the origin (and hence not differentiable).

Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

- a) Show that partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points.
 b) Show that for all $u \in \mathbb{R}^2$ with $\|u\| = 1$, the directional derivative $D_u f$ exists at all points.
 c) Show that f is continuous at the origin.
 d) Show that f is not differentiable at the origin.

Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one, onto, differentiable at all points, and such that f^{-1} is also differentiable at all points.

- a) Show that $f'(p)$ is invertible at all points p and compute $(f^{-1})'(f(p))$. Hint: consider $p = f^{-1}(f(p))$.
 b) Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function differentiable at $q \in \mathbb{R}^n$ and such that $g(q) = q$. Suppose $f(p) = q$ for some $p \in \mathbb{R}^n$. Show $J_g(q) = J_{f^{-1} \circ g} \circ f(p)$ where J_g is the Jacobian determinant.

Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable and such that $f(x, y) = 0$ if and only if $y = 0$ and such that $\nabla f(0, 0) = (1, 1)$. Prove that $f(x, y) > 0$ whenever $y > 0$, and $f(x, y) < 0$ whenever $y < 0$.

[exercise:mv:maximumcritical] Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}$ is differentiable. Suppose f has a local maximum at $p \in U$. Show that $f'(p) = 0$, that is the zero mapping in $L(\mathbb{R}^n, \mathbb{R})$. That is p is a *critical point* of f .

Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable and suppose that whenever $x^2 + y^2 = 1$, then $f(x, y) = 0$. Prove that there exists at least one point (x_0, y_0) such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.

Define $f(x, y) := (x - y^2)(2y^2 - x)$. Show

- a) $(0, 0)$ is a critical point, that is $f'(0, 0) = 0$, that is the zero linear map in $L(\mathbb{R}^2, \mathbb{R})$.
 b) For every direction, that is (x, y) such that $x^2 + y^2 = 1$ the “restriction of f to the line containing the points $(0, 0)$ and (x, y) ”, that is a function $g(t) := f(tx, ty)$ has a local maximum at $t = 0$.
 c) f does not have a local maximum at $(0, 0)$.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}^n$ is differentiable and $\|f(t)\| = 1$ for all t (that is, we have a curve in the unit sphere). Then show that for all t , treating f' as a vector we have, $f'(t) \cdot f(t) = 0$.

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(x, y) := (x, y + \phi(x))$ for some differentiable function ϕ of one variable. Show f is differentiable and find f' .

Continuity and the derivative

Note: 1–2 lectures

Bounding the derivative

Let us prove a “mean value theorem” for vector valued functions.

If $\phi: [a, b] \rightarrow \mathbb{R}^n$ is differentiable on (a, b) and continuous on $[a, b]$, then there exists a $t_0 \in (a, b)$ such that

$$\|\phi(b) - \phi(a)\| \leq (b - a) \|\phi'(t_0)\|.$$

By mean value theorem on the function $(\phi(b) - \phi(a)) \cdot \phi(t)$ (the dot is the scalar dot product again) we obtain there is a $t_0 \in (a, b)$ such that

$$(\varphi(b) - \varphi(a)) \cdot \varphi(b) - (\varphi(b) - \varphi(a)) \cdot \varphi(a) = \|\varphi(b) - \varphi(a)\|^2 = (b - a)(\varphi(b) - \varphi(a)) \cdot \varphi'(t_0)$$

where we treat φ' as a simply a column vector of numbers by abuse of notation. Note that in this case, if we think of $\varphi'(t)$ as simply a vector, then by , $\|\varphi'(t)\|_{L(\mathbb{R}, \mathbb{R}^n)} = \|\varphi'(t)\|_{\mathbb{R}^n}$. That is, the euclidean norm of the vector is the same as the operator norm of $\varphi'(t)$.

By Cauchy-Schwarz inequality

$$\|\varphi(b) - \varphi(a)\|^2 = (b - a)(\varphi(b) - \varphi(a)) \cdot \varphi'(t_0) \leq (b - a)\|\varphi(b) - \varphi(a)\| \|\varphi'(t_0)\|. \text{ \textcolor{red}{qedhere}}$$

Recall that a set U is convex if whenever $x, y \in U$, the line segment from x to y lies in U .

[mv:prop:convexlip] Let $U \subset \mathbb{R}^n$ be a convex open set, $f: U \rightarrow \mathbb{R}^m$ a differentiable function, and an M such that

$$\|f'(x)\| \leq M$$

for all $x \in U$. Then f is Lipschitz with constant M , that is

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

for all $x, y \in U$.

Fix x and y in U and note that $(1 - t)x + ty \in U$ for all $t \in [0, 1]$ by convexity. Next

$$\frac{d}{dt} [f((1 - t)x + ty)] = f'((1 - t)x + ty)(y - x).$$

By the mean value theorem above we get for some $t_0 \in (0, 1)$

$$\|f(x) - f(y)\| \leq \left\| \frac{d}{dt} \Big|_{t=t_0} [f((1 - t)x + ty)] \right\| \leq \|f'((1 - t_0)x + t_0y)\| \|y - x\| \leq M\|y - x\|. \text{ \textcolor{red}{qedhere}}$$

If U is not convex the proposition is not true. To see this fact, take the set

$$U = \{(x, y): 0.9 < x^2 + y^2 < 1.1\} \setminus \{(x, 0): x < 0\}.$$

Let $f(x, y)$ be the angle that the line from the origin to (x, y) makes with the positive x axis. You can even write the formula for f :

$$f(x, y) = 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right).$$

Think spiral staircase with room in the middle. See .

The function is differentiable, and the derivative is bounded on U , which is not hard to see. Thinking of what happens near where the negative x -axis cuts the annulus in half, we see that the conclusion of the proposition cannot hold.

Let us solve the differential equation $f' = 0$.

If $U \subset \mathbb{R}^n$ is connected and $f: U \rightarrow \mathbb{R}^m$ is differentiable and $f'(x) = 0$, for all $x \in U$, then f is constant.

For any $x \in U$, there is a ball $B(x, \delta) \subset U$. The ball $B(x, \delta)$ is convex. Since $\|f'(y)\| \leq 0$ for all $y \in B(x, \delta)$, then by the theorem, $\|f(x) - f(y)\| \leq 0\|x - y\| = 0$. So $f(x) = f(y)$ for all $y \in B(x, \delta)$.

This means that $f^{-1}(c)$ is open for any $c \in \mathbb{R}^m$. Suppose $f^{-1}(c)$ is nonempty. The two sets

$$U' = f^{-1}(c), \quad U'' = f^{-1}(\mathbb{R}^m \setminus \{c\}) = \bigcup_{\substack{a \in \mathbb{R}^m \\ a \neq c}} f^{-1}(a)$$

are open disjoint, and further $U = U' \cup U''$. So as U' is nonempty, and U is connected, we have that $U'' = \emptyset$. So $f(x) = c$ for all $x \in U$.

Continuously differentiable functions

We say $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *continuously differentiable*, or $C^1(U)$ if f is differentiable and $f': U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

[mv:prop:contdiffpartials] Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^m$. The function f is continuously differentiable if and only if all the partial derivatives exist and are continuous.

Without continuity the theorem does not hold. Just because partial derivatives exist does not mean that f is differentiable, in fact, f may not even be continuous. See the exercises for the last section and also for this section.

We have seen that if f is differentiable, then the partial derivatives exist. Furthermore, the partial derivatives are the entries of the matrix of $f'(x)$. So if $f': U \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then the entries are continuous, hence the partial derivatives are continuous.

To prove the opposite direction, suppose the partial derivatives exist and are continuous. Fix $x \in U$. If we show that $f'(x)$ exists we are done, because the entries of the matrix $f'(x)$ are then the partial derivatives and if the entries are continuous functions, the matrix valued function f' is continuous.

Let us do induction on dimension. First let us note that the conclusion is true when $n = 1$. In this case the derivative is just the regular derivative (exercise: you should check that the fact that the function is vector valued is not a problem).

Suppose the conclusion is true for $\mathbb{R}^{n-1}$, that is, if we restrict to the first $n - 1$ variables, the conclusion is true. It is easy to see that the first $n - 1$ partial derivatives of f restricted to the set where the last coordinate is fixed are the same as those for f . In the following we think of $\mathbb{R}^{n-1}$ as a subset of $\mathbb{R}^n$, that is the set in $\mathbb{R}^n$ where $x_n = 0$. Let

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_{n-1}}(x) \end{bmatrix}, \quad v = \begin{bmatrix} \frac{\partial f_1}{\partial x_n}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that for any $k \in \mathbb{R}^{n-1}$ with $\|k\| < \delta$ we have

$$\frac{\|f(x+k) - f(x) - A_1 k\|}{\|k\|} < \epsilon.$$

By continuity of the partial derivatives, suppose δ is small enough so that

$$\left| \frac{\partial f_j}{\partial x_n}(x+h) - \frac{\partial f_j}{\partial x_n}(x) \right| < \epsilon,$$

for all j and all h with $\|h\| < \delta$.

Let $h = h_1 + te_n$ be a vector in $\mathbb{R}^n$ where $h_1 \in \mathbb{R}^{n-1}$ such that $\|h\| < \delta$. Then $\|h_1\| \leq \|h\| < \delta$. Note that $Ah = A_1 h_1 + tv$.

$$\begin{aligned} \|f(x+h) - f(x) - Ah\| &= \|f(x+h_1+te_n) - f(x+h_1) - tv + f(x+h_1) - f(x) - A_1 h_1\| \\ &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \|f(x+h_1) - f(x) - A_1 h_1\| \\ &\leq \|f(x+h_1+te_n) - f(x+h_1) - tv\| + \epsilon \|h_1\|. \end{aligned}$$

As all the partial derivatives exist, by the mean value theorem, for each j there is some $\theta_j \in [0, t]$ (or $[t, 0]$ if $t < 0$), such that

$$f_j(x + h_1 + te_n) - f_j(x + h_1) = t \frac{\partial f_j}{\partial x_n}(x + h_1 + \theta_j e_n).$$

Note that if $\|h\| < \delta$, then $\|h_1 + \theta_j e_n\| \leq \|h\| < \delta$. So to finish the estimate

$$\begin{aligned} \|f(x + h) - f(x) - Ah\| &\leq \|f(x + h_1 + te_n) - f(x + h_1) - tv\| + \epsilon \|h_1\| \\ &\leq \sqrt{\sum_{j=1}^m \left(t \frac{\partial f_j}{\partial x_n}(x + h_1 + \theta_j e_n) - t \frac{\partial f_j}{\partial x_n}(x) \right)^2} + \epsilon \|h_1\| \\ &\leq \sqrt{m} \epsilon |t| + \epsilon \|h_1\| \\ &\leq (\sqrt{m} + 1) \epsilon \|h\|. \end{aligned}$$

Exercises

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$f(x, y) := \begin{cases} (x^2 + y^2) \sin((x^2 + y^2)^{-1}) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else.} \end{cases}$$

Show that f is differentiable at the origin, but that it is not continuously differentiable.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function from , that is,

$$f(x, y) := \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at all points and show that these are not continuous functions.

Let $B(0, 1) \subset \mathbb{R}^2$ be the unit ball (disc), that is, the set given by $x^2 + y^2 < 1$. Suppose $f: B(0, 1) \rightarrow \mathbb{R}$ is a differentiable function

such that $|f(0, 0)| \leq 1$, and $\left| \frac{\partial f}{\partial x} \right| \leq 1$ and $\left| \frac{\partial f}{\partial y} \right| \leq 1$ for all points in $B(0, 1)$.

- Find an $M \in \mathbb{R}$ such that $\|f'(x, y)\| \leq M$ for all $(x, y) \in B(0, 1)$.
- Find a $B \in \mathbb{R}$ such that $|f(x, y)| \leq B$ for all $(x, y) \in B(0, 1)$.

Define $\varphi: [0, 2\pi] \rightarrow \mathbb{R}^2$ by $\varphi(t) = (\sin(t), \cos(t))$. Compute $\varphi'(t)$ for all t . Compute $\|\varphi'(t)\|$ for all t . Notice that $\varphi'(t)$ is never zero, yet $\varphi(0) = \varphi(2\pi)$, therefore, Rolle's theorem is not true in more than one dimension.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points and there exists an $M \in \mathbb{R}$ such that $\left| \frac{\partial f}{\partial x} \right| \leq M$ and $\left| \frac{\partial f}{\partial y} \right| \leq M$ at all points. Show that f is continuous.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function and $M \in \mathbb{R}$, such that for every $(x, y) \in \mathbb{R}^2$, the function $g(t) := f(xt, yt)$ is differentiable and $|g'(t)| \leq M$.

- Show that f is continuous at $(0, 0)$.
- Find an example of such an f which is not continuous at every other point of $\mathbb{R}^2$ (Hint: Think back to how did we construct a nowhere continuous function on $[0, 1]$).

Inverse and implicit function theorem

Note: 2–3 lectures

To prove the inverse function theorem we use the contraction mapping principle we have seen in and that we have used to prove Picard's theorem. Recall that a mapping $f: X \rightarrow X'$ between two metric spaces (X, d) and (X', d') is called a contraction if there exists a $k < 1$ such that

$$d'(f(x), f(y)) \leq kd(x, y) \quad \text{for all } x, y \in X.$$

The contraction mapping principle says that if $f: X \rightarrow X$ is a contraction and X is a complete metric space, then there exists a unique fixed point, that is, there exists a unique $x \in X$ such that $f(x) = x$.

Intuitively if a function is differentiable, then it locally “behaves like” the derivative (which is a linear function). The idea of the inverse function theorem is that if a function is differentiable and the derivative is invertible, the function is (locally) invertible.

[thm:inverse] Let $U \subset \mathbb{R}^n$ be a set and let $f: U \rightarrow \mathbb{R}^n$ be a continuously differentiable function. Also suppose $p \in U$, $f(p) = q$, and $f'(p)$ is invertible (that is, $J_f(p) \neq 0$). Then there exist open sets $V, W \subset \mathbb{R}^n$ such that $p \in V \subset U$, $f(V) = W$ and $f|_V$ is one-to-one and onto. Furthermore, the inverse $g(y) = (f|_V)^{-1}(y)$ is continuously differentiable and

$$g'(y) = (f'(x))^{-1}, \quad \text{for all } x \in V, y = f(x).$$

Write $A = f'(p)$. As f' is continuous, there exists an open ball V around p such that

$$\|A - f'(x)\| < \frac{1}{2\|A^{-1}\|} \quad \text{for all } x \in V.$$

Note that $f'(x)$ is invertible for all $x \in V$.

Given $y \in \mathbb{R}^n$ we define $\varphi_y: V \rightarrow \mathbb{R}^n$

$$\varphi_y(x) = x + A^{-1}(y - f(x)).$$

As A^{-1} is one-to-one, then $\varphi_y(x) = x$ (x is a fixed point) if only if $y - f(x) = 0$, or in other words $f(x) = y$. Using chain rule we obtain

$$\varphi_y'(x) = I - A^{-1}f'(x) = A^{-1}(A - f'(x)).$$

So for $x \in V$ we have

$$\|\varphi_y'(x)\| \leq \|A^{-1}\| \|A - f'(x)\| < \frac{1}{2}.$$

As V is a ball it is convex, and hence

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in V.$$

In other words φ_y is a contraction defined on V , though we so far do not know what is the range of φ_y . We cannot apply the fixed point theorem, but we can say that φ_y has at most one fixed point (note proof of uniqueness in the contraction mapping principle). That is, there exists at most one $x \in V$ such that $f(x) = y$, and so $f|_V$ is one-to-one.

Let $W = f(V)$. We need to show that W is open. Take a $y_1 \in W$, then there is a unique $x_1 \in V$ such that $f(x_1) = y_1$. Let $r > 0$ be small enough such that the closed ball $C(x_1, r) \subset V$ (such $r > 0$ exists as V is open).

Suppose y is such that

$$\|y - y_1\| < \frac{r}{2\|A^{-1}\|}.$$

If we show that $y \in W$, then we have shown that W is open. Define $\varphi_y(x) = x + A^{-1}(y - f(x))$ as before. If $x \in C(x_1, r)$, then

$$\begin{aligned}\|\varphi_y(x) - x_1\| &\leq \|\varphi_y(x) - \varphi_y(x_1)\| + \|\varphi_y(x_1) - x_1\| \\ &\leq \frac{1}{2}\|x - x_1\| + \|A^{-1}(y - y_1)\| \\ &\leq \frac{1}{2}r + \|A^{-1}\|\|y - y_1\| \\ &< \frac{1}{2}r + \|A^{-1}\|\frac{r}{2\|A^{-1}\|} = r.\end{aligned}$$

So φ_y takes $C(x_1, r)$ into $B(x_1, r) \subset C(x_1, r)$. It is a contraction on $C(x_1, r)$ and $C(x_1, r)$ is complete (closed subset of $\mathbb{R}^n$ is complete). Apply the contraction mapping principle to obtain a fixed point x , i.e. $\varphi_y(x) = x$. That is $f(x) = y$. So $y \in f(C(x_1, r)) \subset f(V) = W$. Therefore W is open.

Next we need to show that g is continuously differentiable and compute its derivative. First let us show that it is differentiable. Let $y \in W$ and $k \in \mathbb{R}^n$, $k \neq 0$, such that $y + k \in W$. Then there are unique $x \in V$ and $h \in \mathbb{R}^n$, $h \neq 0$ and $x + h \in V$, such that $f(x) = y$ and $f(x + h) = y + k$ as $f|_V$ is a one-to-one and onto mapping of V onto W . In other words, $g(y) = x$ and $g(y + k) = x + h$. We can still squeeze some information from the fact that φ_y is a contraction.

$$\varphi_y(x + h) - \varphi_y(x) = h + A^{-1}(f(x) - f(x + h)) = h - A^{-1}k.$$

So

$$\|h - A^{-1}k\| = \|\varphi_y(x + h) - \varphi_y(x)\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2}.$$

By the inverse triangle inequality $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\|$ so

$$\|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\|\|k\|.$$

In particular, as k goes to 0, so does h .

As $x \in V$, then $f'(x)$ is invertible. Let $B = (f'(x))^{-1}$, which is what we think the derivative of g at y is. Then

$$\begin{aligned}\frac{\|g(y + k) - g(y) - Bk\|}{\|k\|} &= \frac{\|h - Bk\|}{\|k\|} \\ &= \frac{\|h - B(f(x + h) - f(x))\|}{\|k\|} \\ &= \frac{\|B(f(x + h) - f(x) - f'(x)h)\|}{\|k\|} \\ &\leq \|B\| \frac{\|h\|}{\|k\|} \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|} \\ &\leq 2\|B\|\|A^{-1}\| \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|}.\end{aligned}$$

As k goes to 0, so does h . So the right hand side goes to 0 as f is differentiable, and hence the left hand side also goes to 0. And B is precisely what we wanted $g'(y)$ to be.

We have g is differentiable, let us show it is $C^1(W)$. Now, $g: W \rightarrow V$ is continuous (it is differentiable), f' is a continuous function from V to $L(R^n)$, and $X \rightarrow X^{-1}$ is a continuous function. $g'(y) = (f'(g(y)))^{-1}$ is the composition of these three continuous functions and hence is continuous.

Suppose $U \subset R^n$ is open and $f: U \rightarrow R^n$ is a continuously differentiable mapping such that $f'(x)$ is invertible for all $x \in U$. Then given any open set $V \subset U$, $f(V)$ is open. (f is an open mapping).

Without loss of generality, suppose $U = V$. For each point $y \in f(V)$, we pick $x \in f^{-1}(y)$ (there could be more than one such point), then by the inverse function theorem there is a neighborhood of x in V that maps onto an neighborhood of y . Hence $f(V)$ is open.

The theorem, and the corollary, is not true if $f'(x)$ is not invertible for some x . For example, the map $f(x, y) = (x, xy)$, maps R^2 onto the set $R^2 \setminus \{(0, y): y \neq 0\}$, which is neither open nor closed. In fact $f^{-1}(0, 0) = \{(0, y): y \in R\}$. This bad behavior only occurs on the y -axis, everywhere else the function is locally invertible. If we avoid the y -axis, f is even one-to-one.

Also note that just because $f'(x)$ is invertible everywhere does not mean that f is one-to-one globally. It is "locally" one-to-one but perhaps not "globally." For an example, take the map $f: R^2 \setminus \{0\} \rightarrow R^2$ defined by $f(x, y) = (x^2 - y^2, 2xy)$. It is left to student to show that f is differentiable and the derivative is invertible

On the other hand, the mapping is 2-to-1 globally. For every (a, b) that is not the origin, there are exactly two solutions to $x^2 - y^2 = a$ and $2xy = b$. We leave it to the student to show that there is at least one solution, and then notice that replacing x and y with $-x$ and $-y$ we obtain another solution.

The invertibility of the derivative is not a necessary condition, just sufficient, for having a continuous inverse and being an open mapping. For example the function $f(x) = x^3$ is an open mapping from R to R and is globally one-to-one with a continuous inverse, although the inverse is not differentiable at $x = 0$.

Implicit function theorem

The inverse function theorem is really a special case of the implicit function theorem which we prove next. Although somewhat ironically we prove the implicit function theorem using the inverse function theorem. What we were showing in the inverse function theorem was that the equation $x - f(y) = 0$ was solvable for y in terms of x if the derivative in terms of y was invertible, that is if $f'(y)$ was invertible. That is there was locally a function g such that $x - f(g(x)) = 0$.

OK, so how about we look at the equation $f(x, y) = 0$. Obviously this is not solvable for y in terms of x in every case. For example, when $f(x, y)$ does not actually depend on y . For a slightly more complicated example, notice that $x^2 + y^2 - 1 = 0$ defines the unit circle, and we can locally solve for y in terms of x when 1) we are near a point which lies on the unit circle and 2) when we are not at a point where the circle has a vertical tangency, or in other words where $\frac{\partial f}{\partial y} = 0$.

To make things simple we fix some notation. We let $(x, y) \in R^{n+m}$ denote the coordinates $(x_1, \dots, x_n, y_1, \dots, y_m)$. A linear transformation $A \in L(R^{n+m}, R^m)$ can then be written as $A = [A_x A_y]$ so that $A(x, y) = A_x x + A_y y$, where $A_x \in L(R^n, R^m)$ and $A_y \in L(R^m, R^m)$.

Let $A = [A_x A_y] \in L(R^{n+m}, R^m)$ and suppose A_y is invertible. If $B = -(A_y)^{-1}A_x$, then

$$0 = A(x, Bx) = A_x x + A_y Bx.$$

The proof is obvious. We simply solve and obtain $y = Bx$. Let us show that the same can be done for C^1 functions.

[thm:implicit] Let $U \subset R^{n+m}$ be an open set and let $f: U \rightarrow R^m$ be a $C^1(U)$ mapping. Let $(p, q) \in U$ be a point such that $f(p, q) = 0$ and such that

$$\frac{\partial(f_1, \dots, f_m)}{\partial(y_1, \dots, y_m)}(p, q) \neq 0.$$

Then there exists an open set $W \subset R^n$ with $p \in W$, an open set $W' \subset R^m$ with $q \in W'$, with $W \times W' \subset U$, and a $C^1(W)$ mapping $g: W \rightarrow W'$, with $g(p) = q$, and for all $x \in W$, the point $g(x)$ is the unique point in W' such that

$$f(x, g(x)) = 0.$$

Furthermore, if $[A_x A_y] = f'(p, q)$, then

$$g'(p) = -(A_y)^{-1}A_x.$$

The condition $\frac{\partial(f_1, \dots, f_m)}{\partial(y_1, \dots, y_m)}(p, q) = \det(A_y) \neq 0$ simply means that A_y is invertible.

Define $F: U \rightarrow \mathbb{R}^{n+m}$ by $F(x, y) := (x, f(x, y))$. It is clear that F is C^1 , and we want to show that the derivative at (p, q) is invertible.

Let us compute the derivative. We know that

$$\frac{\|f(p+h, q+k) - f(p, q) - A_x h - A_y k\|}{\|(h, k)\|}$$

goes to zero as $\|(h, k)\| = \sqrt{\|h\|^2 + \|k\|^2}$ goes to zero. But then so does

$$\frac{\|(h, f(p+h, q+k) - f(p, q)) - (h, A_x h + A_y k)\|}{\|(h, k)\|} = \frac{\|f(p+h, q+k) - f(p, q) - A_x h - A_y k\|}{\|(h, k)\|}.$$

So the derivative of F at (p, q) takes (h, k) to $(h, A_x h + A_y k)$. If $(h, A_x h + A_y k) = (0, 0)$, then $h = 0$, and so $A_y k = 0$. As A_y is one-to-one, then $k = 0$. Therefore $F'(p, q)$ is one-to-one or in other words invertible and we apply the inverse function theorem.

That is, there exists some open set $V \subset \mathbb{R}^{n+m}$ with $(p, 0) \in V$, and an inverse mapping $G: V \rightarrow \mathbb{R}^{n+m}$, that is $F(G(x, s)) = (x, s)$ for all $(x, s) \in V$ (where $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$). Write $G = (G_1, G_2)$ (the first n and the second m components of G). Then

$$F(G_1(x, s), G_2(x, s)) = (G_1(x, s), f(G_1(x, s), G_2(x, s))) = (x, s).$$

So $x = G_1(x, s)$ and $f(G_1(x, s), G_2(x, s)) = f(x, G_2(x, s)) = s$. Plugging in $s = 0$ we obtain

$$f(x, G_2(x, 0)) = 0.$$

The set $G(V)$ contains a whole neighborhood of the point (p, q) and therefore there are some open The set V is open and hence there exist some open sets $\tilde{W}$ and W' such that $\tilde{W} \times W' \subset G(V)$ with $p \in \tilde{W}$ and $q \in W'$. Then take $W = \{x \in \tilde{W} : G_2(x, 0) \in W'\}$. The function that takes x to $G_2(x, 0)$ is continuous and therefore W is open. We define $g: W \rightarrow \mathbb{R}^m$ by $g(x) := G_2(x, 0)$ which is the g in the theorem. The fact that $g(x)$ is the unique point in W' follows because $W \times W' \subset G(V)$ and G is one-to-one and onto $G(V)$.

Next differentiate

$$x \mapsto f(x, g(x)),$$

at p , which should be the zero map. The derivative is done in the same way as above. We get that for all $h \in \mathbb{R}^n$

$$0 = A(h, g'(p)h) = A_x h + A_y g'(p)h,$$

and we obtain the desired derivative for g as well.

In other words, in the context of the theorem we have m equations in $n + m$ unknowns.

$$\begin{aligned} f_1(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \\ &\vdots \\ f_m(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \end{aligned}$$

And the condition guaranteeing a solution is that this is a C^1 mapping (that all the components are C^1 , or in other words all the partial derivatives exist and are continuous), and the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{bmatrix}$$

is invertible at (p, q) .

Consider the set $x^2 + y^2 - (z + 1)^3 = -1$, $e^x + e^y + e^z = 3$ near the point $(0, 0, 0)$. The function we are looking at is

$$f(x, y, z) = (x^2 + y^2 - (z + 1)^3 + 1, e^x + e^y + e^z - 3).$$

We find that

$$f' = \begin{bmatrix} 2x & 2y & -3(z + 1)^2 \\ e^x & e^y & e^z \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2(0) & -3(0 + 1)^2 \\ e^0 & e^0 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

is invertible. Hence near $(0, 0, 0)$ we can find y and z as C^1 functions of x such that for x near 0 we have

$$x^2 + y(x)^2 - (z(x) + 1)^3 = -1, \quad e^x + e^{y(x)} + e^{z(x)} = 3.$$

The theorem does not tell us how to find $y(x)$ and $z(x)$ explicitly, it just tells us they exist. In other words, near the origin the set of solutions is a smooth curve in $\mathbb{R}^3$ that goes through the origin.

We remark that there are versions of the theorem for arbitrarily many derivatives. If f has k continuous derivatives, then the solution also has k continuous derivatives.

Exercises

Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

- Solve for y in terms of x near $(0, 1)$.
- Solve for y in terms of x near $(0, -1)$.
- Solve for x in terms of y near $(-1, 0)$.

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(x, y) := (x, y + h(x))$ for some continuously differentiable function h of one variable.

- Show that f is one-to-one and onto.
- Compute f' .
- Show that f' is invertible at all points, and compute its inverse.

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ by $f(x, y) := (e^x \cos(y), e^x \sin(y))$.

- Show that f is onto.
- Show that f' is invertible at all points.

c) Show that f is not one-to-one, in fact for every $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, there exist infinitely many different points $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (a, b)$.

Therefore, invertible derivative at every point does not mean that f is invertible globally.

Find a map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that is one-to-one, onto, continuously differentiable, but $f'(0) = 0$. Hint: Generalize $f(x) = x^3$ from one to n dimensions.

Consider $z^2 + xz + y = 0$ in $\mathbb{R}^3$. Find an equation $D(x, y) = 0$, such that if $D(x_0, y_0) \neq 0$ and $z^2 + x_0z + y_0 = 0$ for some $z \in \mathbb{R}$, then for points near (x_0, y_0) there exist exactly two distinct continuously differentiable functions $r_1(x, y)$ and $r_2(x, y)$ such that $z = r_1(x, y)$ and $z = r_2(x, y)$ solve $z^2 + xz + y = 0$. Do you recognize the expression D from algebra?

Suppose $f: (a, b) \rightarrow \mathbb{R}^2$ is continuously differentiable and $\frac{\partial f}{\partial x}(t) \neq 0$ for all $t \in (a, b)$. Prove that there exists an interval (c, d) and a continuously differentiable function $g: (c, d) \rightarrow \mathbb{R}$ such that $(x, y) \in f((a, b))$ if and only if $x \in (c, d)$ and $y = g(x)$. In other words, the set $f((a, b))$ is a graph of g .

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) := \begin{cases} (x^2 \sin(\frac{1}{x}) + \frac{x}{2}, y) & \text{if } x \neq 0, \\ (0, y) & \text{if } x = 0. \end{cases}$$

a) Show that f is differentiable everywhere.

b) Show that $f'(0, 0)$ is invertible.

c) Show that f is not one-to-one in any neighborhood of the origin (it is not locally invertible, that is, the inverse theorem does not work).

d) Show that f is not continuously differentiable.

[mv:exercise:polarcoordinates] Define a mapping $F(r, \theta) := (r \cos(\theta), r \sin(\theta))$.

a) Show that F is continuously differentiable (for all $(r, \theta) \in \mathbb{R}^2$).

b) Compute $F'(0, \theta)$ for any θ .

c) Show that if $r \neq 0$, then $F'(r, \theta)$ is invertible, therefore an inverse of F exists locally as long as $r \neq 0$.

d) Show that $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is onto, and for each point $(x, y) \in \mathbb{R}^2$, the set $F^{-1}(x, y)$ is infinite.

e) Show that $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an open map, despite not satisfying the condition of the inverse function theorem.

f) Show that $F|_{(0, \infty) \times [0, 2\pi)}$ is one to one and onto $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Higher order derivatives

Note: less than 1 lecture, depends on the optional §4.3 of volume I

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \rightarrow \mathbb{R}$ a function. Denote by $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ our coordinates. Suppose $\frac{\partial f}{\partial x_j}$ exists everywhere in U , then we note that it is also a function $\frac{\partial f}{\partial x_j}: U \rightarrow \mathbb{R}$. Therefore it makes sense to talk about its partial derivatives.

We denote the partial derivative of $\frac{\partial f}{\partial x_j}$ with respect to x_k by

$$\frac{\partial^2 f}{\partial x_k \partial x_j} := \frac{\partial \left(\frac{\partial f}{\partial x_j} \right)}{\partial x_k}.$$

If $k = j$, then we write $\frac{\partial^2 f}{\partial x_j^2}$ for simplicity.

We define higher order derivatives inductively. Suppose $j_1, j_2, \dots, j_\ell$ are integers between 1 and n , and suppose

$$\frac{\partial^{\ell-1}f}{\partial x_{j_{\ell-1}}\partial x_{j_{\ell-2}}\cdots\partial x_{j_1}}$$

exists and is differentiable in the variable x_{j_ℓ} , then the partial derivative with respect to that variable is denoted by

$$\frac{\partial^\ell f}{\partial x_{j_\ell}\partial x_{j_{\ell-1}}\cdots\partial x_{j_1}} := \frac{\partial\left(\frac{\partial^{\ell-1}f}{\partial x_{j_{\ell-1}}\partial x_{j_{\ell-2}}\cdots\partial x_{j_1}}\right)}{\partial x_{j_\ell}}.$$

Such a derivative is called a *partial derivative of order ℓ* .

Remark that sometimes the notation $f_{x_j x_k}$ is used for $\frac{\partial^2 f}{\partial x_k \partial x_j}$. This notation swaps the order of derivatives, which may be important.

If $U \subset \mathbb{R}^n$ is an open set and $f: U \rightarrow \mathbb{R}$ a function. We say f is *k-times continuously differentiable function*, or a C^k function, if all partial derivatives of all orders up to and including order k exist and are continuous.

So a continuously differentiable, or C^1 , function is one where all partial derivatives exist and are continuous, which agrees with our previous definition due to . We could have required only that the k th order partial derivatives exist and are continuous, as the existence of lower order derivatives is clearly necessary to even define k th order partial derivatives, and these lower order derivatives will be continuous as they will be differentiable functions.

When the partial derivatives are continuous, we can swap their order.

[mv:prop:swapders] Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}$ is a C^2 function, and j and k are two integers between 1 and n . Then

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}.$$

Fix a point $p \in U$, and let e_j and e_k be the standard basis vectors and let s and t be two small nonzero real numbers. We pick s and t small enough so that $p + s_0 e_j + t_0 e_k \in U$ for all s_0 and t_0 with $|s_0| \leq |s|$ and $|t_0| \leq |t|$. This is possible since U is open and so contains a small ball (or a box if you wish).

Using the mean value theorem on the partial derivative in x_k of the function $f(p + se_j) - f(p)$, we find a t_0 between 0 and t such that

$$\frac{f(p + se_j + te_k) - f(p + te_k) - f(p + se_j) + f(p)}{t} = \frac{\partial f}{\partial x_k}(p + se_j + t_0 e_k) - \frac{\partial f}{\partial x_k}(p + t_0 e_k).$$

Next there exists a number s_0 between 0 and s such that

$$\frac{\frac{\partial f}{\partial x_k}(p + se_j + t_0 e_k) - \frac{\partial f}{\partial x_k}(p + t_0 e_k)}{s} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p + s_0 e_j + t_0 e_k).$$

In other words

$$g(s, t) := \frac{f(p + se_j + te_k) - f(p + te_k) - f(p + se_j) + f(p)}{st} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p + s_0 e_j + t_0 e_k).$$

Taking a limit as $(s, t) \in \mathbb{R}^2$ goes to zero we find that (s_0, t_0) also goes to zero and by continuity of the second partial derivatives we find that

$$\lim_{(s, t) \rightarrow 0} g(s, t) = \frac{\partial^2 f}{\partial x_j \partial x_k}(p).$$

We now reverse the ordering, starting with the function $f(p + te_k) - f(p)$ we find an s_1 between 0 and s such that

$$\frac{f(p + te_k + se_j) - f(p + se_j) - f(p + te_k) + f(p)}{s} = \frac{\partial f}{\partial x_j}(p + te_k + s_1 e_j) - \frac{\partial f}{\partial x_j}(p + s_1 e_j).$$

And we find a t_1 between 0 and t

$$\frac{\frac{\partial f}{\partial x_j}(p + te_k + s_1 e_j) - \frac{\partial f}{\partial x_j}(p + s_1 e_j)}{t} = \frac{\partial^2 f}{\partial x_k \partial x_j}(p + t_1 e_k + s_1 e_j).$$

Again we find that $g(s, t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p + t_1 e_k + s_1 e_j)$ and therefore

$$\lim_{(s,t) \rightarrow 0} g(s, t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p).$$

And therefore the two partial derivatives are equal.

The proposition does not hold if the derivatives are not continuous. See the exercises. Notice also that we did not really need a C^2 function we only needed the two second order partial derivatives involved to be continuous functions.

Exercises

Suppose $f: U \rightarrow \mathbb{R}$ is a C^2 function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Use the proof of to find an expression in terms of just the values of f (analogue of the difference quotient for the first derivative), whose limit is $\frac{\partial^2 f}{\partial x_j \partial x_k}(p)$.

Define

$$f(x, y) := \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that

a) The first order partial derivatives exist and are continuous.

b) The partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist, but are not continuous at the origin, and $\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)$.

Suppose $f: U \rightarrow \mathbb{R}$ is a C^k function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Suppose $j_1, j_2, \dots, j_k$ are integers between 1 and n , and suppose $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ is a permutation of $(1, 2, \dots, k)$. Prove

$$\frac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \dots \partial x_{j_1}}(p) = \frac{\partial^k f}{\partial x_{j_{\sigma_k}} \partial x_{j_{\sigma_{k-1}}} \dots \partial x_{j_{\sigma_1}}}(p).$$

Suppose $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^k function such that $\varphi(0, \theta) = \varphi(0, \psi)$ for all $\theta, \psi \in \mathbb{R}$ and $\varphi(r, \theta) = \varphi(r, \theta + 2\pi)$ for all $r, \theta \in \mathbb{R}$. Let $F(r, \theta) = (r \cos(\theta), r \sin(\theta))$ from . Show that a function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$, given $g(x, y) := \varphi(F^{-1}(x, y))$ is well defined (notice that $F^{-1}(x, y)$ can only be defined locally), and when restricted to $\mathbb{R}^2 \setminus \{0\}$ it is a C^k function.

One dimensional integrals in several variables

Differentiation under the integral

Note: less than 1 lecture

Let $f(x, y)$ be a function of two variables and define

$$g(y) := \int_a^b f(x, y) dx.$$

Suppose f is differentiable in y . The question we ask is when can we “differentiate under the integral”, that is, when is it true that g is differentiable and its derivative

$$g'(y) \stackrel{?}{=} \int_a^b \frac{\partial f}{\partial y}(x, y) dx.$$

Differentiation is a limit and therefore we are really asking when do the two limiting operations of integration and differentiation commute. As we have seen, this is not always possible, some sort of uniformity is necessary. In particular, the first question we would face is the integrability of $\frac{\partial f}{\partial y}$, but the formula can fail even if $\frac{\partial f}{\partial y}$ is integrable for all y .

Let us prove a simple, but the most useful version of this theorem.

Suppose $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a continuous function, such that $\frac{\partial f}{\partial y}$ exists for all $(x, y) \in [a, b] \times [c, d]$ and is continuous. Define

$$g(y) := \int_a^b f(x, y) dx.$$

Then $g: [c, d] \rightarrow \mathbb{R}$ is differentiable and

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) dx.$$

The continuity requirements for f and $\frac{\partial f}{\partial y}$ can be weakened, but not dropped outright. The main point is for $\frac{\partial f}{\partial y}$ to exist and be continuous for a small interval in the y direction. In applications, the $[c, d]$ can be a small interval around the point where you need to differentiate.

Fix $y \in [c, d]$ and let $\epsilon > 0$ be given. As $\frac{\partial f}{\partial y}$ is continuous on $[a, b] \times [c, d]$ it is uniformly continuous. In particular, there exists $\delta > 0$ such that whenever $y_1 \in [c, d]$ with $|y_1 - y| < \delta$ and all $x \in [a, b]$ we have

$$\left| \frac{\partial f}{\partial y}(x, y_1) - \frac{\partial f}{\partial y}(x, y) \right| < \epsilon.$$

Suppose h is such that $y + h \in [c, d]$ and $|h| < \delta$. Fix x for a moment and apply mean value theorem to find a y_1 between y and $y + h$ such that

$$\frac{f(x, y + h) - f(x, y)}{h} = \frac{\partial f}{\partial y}(x, y_1).$$

If $|h| < \delta$, then

$$\left| \frac{f(x, y + h) - f(x, y)}{h} - \frac{\partial f}{\partial y}(x, y) \right| = \left| \frac{\partial f}{\partial y}(x, y_1) - \frac{\partial f}{\partial y}(x, y) \right| < \epsilon.$$

This argument worked for every $x \in [a, b]$. Therefore, as a function of x

$$x \mapsto \frac{f(x, y + h) - f(x, y)}{h} \quad \text{converges uniformly to} \quad x \mapsto \frac{\partial f}{\partial y}(x, y) \quad \text{as } h \rightarrow 0.$$

We only defined uniform convergence for sequences although the idea is the same. If you wish you can replace h with $\frac{1}{n}$ above and let $n \rightarrow \infty$.

Now consider the difference quotient

$$\frac{g(y+h) - g(y)}{h} = \frac{\int_a^b f(x, y+h) dx - \int_a^b f(x, y) dx}{h} = \int_a^b \frac{f(x, y+h) - f(x, y)}{h} dx.$$

Uniform convergence can be taken underneath the integral and therefore

$$\lim_{h \rightarrow 0} \frac{g(y+h) - g(y)}{h} = \int_a^b \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx. \quad \text{qed here}$$

Let

$$f(y) = \int_0^1 \sin(x^2 - y^2) dx.$$

Then

$$f'(y) = \int_0^1 -2y \cos(x^2 - y^2) dx.$$

Suppose we start with

$$\int_0^1 \frac{x-1}{\ln(x)} dx.$$

The function under the integral extends to be continuous on $[0, 1]$, and hence the integral exists, see exercise below. Trouble is finding it. Introduce a parameter y and define a function:

$$g(y) := \int_0^1 \frac{x^y - 1}{\ln(x)} dx.$$

The function $\frac{x^y - 1}{\ln(x)}$ also extends to a continuous function of x and y for $(x, y) \in [0, 1] \times [0, 1]$. Therefore g is a continuous function of on $[0, 1]$. In particular, $g(0) = 0$. For any $\epsilon > 0$, the y derivative of the integrand, x^y , is continuous on $[0, 1] \times [\epsilon, 1]$. Therefore, for $y > 0$ we may differentiate under the integral sign

$$g'(y) = \int_0^1 \frac{\ln(x)x^y}{\ln(x)} dx = \int_0^1 x^y dx = \frac{1}{y+1}.$$

We need to figure out $g(1)$, knowing $g'(y) = \frac{1}{y+1}$ and $g(0) = 0$. By elementary calculus we find $g(1) = \int_0^1 g'(y) dy = \ln(2)$. Therefore

$$\int_0^1 \frac{x-1}{\ln(x)} dx = \ln(2).$$

Prove the two statements that were asserted in the example.

- Prove $\frac{x-1}{\ln(x)}$ extends to a continuous function of $[0, 1]$.
- Prove $\frac{x^y-1}{\ln(x)}$ extends to be a continuous function on $[0, 1] \times [0, 1]$.

Exercises

Suppose $h: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$ is which is continuously differentiable and compactly supported. That is there exists some $M > 0$ such that $g(x) = 0$ whenever $|x| \geq M$. Define

$$f(x) := \int_{-\infty}^{\infty} h(y)g(x-y) dy.$$

Show that f is differentiable.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is an infinitely differentiable function (all derivatives exist) such that $f(0) = 0$. Then show that there exists another infinitely differentiable function $g(x)$ such that $f(x) = xg(x)$. Finally show that if $f'(0) \neq 0$, then $g(0) \neq 0$. Hint: first write $f(x) = \int_0^x f'(s)ds$ and then rewrite the integral to go from 0 to 1.

Compute $\int_0^1 e^{tx} dx$. Derive the formula for $\int_0^1 x^n e^x dx$ not using integration by parts, but by differentiation underneath the integral.

Let $U \subset \mathbb{R}^n$ be an open set and suppose $f(x, y_1, y_2, \dots, y_n)$ is a continuous function defined on $[0, 1] \times U \subset \mathbb{R}^{n+1}$. Suppose $\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \dots, \frac{\partial f}{\partial y_n}$ exist and are continuous on $[0, 1] \times U$. Then prove that $F: U \rightarrow \mathbb{R}$ defined by

$$F(y_1, y_2, \dots, y_n) := \int_0^1 f(x, y_1, y_2, \dots, y_n) dx$$

is continuously differentiable.

Work out the following counterexample: Let

$$f(x, y) := \begin{cases} \frac{xy^3}{(x^2+y^2)^2} & \text{if } x \neq 0 \text{ or } y \neq 0, \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

a) Prove that for any fixed y the function $x \mapsto f(x, y)$ is Riemann integrable on $[0, 1]$ and

$$g(y) = \int_0^1 f(x, y) dx = \frac{y}{2y^2 + 2}.$$

Therefore $g'(y)$ exists and we get the continuous function

$$g'(y) = \frac{1-y^2}{2(y^2+1)^2}.$$

b) Prove $\frac{\partial f}{\partial y}$ exists at all x and y and compute it.

c) Show that for all y

$$\int_0^1 \frac{\partial f}{\partial y}(x, y) dx$$

exists but

$$g'(0) \neq \int_0^1 \frac{\partial f}{\partial y}(x, 0) dx.$$

Work out the following counterexample: Let

$$f(x, y) := \begin{cases} xy^2 \sin\left(\frac{1}{x^3 y}\right) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

a) Prove f is continuous on $[0, 1] \times [a, b]$ for any interval $[a, b]$. Therefore the following function is well defined on $[a, b]$

$$g(y) = \int_0^1 f(x, y) dx.$$

b) Prove $\frac{\partial f}{\partial y}$ exists for all (x, y) in $[0, 1] \times [a, b]$, but is not continuous.

c) Show that $\int_0^1 \frac{\partial f}{\partial y}(x, y) dx$ does not exist if $y \neq 0$ even if we take improper integrals.

Path integrals

Note: 2–3 lectures

Piecewise smooth paths

A continuously differentiable function $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is called a *smooth path* or a *continuously differentiable path*⁴ if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

The function γ is called a *piecewise smooth path* or a *piecewise continuously differentiable path* if there exist finitely many points $t_0 = a < t_1 < t_2 < \dots < t_k = b$ such that the restriction of the function $\gamma|_{[t_{j-1}, t_j]}$ is smooth path.

We say γ is a *simple path* if $\gamma|_{(a, b)}$ is a one-to-one function. A γ is a *closed path* if $\gamma(a) = \gamma(b)$, that is if the path starts and ends in the same point.

Since γ is a function of one variable, we have seen before that treating $\gamma'(t)$ as a matrix is equivalent to treating it as a vector since it is an $n \times 1$ matrix, that is, a column vector. In fact, by an earlier exercise, even the operator norm of $\gamma'(t)$ is equal to the euclidean norm. Therefore, we will write $\gamma'(t)$ as a vector as is usual, and then $\gamma'(t)$ is just the vector of the derivatives of its components, so if $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$, then $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t), \dots, \gamma_n'(t))$.

One can often get by with only smooth paths, but for computations, the simplest paths to write down are often piecewise smooth. Note that a piecewise smooth function (or path) is automatically continuous (exercise).

Generally, it is the direct image $\gamma([a, b])$ that is what we are interested in, although how we parametrize it with γ is also important to some degree. We informally talk about a curve, and often we really mean the set $\gamma([a, b])$, just as before depending on context.

[mv:example:unitsquarepath] Let $\gamma: [0, 4] \rightarrow \mathbb{R}^2$ be defined by

$$\gamma(t) := \begin{cases} (t, 0) & \text{if } t \in [0, 1], \\ (1, t - 1) & \text{if } t \in (1, 2], \\ (3 - t, 1) & \text{if } t \in (2, 3], \\ (0, 4 - t) & \text{if } t \in (3, 4]. \end{cases}$$

Then the reader can check that the path is the unit square traversed counterclockwise. We can check that for example $\gamma|_{[1, 2]}(t) = (1, t - 1)$ and therefore $(\gamma|_{[1, 2]})'(t) = (0, 1) \neq 0$. It is good to notice at this point that $(\gamma|_{[1, 2]})'(1) = (0, 1)$, $(\gamma|_{[0, 1]})'(1) = (1, 0)$, and $\gamma'(1)$ does not exist. That is, at the corners γ is of course not differentiable, even though the restrictions are differentiable and the derivative depends on which restriction you take.

The condition that $\gamma'(t) \neq 0$ means that the image of γ has no “corners” where γ is continuously differentiable. For example, take the function

$$\gamma(t) := \begin{cases} (t^2, 0) & \text{if } t < 0, \\ (0, t^2) & \text{if } t \geq 0. \end{cases}$$

It is left for the reader to check that γ is continuously differentiable, yet the image $\gamma(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 : (x, y) = (s, 0) \text{ or } (x, y) = (0, s) \text{ for some } s \in \mathbb{R}\}$ has a “corner” at the origin. And that is because $\gamma'(0) = (0, 0)$. More complicated examples with even infinitely many corners exist, see the exercises.

The condition that $\gamma'(t) \neq 0$ even at the endpoints guarantees not only no corners, but also that the path ends nicely, that is, can extend a little bit past the endpoints. Again, see the exercises.

A graph of a continuously differentiable function $f: [a, b] \rightarrow \mathbb{R}$ is a smooth path. That is, define $\gamma: [a, b] \rightarrow \mathbb{R}^2$ by

$$\gamma(t) := (t, f(t)).$$

Then $\gamma'(t) = (1, f'(t))$, which is never zero.

There are other ways of parametrizing the path. That is, having a different path with the same image. For example, the function that takes t to $(1 - t)a + tb$, takes the interval $[0, 1]$ to $[a, b]$. So let $\alpha: [0, 1] \rightarrow \mathbb{R}^2$ be defined by

$$\alpha(t) := ((1 - t)a + tb, f((1 - t)a + tb)).$$

Then $\alpha'(t) = (b - a, (b - a)f'((1 - t)a + tb))$, which is never zero. Furthermore as sets $\alpha([0, 1]) = \gamma([a, b]) = \{(x, y) \in \mathbb{R}^2 : x \in [a, b] \text{ and } f(x) = y\}$, which is just the graph of f .

The last example leads us to a definition.

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a smooth path and $h: [c, d] \rightarrow [a, b]$ a continuously differentiable bijective function such that $h'(t) \neq 0$ for all $t \in [c, d]$. Then the composition $\gamma \circ h$ is called a *smooth reparametrization* of γ .

Let γ be a piecewise smooth path, and h be a piecewise smooth bijective function. Then the composition $\gamma \circ h$ is called a *piecewise smooth reparametrization* of γ .

If h is strictly increasing, then h is said to *preserve orientation*. If h does not preserve orientation, then h is said to *reverse orientation*.

A reparametrization is another path for the same set. That is, $(\gamma \circ h)([c, d]) = \gamma([a, b])$.

Let us remark that for h , piecewise smooth means that there is some partition $t_0 = c < t_1 < t_2 < \dots < t_k = d$, such that $h|_{[t_{j-1}, t_j]}$ is continuously differentiable and $(h|_{[t_{j-1}, t_j]})'(t) \neq 0$ for all $t \in [t_{j-1}, t_j]$. Since h is bijective, it is either strictly increasing or strictly decreasing. Therefore either $(h|_{[t_{j-1}, t_j]})'(t) > 0$ for all t or $(h|_{[t_{j-1}, t_j]})'(t) < 0$ for all t .

[prop:reparamapiecewisessmooth] If $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is a piecewise smooth path, and $\gamma \circ h: [c, d] \rightarrow \mathbb{R}^n$ is a piecewise smooth reparametrization, then $\gamma \circ h$ is a piecewise smooth path.

Let us assume that h preserves orientation, that is, h is strictly increasing. If $h: [c, d] \rightarrow [a, b]$ gives a piecewise smooth reparametrization, then for some partition $r_0 = c < r_1 < r_2 < \dots < r_\ell = d$, we have $h|_{[t_{j-1}, t_j]}$ is continuously differentiable with positive derivative.

Let $t_0 = a < t_1 < t_2 < \dots < t_k = b$ be the partition from the definition of piecewise smooth for γ together with the points $\{h(r_0), h(r_1), h(r_2), \dots, h(r_\ell)\}$. Let $s_j := h^{-1}(t_j)$. Then $s_0 = c < s_1 < s_2 < \dots < s_k = d$. For $t \in [s_{j-1}, s_j]$ notice that $h(t) \in [t_{j-1}, t_j]$, $h|_{[s_{j-1}, s_j]}$ is continuously differentiable, and $\varphi|_{[t_{j-1}, t_j]}$ is also continuously differentiable. Then

$$(\gamma \circ h)|_{[s_{j-1}, s_j]}(t) = \gamma|_{[t_{j-1}, t_j]}(h|_{[s_{j-1}, s_j]}(t)).$$

The function $(\gamma \circ h)|_{[s_{j-1}, s_j]}$ is therefore continuously differentiable and by the chain rule

$$((\gamma \circ h)|_{[s_{j-1}, s_j]})'(t) = (\gamma|_{[t_{j-1}, t_j]})'(h(t))(h|_{[s_{j-1}, s_j]})'(t) \neq 0.$$

Therefore $\gamma \circ h$ is a piecewise smooth path. The case for an orientation reversing h is left as an exercise.

If two paths are simple and their images are the same, it is left as an exercise that there exists a reparametrization.

Path integral of a one-form

If $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ are our coordinates, and given n real-valued continuous functions $f_1, f_2, \dots, f_n$ defined on some set $S \subset \mathbb{R}^n$ we define a so-called *one-form*:

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \dots + \omega_n dx_n.$$

We could represent ω as a continuous function from S to $\mathbb{R}^n$, although it is better to think of it as a different object.

For example,

$$\omega(x, y) = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is a one-form defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a smooth path and

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \dots + \omega_n dx_n,$$

a one-form defined on the direct image $\gamma([a, b])$. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ be the components of γ . Define:

$$\begin{aligned} \int_{\gamma} \omega &:= \int_a^b (\omega_1(\gamma(t))\gamma_1'(t) + \omega_2(\gamma(t))\gamma_2'(t) + \dots + \omega_n(\gamma(t))\gamma_n'(t)) dt \\ &= \int_a^b \left(\sum_{j=1}^n \omega_j(\gamma(t))\gamma_j'(t) \right) dt. \end{aligned}$$

If γ is piecewise smooth, take the corresponding partition $t_0 = a < t_1 < t_2 < \dots < t_k = b$, where we assume the partition is the minimal one, that is γ is not differentiable at $t_2, t_3, \dots, t_{k-1}$. Each $\gamma|_{[t_{j-1}, t_j]}$ is a smooth path and we define

$$\int_{\gamma} \omega := \int_{\gamma|_{[t_0, t_1]}} \omega + \int_{\gamma|_{[t_1, t_2]}} \omega + \dots + \int_{\gamma|_{[t_{n-1}, t_n]}} \omega.$$

The notation makes sense from the formula you remember from calculus, let us state it somewhat informally: if $x_j(t) = \gamma_j(t)$, then $dx_j = \gamma_j'(t)dt$.

Paths can be cut up or concatenated as follows. The proof is a direct application of the additivity of the Riemann integral, and is left as an exercise. The proposition also justifies why we defined the integral over a piecewise smooth path in the way we did, and it further justifies that we may as well have taken any partition not just the minimal one in the definition.

[mv:prop:pathconcat] Let $\gamma: [a, c] \rightarrow \mathbb{R}^n$ be a piecewise smooth path. For some $b \in (a, c)$, define the piecewise smooth paths $\alpha = \gamma|_{[a, b]}$ and $\beta = \gamma|_{[b, c]}$. For a one-form ω defined on the image of γ we have

$$\int_{\gamma} \omega = \int_{\alpha} \omega + \int_{\beta} \omega.$$

[example:mv:irrotoneformint] Let the one-form ω and the path $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ be defined by

$$\omega(x, y) := \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy, \quad \gamma(t) := (\cos(t), \sin(t)).$$

Then $\int_{\gamma} \omega = \int_0^{2\pi} \frac{-\sin(t)}{(\cos(t))^2 + (\sin(t))^2} dt$

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defined by $\alpha(t) := (\cos(2\pi t), \sin(2\pi t))$, that is α is a smooth reparametrization of γ . Then $\int_{\alpha} \omega = \int_0^1 \frac{-\sin(2\pi t)}{(\cos(2\pi t))^2 + (\sin(2\pi t))^2} dt$

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as $\beta(t) := (\cos(-t), \sin(-t))$. Then

$$\begin{aligned} \int_{\beta} \omega &= \int_0^{2\pi} \left(\frac{-\sin(-t)}{(\cos(-t))^2 + (\sin(-t))^2} (\sin(-t)) + \frac{\cos(-t)}{(\cos(-t))^2 + (\sin(-t))^2} (-\cos(-t)) \right) dt \\ &= \int_0^{2\pi} (-1) dt = -2\pi. \end{aligned}$$

Now, α was an orientation preserving reparametrization of γ , and the integral was the same. On the other hand β is an orientation reversing reparametrization and the integral was minus the original.

The previous example is not a fluke. The path integral does not depend on the parametrization of the curve, the only thing that matters is the direction in which the curve is traversed.

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h: [c, d] \rightarrow \mathbb{R}^n$ a piecewise smooth reparametrization. Suppose ω is a one-form defined on the set $\gamma([a, b])$. Then

$$\int_{\gamma \circ h} \omega = \begin{cases} \int_{\gamma} \omega & \text{if } h \text{ preserves orientation,} \\ -\int_{\gamma} \omega & \text{if } h \text{ reverses orientation.} \end{cases}$$

Assume first that γ and h are both smooth. Write the one-form as $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \dots + \omega_n dx_n$. Suppose first that h is orientation preserving. Using the definition of the path integral and the change of variables formula for the Riemann integral,

$$\begin{aligned} \int_{\gamma} \omega &= \int_a^b \left(\sum_{j=1}^n \omega_j(\gamma(t)) \gamma_j'(t) \right) dt \\ &= \int_c^d \left(\sum_{j=1}^n \omega_j(\gamma(h(\tau))) \gamma_j'(h(\tau)) \right) h'(\tau) d\tau \\ &= \int_c^d \left(\sum_{j=1}^n \omega_j(\gamma(h(\tau))) (\gamma \circ h)'(\tau) \right) d\tau = \int_{\gamma \circ h} \omega. \end{aligned}$$

If h is orientation reversing it will swap the order of the limits on the integral introducing a minus sign. The details, along with finishing the proof for piecewise smooth paths is left to the reader as .

Due to this proposition (and the exercises), if we have a set $\Gamma \subset \mathbb{R}^n$ that is the image of a simple piecewise smooth path $\gamma([a, b])$, then if we somehow indicate the orientation, that is, which direction we traverse the curve, in other words where we start and where we finish. Then we just write

$$\int_{\Gamma} \omega,$$

without mentioning the specific γ . Furthermore, for a simple closed path, it does not even matter where we start the parametrization. See the exercises.

Recall that *simple* means that γ restricted to (a, b) is one-to-one, that is, it is one-to-one except perhaps at the endpoints. We also often relax the simple path condition a little bit. For example, as long as $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is one-to-one except at finitely many points. That is, there are only finitely many points $p \in \mathbb{R}^n$ such that $\gamma^{-1}(p)$ is more than one point. See the exercises. The issue about the injectivity problem is illustrated by the following example.

Suppose $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ is given by $\gamma(t) := (\cos(t), \sin(t))$ and $\beta: [0, 2\pi] \rightarrow \mathbb{R}^2$ is given by $\beta(t) := (\cos(2t), \sin(2t))$. Notice that $\gamma([0, 2\pi]) = \beta([0, 2\pi])$, and we travel around the same curve, the unit circle. But γ goes around the unit circle once in the counter clockwise direction, and β goes around the unit circle twice (in the same direction). Then

$$\begin{aligned}\int_{\gamma} -y \, dx + x \, dy &= \int_0^{2\pi} ((-\sin(t))(-\sin(t)) + \cos(t)\cos(t)) \, dt = 2\pi, \\ \int_{\beta} -y \, dx + x \, dy &= \int_0^{2\pi} ((-\sin(2t))(-2\sin(2t)) + \cos(t)(2\cos(t))) \, dt = 4\pi.\end{aligned}$$

It is sometimes convenient to define a path integral over $\gamma: [a, b] \rightarrow \mathbb{R}^n$ that is not a path. We define

$$\int_{\gamma} \omega := \int_a^b \left(\sum_{j=1}^n \omega_j(\gamma(t)) \gamma_j'(t) \right) dt$$

for any γ which is continuously differentiable. A case which comes up naturally is when γ is constant. In this case $\gamma'(t) = 0$ for all t and $\gamma([a, b])$ is a single point, which we regard as a “curve” of length zero. Then, $\int_{\gamma} \omega = 0$.

Line integral of a function

Sometimes we wish to simply integrate a function against the so-called arc-length measure.

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is a smooth path, and f is a continuous function defined on the image $\gamma([a, b])$. Then define

$$\int_{\gamma} f \, ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

The definition for a piecewise smooth path is similar as before and is left to the reader.

The geometric idea of this integral is to find the “area under the graph of a function” as we move around the path γ . The line integral of a function is also independent of the parametrization, and in this case, the orientation does not matter.

[mv:prop:lineintreparam] Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h: [c, d] \rightarrow \mathbb{R}^n$ a piecewise smooth reparametrization. Suppose f is a continuous function defined on the set $\gamma([a, b])$. Then

$$\int_{\gamma \circ h} f \, ds = \int_{\gamma} f \, ds.$$

Suppose first that h is orientation preserving and γ and h are both smooth. Then as before

$$\begin{aligned}\int_{\gamma} f \, ds &= \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt \\ &= \int_c^d f(\gamma(h(\tau))) \|\gamma'(h(\tau))\| h'(\tau) \, d\tau \\ &= \int_c^d f(\gamma(h(\tau))) \|\gamma'(h(\tau)) h'(\tau)\| \, d\tau \\ &= \int_c^d f((\gamma \circ h)(\tau)) \|(\gamma \circ h)'(\tau)\| \, d\tau \\ &= \int_{\gamma \circ h} f \, ds.\end{aligned}$$

If h is orientation reversing it will swap the order of the limits on the integral but you also have to introduce a minus sign in order to take h' inside the norm. The details, along with finishing the proof for piecewise smooth paths is left to the reader as .

Similarly as before, because of this proposition (and the exercises), if γ is simple, it does not matter which parametrization we use. Therefore, if $\Gamma = \gamma([a, b])$ we can simply write

$$\int_{\Gamma} f ds.$$

In this case we also do not need to worry about orientation, either way we get the same thing.

Let $f(x, y) = x$. Let $C \subset \mathbb{R}^2$ be half of the unit circle for $x \geq 0$. We wish to compute

$$\int_C f ds.$$

Parametrize the curve C via $\gamma: [-\pi/2, \pi/2] \rightarrow \mathbb{R}^2$ defined as $\gamma(t) := (\cos(t), \sin(t))$. Then $\gamma'(t) = (-\sin(t), \cos(t))$, and

$$\int_C f ds = \int_{\gamma} f ds = \int_{-\pi/2}^{\pi/2} \cos(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2} dt = \int_{-\pi/2}^{\pi/2} \cos(t) dt = 2.$$

Suppose $\Gamma \subset \mathbb{R}^n$ is parametrized by a simple piecewise smooth path $\gamma: [a, b] \rightarrow \mathbb{R}^n$, that is $\gamma([a, b]) = \Gamma$. Then we define the *length* by

$$\ell(\Gamma) := \int_{\Gamma} ds = \int_{\gamma} ds = \int_a^b \|\gamma'(t)\| dt.$$

Let $x, y \in \mathbb{R}^n$ be two points and write $[x, y]$ as the straight line segment between the two points x and y . We parametrize $[x, y]$ by $\gamma(t) := (1-t)x + ty$ for t running between 0 and 1. We find $\gamma'(t) = y - x$ and therefore

$$\ell([x, y]) = \int_{[x, y]} ds = \int_0^1 \|y - x\| dt = \|y - x\|.$$

So the length of $[x, y]$ is the distance between x and y in the euclidean metric.

A simple piecewise smooth path $\gamma: [0, r] \rightarrow \mathbb{R}^n$ is said to be an *arc-length parametrization* if

$$\ell(\gamma([0, t])) = \int_0^t \|\gamma'(\tau)\| d\tau = t.$$

You can think of such a parametrization as moving around your curve at speed 1.

Exercises

Show that if $\varphi: [a, b] \rightarrow \mathbb{R}^n$ is piecewise smooth as we defined it, then φ is a continuous function.

Finish the proof of for orientation reversing reparametrizations.

Prove .

[mv:exercise:pathpiece] Finish the proof of for a) orientation reversing reparametrizations, and b) piecewise smooth paths and reparametrizations.

[mv:exercise:linepiece] Finish the proof of for a) orientation reversing reparametrizations, and b) piecewise smooth paths and reparametrizations.

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is a piecewise smooth path, and f is a continuous function defined on the image $\gamma([a, b])$. Provide a definition of $\int_{\gamma} f ds$.

Directly using the definitions compute:

a) the arc-length of the unit square from using the given parametrization.

- b) the arc-length of the unit circle using the parametrization $\gamma: [0, 1] \rightarrow \mathbb{R}^2$, $\gamma(t) := (\cos(2\pi t), \sin(2\pi t))$.
 c) the arc-length of the unit circle using the parametrization $\beta: [0, 2\pi] \rightarrow \mathbb{R}^2$, $\beta(t) := (\cos(t), \sin(t))$.

Suppose $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ is a smooth path, and ω is a one-form defined on the image $\gamma([a, b])$. For $r \in [0, 1]$, let $\gamma_r: [0, r] \rightarrow \mathbb{R}^n$ be defined as simply the restriction of γ to $[0, r]$. Show that the function $h(r) := \int_{\gamma_r} \omega$ is a continuously differentiable function on $[0, 1]$.

Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is a smooth path. Show that there exists an $\epsilon > 0$ and a smooth function $\tilde{\gamma}: (a - \epsilon, b + \epsilon) \rightarrow \mathbb{R}^n$ with $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in [a, b]$ and $\tilde{\gamma}'(t) \neq 0$ for all $t \in (a - \epsilon, b + \epsilon)$. That is, prove that a smooth path extends some small distance past the end points.

Suppose $\alpha: [a, b] \rightarrow \mathbb{R}^n$ and $\beta: [c, d] \rightarrow \mathbb{R}^n$ are piecewise smooth paths such that $\Gamma := \alpha([a, b]) = \beta([c, d])$. Show that there exist finitely many points $\{p_1, p_2, \dots, p_k\} \in \Gamma$, such that the sets $\alpha^{-1}(\{p_1, p_2, \dots, p_k\})$ and $\beta^{-1}(\{p_1, p_2, \dots, p_k\})$ are partitions of $[a, b]$ and $[c, d]$, such that on any subinterval the paths are smooth (that is, they are partitions as in the definition of piecewise smooth path).

- a) Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ and $\alpha: [c, d] \rightarrow \mathbb{R}^n$ are two smooth paths which are one-to-one and $\gamma([a, b]) = \alpha([c, d])$. Then there exists a smooth reparametrization $h: [a, b] \rightarrow [c, d]$ such that $\gamma = \alpha \circ h$. Hint: It should be not hard to find some h . The trick is to show it is continuously differentiable with a nonvanishing derivative. You will want to apply the implicit function theorem and it may at first seem the dimensions don't seem to work out.
 b) Prove the same thing as part a, but now for simple closed paths with the further assumption that $\gamma(a) = \gamma(b) = \alpha(c) = \alpha(d)$.
 c) Prove parts a) and b) but for piecewise smooth paths, obtaining piecewise smooth reparametrizations. Hint: The trick is to find two partitions such that when restricted to a subinterval of the partition both paths have the same image and are smooth, see the above exercise.

Suppose $\alpha: [a, b] \rightarrow \mathbb{R}^n$ and $\beta: [b, c] \rightarrow \mathbb{R}^n$ are piecewise smooth paths with $\alpha(b) = \beta(b)$. Let $\gamma: [a, c] \rightarrow \mathbb{R}^n$ be defined by

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [a, b], \\ \beta(t) & \text{if } t \in (b, c]. \end{cases}$$

Show that γ is a piecewise smooth path, and that if ω is a one-form defined on the curve given by γ , then

$$\int_{\gamma} \omega = \int_{\alpha} \omega + \int_{\beta} \omega.$$

[mv:exercise:closedcurveintegral] Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ and $\beta: [c, d] \rightarrow \mathbb{R}^n$ are two simple piecewise smooth closed paths. That is $\gamma(a) = \gamma(b)$ and $\beta(c) = \beta(d)$ and the restrictions $\gamma|_{(a,b)}$ and $\beta|_{(c,d)}$ are one-to-one. Suppose $\Gamma = \gamma([a, b]) = \beta([c, d])$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either

$$\int_{\gamma} \omega = \int_{\beta} \omega, \quad \text{or} \quad \int_{\gamma} \omega = - \int_{\beta} \omega.$$

In particular, the notation $\int_{\Gamma} \omega$ makes sense if we indicate the direction in which the integral is evaluated. Hint: see previous three exercises.

[mv:exercise:curveintegral] Suppose $\gamma: [a, b] \rightarrow \mathbb{R}^n$ and $\beta: [c, d] \rightarrow \mathbb{R}^n$ are two piecewise smooth paths which are one-to-one except at finitely many points. That is, there is at most finitely many points $p \in \mathbb{R}^n$ such that $\gamma^{-1}(p)$ or $\beta^{-1}(p)$ contains more than one point. Suppose $\Gamma = \gamma([a, b]) = \beta([c, d])$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either

$$\int_{\gamma} \omega = \int_{\beta} \omega, \quad \text{or} \quad \int_{\gamma} \omega = - \int_{\beta} \omega.$$

In particular, the notation $\int_{\Gamma} \omega$ makes sense if we indicate the direction in which the integral is evaluated.
 Hint: same hint as the last exercise.

Define $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ by $\gamma(t) := \left(t^3 \sin\left(\frac{1}{t}\right), t \left(3t^2 \sin\left(\frac{1}{t}\right) - t \cos\left(\frac{1}{t}\right) \right)^2 \right)$ for $t \neq 0$ and $\gamma(0) = (0, 0)$. Show that:

- γ is continuously differentiable on $[0, 1]$.
- Show that there exists an infinite sequence $\{t_n\}$ in $[0, 1]$ converging to 0, such that $\gamma'(t_n) = (0, 0)$.
- Show that the points $\gamma(t_n)$ lie on the line $y = 0$ and such that the x -coordinate of $\gamma(t_n)$ alternates between positive and negative (if they do not alternate you only found a subsequence and you need to find them all).
- Show that there is no piecewise smooth α whose image equals $\gamma([0, 1])$. Hint: look at part c) and show that α' must be zero where it reaches the origin.
- (Computer) if you know a plotting software that allows you to plot parametric curves, make a plot of the curve, but only for t in the range $[0, 0.1]$ otherwise you will not see the behavior. In particular, you should notice that $\gamma([0, 1])$ has infinitely many "corners" near the origin.

Path independence

Note: 2 lectures

Path independent integrals

Let $U \subset \mathbb{R}^n$ be a set and ω a one-form defined on U . The integral of ω is said to be *path independent* if for any two points $x, y \in U$ and any two piecewise smooth paths $\gamma: [a, b] \rightarrow U$ and $\beta: [c, d] \rightarrow U$ such that $\gamma(a) = \beta(c) = x$ and $\gamma(b) = \beta(d) = y$ we have

$$\int_{\gamma} \omega = \int_{\beta} \omega.$$

In this case we simply write

$$\int_x^y \omega := \int_{\gamma} \omega = \int_{\beta} \omega.$$

Not every one-form gives a path independent integral. In fact, most do not.

Let $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ be the path $\gamma(t) = (t, 0)$ going from $(0, 0)$ to $(1, 0)$. Let $\beta: [0, 1] \rightarrow \mathbb{R}^2$ be the path $\beta(t) = (t, (1-t)t)$ also going between the same points. Then

$$\begin{aligned} \int_{\gamma} y \, dx &= \int_0^1 \gamma_2(t) \gamma_1'(t) \, dt = \int_0^1 0(1) \, dt = 0, \\ \int_{\beta} y \, dx &= \int_0^1 \beta_2(t) \beta_1'(t) \, dt = \int_0^1 (1-t)t(1) \, dt = \frac{1}{6}. \end{aligned}$$

So the integral of $y \, dx$ is not path independent. In particular, $\int_{(0,0)}^{(1,0)} y \, dx$ does not make sense.

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \rightarrow \mathbb{R}$ a continuously differentiable function. Then the one-form

$$df := \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

is called the *total derivative* of f .

An open set $U \subset \mathbb{R}^n$ is said to be *path connected* ⁵ if for every two points x and y in U , there exists a piecewise smooth path starting at x and ending at y .

We will leave as an exercise that every connected open set is path connected.

[mv:prop:pathinddf] Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a one-form defined on U . Then

$$\int_x^y \omega$$

is path independent (for all $x, y \in U$) if and only if there exists a continuously differentiable $f: U \rightarrow \mathbb{R}$ such that $\omega = df$.

In fact, if such an f exists, then for any two points $x, y \in U$

$$\int_x^y \omega = f(y) - f(x).$$

In other words if we fix $p \in U$, then $f(x) = C + \int_p^x \omega$.

First suppose that the integral is path independent. Pick $p \in U$ and define

$$f(x) := \int_p^x \omega.$$

Write $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n$. We wish to show that for every $j = 1, 2, \dots, n$, the partial derivative $\frac{\partial f}{\partial x_j}$ exists and is equal to ω_j .

Let e_j be an arbitrary standard basis vector. Compute

$$\frac{f(x + he_j) - f(x)}{h} = \frac{1}{h} \left(\int_p^{x+he_j} \omega - \int_p^x \omega \right) = \frac{1}{h} \int_x^{x+he_j} \omega,$$

which follows by path independence as $\int_p^{x+he_j} \omega = \int_p^x \omega + \int_x^{x+he_j} \omega$, because we could have picked a path from p to $x + he_j$ that also happens to pass through x , and then cut this path in two.

Since U is open, suppose h is so small so that all points of distance $|h|$ or less from x are in U . As the integral is path independent, pick the simplest path possible from x to $x + he_j$, that is $\gamma(t) = x + the_j$ for $t \in [0, 1]$. The path is in U . Notice $\gamma'(t) = he_j$ has only one nonzero component and that is the j th component, which is h . Therefore

$$\frac{1}{h} \int_x^{x+he_j} \omega = \frac{1}{h} \int_\gamma \omega = \frac{1}{h} \int_0^1 \omega_j(x + the_j) h dt = \int_0^1 \omega_j(x + the_j) dt.$$

We wish to take the limit as $h \rightarrow 0$. The function ω_j is continuous. So given $\epsilon > 0$, h can be small enough so that $|\omega_j(x) - \omega_j(y)| < \epsilon$, whenever $\|x - y\| \leq |h|$. Therefore, $|\omega_j(x + the_j) - \omega_j(x)| < \epsilon$ for all $t \in [0, 1]$, and we estimate

$$\left| \int_0^1 \omega_j(x + the_j) dt - \omega_j(x) \right| = \left| \int_0^1 (\omega_j(x + the_j) - \omega_j(x)) dt \right| \leq \epsilon.$$

That is,

$$\lim_{h \rightarrow 0} \frac{f(x + he_j) - f(x)}{h} = \omega_j(x),$$

which is what we wanted that is $df = \omega$. As ω_j are continuous for all j , we find that f has continuous partial derivatives and therefore is continuously differentiable.

For the other direction suppose f exists such that $df = \omega$. Suppose we take a smooth path $\gamma: [a, b] \rightarrow U$ such that $\gamma(a) = x$ and $\gamma(b) = y$, then

$$\begin{aligned} \int_\gamma df &= \int_a^b \left(\frac{\partial f}{\partial x_1}(\gamma(t)) \gamma_1'(t) + \frac{\partial f}{\partial x_2}(\gamma(t)) \gamma_2'(t) + \cdots + \frac{\partial f}{\partial x_n}(\gamma(t)) \gamma_n'(t) \right) dt \\ &= \int_a^b \frac{d}{dt} [f(\gamma(t))] dt \\ &= f(y) - f(x). \end{aligned}$$

The value of the integral only depends on x and y , not the path taken. Therefore the integral is path independent. We leave checking this for a piecewise smooth path as an exercise to the reader.

Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a 1-form defined on U . Then $\omega = df$ for some continuously differentiable $f: U \rightarrow \mathbb{R}$ if and only if

$$\int_{\gamma} \omega = 0$$

for every piecewise smooth closed path $\gamma: [a, b] \rightarrow U$.

Suppose first that $\omega = df$ and let γ be a piecewise smooth closed path. Then we from above we have that

$$\int_{\gamma} \omega = f(\gamma(b)) - f(\gamma(a)) = 0,$$

because $\gamma(a) = \gamma(b)$ for a closed path.

Now suppose that for every piecewise smooth closed path γ , $\int_{\gamma} \omega = 0$. Let x, y be two points in U and let $\alpha: [0, 1] \rightarrow U$ and $\beta: [0, 1] \rightarrow U$ be two piecewise smooth paths with $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$. Then let $\gamma: [0, 2] \rightarrow U$ be defined by

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [0, 1], \\ \beta(2-t) & \text{if } t \in (1, 2]. \end{cases}$$

This is a piecewise smooth closed path and so

$$0 = \int_{\gamma} \omega = \int_{\alpha} \omega - \int_{\beta} \omega.$$

This follows first by , and then noticing that the second part is β travelled backwards so that we get minus the β integral. Thus the integral of ω on U is path independent.

There is a local criterion, a differential equation, that guarantees path independence. That is, under the right condition there exists an *antiderivative* f whose total derivative is the given one-form ω . However, since the criterion is local, we only get the result locally. We can define the antiderivative in any so-called *simply connected* domain, which informally is a domain where any path between two points can be “continuously deformed” into any other path between those two points. To make matters simple, the usual way this result is proved is for so-called star-shaped domains.

Let $U \subset \mathbb{R}^n$ be an open set and $p \in U$. We say U is a *star shaped domain* with respect to p if for any other point $x \in U$, the line segment between p and x is in U , that is, if $(1-t)p + tx \in U$ for all $t \in [0, 1]$. If we say simply *star shaped*, then U is star shaped with respect to some $p \in U$.

Notice the difference between star shaped and convex. A convex domain is star shaped, but a star shaped domain need not be convex.

Let $U \subset \mathbb{R}^n$ be a star shaped domain and ω a continuously differentiable one-form defined on U . That is, if

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \cdots + \omega_n dx_n,$$

then $\omega_1, \omega_2, \dots, \omega_n$ are continuously differentiable functions. Suppose that for every j and k

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j},$$

then there exists a twice continuously differentiable function $f: U \rightarrow \mathbb{R}$ such that $df = \omega$.

The condition on the derivatives of ω is precisely the condition that the second partial derivatives commute. That is, if $df = \omega$, and f is twice continuously differentiable, then

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial \omega_k}{\partial x_j}.$$

The condition is therefore clearly necessary. The lemma says that it is sufficient for a star shaped U .

Suppose U is star shaped with respect to $y = (y_1, y_2, \dots, y_n) \in U$.

Given $x = (x_1, x_2, \dots, x_n) \in U$, define the path $\gamma: [0, 1] \rightarrow U$ as $\gamma(t) := (1-t)y + tx$, so $\gamma'(t) = x - y$. Then let

$$f(x) := \int_{\gamma} \omega = \int_0^1 \left(\sum_{k=1}^n \omega_k((1-t)y + tx)(x_k - y_k) \right) dt.$$

We differentiate in x_j under the integral. We can do that since everything, including the partials themselves are continuous.

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x) &= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_k}{\partial x_j}((1-t)y + tx)t(x_k - y_k) \right) + \omega_j((1-t)y + tx) \right) dt \\ &= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_j}{\partial x_k}((1-t)y + tx)t(x_k - y_k) \right) + \omega_j((1-t)y + tx) \right) dt \\ &= \int_0^1 \frac{d}{dt} [t\omega_j((1-t)y + tx)] dt \\ &= \omega_j(x). \end{aligned}$$

And this is precisely what we wanted.

Without some hypothesis on U the theorem is not true. Let

$$\omega(x, y) := \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

be defined on $\mathbb{R}^2 \setminus \{0\}$. It is easy to see that

$$\frac{\partial}{\partial y} \left[\frac{-y}{x^2 + y^2} \right] = \frac{\partial}{\partial x} \left[\frac{x}{x^2 + y^2} \right].$$

However, there is no $f: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ such that $df = \omega$. We saw in if we integrate from $(1, 0)$ to $(1, 0)$ along the unit circle, that is $\gamma(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$ we got 2π and not 0 as it should be if the integral is path independent or in other words if there would exist an f such that $df = \omega$.

Vector fields

A common object to integrate is a so-called vector field. That is an assignment of a vector at each point of a domain.

Let $U \subset \mathbb{R}^n$ be a set. A continuous function $v: U \rightarrow \mathbb{R}^n$ is called a *vector field*. Write $v = (v_1, v_2, \dots, v_n)$.

Given a smooth path $\gamma: [a, b] \rightarrow \mathbb{R}^n$ with $\gamma([a, b]) \subset U$ we define the path integral of the vectorfield v as

$$\int_{\gamma} v \cdot d\gamma := \int_a^b v(\gamma(t)) \cdot \gamma'(t) dt,$$

where the dot in the definition is the standard dot product. Again the definition of a piecewise smooth path is done by integrating over each smooth interval and adding the result.

If we unravel the definition we find that

$$\int_{\gamma} v \cdot d\gamma = \int_{\gamma} v_1 dx_1 + v_2 dx_2 + \dots + v_n dx_n.$$

Therefore what we know about integration of one-forms carries over to the integration of vector fields. For example path independence for integration of vector fields is simply that

$$\int_x^y \mathbf{v} \cdot d\mathbf{y}$$

is path independent (so for any γ) if and only if $\mathbf{v} = \nabla f$, that is the gradient of a function. The function f is then called the *potential* for $\mathbf{v}$.

A vector field $\mathbf{v}$ whose path integrals are path independent is called a *conservative vector field*. The naming comes from the fact that such vector fields arise in physical systems where a certain quantity, the energy is conserved.

Exercises

Find an $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $df = xe^{x^2+y^2}dx + ye^{x^2+y^2}dy$.

Find an $\omega_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that there exists a continuously differentiable $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ for which $df = e^{xy}dx + \omega_2 dy$.

Finish the proof of , that is, we only proved the second direction for a smooth path, not a piecewise smooth path.

Show that a star shaped domain $U \subset \mathbb{R}^n$ is path connected.

Show that $U := \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x \leq 0, y = 0\}$ is star shaped and find all points $(x_0, y_0) \in U$ such that U is star shaped with respect to (x_0, y_0) .

Suppose U_1 and U_2 are two open sets in $\mathbb{R}^n$ with $U_1 \cap U_2$ nonempty and connected. Suppose there exists an $f_1: U_1 \rightarrow \mathbb{R}$ and $f_2: U_2 \rightarrow \mathbb{R}$, both twice continuously differentiable such that $df_1 = df_2$ on $U_1 \cap U_2$. Then there exists a twice differentiable function $F: U_1 \cup U_2 \rightarrow \mathbb{R}$ such that $dF = df_1$ on U_1 and $dF = df_2$ on U_2 .

Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a simple nonclosed piecewise smooth path (so γ is one-to-one). Suppose ω is a continuously differentiable one-form defined on some open set V with $\gamma([a, b]) \subset V$ and $\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j}$ for all j and k . Prove that there exists an open set U with $\gamma([a, b]) \subset U \subset V$ and a twice continuously differentiable function $f: U \rightarrow \mathbb{R}$ such that $df = \omega$.

Hint 1: $\gamma([a, b])$ is compact.

Hint 2: Show that you can cover the curve by finitely many balls in sequence so that the k th ball only intersects the $(k-1)$ th ball.

Hint 3: See previous exercise.

a) Show that a connected open set is path connected. Hint: Start with two points x and y in a connected set U , and let $U_x \subset U$ be the set of points that are reachable by a path from x and similarly for U_y . Show that both sets are open, since they are nonempty ($x \in U_x$ and $y \in U_y$) it must be that $U_x = U_y = U$.

b) Prove the converse that is, a path connected set $U \subset \mathbb{R}^n$ is connected. Hint: for contradiction assume there exist two open and disjoint nonempty open sets and then assume there is a piecewise smooth (and therefore continuous) path between a point in one to a point in the other.

Usually path connectedness is defined using just continuous paths rather than piecewise smooth paths. Prove that the definitions are equivalent, in other words prove the following statement:

Suppose $U \subset \mathbb{R}^n$ is such that for any $x, y \in U$, there exists a continuous function $\gamma: [a, b] \rightarrow U$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then U is path connected (in other words, then there exists a piecewise smooth path).

Take

$$\omega(x, y) = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Let $\gamma: [a, b] \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ be a closed piecewise smooth path. Let $R := \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$. Suppose $R \cap \gamma([a, b])$ is a finite set of k points. Then

$$\int_\gamma \omega = 2\pi\ell$$

for some integer ℓ with $|\ell| \leq k$.

Hint 1: First prove that for a path β that starts and end on R but does not intersect it otherwise, you find that $\int_{\beta} \omega$ is -2π , 0 , or 2π .

Hint 2: You proved above that $\mathbb{R}^2 \setminus R$ is star shaped.

Note: The number ℓ is called the *winding number* it measures how many times does γ wind around the origin in the clockwise direction.

Multivariable integral

Riemann integral over rectangles

Note: 2–3 lectures

As in , we define the Riemann integral using the Darboux upper and lower integrals. The ideas in this section are very similar to integration in one dimension. The complication is mostly notational. The differences between one and several dimensions will grow more pronounced in the sections following.

Rectangles and partitions

Let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be such that $a_k \leq b_k$ for all k . A set of the form $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ is called a *closed rectangle*. In this setting it is sometimes useful to allow $a_k = b_k$, in which case we think of $[a_k, b_k] = \{a_k\}$ as usual. If $a_k < b_k$ for all k , then a set of the form $(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n)$ is called an *open rectangle*.

For an open or closed rectangle $R := [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$ or $R := (a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n) \subset \mathbb{R}^n$, we define the *n-dimensional volume* by

$$V(R) := (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

A *partition* P of the closed rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ is a finite set of partitions $P_1, P_2, \dots, P_n$ of the intervals $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$. We write $P = (P_1, P_2, \dots, P_n)$. That is, for every k there is an integer ℓ_k and the finite set of numbers $P_k = \{x_{k,0}, x_{k,1}, x_{k,2}, \dots, x_{k,\ell_k}\}$ such that

$$a_k = x_{k,0} < x_{k,1} < x_{k,2} < \dots < x_{k,\ell_k-1} < x_{k,\ell_k} = b_k.$$

Picking a set of n integers $j_1, j_2, \dots, j_n$ where $j_k \in \{1, 2, \dots, \ell_k\}$ we get the *subrectangle*

$$[x_{1,j_1-1}, x_{1,j_1}] \times [x_{2,j_2-1}, x_{2,j_2}] \times \dots \times [x_{n,j_n-1}, x_{n,j_n}].$$

For simplicity, we order the subrectangles somehow and we say $\{R_1, R_2, \dots, R_N\}$ are the subrectangles corresponding to the partition P of R . More simply, we say they are the subrectangles of P . In other words, we subdivided the original rectangle into many smaller subrectangles. See . It is not difficult to see that these subrectangles cover our original R , and their volume sums to that of R . That is,

$$R = \bigcup_{j=1}^N R_j, \quad \text{and} \quad V(R) = \sum_{j=1}^N V(R_j).$$

When

$$R_k = [x_{1,j_1-1}, x_{1,j_1}] \times [x_{2,j_2-1}, x_{2,j_2}] \times \dots \times [x_{n,j_n-1}, x_{n,j_n}],$$

then

$$V(R_k) = \Delta x_{1,j_1} \Delta x_{2,j_2} \cdots \Delta x_{n,j_n} = (x_{1,j_1} - x_{1,j_1-1})(x_{2,j_2} - x_{2,j_2-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $f: R \rightarrow \mathbb{R}$ be a bounded function. Let P be a partition of $[a, b]$ and suppose that there are N subrectangles $R_1, R_2, \dots, R_N$. Define

$$\begin{aligned}
 m_i &:= \inf \{f(x) : x \in R_i\}, \\
 M_i &:= \sup \{f(x) : x \in R_i\}, \\
 L(P, f) &:= \sum_{i=1}^N m_i V(R_i), \\
 U(P, f) &:= \sum_{i=1}^N M_i V(R_i).
 \end{aligned}$$

We call $L(P, f)$ the *lower Darboux sum* and $U(P, f)$ the *upper Darboux sum*.

The indexing in the definition may be complicated, but fortunately we generally do not need to go back directly to the definition often. We start proving facts about the Darboux sums analogous to the one-variable results.

[mv:sumulbound:prop] Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f: R \rightarrow \mathbb{R}$ is a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. For any partition P of R we have

$$mV(R) \leq L(P, f) \leq U(P, f) \leq MV(R).$$

Let P be a partition. Then for all i we have $m \leq m_i$ and $M_i \leq M$. Also $m_i \leq M_i$ for all i . Finally $\sum_{i=1}^N V(R_i) = V(R)$. Therefore,

$$\begin{aligned}
 mV(R) &= m \left(\sum_{i=1}^N V(R_i) \right) = \sum_{i=1}^N mV(R_i) \leq \sum_{i=1}^N m_i V(R_i) \leq \\
 &\leq \sum_{i=1}^N M_i V(R_i) \leq \sum_{i=1}^N M V(R_i) = M \left(\sum_{i=1}^N V(R_i) \right) = MV(R). \quad \text{\textcolor{red}{qedhere}}
 \end{aligned}$$

Upper and lower integrals

By the set of upper and lower Darboux sums are bounded sets and we can take their infima and suprema. As before, we now make the following definition.

If $f: R \rightarrow \mathbb{R}$ is a bounded function on a closed rectangle $R \subset \mathbb{R}^n$. Define

$$\int_R f := \sup \{L(P, f) : P \text{ a partition of } R\}, \quad \int_R f := \inf \{U(P, f) : P \text{ a partition of } R\}.$$

We call $\int$ the *lower Darboux integral* and $\int$ the *upper Darboux integral*.

As in one dimension we have refinements of partitions.

Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $P = (P_1, P_2, \dots, P_n)$ and $\tilde{P} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n)$ be partitions of R . We say $\tilde{P}$ a *refinement* of P if, as sets, $P_k \subset \tilde{P}_k$ for all $k = 1, 2, \dots, n$.

It is not difficult to see that if $\tilde{P}$ is a refinement of P , then subrectangles of P are unions of subrectangles of $\tilde{P}$. Simply put, in a refinement we take the subrectangles of P , and we cut them into smaller subrectangles. See .

[mv:prop:refinement] Suppose $R \subset \mathbb{R}^n$ is a closed rectangle, P is a partition of R and $\tilde{P}$ is a refinement of P . If $f: R \rightarrow \mathbb{R}$ be a bounded function, then

$$L(P, f) \leq L(\tilde{P}, f) \quad \text{and} \quad U(\tilde{P}, f) \leq U(P, f).$$

We prove the first inequality, the second follows similarly. Let $R_1, R_2, \dots, R_N$ be the subrectangles of P and $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{\tilde{N}}$ be the subrectangles of $\tilde{R}$. Let I_k be the set of all indices j such that $\tilde{R}_j \subset R_k$. For example, using the examples in figures [\[mv:figrect\]](#) and [\[mv:figrectpart\]](#), $I_4 = \{6, 7, 8, 9\}$ and $R_4 = \tilde{R}_6 \cup \tilde{R}_7 \cup \tilde{R}_8 \cup \tilde{R}_9$. We notice in general that

$$R_k = \bigcup_{j \in I_k} \tilde{R}_j, \quad V(R_k) = \sum_{j \in I_k} V(\tilde{R}_j).$$

Let $m_j := \inf \{f(x) : x \in R_j\}$, and $\tilde{m}_j := \inf \{f(x) : x \in \tilde{R}_j\}$ as usual. Notice also that if $j \in I_k$, then $m_k \leq \tilde{m}_j$. Then

$$L(P, f) = \sum_{k=1}^N m_k V(R_k) = \sum_{k=1}^N \sum_{j \in I_k} m_k V(\tilde{R}_j) \leq \sum_{k=1}^N \sum_{j \in I_k} \tilde{m}_j V(\tilde{R}_j) = \sum_{j=1}^{\tilde{N}} \tilde{m}_j V(\tilde{R}_j) = L(\tilde{P}, f). \quad \text{\textcolor{red}{qedhere}}$$

The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

[\[mv:intulbound:prop\]](#) Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. Then

$$mV(R) \leq \int_R f \leq \int_R f \leq MV(R).$$

For any partition P , via ,

$$mV(R) \leq L(P, f) \leq U(P, f) \leq MV(R).$$

Taking supremum of $L(P, f)$ and infimum of $U(P, f)$ over all P , we obtain the first and the last inequality.

The key inequality in [\[mv:intulbound:eq\]](#) is the middle one. Let $P = (P_1, P_2, \dots, P_n)$ and $Q = (Q_1, Q_2, \dots, Q_n)$ be partitions of R . Define $\tilde{P} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n)$ by letting $\tilde{P}_k = P_k \cup Q_k$. Then $\tilde{P}$ is a partition of R as can easily be checked, and $\tilde{P}$ is a refinement of P and a refinement of Q . By , $L(P, f) \leq L(\tilde{P}, f)$ and $U(\tilde{P}, f) \leq U(Q, f)$. Therefore,

$$L(P, f) \leq L(\tilde{P}, f) \leq U(\tilde{P}, f) \leq U(Q, f).$$

In other words, for two arbitrary partitions P and Q we have $L(P, f) \leq U(Q, f)$. Via Proposition 1.2.7 from volume I, we obtain

$$\sup \{L(P, f) : P \text{ a partition of } R\} \leq \inf \{U(P, f) : P \text{ a partition of } R\}.$$

In other words $\int_R f \leq \int_R f$.

The Riemann integral

We have all we need to define the Riemann integral in n -dimensions over rectangles. Again, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $f: R \rightarrow \mathbb{R}$ be a bounded function such that

$$\int_R f(x) \, dx = \int_R f(x) \, dx.$$

Then f is said to be *Riemann integrable*, and we sometimes say simply *integrable*. The set of Riemann integrable functions on R is denoted by $\mathcal{R}(R)$. When $f \in \mathcal{R}(R)$ we define the *Riemann integral*

$$\int_R f := \int_R f = \int_R f.$$

When the variable $x \in \mathbb{R}^n$ needs to be emphasized we write

$$\int_R f(x) \, dx, \quad \int_R f(x_1, \dots, x_n) \, dx_1 \cdots dx_n, \quad \text{or} \quad \int_R f(x) \, dV.$$

If $R \subset \mathbb{R}^2$, then often instead of volume we say area, and hence write

$$\int_R f(x) \, dA.$$

implies immediately the following proposition.

[mv:intbound:prop] Let $f: R \rightarrow \mathbb{R}$ be a Riemann integrable function on a closed rectangle $R \subset \mathbb{R}^n$. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in R$. Then

$$mV(R) \leq \int_R f \leq MV(R).$$

A constant function is Riemann integrable. Suppose $f(x) = c$ for all x on R . Then

$$cV(R) \leq \int_R f \leq \int_R f \leq cV(R).$$

So f is integrable, and furthermore $\int_R f = cV(R)$.

The proofs of linearity and monotonicity are almost completely identical as the proofs from one variable. We therefore leave it as an exercise to prove the next two propositions.

[mv:intlinearity:prop] Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathcal{R}(R)$ and $\alpha \in \mathbb{R}$.

1. αf is in $\mathcal{R}(R)$ and

$$\int_R \alpha f = \alpha \int_R f.$$

2. $f + g$ is in $\mathcal{R}(R)$ and

$$\int_R (f + g) = \int_R f + \int_R g.$$

Let $R \subset \mathbb{R}^n$ be a closed rectangle, let f and g be in $\mathcal{R}(R)$, and suppose $f(x) \leq g(x)$ for all $x \in R$. Then

$$\int_R f \leq \int_R g.$$

Checking for integrability using the definition often involves the following technique, as in the single variable case.

[mv:prop:upperlowerepsilon] Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. Then $f \in \mathcal{R}(R)$ if and only if for every $\epsilon > 0$, there exists a partition P of R such that

$$U(P, f) - L(P, f) < \epsilon.$$

First, if f is integrable, then clearly the supremum of $L(P, f)$ and infimum of $U(P, f)$ must be equal and hence the infimum of $U(P, f) - L(P, f)$ is zero. Therefore for every $\epsilon > 0$ there must be some partition P such that $U(P, f) - L(P, f) < \epsilon$.

For the other direction, given an $\epsilon > 0$ find P such that $U(P, f) - L(P, f) < \epsilon$.

$$\int_R f - \int_R f \leq U(P, f) - L(P, f) < \epsilon.$$

As $\int_R f \geq \int_R f$ and the above holds for every $\epsilon > 0$, we conclude $\int_R f = \int_R f$ and $f \in \mathcal{R}(R)$.

For simplicity if $f: S \rightarrow \mathbb{R}$ is a function and $R \subset S$ is a closed rectangle, then if the restriction $f|_R$ is integrable we say f is integrable on R , or $f \in \mathcal{R}(R)$ and we write

$$\int_R f := \int_R f|_R.$$

[mv:prop:integralsmallerset] For a closed rectangle $S \subset \mathbb{R}^n$, if $f: S \rightarrow \mathbb{R}$ is integrable and $R \subset S$ is a closed rectangle, then f is integrable over R .

Given $\epsilon > 0$, we find a partition P of S such that $U(P, f) - L(P, f) < \epsilon$. By making a refinement of P if necessary, we assume that the endpoints of R are in P . In other words, R is a union of subrectangles of P . The subrectangles of P divide into two collections, ones that are subsets of R and ones whose intersection with the interior of R is empty. Suppose $R_1, R_2, \dots, R_K$ are the subrectangles that are subsets of R and let $R_{K+1}, \dots, R_N$ be the rest. Let $\tilde{P}$ be the partition of R composed of those subrectangles of P contained in R . Using the same notation as before,

$$\begin{aligned} \epsilon &> U(P, f) - L(P, f) = \sum_{k=1}^K (M_k - m_k)V(R_k) + \sum_{k=K+1}^N (M_k - m_k)V(R_k) \\ &\geq \sum_{k=1}^K (M_k - m_k)V(R_k) = U(\tilde{P}, f|_R) - L(\tilde{P}, f|_R). \end{aligned}$$

Therefore, $f|_R$ is integrable.

Integrals of continuous functions

Although later we will prove a much more general result, it is useful to start with integrability of continuous functions. First we wish to measure the fineness of partitions. In one variable we measured the length of a subinterval, in several variables, we similarly measure the sides of a subrectangle. We say a rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ has *longest side at most* α if $b_k - a_k \leq \alpha$ for all $k = 1, 2, \dots, n$.

[prop:diameterrectangle] If a rectangle $R \subset \mathbb{R}^n$ has longest side at most α . Then for any $x, y \in R$,

$$\begin{aligned} \|x - y\| &\leq \sqrt{n} \alpha. \\ \|x - y\| &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2} \\ &\leq \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \dots + (b_n - a_n)^2} \\ &\leq \sqrt{\alpha^2 + \alpha^2 + \dots + \alpha^2} = \sqrt{n} \alpha. \end{aligned}$$

[mv:thm:contintrect] Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a continuous function, then $f \in \mathcal{R}(R)$.

The proof is analogous to the one variable proof with some complications. The set R is a closed and bounded subset of $\mathbb{R}^n$, and hence compact. So f is not just continuous, but in fact uniformly continuous by Theorem 7.5 from volume I. Let $\epsilon > 0$ be given.

Find a $\delta > 0$ such that $\|x - y\| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{V(R)}$.

Let P be a partition of R , such that longest side of any subrectangle is strictly less than $\frac{\delta}{\sqrt{n}}$. If $x, y \in R_k$ for some subrectangle R_k of P we have, by the proposition above, $\|x - y\| < \sqrt{n} \frac{\delta}{\sqrt{n}} = \delta$. Therefore

$$f(x) - f(y) \leq |f(x) - f(y)| < \frac{\epsilon}{V(R)}.$$

As f is continuous on R_k , it attains a maximum and a minimum on this subrectangle. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_k$ and $f(y) = m_k$ in the notation from the definition of the integral. Therefore,

$$M_k - m_k = f(x) - f(y) < \frac{\epsilon}{V(R)}.$$

And so

$$\begin{aligned} U(P, f) - L(P, f) &= \left(\sum_{k=1}^N M_k V(R_k) \right) - \left(\sum_{k=1}^N m_k V(R_k) \right) \\ &= \sum_{k=1}^N (M_k - m_k) V(R_k) \\ &< \frac{\epsilon}{V(R)} \sum_{k=1}^N V(R_k) = \epsilon. \end{aligned}$$

Via application of we find that $f \in \mathcal{R}(R)$.

Integration of functions with compact support

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \rightarrow \mathbb{R}$ be a function. We say the *support* of f is the set

$$\text{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}},$$

where the closure is with respect to the subspace topology on U . Recall that taking the closure with respect to the subspace topology is the same as $\{x \in U : f(x) \neq 0\} \cap U$, now taking the closure with respect to the ambient euclidean space $\mathbb{R}^n$. In particular, $\text{supp}(f) \subset U$. That is, the support is the closure (in U) of the set of points where the function is nonzero. Its complement in U is open. If $x \in U$ and x is not in the support of f , then f is constantly zero in a whole neighborhood of x .

A function f is said to have *compact support* if $\text{supp}(f)$ is a compact set.

Suppose $B(0, 1) \subset \mathbb{R}^2$ is the unit disc. The function $f: B(0, 1) \rightarrow \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} 0 & \text{if } \sqrt{x^2 + y^2} > \frac{1}{2}, \\ \frac{1}{2} - \sqrt{x^2 + y^2} & \text{if } \sqrt{x^2 + y^2} \leq \frac{1}{2}, \end{cases}$$

is continuous on $B(0, 1)$ and its support is the smaller closed ball $C(0, \frac{1}{2})$. As that is a compact set, f has compact support.

Similarly $g: B(0, 1) \rightarrow \mathbb{R}$ defined by

$$g(x, y) := \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x > 0, \end{cases}$$

is continuous on $B(0, 1)$, but its support is the set $\{(x, y) \in B(0, 1) : x \geq 0\}$. In particular, g is not compactly supported.

We will mostly consider the case when $U = \mathbb{R}^n$. In light of the following exercise, this is not an oversimplification.

Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}$ is continuous and of compact support. Show that the function $\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous.

On the other hand for the unit disc $B(0, 1) \subset \mathbb{R}^2$, the function continuous $f: B(0, 1) \rightarrow \mathbb{R}$ defined by $f(x, y) := \sin\left(\frac{1}{1-x^2-y^2}\right)$, does not have compact support; as f is not constantly zero on neighborhood of any point in $B(0, 1)$, we know that the support is the entire disc $B(0, 1)$. The function clearly does not extend as above to a continuous function. In fact it is not difficult to show that it cannot be extended in any way whatsoever to be continuous on all of $\mathbb{R}^2$ (the boundary of the disc is the problem).

[mv:prop:rectanglessupp] Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function with compact support. If R and S are closed rectangles such that $\text{supp}(f) \subset R$ and $\text{supp}(f) \subset S$, then

$$\int_S f = \int_R f.$$

As f is continuous, it is automatically integrable on the rectangles R , S , and $R \cap S$. Then says $\int_S f = \int_{S \cap R} f = \int_R f$.

Because of this proposition, when $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has compact support and is integrable over a rectangle R containing the support we write

$$\int f := \int_R f \quad \text{or} \quad \int_{\mathbb{R}^n} f := \int_R f.$$

For example, if f is continuous and of compact support, then $\int_{\mathbb{R}^n} f$ exists.

Exercises

Prove .

Suppose R is a rectangle with the length of one of the sides equal to 0. For any bounded function f , show that $f \in \mathcal{R}(R)$ and $\int_R f = 0$.

[mv:zerosiderectangle] Suppose R is a rectangle with the length of one of the sides equal to 0, and suppose S is a rectangle with $R \subset S$. If f is a bounded function such that $f(x) = 0$ for $x \in R \setminus S$, show that $f \in \mathcal{R}(R)$ and $\int_R f = 0$.

Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is such that $f(x) := 0$ if $x \neq 0$ and $f(0) := 1$. Show that f is integrable on $R := [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1]$ directly using the definition, and find $\int_R f$.

[mv:zeroinside] Suppose R is a closed rectangle and $h: R \rightarrow \mathbb{R}$ is a bounded function such that $h(x) = 0$ if $x \notin \partial R$ (the boundary of R). Let S be any closed rectangle. Show that $h \in \mathcal{R}(S)$ and

$$\int_S h = 0.$$

Hint: Write h as a sum of functions as in .

[mv:zerooutside] Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose $f: R \rightarrow \mathbb{R}$ is in $\mathcal{R}(R')$ and $f(x) = 0$ for $x \in R \setminus R'$. Show that $f \in \mathcal{R}(R)$ and

$$\int_{R'} f = \int_R f.$$

Do this in the following steps.

a) First do the proof assuming that furthermore $f(x) = 0$ whenever $x \in R \setminus R'$.

- b) Write $f(x) = g(x) + h(x)$ where $g(x) = 0$ whenever $x \in R \setminus R'$, and $h(x)$ is zero except perhaps on $\partial R'$. Then show $\int_R h = \int_{R'} h = 0$ (see).
- c) Show $\int_{R'} f = \int_R f$.

Suppose $R' \subset \mathbb{R}^n$ and $R'' \subset \mathbb{R}^n$ are two rectangles such that $R = R' \cup R''$ is a rectangle, and $R' \cap R''$ is rectangle with one of the sides having length 0 (that is $V(R' \cap R'') = 0$). Let $f: R \rightarrow \mathbb{R}$ be a function such that $f \in \mathcal{R}(R')$ and $f \in \mathcal{R}(R'')$. Show that $f \in \mathcal{R}(R)$ and

$$\int_R f = \int_{R'} f + \int_{R''} f.$$

Hint: see previous exercise.

Prove a stronger version of . Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function with compact support but not necessarily continuous. Prove that if R is a closed rectangle such that $\text{supp}(f) \subset R$ and f is integrable over R , then for any other closed rectangle S with $\text{supp}(f) \subset S$, the function f is integrable over S and $\int_S f = \int_R f$. Hint: See .

Suppose R and S are closed rectangles of $\mathbb{R}^n$. Define $f: \mathbb{R}^n \rightarrow \mathbb{R}$ as $f(x) := 1$ if $x \in R$, and $f(x) := 0$ otherwise. Prove f is integrable over S and compute $\int_S f$. Hint: Consider $S \cap R$.

Let $R = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

- a) Suppose $f: R \rightarrow \mathbb{R}$ is defined by

$$f(x, y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

Show that $f \in \mathcal{R}(R)$ and compute $\int_R f$.

- b) Suppose $f: R \rightarrow \mathbb{R}$ is defined by

$$f(x, y) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}, \\ 0 & \text{else.} \end{cases}$$

Show that $f \notin \mathcal{R}(R)$.

Suppose R is a closed rectangle, and suppose S_j are closed rectangles such that $S_j \subset R$ and $S_j \subset S_{j+1}$ for all j . Suppose $f: R \rightarrow \mathbb{R}$ is bounded and $f \in \mathcal{R}(S_j)$ for all j . Show that $f \in \mathcal{R}(R)$ and

$$\lim_{j \rightarrow \infty} \int_{S_j} f = \int_R f.$$

Suppose $f: [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}$ is a Riemann integrable function such $f(x) = -f(-x)$. Using the definition prove

$$\int_{[-1, 1] \times [-1, 1]} f = 0.$$

Iterated integrals and Fubini theorem

Note: 1–2 lectures

The Riemann integral in several variables is hard to compute from the definition. For one-dimensional Riemann integral we have the fundamental theorem of calculus and we can compute many integrals without having to appeal to the definition of the integral. We will rewrite a Riemann integral in several variables into several one-dimensional Riemann integrals by iterating. However, if $f: [0, 1]^2 \rightarrow \mathbb{R}$ is a Riemann integrable function, it is not immediately clear if the three expressions

$$\int_{[0, 1]^2} f, \quad \int_0^1 \int_0^1 f(x, y) dx dy, \quad \text{and} \quad \int_0^1 \int_0^1 f(x, y) dy dx$$

are equal, or if the last two are even well-defined.

Define

$$f(x, y) := \begin{cases} 1 & \text{if } x = \frac{1}{2} \text{ and } y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is Riemann integrable on $R := [0, 1]^2$ and $\int_R f = 0$. Furthermore, $\int_0^1 \int_0^1 f(x, y) dx dy = 0$. However

$$\int_0^1 f\left(\frac{1}{2}, y\right) dy$$

does not exist, so we cannot even write $\int_0^1 \int_0^1 f(x, y) dy dx$.

Proof: Let us start with integrability of f . We simply take the partition of $[0, 1]^2$ where the partition in the x direction is $\{0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, 1\}$ and in the y direction $\{0, 1\}$. The subrectangles of the partition are

$$R_1 := [0, \frac{1}{2} - \epsilon] \times [0, 1], \quad R_2 := [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \times [0, 1], \quad R_3 := [\frac{1}{2} + \epsilon, 1] \times [0, 1].$$

We have $m_1 = M_1 = 0$, $m_2 = 0$, $M_2 = 1$, and $m_3 = M_3 = 0$. Therefore,

$$L(P, f) = m_1 V(R_1) + m_2 V(R_2) + m_3 V(R_3) = 0\left(\frac{1}{2} - \epsilon\right) + 0(2\epsilon) + 0\left(\frac{1}{2} - \epsilon\right) = 0,$$

and

$$U(P, f) = M_1 V(R_1) + M_2 V(R_2) + M_3 V(R_3) = 0\left(\frac{1}{2} - \epsilon\right) + 1(2\epsilon) + 0\left(\frac{1}{2} - \epsilon\right) = 2\epsilon.$$

The upper and lower sum are arbitrarily close and the lower sum is always zero, so the function is integrable and $\int_R f = 0$.

For any y , the function that takes x to $f(x, y)$ is zero except perhaps at a single point $x = \frac{1}{2}$. We know that such a function is integrable and $\int_0^1 f(x, y) dx = 0$. Therefore, $\int_0^1 \int_0^1 f(x, y) dx dy = 0$.

However if $x = \frac{1}{2}$, the function that takes y to $f(\frac{1}{2}, y)$ is the nonintegrable function that is 1 on the rationals and 0 on the irrationals. See Example 5.1.4 from volume I.

We will solve this problem of undefined inside integrals by using the upper and lower integrals, which are always defined.

We split the coordinates of $\mathbb{R}^{n+m}$ into two parts. That is, we write the coordinates on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ as (x, y) where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a function $f(x, y)$ we write

$$f_x(y) := f(x, y)$$

when x is fixed and we wish to speak of the function in terms of y . We write

$$f^y(x) := f(x, y)$$

when y is fixed and we wish to speak of the function in terms of x .

[my:fubinivA] Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f: R \times S \rightarrow \mathbb{R}$ be integrable. The functions $g: R \rightarrow \mathbb{R}$ and $h: S \rightarrow \mathbb{R}$ defined by

$$g(x) := \int_S f_x \quad \text{and} \quad h(y) := \int_R f^y$$

are integrable over R and

$$\int_R g = \int_S h = \int_{R \times S} f.$$

In other words

$$\int_{R \times S} f = \int_R \left(\int_S f(x, y) dy \right) dx = \int_R \left(\int_S f(x, y) dy \right) dx.$$

If it turns out that f_x is integrable for all x , for example when f is continuous, then we obtain the more familiar

$$\int_{R \times S} f = \int_R \int_S f(x, y) dy dx.$$

Any partition of $R \times S$ is a concatenation of a partition of R and a partition of S . That is, write a partition of $R \times S$ as $(P, P') = (P_1, P_2, \dots, P_n, P'_1, P'_2, \dots, P'_m)$, where $P = (P_1, P_2, \dots, P_n)$ and $P' = (P'_1, P'_2, \dots, P'_m)$ are partitions of R and S respectively. Let $R_1, R_2, \dots, R_N$ be the subrectangles of P and $R'_1, R'_2, \dots, R'_K$ be the subrectangles of P' . Then the subrectangles of (P, P') are $R_j \times R'_k$ where $1 \leq j \leq N$ and $1 \leq k \leq K$.

Let

$$m_{j,k} := \inf_{(x,y) \in R_j \times R'_k} f(x, y).$$

We notice that $V(R_j \times R'_k) = V(R_j)V(R'_k)$ and hence

$$L((P, P'), f) = \sum_{j=1}^N \sum_{k=1}^K m_{j,k} V(R_j \times R'_k) = \sum_{j=1}^N \left(\sum_{k=1}^K m_{j,k} V(R'_k) \right) V(R_j).$$

If we let

$$m_k(x) := \inf_{y \in R'_k} f(x, y) = \inf_{y \in R'_k} f_x(y),$$

then of course if $x \in R_j$, then $m_{j,k} \leq m_k(x)$. Therefore

$$\sum_{k=1}^K m_{j,k} V(R'_k) \leq \sum_{k=1}^K m_k(x) V(R'_k) = L(P', f_x) \leq \int_S f_x = g(x).$$

As we have the inequality for all $x \in R_j$ we have

$$\sum_{k=1}^K m_{j,k} V(R'_k) \leq \inf_{x \in R_j} g(x).$$

We thus obtain

$$L((P, P'), f) \leq \sum_{j=1}^N \left(\inf_{x \in R_j} g(x) \right) V(R_j) = L(P, g).$$

Similarly $U((P, P'), f) \geq U(P, g)$, and the proof of this inequality is left as an exercise.

Putting this together we have

$$L((P, P'), f) \leq L(P, g) \leq U(P, g) \leq U(P, h) \leq U((P, P'), f).$$

And since f is integrable, it must be that g is integrable as

$$U(P, g) - L(P, g) \leq U((P, P'), f) - L((P, P'), f),$$

and we can make the right hand side arbitrarily small. As for any partition we have $L((P, P'), f) \leq L(P, g) \leq U((P, P'), f)$ we must have that $\int_R g = \int_{R \times S} f$.

Similarly we have

$$L((P, P'), f) \leq L(P, g) \leq L(P, h) \leq U(P, h) \leq U((P, P'), f),$$

and hence

$$U(P, h) - L(P, h) \leq U((P, P'), f) - L((P, P'), f).$$

So if f is integrable so is h , and as $L((P, P'), f) \leq L(P, h) \leq U((P, P'), f)$ we must have that $\int_R h = \int_{R \times S} f$.

We can also do the iterated integration in opposite order. The proof of this version is almost identical to version A, and we leave it as an exercise to the reader.

[mv:fubiniB] Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f: R \times S \rightarrow \mathbb{R}$ be integrable. The functions $g: S \rightarrow \mathbb{R}$ and $h: S \rightarrow \mathbb{R}$ defined by

$$g(y) := \int_R f^y \quad \text{and} \quad h(y) := \int_R f^y$$

are integrable over S and

$$\int_S g = \int_S h = \int_{R \times S} f.$$

That is we also have

$$\int_{R \times S} f = \int_S \left(\int_R f(x, y) dx \right) dy = \int_S \left(\int_R f(x, y) dy \right) dx.$$

Next suppose that f_x and f^y are integrable for simplicity. For example, suppose that f is continuous. Then by putting the two versions together we obtain the familiar

$$\int_{R \times S} f = \int_R \int_S f(x, y) dy dx = \int_S \int_R f(x, y) dx dy.$$

Often the Fubini theorem is stated in two dimensions for a continuous function $f: R \rightarrow \mathbb{R}$ on a rectangle $R = [a, b] \times [c, d]$. Then the Fubini theorem states that

$$\int_R f = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

And the Fubini theorem is commonly thought of as the theorem that allows us to swap the order of iterated integrals.

Repeatedly applying Fubini theorem gets us the following corollary: Let $R := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a closed rectangle and let $f: R \rightarrow \mathbb{R}$ be continuous. Then

$$\int_R f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n) dx_n dx_{n-1} \cdots dx_1.$$

Clearly we can also switch the order of integration to any order we please. We can also relax the continuity requirement by making sure that all the intermediate functions are integrable, or by using upper or lower integrals.

Exercises

Compute $\int_0^1 \int_{-1}^1 x e^{xy} dx dy$ in a simple way.

Prove the assertion $U((P, P'), f) \geq U(P, h)$ from the proof of .

Prove .

Let $R = [a, b] \times [c, d]$ and $f(x, y)$ is an integrable function on R such that for any fixed y , the function that takes x to $f(x, y)$ is zero except at finitely many points. Show

$$\int_R f = 0.$$

Let $R = [a, b] \times [c, d]$ and $f(x, y) := g(x)h(y)$ for two continuous functions $g: [a, b] \rightarrow \mathbb{R}$ and $h: [c, d] \rightarrow \mathbb{R}$. Prove

$$\int_R f = \left(\int_a^b g \right) \left(\int_c^d h \right).$$

Compute

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy \quad \text{and} \quad \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy dx.$$

You will need to interpret the integrals as improper, that is, the limit of $\int_\epsilon^1$ as $\epsilon \rightarrow 0$.

Suppose $f(x, y) := g(x)$ where $g: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable. Show that f is Riemann integrable for any $R = [a, b] \times [c, d]$ and

$$\int_R f = (d - c) \int_a^b g.$$

Define $f: [-1, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} x & \text{if } y \in \mathbb{Q}, \\ 0 & \text{else.} \end{cases}$$

Show

a) $\int_0^1 \int_{-1}^1 f(x, y) dx dy$ exists, but $\int_{-1}^1 \int_0^1 f(x, y) dy dx$ does not.

b) Compute $\int_{-1}^1 \int_0^1 f(x, y) dy dx$ and $\int_{-1}^1 \int_0^1 f(x, y) dx dy$.

c) Show f is not Riemann integrable on $[-1, 1] \times [0, 1]$ (use Fubini).

Define $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q}, y \in \mathbb{Q}, \text{ and } y = \frac{p}{q} \text{ in lowest terms,} \\ 0 & \text{else.} \end{cases}$$

Show

a) Show f is Riemann integrable on $[0, 1] \times [0, 1]$.

b) Find $\int_0^1 f(x, y) dx$ and $\int_0^1 f(x, y) dy$ for all $y \in [0, 1]$, and show they are unequal for all $y \in \mathbb{Q}$.

c) $\int_0^1 \int_0^1 f(x, y) dy dx$ exists, but $\int_0^1 \int_0^1 f(x, y) dx dy$ does not.

Note: By Fubini, $\int_0^1 \int_0^1 f(x, y) dy dx$ and $\int_0^1 \int_0^1 f(x, y) dx dy$ do exist and equal the integral of f on R .

Outer measure and null sets

Note: 2 lectures

Outer measure and null sets

Before we characterize all Riemann integrable functions, we need to make a slight detour. We introduce a way of measuring the size of sets in $\mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be a subset. Define the *outer measure* of S as

$$m^*(S) := \inf \sum_{j=1}^{\infty} V(R_j),$$

where the infimum is taken over all sequences $\{R_j\}$ of open rectangles such that $S \subset \bigcup_{j=1}^{\infty} R_j$. In particular, S is of *measure zero* or a *null set* if $m^*(S) = 0$.

The theory of measures on $\mathbb{R}^n$ is a very complicated subject. We will only require measure-zero sets and so we focus on these. The set S is of measure zero if for every $\epsilon > 0$ there exist a sequence of open rectangles $\{R_j\}$ such that

$$S \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \epsilon.$$

Furthermore, if S is measure zero and $S' \subset S$, then S' is of measure zero. We can in fact use the same exact rectangles.

It is sometimes more convenient to use balls instead of rectangles. In fact we can choose balls no bigger than a fixed radius.

[mv:prop:ballsnul] Let $\delta > 0$ be given. A set $S \subset \mathbb{R}^n$ is measure zero if and only if for every $\epsilon > 0$, there exists a sequence of open balls $\{B_j\}$, where the radius of B_j is $r_j < \delta$ such that

$$S \subset \bigcup_{j=1}^{\infty} B_j \quad \text{and} \quad \sum_{j=1}^{\infty} r_j^n < \epsilon.$$

Note that the “volume” of B_j is proportional to r_j^n .

If R is a (closed or open) cube (rectangle with all sides equal) of side s , then R is contained in a closed ball of radius $\sqrt{n}s$ by , and therefore in an open ball of size $2\sqrt{n}s$.

Let s be a number that is less than the smallest side of R and also so that $2\sqrt{n}s < \delta$. We claim R is contained in a union of closed cubes $C_1, C_2, \dots, C_k$ of sides s such that

$$\sum_{j=1}^k V(C_j) \leq 2^n V(R).$$

It is clearly true (without the 2^n) if R has sides that are integer multiples of s . So if a side is of length $(\ell + \alpha)s$, for $\ell \in \mathbb{N}$ and $0 \leq \alpha < 1$, then $(\ell + \alpha)s \leq 2\ell s$. Increasing the side to $2\ell s$ we obtain a new larger rectangle of volume at most 2^n times larger, but whose sides are multiples of s .

So suppose that there exist $\{R_j\}$ as in the definition such that [\[mv:eq:nullR\]](#) is true. As we have seen above, we can choose closed cubes $\{C_k\}$ with C_k of side s_k as above that cover all the rectangles $\{R_j\}$ and so that

$$\sum_{k=1}^{\infty} s_k^n = \sum_{k=1}^{\infty} V(C_k) \leq 2^n \sum_{j=1}^{\infty} V(R_k) < 2^n \epsilon.$$

Covering C_k with balls B_k of radius $r_k = 2\sqrt{n}s_k$ we obtain

$$\sum_{k=1}^{\infty} r_k^n < 2^{2n} n \epsilon.$$

And as $S \subset \bigcup_j R_j \subset \bigcup_k C_k \subset \bigcup_k B_k$, we are finished.

Suppose we have the ball condition above for some $\epsilon > 0$. Without loss of generality assume that all $r_j < 1$. Each B_j is contained in a cube R_j of side $2r_j$. So $V(R_j) = (2r_j)^n < 2^n r_j^n$. Therefore

$$S \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} 2^n r_j^n < 2^n \epsilon. \quad \text{\textcolor{red}{qedhere}}$$

The definition of outer measure could have been done with open balls as well, not just null sets. We leave this generalization to the reader.

Examples and basic properties

The set $Q^n \subset R^n$ of points with rational coordinates is a set of measure zero.

Proof: The set Q^n is countable and therefore let us write it as a sequence $q_1, q_2, \dots$. For each q_j find an open rectangle R_j with $q_j \in R_j$ and $V(R_j) < \epsilon 2^{-j}$. Then

$$Q^n \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} \epsilon 2^{-j} = \epsilon.$$

The example points to a more general result.

A countable union of measure zero sets is of measure zero.

Suppose

$$S = \bigcup_{j=1}^{\infty} S_j,$$

where S_j are all measure zero sets. Let $\epsilon > 0$ be given. For each j there exists a sequence of open rectangles $\{R_{j,k}\}_{k=1}^{\infty}$ such that

$$S_j \subset \bigcup_{k=1}^{\infty} R_{j,k}$$

and

$$\sum_{k=1}^{\infty} V(R_{j,k}) < 2^{-j} \epsilon.$$

Then

$$S \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} R_{j,k}.$$

As $V(R_{j,k})$ is always positive, the sum over all j and k can be done in any order. In particular, it can be done as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} V(R_{j,k}) < \sum_{j=1}^{\infty} 2^{-j} \epsilon = \epsilon. \text{ \textcolor{red}{qedhere}}$$

The next example is not just interesting, it will be useful later.

[mv:example:planenull] Let $P := \{x \in \mathbb{R}^n : x_k = c\}$ for a fixed $k = 1, 2, \dots, n$ and a fixed constant $c \in \mathbb{R}$. Then P is of measure zero.

Proof: First fix s and let us prove that

$$P_s := \{x \in \mathbb{R}^n : x_k = c, |x_j| \leq s \text{ for all } j \neq k\}$$

is of measure zero. Given any $\epsilon > 0$ define the open rectangle

$$R := \{x \in \mathbb{R}^n : c - \epsilon < x_k < c + \epsilon, |x_j| < s + 1 \text{ for all } j \neq k\}.$$

It is clear that $P_s \subset R$. Furthermore

$$V(R) = 2\epsilon(2(s+1))^{n-1}.$$

As s is fixed, we can make $V(R)$ arbitrarily small by picking ϵ small enough.

Next we note that

$$P = \bigcup_{j=1}^{\infty} P_j$$

and a countable union of measure zero sets is measure zero.

If $a < b$, then $m^*([a, b]) = b - a$.

Proof: In the case of $\mathbb{R}$, open rectangles are open intervals. Since $[a, b] \subset (a - \epsilon, b + \epsilon)$ for all $\epsilon > 0$. Hence, $m^*([a, b]) \leq b - a$.

Let us prove the other inequality. Suppose $\{(a_j, b_j)\}$ are open intervals such that

$$[a, b] \subset \bigcup_{j=1}^{\infty} (a_j, b_j).$$

We wish to bound $\sum (b_j - a_j)$ from below. Since $[a, b]$ is compact, then there are only finitely many open intervals that still cover $[a, b]$. As throwing out some of the intervals only makes the sum smaller, we only need to take the finite number of intervals still covering $[a, b]$. If $(a_i, b_i) \subset (a_j, b_j)$, then we can throw out (a_i, b_i) as well. Therefore we have $[a, b] \subset \bigcup_{j=1}^k (a_j, b_j)$ for some k , and we assume that the intervals are sorted such that $a_1 < a_2 < \dots < a_k$. Note that since (a_2, b_2) is not contained in (a_1, b_1) we have that $a_1 < a_2 < b_1 < b_2$. Similarly $a_j < a_{j+1} < b_j < b_{j+1}$. Furthermore, $a_1 < a$ and $b_k > b$. Thus,

$$m^*([a, b]) \geq \sum_{j=1}^k (b_j - a_j) \geq \sum_{j=1}^{k-1} (a_{j+1} - a_j) + (b_k - a_k) = b_k - a_1 > b - a.$$

[mv:prop:compactnull] Suppose $E \subset \mathbb{R}^n$ is a compact set of measure zero. Then for every $\epsilon > 0$, there exist finitely many open rectangles $R_1, R_2, \dots, R_k$ such that

$$E \subset R_1 \cup R_2 \cup \dots \cup R_k \quad \text{and} \quad \sum_{j=1}^k V(R_j) < \epsilon.$$

Also for any $\delta > 0$, there exist finitely many open balls $B_1, B_2, \dots, B_k$ of radii $r_1, r_2, \dots, r_k < \delta$ such that

$$E \subset B_1 \cup B_2 \cup \dots \cup B_k \quad \text{and} \quad \sum_{j=1}^k r_j^n < \epsilon.$$

Find a sequence of open rectangles $\{R_j\}$ such that

$$E \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \epsilon.$$

By compactness, there are finitely many of these rectangles that still contain E . That is, there is some k such that $E \subset R_1 \cup R_2 \cup \dots \cup R_k$. Hence

$$\sum_{j=1}^k V(R_j) \leq \sum_{j=1}^{\infty} V(R_j) < \epsilon.$$

The proof that we can choose balls instead of rectangles is left as an exercise.

[example:cantor] So that the reader is not under the impression that there are only very few measure zero sets and that these are simple, let us give an uncountable, compact, measure zero subset in $[0, 1]$. For any $x \in [0, 1]$ write the representation in ternary notation

$$x = \sum_{j=1}^{\infty} d_j 3^{-j}.$$

See §1.5 in volume I, in particular Exercise 1.5.4. Define the *Cantor set* C as

$$C := \left\{ x \in [0, 1] : x = \sum_{j=1}^{\infty} d_j 3^{-j}, \text{ where } d_j = 0 \text{ or } d_j = 2 \text{ for all } j \right\}.$$

That is, x is in C if it has a ternary expansion in only 0's and 2's. If x has two expansions, as long as one of them does not have any 1's, then x is in C . Define $C_0 := [0, 1]$ and

$$C_k := \left\{ x \in [0, 1] : x = \sum_{j=1}^{\infty} d_j 3^{-j}, \text{ where } d_j = 0 \text{ or } d_j = 2 \text{ for all } j = 1, 2, \dots, k \right\}.$$

Clearly,

$$C = \bigcap_{k=1}^{\infty} C_k.$$

We leave as an exercise to prove that:

1. Each C_k is a finite union of closed intervals. It is obtained by taking C_{k-1} , and from each closed interval removing the “middle third”.

2. Therefore, each C_k is closed.
3. Furthermore, $m^*(C_k) = 1 - \sum_{n=1}^k \frac{2^n}{3^{n+1}}$.
4. Hence, $m^*(C) = 0$.
5. The set C is in one to one correspondence with $[0, 1]$, in other words, uncountable.

See .

Images of null sets

Before we look at images of measure zero sets, let us see what a continuously differentiable function does to a ball.

[lemma:ballmapder] Suppose $U \subset \mathbb{R}^n$ is an open set, $B \subset U$ is an open or closed ball of radius at most r , $f: B \rightarrow \mathbb{R}^n$ is continuously differentiable and suppose $\|f'(x)\| \leq M$ for all $x \in B$. Then $f(B) \subset B'$, where B' is a ball of radius at most Mr .

Without loss of generality assume B is a closed ball. The ball B is convex, and hence via , that $\|f(x) - f(y)\| \leq M\|x - y\|$ for all x, y in B . In particular, suppose $B = C(y, r)$, then $f(B) \subset C(f(y), Mr)$.

The image of a measure zero set using a continuous map is not necessarily a measure zero set. However if we assume the mapping is continuously differentiable, then the mapping cannot “stretch” the set too much.

[prop:imagenull] Suppose $U \subset \mathbb{R}^n$ is an open set and $f: U \rightarrow \mathbb{R}^n$ is a continuously differentiable mapping. If $E \subset U$ is a measure zero set, then $f(E)$ is measure zero.

We leave the proof for a general measure zero set as an exercise, and we now prove the proposition for a compact measure zero set. Therefore let us suppose E is compact.

First let us replace U by a smaller open set to make $\|f'(x)\|$ bounded. At each point $x \in E$ pick an open ball $B(x, r_x)$ such that the closed ball $C(x, r_x) \subset U$. By compactness we only need to take finitely many points $x_1, x_2, \dots, x_q$ to still cover E . Define

$$U' := \bigcup_{j=1}^q B(x_j, r_{x_j}), \quad K := \bigcup_{j=1}^q C(x_j, r_{x_j}).$$

We have $E \subset U' \subset K \subset U$. The set K is compact. The function that takes x to $\|f'(x)\|$ is continuous, and therefore there exists an $M > 0$ such that $\|f'(x)\| \leq M$ for all $x \in K$. So without loss of generality we may replace U by U' and from now on suppose that $\|f'(x)\| \leq M$ for all $x \in U$.

At each point $x \in E$ pick a ball $B(x, \delta_x)$ of maximum radius so that $B(x, \delta_x) \subset U$. Let $\delta = \inf_{x \in E} \delta_x$. Take a sequence $\{x_j\} \subset E$ so that $\delta_{x_j} \rightarrow \delta$. As E is compact, we can pick the sequence to be convergent to some $y \in E$. Once $\|x_j - y\| < \frac{\delta_y}{2}$, then $\delta_{x_j} > \frac{\delta_y}{2}$ by the triangle inequality. Therefore $\delta > 0$.

Given $\epsilon > 0$, there exist balls $B_1, B_2, \dots, B_k$ of radii $r_1, r_2, \dots, r_k < \delta$ such that

$$E \subset B_1 \cup B_2 \cup \dots \cup B_k \quad \text{and} \quad \sum_{j=1}^k r_j^n < \epsilon.$$

Suppose $B'_1, B'_2, \dots, B'_k$ are the balls of radius $Mr_1, Mr_2, \dots, Mr_k$ from , such that $f(B_j) \subset B'_j$ for all j .

$$f(E) \subset f(B_1) \cup f(B_2) \cup \dots \cup f(B_k) \subset B'_1 \cup B'_2 \cup \dots \cup B'_k \quad \text{and} \quad \sum_{j=1}^k Mr_j^n < M\epsilon. \quad \text{qedhere}$$

Exercises

Finish the proof of , that is, show that you can use balls instead of rectangles.

If $A \subset B$, then $m^*(A) \leq m^*(B)$.

Suppose $X \subset \mathbb{R}^n$ is a set such that for every $\epsilon > 0$ there exists a set Y such that $X \subset Y$ and $m^*(Y) \leq \epsilon$. Prove that X is a measure zero set.

Show that if $R \subset \mathbb{R}^n$ is a closed rectangle, then $m^*(R) = V(R)$.

The closure of a measure zero set can be quite large. Find an example set $S \subset \mathbb{R}^n$ that is of measure zero, but whose closure $\bar{S} = \mathbb{R}^n$.

Prove the general case of without using compactness:

a) Mimic the proof to first prove that the proposition holds if E is *relatively compact*; a set $E \subset U$ is relatively compact if the closure of E in the subspace topology on U is compact, or in other words if there exists a compact set K with $K \subset U$ and $E \subset K$. Hint: The bound on the size of the derivative still holds, but you need to use countably many balls in the second part of the proof. Be careful as the closure of E need no longer be measure zero.

b) Now prove it for any null set E .

$$\{x \in U : d(x, y) \geq m\}$$

Hint: First show that $\{x \in U : d(x, y) \geq m\}$ is a compact set for any $m > 0$.

Let $U \subset \mathbb{R}^n$ be an open set and let $f: U \rightarrow \mathbb{R}$ be a continuously differentiable function. Let $G := \{(x, y) \in U \times \mathbb{R} : y = f(x)\}$ be the graph of f . Show that G is of measure zero.

Given a closed rectangle $R \subset \mathbb{R}^n$, show that for any $\epsilon > 0$ there exists a number $s > 0$ and finitely many open cubes $C_1, C_2, \dots, C_k$ of side s such that $R \subset C_1 \cup C_2 \cup \dots \cup C_k$ and

$$\sum_{j=1}^k V(C_j) \leq V(R) + \epsilon.$$

Show that there exists a number $k = k(n, r, \delta)$ depending only on n, r and δ such the following holds. Given $B(x, r) \subset \mathbb{R}^n$ and $\delta > 0$, there exist k open balls $B_1, B_2, \dots, B_k$ of radius at most δ such that $B(x, r) \subset B_1 \cup B_2 \cup \dots \cup B_k$. Note that you can find k that really only depends on n and the ratio $\frac{r}{\delta}$.

Prove the statements of . That is, prove:

a) Each C_k is a finite union of closed intervals, and so C is closed.

b) $m^*(C_k) = 1 - \sum_{n=1}^k \frac{2^n}{3^{n+1}}.$

c) $m^*(C) = 0.$

d) The set C is in one to one correspondence with $[0, 1]$.

The set of Riemann integrable functions

Note: 1 lecture

Oscillation and continuity

Let $S \subset \mathbb{R}^n$ be a set and $f: S \rightarrow \mathbb{R}$ a function. Instead of just saying that f is or is not continuous at a point $x \in S$, we need to be able to quantify how discontinuous f is at a function is at x . For any $\delta > 0$ define the oscillation of f on the δ -ball in subset topology that is $B_S(x, \delta) = B_{\mathbb{R}^n}(x, \delta) \cap S$ as

$$o(f, x, \delta) := \sup_{y \in B_S(x, \delta)} f(y) - \inf_{y \in B_S(x, \delta)} f(y) = \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)).$$

That is, $o(f, x, \delta)$ is the length of the smallest interval that contains the image $f(B_S(x, \delta))$. Clearly $o(f, x, \delta) \geq 0$ and notice $o(f, x, \delta) \leq o(f, x, \delta')$ whenever $\delta < \delta'$. Therefore, the limit as $\delta \rightarrow 0$ from the right exists and we define the *oscillation* of a function f at x as

$$o(f, x) := \lim_{\delta \rightarrow 0^+} o(f, x, \delta) = \inf_{\delta > 0} o(f, x, \delta).$$

$f: S \rightarrow \mathbb{R}$ is continuous at $x \in S$ if and only if $o(f, x) = 0$.

First suppose that f is continuous at $x \in S$. Then given any $\epsilon > 0$, there exists a $\delta > 0$ such that for $y \in B_S(x, \delta)$ we have $|f(x) - f(y)| < \epsilon$. Therefore if $y_1, y_2 \in B_S(x, \delta)$, then

$$f(y_1) - f(y_2) = f(y_1) - f(x) - (f(y_2) - f(x)) < \epsilon + \epsilon = 2\epsilon.$$

We take the supremum over y_1 and y_2

$$o(f, x, \delta) = \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) \leq 2\epsilon.$$

Hence, $o(x, f) = 0$.

On the other hand suppose that $o(x, f) = 0$. Given any $\epsilon > 0$, find a $\delta > 0$ such that $o(f, x, \delta) < \epsilon$. If $y \in B_S(x, \delta)$, then

$$|f(x) - f(y)| \leq \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) = o(f, x, \delta) < \epsilon. \text{ \textcolor{red}{qedhere}}$$

[prop:secclosed] Let $S \subset \mathbb{R}^n$ be closed, $f: S \rightarrow \mathbb{R}$, and $\epsilon > 0$. The set $\{x \in S: o(f, x) \geq \epsilon\}$ is closed.

Equivalently we want to show that $G = \{x \in S: o(f, x) < \epsilon\}$ is open in the subset topology. As $\inf_{\delta > 0} o(f, x, \delta) < \epsilon$, find a $\delta > 0$ such that

$$o(f, x, \delta) < \epsilon$$

Take any $\xi \in B_S(x, \textcolor{red}{\frac{1}{2}\delta})$. Notice that $B_S(\xi, \textcolor{red}{\frac{1}{2}\delta}) \subset B_S(x, \delta)$. Therefore,

$$o(f, \xi, \textcolor{red}{\frac{1}{2}\delta}) = \sup_{y_1, y_2 \in B_S(\xi, \textcolor{red}{\frac{1}{2}\delta})} (f(y_1) - f(y_2)) \leq \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) = o(f, x, \delta) < \epsilon.$$

So $o(f, \xi) < \epsilon$ as well. As this is true for all $\xi \in B_S(x, \textcolor{red}{\frac{1}{2}\delta})$ we get that G is open in the subset topology and $S \setminus G$ is closed as is claimed.

The set of Riemann integrable functions

We have seen that continuous functions are Riemann integrable, but we also know that certain kinds of discontinuities are allowed. It turns out that as long as the discontinuities happen on a set of measure zero, the function is integrable and vice versa.

Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. Then f is Riemann integrable if and only if the set of discontinuities of f is of measure zero (a null set).

Let $S \subset R$ be the set of discontinuities of f . That is $S = \{x \in R: o(f, x) > 0\}$. The trick to this proof is to isolate the bad set into a small set of subrectangles of a partition. There are only finitely many subrectangles of a partition, so we will wish to use compactness. If S is closed, then it would be compact and we could cover it by small rectangles as it is of measure zero. Unfortunately, in general S is not closed so we need to work a little harder.

For every $\epsilon > 0$, define

$$S_\epsilon := \{x \in R: o(f, x) \geq \epsilon\}.$$

By S_ϵ is closed and as it is a subset of R , which is bounded, S_ϵ is compact. Furthermore, $S_\epsilon \subset S$ and S is of measure zero. Via there are finitely many open rectangles $O_1, O_2, \dots, O_k$ that cover S_ϵ and $\sum V(O_j) < \epsilon$.

The set $T = R \setminus (O_1 \cup \dots \cup O_k)$ is closed, bounded, and therefore compact. Furthermore for $x \in T$, we have $o(f, x) < \epsilon$. Hence for each $x \in T$, there exists a small closed rectangle T_x with x in the interior of T_x , such that

$$\sup_{y \in T_x} f(y) - \inf_{y \in T_x} f(y) < 2\epsilon.$$

The interiors of the rectangles T_x cover T . As T is compact there exist finitely many such rectangles $T_1, T_2, \dots, T_m$ that cover T .

Take the rectangles $T_1, T_2, \dots, T_m$ and $O_1, O_2, \dots, O_k$ and construct a partition out of their endpoints. That is construct a partition P of R with subrectangles $R_1, R_2, \dots, R_p$ such that every R_j is contained in T_ℓ for some ℓ or the closure of O_ℓ for some ℓ . Order the rectangles so that $R_1, R_2, \dots, R_q$ are those that are contained in some T_ℓ , and $R_{q+1}, R_{q+2}, \dots, R_p$ are the rest. In particular,

$$\sum_{j=1}^q V(R_j) \leq V(R) \quad \text{and} \quad \sum_{j=q+1}^p V(R_j) \leq \epsilon.$$

Let m_j and M_j be the inf and sup of f over R_j as before. If $R_j \subset T_\ell$ for some ℓ , then $(M_j - m_j) < 2\epsilon$. Let $B \in \mathbb{R}$ be such that $|f(x)| \leq B$ for all $x \in R$, so $(M_j - m_j) < 2B$ over all rectangles. Then

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{j=1}^p (M_j - m_j) V(R_j) \\ &= \left(\sum_{j=1}^q (M_j - m_j) V(R_j) \right) + \left(\sum_{j=q+1}^p (M_j - m_j) V(R_j) \right) \\ &\leq \left(\sum_{j=1}^q 2\epsilon V(R_j) \right) + \left(\sum_{j=q+1}^p 2BV(R_j) \right) \\ &\leq 2\epsilon V(R) + 2B\epsilon = \epsilon(2V(R) + 2B). \end{aligned}$$

Clearly, we can make the right hand side as small as we want and hence f is integrable.

For the other direction, suppose f is Riemann integrable over R . Let S be the set of discontinuities again and now let

$$S_k := \{x \in R : \omega(f, x) \geq \frac{1}{k}\}.$$

Fix a $k \in \mathbb{N}$. Given an $\epsilon > 0$, find a partition P with subrectangles $R_1, R_2, \dots, R_p$ such that

$$U(P, f) - L(P, f) = \sum_{j=1}^p (M_j - m_j) V(R_j) < \epsilon$$

Suppose $R_1, R_2, \dots, R_p$ are ordered so that the interiors of $R_1, R_2, \dots, R_q$ intersect S_k , while the interiors of $R_{q+1}, R_{q+2}, \dots, R_p$ are disjoint from S_k . If $x \in R_j \cap S_k$ and x is in the interior of R_j so sufficiently small balls are completely inside R_j , then by definition of S_k we have $M_j - m_j \geq \frac{1}{k}$. Then

$$\epsilon > \sum_{j=1}^p (M_j - m_j) V(R_j) \geq \sum_{j=1}^q (M_j - m_j) V(R_j) \geq \frac{1}{k} \sum_{j=1}^q V(R_j)$$

In other words $\sum_{j=1}^q V(R_j) < k\epsilon$. Let G be the set of all boundaries of all the subrectangles of P . The set G is of measure zero (see).

Let R_j° denote the interior of R_j , then

$$S_k \subset R_1^\circ \cup R_2^\circ \cup \dots \cup R_q^\circ \cup G.$$

As G can be covered by open rectangles arbitrarily small volume, S_k must be of measure zero. As

$$S = \bigcup_{k=1}^{\infty} S_k$$

and a countable union of measure zero sets is of measure zero, S is of measure zero.

Exercises

Suppose $f: (a, b) \times (c, d) \rightarrow \mathbb{R}$ is a bounded continuous function. Show that the integral of f over $R = [a, b] \times [c, d]$ makes sense and is uniquely defined. That is, set f to be anything on the boundary of R and compute the integral.

Suppose $R \subset \mathbb{R}^n$ is a closed rectangle. Show that $\mathcal{R}(R)$, the set of Riemann integrable functions, is an algebra. That is, show that if $f, g \in \mathcal{R}(R)$ and $a \in \mathbb{R}$, then $af \in \mathcal{R}(R)$, $f + g \in \mathcal{R}(R)$ and $fg \in \mathcal{R}(R)$.

Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f: R \rightarrow \mathbb{R}$ is a bounded function which is zero except on a closed set $E \subset R$ of measure zero. Show that $\int_R f$ exists and compute it.

Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f: R \rightarrow \mathbb{R}$ and $g: R \rightarrow \mathbb{R}$ are two Riemann integrable functions. Suppose $f = g$ except for a closed set $E \subset R$ of measure zero. Show that $\int_R f = \int_R g$.

Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f: R \rightarrow \mathbb{R}$ is a bounded function.

a) Suppose there exists a closed set $E \subset R$ of measure zero such that $f|_{R \setminus E}$ is continuous. Then $f \in \mathcal{R}(R)$.

b) Find an example where $E \subset R$ is a set of measure zero (but not closed) such that $f|_{R \setminus E}$ is continuous and $f \notin \mathcal{R}(R)$.

Jordan measurable sets

Note: 1 lecture

Volume and Jordan measurable sets

Given a bounded set $S \subset \mathbb{R}^n$ its *characteristic function* or *indicator function* is

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

A bounded set S is *Jordan measurable* if for some closed rectangle R such that $S \subset R$, the function χ_S is in $\mathcal{R}(R)$. Take two closed rectangles R and R' with $S \subset R$ and $S \subset R'$, then $R \cap R'$ is a closed rectangle also containing S . By and, $\chi_S \in \mathcal{R}(R \cap R')$ and so $\chi_S \in \mathcal{R}(R')$. Thus

$$\int_R \chi_S = \int_{R'} \chi_S = \int_{R \cap R'} \chi_S.$$

We define the *n-dimensional volume* of the bounded Jordan measurable set S as

$$V(S) := \int_R \chi_S,$$

where R is any closed rectangle containing S .

A bounded set $S \subset \mathbb{R}^n$ is Jordan measurable if and only if the boundary ∂S is a measure zero set.

Suppose R is a closed rectangle such that S is contained in the interior of R . If $x \in \partial S$, then for every $\delta > 0$, the sets $S \cap B(x, \delta)$ (where χ_S is 1) and the sets $(R \setminus S) \cap B(x, \delta)$ (where χ_S is 0) are both nonempty. So χ_S is not continuous at x . If x is either in the

interior of S or in the complement of the closure $\bar{S}$, then χ_S is either identically 1 or identically 0 in a whole neighborhood of x and hence χ_S is continuous at x . Therefore, the set of discontinuities of χ_S is precisely the boundary ∂S . The proposition then follows.

[prop:jordanmeas] Suppose S and T are bounded Jordan measurable sets. Then

1. The closure $\bar{S}$ is Jordan measurable.
2. The interior S° is Jordan measurable.

3. $S \cup T$ is Jordan measurable.
4. $S \cap T$ is Jordan measurable.
5. $S \setminus T$ is Jordan measurable.

The proof of the proposition is left as an exercise. Next, we find that the volume that we defined above coincides with the outer measure we defined above.

If $S \subset \mathbb{R}^n$ is Jordan measurable, then $V(S) = m^*(S)$.

Given $\epsilon > 0$, let R be a closed rectangle that contains S . Let P be a partition of R such that

$$U(P, \chi_S) \leq \int_R \chi_S + \epsilon = V(S) + \epsilon \quad \text{and} \quad L(P, \chi_S) \geq \int_R \chi_S - \epsilon = V(S) - \epsilon.$$

Let $R_1, \dots, R_k$ be all the subrectangles of P such that χ_S is not identically zero on each R_j . That is, there is some point $x \in R_j$ such that $x \in S$. Let O_j be an open rectangle such that $R_j \subset O_j$ and $V(O_j) < V(R_j) + \frac{\epsilon}{k}$. Notice that $S \subset \bigcup_j O_j$. Then

$$U(P, \chi_S) = \sum_{j=1}^k V(R_j) > \left(\sum_{j=1}^k V(O_j) \right) - \epsilon \geq m^*(S) - \epsilon.$$

As $U(P, \chi_S) \leq V(S) + \epsilon$, then $m^*(S) - \epsilon \leq V(S) + \epsilon$, or in other words $m^*(S) \leq V(S)$.

Let $R'_1, \dots, R'_\ell$ be all the subrectangles of P such that χ_S is identically one on each R'_j . In other words, these are the subrectangles contained in S . The interiors of the subrectangles R'_j are disjoint and $V(R'_j) = V(R'_j \cap S)$. It is easy to see from definition that

$$m^*\left(\bigcup_{j=1}^{\ell} R'_j\right) = \sum_{j=1}^{\ell} V(R'_j).$$

Hence

$$m^*(S) \geq m^*\left(\bigcup_{j=1}^{\ell} R'_j\right) \geq m^*\left(\bigcup_{j=1}^{\ell} R'_j \cap S\right) = \sum_{j=1}^{\ell} V(R'_j \cap S) = \sum_{j=1}^{\ell} V(R'_j) = L(P, f) \geq V(S) - \epsilon.$$

Therefore $m^*(S) \geq V(S)$ as well.

Integration over Jordan measurable sets

In one variable there is really only one type of reasonable set to integrate over: an interval. In several variables we have many common types of sets we might want to integrate over and these are not described so easily.

Let $S \subset \mathbb{R}^n$ be a bounded Jordan measurable set. A bounded function $f: S \rightarrow \mathbb{R}$ is said to be Riemann integrable on S , or $f \in \mathcal{R}(S)$, if for a closed rectangle R such that $S \subset R$, the function $\tilde{f}: R \rightarrow \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

is in $\mathcal{R}(R)$. In this case we write

$$\int_S f := \int_R \tilde{f}.$$

When f is defined on a larger set and we wish to integrate over S , then we apply the definition to the restriction $f|_S$. In particular, if $f: R \rightarrow \mathbb{R}$ for a closed rectangle R , and $S \subset R$ is a Jordan measurable subset, then

$$\int_S f = \int_R f \chi_S.$$

If $S \subset \mathbb{R}^n$ is a Jordan measurable set and $f: S \rightarrow \mathbb{R}$ is a bounded continuous function, then f is integrable on S .

Define the function $\tilde{f}$ as above for some closed rectangle R with $S \subset R$. If $x \in R \setminus S$, then $\tilde{f}$ is identically zero in a neighborhood of x . Similarly if x is in the interior of S , then $\tilde{f} = f$ on a neighborhood of x and f is continuous at x . Therefore, $\tilde{f}$ is only ever possibly discontinuous at ∂S , which is a set of measure zero, and we are finished.

Images of Jordan measurable subsets

Finally, images of Jordan measurable sets are Jordan measurable under nice enough mappings. For simplicity, let us assume that the Jacobian never vanishes.

Suppose $S \subset \mathbb{R}^n$ is a closed bounded Jordan measurable set, and $S \subset U$ for an open set $U \subset \mathbb{R}^n$. Suppose $g: U \rightarrow \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that J_g is never zero on S . Then $g(S)$ is Jordan measurable.

Let $T = g(S)$. We claim that the boundary ∂T is contained in the set $g(\partial S)$. Suppose the claim is proved. As S is Jordan measurable, then ∂S is measure zero. Then $g(\partial S)$ is measure zero by . As $\partial T \subset g(\partial S)$, then T is Jordan measurable.

It is therefore left to prove the claim. First, S is closed and bounded and hence compact. By Lemma 7.5.4 from volume I, $T = g(S)$ is also compact and therefore closed. In particular, $\partial T \subset T$. Suppose $y \in \partial T$, then there must exist an $x \in S$ such that $g(x) = y$, and by hypothesis $J_g(x) \neq 0$.

We now use the inverse function theorem . We find a neighborhood $V \subset U$ of x and an open set W such that the restriction $f|_V$ is a one-to-one and onto function from V to W with a continuously differentiable inverse. In particular, $g(x) = y \in W$. As $y \in \partial T$, there exists a sequence $\{y_k\}$ in W with $\lim y_k = y$ and $y_k \notin T$. As $g|_V$ is invertible and in particular has a continuous inverse, there exists a sequence $\{x_k\}$ in V such that $g(x_k) = y_k$ and $\lim x_k = x$. Since $y_k \notin T = g(S)$, clearly $x_k \notin S$. Since $x \in S$, we conclude that $x \in \partial S$. The claim is proved, $\partial T \subset g(\partial S)$.

Exercises

Prove .

Prove that a bounded convex set is Jordan measurable. Hint: induction on dimension.

[exercise:intovertypelset] Let $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ be continuous functions and such that for all $x \in (a, b)$, $f(x) < g(x)$. Let

$$U := \{(x, y) \in \mathbb{R}^2 : a < x < b \text{ and } f(x) < y < g(x)\}.$$

- Show that U is Jordan measurable.
- If $f: U \rightarrow \mathbb{R}$ is Riemann integrable on U , then

$$\int_U f = \int_a^b \int_{g(x)}^{f(x)} f(x, y) dy dx.$$

Let us construct an example of a non-Jordan measurable open set. For simplicity we work first in one dimension. Let $\{r_j\}$ be an enumeration of all rational numbers in $(0, 1)$. Let (a_j, b_j) be open intervals such that $(a_j, b_j) \subset (0, 1)$ for all j , $r_j \in (a_j, b_j)$, and $\sum_{j=1}^{\infty} (b_j - a_j) < \frac{1}{2}$. Now let $U := \bigcup_{j=1}^{\infty} (a_j, b_j)$. Show that

- The open intervals (a_j, b_j) as above actually exist.
- $\partial U = [0, 1] \setminus U$.
- ∂U is not of measure zero, and therefore U is not Jordan measurable.
- Show that $W := ((0, 1) \times (0, 2)) \setminus (U \times [0, 1]) \subset \mathbb{R}^2$ is a connected bounded open set in $\mathbb{R}^2$ that is not Jordan measurable.

Green's theorem

Note: 1 lecture

One of the most important theorems of analysis in several variables is the so-called generalized Stokes' theorem, a generalization of the fundamental theorem of calculus. Perhaps the most often used version is the version in two dimensions, called Green's theorem, which we prove here.

Let $U \subset \mathbb{R}^2$ be a bounded connected open set. Suppose the boundary ∂U is a finite union of (the images of) simple piecewise smooth paths such that near each point $p \in \partial U$ every neighborhood V of p contains points of $\mathbb{R}^2 \setminus U$. Then U is called a *bounded domain with piecewise smooth boundary* in $\mathbb{R}^2$.

The condition about points outside the closure means that locally ∂U separates $\mathbb{R}^2$ into “inside” and “outside”. The condition prevents ∂U from being just a “cut” inside U . Therefore as we travel along the path in a certain orientation, there is a well defined left and a right, and either it is U on the left and the complement of U on the right, or vice-versa. Thus by orientation on U we mean the direction along which we travel along the paths. It is easy to switch orientation if needed by reparametrizing the path.

If $U \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary, let ∂U be oriented and $\gamma: [a, b] \rightarrow \mathbb{R}^2$ is a parametrization of ∂U giving the orientation. Write $\gamma(t) = (x(t), y(t))$. If the vector $n(t) := (-y'(t), x'(t))$ points into the domain, that is, $\epsilon n(t) + \gamma(t)$ is in U for all small enough $\epsilon > 0$, then ∂U is *positively oriented*. Otherwise it is *negatively oriented*.

The vector $n(t)$ turns $\gamma'(t)$ counterclockwise by 90° , that is to the left. A boundary is positively oriented, if when we travel along the boundary in the direction of its orientation, the domain is “on our left”. For example, if U is a bounded domain with “no holes”, that is ∂U is connected, then the positive orientation means we are travelling counterclockwise around ∂U . If we do have “holes”, then we travel around them clockwise.

Let $U \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary, then U is Jordan measurable.

We need that ∂U is of measure zero. As ∂U is a finite union of simple piecewise smooth paths, which themselves are finite unions of smooth paths we need only show that a smooth path is of measure zero in $\mathbb{R}^2$.

Let $\gamma: [a, b] \rightarrow \mathbb{R}^2$ be a smooth path. It is enough to show that $\gamma((a, b))$ is of measure zero, as adding two points, that is the points $\gamma(a)$ and $\gamma(b)$, to a measure zero set still results in a measure zero set. Define

$$f: (a, b) \times (-1, 1) \rightarrow \mathbb{R}^2, \quad \text{as} \quad f(x, y) := \gamma(x).$$

The set $(a, b) \times \{0\}$ is of measure zero in $\mathbb{R}^2$ and $\gamma((a, b)) = f((a, b) \times \{0\})$. Hence by $\gamma((a, b))$ is measure zero in $\mathbb{R}^2$ and so $\gamma([a, b])$ is also measure zero, and so finally ∂U is also measure zero.

Suppose $U \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary with the boundary positively oriented. Suppose P and Q are continuously differentiable functions defined on some open set that contains the closure $\bar{U}$. Then

$$\int_{\partial U} P dx + Q dy = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

We stated Green’s theorem in general, although we will only prove a special version of it. That is, we will only prove it for a special kind of domain. The general version follows from the special case by application of further geometry, and cutting up the general domain into smaller domains on which to apply the special case. We will not prove the general case.

Let $U \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary. We say U is of *type I* if there exist numbers $a < b$, and continuous functions $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$, such that

$$U := \{(x, y) \in \mathbb{R}^2 : a < x < b \text{ and } f(x) < y < g(x)\}.$$

Similarly, U is of *type II* if there exist numbers $c < d$, and continuous functions $h: [c, d] \rightarrow \mathbb{R}$ and $k: [c, d] \rightarrow \mathbb{R}$, such that

$$U := \{(x, y) \in \mathbb{R}^2 : c < y < d \text{ and } h(y) < x < k(y)\}.$$

Finally, $U \subset \mathbb{R}^2$ is of *type III* if it is both of type I and type II.

We will only prove Green’s theorem for type III domains.

Let f, g, h, k be the functions defined above. By U is Jordan measurable and as U is of type I, then

$$\begin{aligned}\int_U \left(-\frac{\partial P}{\partial y} \right) &= \int_a^b \int_{g(x)}^{f(x)} \left(-\frac{\partial P}{\partial y}(x, y) \right) dy dx \\ &= \int_a^b \left(-P(x, f(x)) + P(x, g(x)) \right) dx \\ &= \int_a^b P(x, g(x)) dx - \int_a^b P(x, f(x)) dx.\end{aligned}$$

Now we wish to integrate $P dx$ along the boundary. The one-form $P dx$ integrates to zero when integrating along the straight vertical lines in the boundary. Therefore it only is integrated along the top and along the bottom. As a parameter, x runs from left to right. If we use the parametrizations that take x to $(x, f(x))$ and to $(x, g(x))$ we recognize path integrals above. However the second path integral is in the wrong direction, the top should be going right to left, and so we must switch orientation.

$$\int_{\partial U} P dx = \int_a^b P(x, g(x)) dx + \int_b^a P(x, f(x)) dx = \int_U \left(-\frac{\partial P}{\partial y} \right).$$

Similarly, U is also of type II. The form $Q dy$ integrates to zero along horizontal lines. So

$$\int_U \frac{\partial Q}{\partial x} = \int_c^d \int_{k(y)}^{h(y)} \frac{\partial Q}{\partial x}(x, y) dx dy = \int_a^b \left(Q(y, h(y)) - Q(y, k(y)) \right) dy = \int_{\partial U} Q dy.$$

Putting the two together we obtain

$$\int_{\partial U} P dx + Q dy = \int_{\partial U} P dx + \int_{\partial U} Q dy = \int_U \left(-\frac{\partial P}{\partial y} \right) + \int_U \frac{\partial Q}{\partial x} = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Let us illustrate the usefulness of Green's theorem on a fundamental result about harmonic functions.

Suppose $U \subset \mathbb{R}^2$ is an open set and $f: U \rightarrow \mathbb{R}$ is harmonic, that is, f is twice continuously differentiable and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. We will prove one of the most fundamental properties of Harmonic functions.

Let $D_r = B(p, r)$ be closed disc such that its closure $C(p, r) \subset U$. Write $p = (x_0, y_0)$. We orient ∂D_r positively. See . Then

$$\begin{aligned}0 &= \frac{1}{2\pi r} \int_{D_r} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \\ &= \frac{1}{2\pi r} \int_{\partial D_r} -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy \\ &= \frac{1}{2\pi r} \int_0^{2\pi} \left(-\frac{\partial f}{\partial y}(x_0 + r\cos(t), y_0 + r\sin(t))(-r\sin(t)) \right. \\ &\quad \left. + \frac{\partial f}{\partial x}(x_0 + r\cos(t), y_0 + r\sin(t))r\cos(t) \right) dt \\ &= \frac{d}{dr} \left[\frac{1}{2\pi} \int_0^{2\pi} f(x_0 + r\cos(t), y_0 + r\sin(t)) dt \right].\end{aligned}$$

Let $g(r) := \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + r\cos(t), y_0 + r\sin(t)) dt$. Then $g'(r) = 0$ for all $r > 0$. The function is constant for $r > 0$ and continuous at $r = 0$ (exercise). Therefore $g(0) = g(r)$ for all $r > 0$. Therefore,

$$g(r) = g(0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + 0\cos(t), y_0 + 0\sin(t)) dt = f(x_0, y_0).$$

We proved the *mean value property* of harmonic functions:

$$f(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + r\cos(t), y_0 + r\sin(t)) dt = \frac{1}{2\pi r} \int_{\partial D_r} f ds.$$

That is, the value at $p = (x_0, y_0)$ is the average over a circle of any radius r centered at (x_0, y_0) .

Exercises

[green:balltype3orient] Prove that a disc $B(p, r) \subset \mathbb{R}^2$ is a type III domain, and prove that the orientation given by the parametrization $\gamma(t) = (x_0 + r\cos(t), y_0 + r\sin(t))$ where $p = (x_0, y_0)$ is the positive orientation of the boundary $\partial B(p, r)$.

Prove that any bounded domain with piecewise smooth boundary that is convex is a type III domain.

Suppose $V \subset \mathbb{R}^2$ is a domain with piecewise smooth boundary that is a type III domain and suppose that $U \subset \mathbb{R}^2$ is a domain such that $V \subset U$. Suppose $f: U \rightarrow \mathbb{R}$ is a twice continuously differentiable function. Prove that $\int_{\partial V} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$.

For a disc $B(p, r) \subset \mathbb{R}^2$, orient the boundary $\partial B(p, r)$ positively:

- Compute $\int_{\partial B(p, r)} -y dx$.
- Compute $\int_{\partial B(p, r)} x dy$.
- Compute $\int_{\partial B(p, r)} \frac{-y}{2} dy + \frac{x}{2} dx$.

Using Green's theorem show that the area of a triangle with vertices (x_1, y_1) , (x_2, y_2) , (x_3, y_3) is $\frac{1}{2} |x_1 y_2 + x_2 y_3 + x_3 y_1 - y_1 x_2 - y_2 x_3 - y_3 x_1|$. Hint: see previous exercise.

Using the mean value property prove the *maximum principle* for harmonic functions: Suppose $U \subset \mathbb{R}^2$ is an connected open set and $f: U \rightarrow \mathbb{R}$ is harmonic. Prove that if f attains a maximum at $p \in U$, then f is constant.

Let $f(x, y) := \ln \sqrt{x^2 + y^2}$.

- Show f is harmonic where defined.
- Show $\lim_{(x, y) \rightarrow 0} f(x, y) = -\infty$.
- Using a circle C_r of radius r around the origin, compute $\frac{1}{2\pi r} \int_{C_r} f ds$. What happens as $r \rightarrow 0$?
- Why can't you use Green's theorem?

-
- Subscripts are used for many purposes, so sometimes we may have several vectors that may also be identified by subscript, such as a finite or infinite sequence of vectors $y_1, y_2, \dots$ ↩
 - If you want a very funky vector space over a different field, $\mathbb{R}$ itself is a vector space over the rational numbers. ↩
 - The matrix from representing $f'(x)$ is sometimes called the *Jacobian matrix*. ↩
 - The word "smooth" is used sometimes for continuously differentiable and sometimes for infinitely differentiable functions in the literature. ↩
 - Normally only a continuous path is used in this definition, but for open sets the two definitions are equivalent. See the exercises. ↩
-

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CHAPTER OVERVIEW

11: Multivariable Integral

Topic hierarchy

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11.1: Riemann integral over Rectangles

Riemann integral over rectangles

Note: FIXME1 lectures

As in chapter FIXME, we define the Riemann integral using the Darboux upper and lower integrals. The ideas in this section are very similar to integration in one dimension. The complication is mostly notational.

Rectangles and partitions

Let $(a^1, a^2, \dots, a^n)$ and $(b^1, b^2, \dots, b^n)$ be such that $a^k \leq b^k$ for all k . A set of the form $[a^1, b^1] \times [a^2, b^2] \times \dots \times [a^n, b^n]$ is called a *closed rectangle*. If $a^k < b^k$, then a set of the form $(a^1, b^1) \times (a^2, b^2) \times \dots \times (a^n, b^n)$ is called an *open rectangle*.

For an open or closed rectangle $R := [a^1, b^1] \times [a^2, b^2] \times \dots \times [a^n, b^n] \subset \mathbb{R}^n$ or $R := (a^1, b^1) \times (a^2, b^2) \times \dots \times (a^n, b^n) \subset \mathbb{R}^n$, we define the *n-dimensional volume* by

$$V(R) := (b^1 - a^1)(b^2 - a^2) \dots (b^n - a^n). \quad (11.1.1)$$

A *partition* P of the closed rectangle $R = [a^1, b^1] \times [a^2, b^2] \times \dots \times [a^n, b^n]$ is a finite set of partitions $P^1, P^2, \dots, P^n$ of the intervals $[a^1, b^1], [a^2, b^2], \dots, [a^n, b^n]$. That is, for every k there is an integer ℓ_k and the finite set of numbers $P^k = \{x_0^k, x_1^k, x_2^k, \dots, x_{\ell_k}^k\}$ such that

$$a^k = x_0^k < x_1^k < x_2^k < \dots < x_{\ell_k-1}^k < x_{\ell_k}^k = b^k. \quad (11.1.2)$$

Picking a set of n integers $j_1, j_2, \dots, j_n$ where $j_k \in \{1, 2, \dots, \ell_k\}$ we get the *subrectangle*

$$[x_{j_1-1}^1, x_{j_1}^1] \times [x_{j_2-1}^2, x_{j_2}^2] \times \dots \times [x_{j_n-1}^n, x_{j_n}^n]. \quad (11.1.3)$$

For simplicity, we order the subrectangles somehow and we say $\{R_1, R_2, \dots, R_N\}$ are the subrectangles corresponding to the partition P of R . In other words we subdivide the original rectangle into many smaller subrectangles. It is not difficult to see that these subrectangles cover our original R , and their volume sums to that of R . That is

$$R = \bigcup_{j=1}^N R_j, \quad \text{and} \quad V(R) = \sum_{j=1}^N V(R_j). \quad (11.1.4)$$

When

$$R_k = [x_{j_1-1}^1, x_{j_1}^1] \times [x_{j_2-1}^2, x_{j_2}^2] \times \dots \times [x_{j_n-1}^n, x_{j_n}^n] \quad (11.1.5)$$

then

$$V(R_k) = \Delta x_{j_1}^1 \Delta x_{j_2}^2 \dots \Delta x_{j_n}^n = (x_{j_1}^1 - x_{j_1-1}^1)(x_{j_2}^2 - x_{j_2-1}^2) \dots (x_{j_n}^n - x_{j_n-1}^n). \quad (11.1.6)$$

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $f: R \rightarrow \mathbb{R}$ be a bounded function. Let P be a partition of $[a, b]$. Let R_i be a subrectangle corresponding to P that has N subrectangles. Define

$$\begin{aligned} m_i &:= \inf\{f(x) : x \in R_i\}, \\ M_i &:= \sup\{f(x) : x \in R_i\}, \\ L(P, f) &:= \sum_{i=1}^N m_i V(R_i), \\ U(P, f) &:= \sum_{i=1}^N M_i V(R_i). \end{aligned}$$

We call $L(P, f)$ the *lower Darboux sum* and $U(P, f)$ the *upper Darboux sum*.

We start proving facts about the Darboux sums analogous to the one-variable results.

[mv:sumulbound:prop] Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f: R \rightarrow \mathbb{R}$ is a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. For any partition P of R we have

$$mV(R) \leq L(P, f) \leq U(P, f) \leq MV(R). \quad (11.1.7)$$

Let P be a partition. Then note that $m \leq m_i$ for all i and $M_i \leq M$ for all i . Also $m_i \leq M_i$ for all i . Finally $\sum_{i=1}^N V(R_i) = V(R)$. Therefore,

$$\begin{aligned} mV(R) &= m \left(\sum_{i=1}^N V(R_i) \right) = \sum_{i=1}^N mV(R_i) \leq \sum_{i=1}^N m_i V(R_i) \leq \\ &\leq \sum_{i=1}^N M_i V(R_i) \leq \sum_{i=1}^N M V(R_i) = M \left(\sum_{i=1}^N V(R_i) \right) = M V(R). \end{aligned} \quad \text{\textcolor{red}{qedhere}}$$

Upper and lower integrals

By the set of upper and lower Darboux sums are bounded sets and we can take their infima and suprema. As before, we now make the following definition.

If $f: R \rightarrow \mathbb{R}$ is a bounded function on a closed rectangle $R \subset \mathbb{R}^n$. Define

$$\int_R f := \sup\{L(P, f) : P \text{ a partition of } R\}, \quad \overline{\int}_R f := \inf\{U(P, f) : P \text{ a partition of } R\}. \quad (11.1.8)$$

We call $\int$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral*.

As in one dimension we have refinements of partitions.

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $P = \{P^1, P^2, \dots, P^n\}$ and $\tilde{P} = \{\tilde{P}^1, \tilde{P}^2, \dots, \tilde{P}^n\}$ be partitions of R . We say $\tilde{P}$ a *refinement* of P if as sets $P^k \subset \tilde{P}^k$ for all $k = 1, 2, \dots, n$.

It is not difficult to see that if $\tilde{P}$ is a refinement of P , then subrectangles of P are unions of subrectangles of $\tilde{P}$. Simply put, in a refinement we took the subrectangles of P and we cut them into smaller subrectangles.

[mv:prop:refinement] Suppose $R \subset \mathbb{R}^n$ is a closed rectangle, P is a partition of R and $\tilde{P}$ is a refinement of P . If $f: R \rightarrow \mathbb{R}$ be a bounded function, then

$$L(P, f) \leq L(\tilde{P}, f) \quad \text{and} \quad U(\tilde{P}, f) \leq U(P, f). \quad (11.1.9)$$

Let $R_1, R_2, \dots, R_N$ be the subrectangles of P and $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_M$ be the subrectangles of $\tilde{P}$. Let I_k be the set of indices j such that $\tilde{R}_j \subset R_k$. We notice that

$$R_k = \bigcup_{j \in I_k} \tilde{R}_j, \quad V(R_k) = \sum_{j \in I_k} V(\tilde{R}_j). \quad (11.1.10)$$

Let $m_j := \inf\{f(x) : x \in R_j\}$, and $\tilde{m}_j := \inf\{f(x) : x \in \tilde{R}_j\}$ as usual. Notice also that if $j \in I_k$, then $m_k \leq \tilde{m}_j$. Then

$$L(P, f) = \sum_{k=1}^N m_k V(R_k) = \sum_{k=1}^N \sum_{j \in I_k} m_k V(\tilde{R}_j) \leq \sum_{k=1}^N \sum_{j \in I_k} \tilde{m}_j V(\tilde{R}_j) = \sum_{j=1}^M \tilde{m}_j V(\tilde{R}_j) = L(\tilde{P}, f). \quad \text{\textcolor{red}{qedhere}} \quad (11.1.11)$$

The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

[mv:intulbound:prop] Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$ we have $m \leq f(x) \leq M$. Then

$$mV(R) \leq \int_R f \leq \overline{\int}_R f \leq MV(R). \quad (11.1.12)$$

For any partition P , via

$$mV(R) \leq L(P, f) \leq U(P, f) \leq MV(R). \quad (11.1.13)$$

By taking suprema of $L(P, f)$ and infima of $U(P, f)$ over all P we obtain the first and the last inequality.

The key of course is the middle inequality in [mv:intulbound:eq]. Let $P_1 = \{P_1^1, P_1^2, \dots, P_1^n\}$ and $P_2 = \{P_2^1, P_2^2, \dots, P_2^n\}$ be partitions of R . Define $\tilde{P} = \{\tilde{P}^1, \tilde{P}^2, \dots, \tilde{P}^n\}$ by letting $\tilde{P}^k = P_1^k \cup P_2^k$. Then $\tilde{P}$ is a partition of R as can easily be checked, and $\tilde{P}$ is a refinement of P_1 and a refinement of P_2 . By, $L(P_1, f) \leq L(\tilde{P}, f)$ and $U(\tilde{P}, f) \leq U(P_2, f)$. Therefore,

$$L(P_1, f) \leq L(\tilde{P}, f) \leq U(\tilde{P}, f) \leq U(P_2, f). \quad (11.1.14)$$

In other words, for two arbitrary partitions P_1 and P_2 we have $L(P_1, f) \leq U(P_2, f)$. Via we obtain

$$\sup\{L(P, f) : P \text{ a partition of } R\} \leq \inf\{U(P, f) : P \text{ a partition of } R\}. \quad (11.1.15)$$

In other words $\int_R f \leq \overline{\int_R f}$.

The Riemann integral

We now have all we need to define the Riemann integral in n -dimensions over rectangles. Again, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $f: R \rightarrow \mathbb{R}$ be a bounded function such that

$$\int_a^b f(x) dx = \overline{\int_a^b f(x) dx}. \quad (11.1.16)$$

Then f is said to be *Riemann integrable*. The set of Riemann integrable functions on R is denoted by $\mathcal{R}(R)$. When $f \in \mathcal{R}(R)$ we define the *Riemann integral*

$$\int_R f := \int_R f = \overline{\int_R f}. \quad (11.1.17)$$

When the variable $x \in \mathbb{R}^n$ needs to be emphasized we write

$$\int_R f(x) dx, \quad (11.1.18)$$

implies immediately the following proposition.

[mv:intbound:prop] Let $f: R \rightarrow \mathbb{R}$ be a Riemann integrable function on a closed rectangle $R \subset \mathbb{R}^n$. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in R$. Then

$$mV(R) \leq \int_a^b f \leq MV(R). \quad (11.1.19)$$

A constant function is Riemann integrable. Suppose $f(x) = c$ for all x on R . Then

$$cV(R) \leq \int_R f \leq \overline{\int_R f} \leq cV(R). \quad (11.1.20)$$

So f is integrable, and furthermore $\int_R f = cV(R)$.

The proofs of linearity and monotonicity are almost completely identical as the proofs from one variable. We therefore leave it as an exercise to prove the next two propositions. (FIXME add the exercise).

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathcal{R}(R)$ and $\alpha \in \mathbb{R}$.

i. αf is in $\mathcal{R}(R)$ and

$$\int_R \alpha f = \alpha \int_R f \quad (11.1.21)$$

ii. $f + g$ is in $\mathcal{R}(R)$ and

$$\int_R (f + g) = \int_R f + \int_R g. \quad (11.1.22)$$

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathcal{R}(R)$ and let $f(x) \leq g(x)$ for all $x \in R$. Then

$$\int_R f \leq \int_R g. \quad (11.1.23)$$

Again for simplicity if $f: S \rightarrow \mathbb{R}$ is a function and $R \subset S$ is a closed rectangle, then if the restriction $f|_R$ is integrable we say f is integrable on R , or $f \in \mathcal{R}(R)$ and we write

$$\int_R f := \int_R f|_R. \quad (11.1.24)$$

For a closed rectangle $S \subset \mathbb{R}^n$, if $f: S \rightarrow \mathbb{R}$ is integrable and $R \subset S$ is a closed rectangle, then f is integrable over R .

Given $\epsilon > 0$, we find a partition P such that $U(P, f) - L(P, f) < \epsilon$. By making a refinement of P we can assume that the endpoints of R are in P , or in other words, R is a union of subrectangles of P . Then the subrectangles of P divide into two collections, ones that are subsets of R and ones whose intersection with the interior of R is empty. Suppose that $R_1, R_2, \dots, R_K$ be the subrectangles that are subsets of R and $R_{K+1}, \dots, R_N$ be the rest. Let $\tilde{P}$ be the partition of R composed of those subrectangles of P contained in R . Then using the same notation as before.

$$\begin{aligned} \epsilon &> U(P, f) - L(P, f) = \sum_{k=1}^K (M_k - m_k) V(R_k) + \sum_{k=K+1}^N (M_k - m_k) V(R_k) \\ &\geq \sum_{k=1}^K (M_k - m_k) V(R_k) = U(\tilde{P}, f|_R) - L(\tilde{P}, f|_R) \end{aligned}$$

Therefore $f|_R$ is integrable.

Integrals of continuous functions

FIXME: We will later on prove a much more general result, but it is useful to start with continuous functions only. Before we get to continuous functions, let us state the following proposition, which has a very easy proof, but it is useful to emphasize as a technique.

Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. If for every $\epsilon > 0$, there exists a partition P of R such that

$$U(P, f) - L(P, f) < \epsilon, \quad (11.1.25)$$

then $f \in \mathcal{R}(R)$.

Given an $\epsilon > 0$ find P as in the hypothesis. Then

$$\overline{\int_R f} - \underline{\int_R f} \leq U(P, f) - L(P, f) < \epsilon. \quad (11.1.26)$$

As $\overline{\int_R f} \geq \underline{\int_R f}$ and the above holds for every $\epsilon > 0$, we conclude $\overline{\int_R f} = \underline{\int_R f}$ and $f \in \mathcal{R}(R)$.

We say a rectangle $R = [a^1, b^1] \times [a^2, b^2] \times \dots \times [a^n, b^n]$ has longest side at most α if $b^k - a^k \leq \alpha$ for all k .

If a rectangle $R \subset \mathbb{R}^n$ has longest side at most α . Then for any $x, y \in R$,

$$\begin{aligned} \|x - y\| &\leq \sqrt{n} \alpha. \\ \|x - y\| &= \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \dots + (x^n - y^n)^2} \\ &\leq \sqrt{(b^1 - a^1)^2 + (b^2 - a^2)^2 + \dots + (b^n - a^n)^2} \\ &\leq \sqrt{\alpha^2 + \alpha^2 + \dots + \alpha^2} = \sqrt{n} \alpha. \end{aligned} \quad (11.1.27)$$

[mv:thm:contintrect] Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a continuous function, then $f \in \mathcal{R}(R)$.

The proof is analogous to the one variable proof with some complications. The set R is closed and bounded and hence compact. So f is not just continuous but in fact uniformly continuous by . Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $\|x - y\| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{V(R)}$.

Let P be a partition of R such that longest side of any subrectangle is strictly less than $\frac{\delta}{\sqrt{n}}$. Then for all $x, y \in R_k$ for a subrectangle R_k of P we have, by the proposition above, $\|x - y\| < \sqrt{n} \frac{\delta}{\sqrt{n}} = \delta$. Therefore

$$f(x) - f(y) \leq |f(x) - f(y)| < \frac{\epsilon}{V(R)}. \quad (11.1.28)$$

As f is continuous on R_k , it attains a maximum and a minimum on this interval. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_k$ and $f(y) = m_k$ in the notation from the definition of the integral. Therefore,

$$M_i - m_i = f(x) - f(y) < \frac{\epsilon}{V(R)}. \quad (11.1.29)$$

And so

$$\begin{aligned} U(P, f) - L(P, f) &= \left(\sum_{k=1}^N M_k V(R_k) \right) - \left(\sum_{k=1}^N m_k V(R_k) \right) \\ &= \sum_{k=1}^N (M_k - m_k) V(R_k) \\ &< \frac{\epsilon}{V(R)} \sum_{k=1}^N V(R_k) = \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary,

$$\overline{\int_a^b f} = \int_a^b f, \quad (11.1.30)$$

and f is Riemann integrable on R .

Integration of functions with compact support

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \rightarrow \mathbb{R}$ be a function. We say the *support* of f be the set

$$\text{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}}. \quad (11.1.31)$$

That is, the support is the closure of the set of points where the function is nonzero. So for a point not in the support we have that f is constantly zero in a whole neighbourhood.

A function f is said to have *compact support* if $\text{supp}(f)$ is a compact set. We will mostly consider the case when $U = \mathbb{R}^n$. In light of the following exercise, this is not an oversimplification.

Suppose $U \subset \mathbb{R}^n$ is open and $f: U \rightarrow \mathbb{R}$ is continuous and of compact support. Show that the function $\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \quad (11.1.32)$$

is continuous.

[mv:prop:rectanglessupp] Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function with compact support. If R is a closed rectangle such that $\text{supp}(f) \subset R^\circ$ where R° is the interior of R , and f is integrable over R , then for any other closed rectangle S with $\text{supp}(f) \subset S^\circ$, the function f is integrable over S and

$$\int_S f = \int_R f. \quad (11.1.33)$$

The intersection of closed rectangles is again a closed rectangle (or empty). Therefore we can take $\tilde{R} = R \cap S$ be the intersection of all rectangles containing $\text{supp}(f)$. If $\tilde{R}$ is the empty set, then $\text{supp}(f)$ is the empty set and f is identically zero and the proposition is trivial. So suppose that $\tilde{R}$ is nonempty. As $\tilde{R} \subset R$, we know that f is integrable over $\tilde{R}$. Furthermore $\tilde{R} \subset S$. Given $\epsilon > 0$, take $\tilde{P}$ to be a partition of $\tilde{R}$ such that

$$U(\tilde{P}, f|_{\tilde{R}}) - L(\tilde{P}, f|_{\tilde{R}}) < \epsilon. \quad (11.1.34)$$

Now add the endpoints of S to $\tilde{P}$ to create a new partition P . Note that the subrectangles of $\tilde{P}$ are subrectangles of P as well. Let $R_1, R_2, \dots, R_K$ be the subrectangles of $\tilde{P}$ and $R_{K+1}, \dots, R_N$ the new subrectangles. Note that since $\text{supp}(f) \subset \tilde{R}$, then for $k = K+1, \dots, N$ we have $\text{supp}(f) \cap R_k = \emptyset$. In other words f is identically zero on R_k . Therefore in the notation used previously we have

$$\begin{aligned}
 U(P, f|_S) - L(P, f|_S) &= \sum_{k=1}^K (M_k - m_k) V(R_k) + \sum_{k=K+1}^N (M_k - m_k) V(R_k) \\
 &= \sum_{k=1}^K (M_k - m_k) V(R_k) + \sum_{k=K+1}^N (0) V(R_k) \\
 &= U(\tilde{P}, f|_{\tilde{R}}) - L(\tilde{P}, f|_{\tilde{R}}) < \epsilon.
 \end{aligned}$$

Similarly we have that $L(P, f|_S) = L(\tilde{P}, f|_{\tilde{R}})$ and therefore

$$\int_S f = \int_{\tilde{R}} f. \quad (11.1.35)$$

Since $\tilde{R} \subset R$ we also get $\int_R f = \int_{\tilde{R}} f$, or in other words $\int_R f = \int_S f$.

Because of this proposition, when $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has compact support and is integrable over a rectangle R containing the support we write

$$\int f := \int_R f \quad \text{or} \quad \int_{\mathbb{R}^n} f := \int_R f. \quad (11.1.36)$$

For example if f is continuous and of compact support then $\int_{\mathbb{R}^n} f$ exists.

Exercises

FIXME

FIXME: Show that integration over a rectangle with one side of size zero results in zero integral.

[mv:exersmallerset] Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose that $f: R \rightarrow \mathbb{R}$ is in $\mathcal{R}(R)$. Show that $f \in \mathcal{R}(R')$.

[mv:zerooutside] Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose that $f: R \rightarrow \mathbb{R}$ is in $\mathcal{R}(R')$ and $f(x) = 0$ for $x \notin R'$. Show that $f \in \mathcal{R}(R)$ and

$$\int_{R'} f = \int_R f. \quad (11.1.37)$$

Hint: see the previous exercise.

Prove a stronger version of . Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function with compact support. Prove that if R is a closed rectangle such that $\text{supp}(f) \subset R$ and f is integrable over R , then for any other closed rectangle S with $\text{supp}(f) \subset S$, the function f is integrable over S and $\int_S f = \int_R f$. *Hint: notice that now the new rectangles that you add as in the proof can intersect $\text{supp}(f)$ on their boundary.*

Suppose that R and S are closed rectangles. Let $f(x) := 1$ if $x \in R$ and $f(x) = 0$ otherwise. Show that f is integrable over S and compute $\int_S f$.

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11.2: Iterated integrals and Fubini theorem

Iterated integrals and Fubini theorem

The Riemann integral in several variables is hard to compute from the definition. For one-dimensional Riemann integral we have the fundamental theorem of calculus (FIXME) and we can compute many integrals without having to appeal to the definition of the integral. We will rewrite a a Riemann integral in several variables into several one dimensional Riemann integrals by iterating. However, if $f: [0, 1]^2 \rightarrow \mathbb{R}$ is a Riemann integrable function, it is not immediately clear if the three expressions

$$\int_{[0,1]^2} f, \quad \int_0^1 \int_0^1 f(x, y) dx dy, \quad \text{and} \quad \int_0^1 \int_0^1 f(x, y) dy dx \quad (11.2.1)$$

are equal, or if the last two are even well-defined.

Define

$$f(x, y) := \begin{cases} 1 & \text{if } x = \frac{1}{2} \text{ and } y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases} \quad (11.2.2)$$

Then f is Riemann integrable on $R := [0, 1]^2$ and $\int_R f = 0$. Furthermore, $\int_0^1 \int_0^1 f(x, y) dx dy = 0$. However

$$\int_0^1 f\left(\frac{1}{2}, y\right) dy \quad (11.2.3)$$

does not exist, so we cannot even write $\int_0^1 \int_0^1 f(x, y) dy dx$.

Proof: Let us start with integrability of f . We simply take the partition of $[0, 1]^2$ where the partition in the x direction is $\{0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, 1\}$ and in the y direction $\{0, 1\}$. The subrectangles of the partition are

$$\begin{aligned} R_1 &:= [0, \frac{1}{2} - \epsilon] \times [0, 1], & R_2 &:= [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \times [0, 1], \\ R_3 &:= [\frac{1}{2} + \epsilon, 1] \times [0, 1]. \end{aligned} \quad (11.2.4)$$

We have $m_1 = M_1 = 0$, $m_2 = 0$, $M_2 = 1$, and $m_3 = M_3 = 0$. Therefore,

$$L(P, f) = m_1(\frac{1}{2} - \epsilon) \cdot 1 + m_2(2\epsilon) \cdot 1 + m_3(\frac{1}{2} - \epsilon) \cdot 1 = 0, \quad (11.2.5)$$

and

$$U(P, f) = M_1(\frac{1}{2} - \epsilon) \cdot 1 + M_2(2\epsilon) \cdot 1 + M_3(\frac{1}{2} - \epsilon) \cdot 1 = 2\epsilon. \quad (11.2.6)$$

The upper and lower sum are arbitrarily close and the lower sum is always zero, so the function is integrable and $\int_R f = 0$.

For any y , the function that takes x to $f(x, y)$ is zero except perhaps at a single point $x = \frac{1}{2}$. We know that such a function is integrable and $\int_0^1 f(x, y) dx = 0$. Therefore, $\int_0^1 \int_0^1 f(x, y) dx dy = 0$.

However if $x = \frac{1}{2}$, the function that takes y to $f(\frac{1}{2}, y)$ is the nonintegrable function that is 1 on the rationals and 0 on the irrationals. See .

We will solve this problem of undefined inside integrals by using the upper and lower integrals, which are always defined.

We split $\mathbb{R}^{n+m}$ into two parts. That is, we write the coordinates on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ as (x, y) where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a function $f(x, y)$ we write

$$f_x(y) := f(x, y) \quad (11.2.7)$$

when x is fixed and we wish to speak of the function in terms of y . We write

$$f^y(x) := f(x, y) \quad (11.2.8)$$

when y is fixed and we wish to speak of the function in terms of x .

[mv:fubinivA] Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f: R \times S \rightarrow \mathbb{R}$ be integrable. The functions $g: R \rightarrow \mathbb{R}$ and $h: S \rightarrow \mathbb{R}$ defined by

$$g(x) := \int_{\underline{S}} f_x \quad \text{and} \quad h(x) := \int_{\overline{S}} f_x \quad (11.2.9)$$

are integrable over R and

$$\int_R g = \int_R h = \int_{R \times S} f. \quad (11.2.10)$$

In other words

$$\int_{R \times S} f = \int_R \left(\int_{\underline{S}} f(x, y) dy \right) dx = \int_R \left(\int_{\overline{S}} f(x, y) dy \right) dx. \quad (11.2.11)$$

If it turns out that f_x is integrable for all x , for example when f is continuous, then we obtain the more familiar

$$\int_{R \times S} f = \int_R \int_S f(x, y) dy dx. \quad (11.2.12)$$

Let P be a partition of R and P' be a partition of S . Let $R_1, R_2, \dots, R_N$ be the subrectangles of P and $R'_1, R'_2, \dots, R'_K$ be the subrectangles of P' . Then $P \times P'$ is the partition whose subrectangles are $R_j \times R'_k$ for all $1 \leq j \leq N$ and all $1 \leq k \leq K$.

Let

$$m_{j,k} := \inf_{(x,y) \in R_j \times R'_k} f(x, y). \quad (11.2.13)$$

We notice that $V(R_j \times R'_k) = V(R_j)V(R'_k)$ and hence

$$L(P \times P', f) = \sum_{j=1}^N \sum_{k=1}^K m_{j,k} V(R_j \times R'_k) = \sum_{j=1}^N \left(\sum_{k=1}^K m_{j,k} V(R'_k) \right) V(R_j). \quad (11.2.14)$$

If we let

$$m_k(x) := \inf_{y \in R'_k} f(x, y) = \inf_{y \in R'_k} f_x(y), \quad (11.2.15)$$

then of course if $x \in R_j$ then $m_{j,k} \leq m_k(x)$. Therefore

$$\sum_{k=1}^K m_{j,k} V(R'_k) \leq \sum_{k=1}^K m_k(x) V(R'_k) = L(P', f_x) \leq \int_{\underline{S}} f_x = g(x). \quad (11.2.16)$$

As we have the inequality for all $x \in R_j$ we have

$$\sum_{k=1}^K m_{j,k} V(R'_k) \leq \inf_{x \in R_j} g(x). \quad (11.2.17)$$

We thus obtain

$$L(P \times P', f) \leq \sum_{j=1}^N \left(\inf_{x \in R_j} g(x) \right) V(R_j) = L(P, g). \quad (11.2.18)$$

Similarly $U(P \times P', f) \geq U(P, h)$, and the proof of this inequality is left as an exercise.

Putting this together we have

$$L(P \times P', f) \leq L(P, g) \leq U(P, g) \leq U(P, h) \leq U(P \times P', f). \quad (11.2.19)$$

And since f is integrable, it must be that g is integrable as

$$U(P, g) - L(P, g) \leq U(P \times P', f) - L(P \times P', f), \quad (11.2.20)$$

and we can make the right hand side arbitrarily small. Furthermore as $L(P \times P', f) \leq L(P, g) \leq U(P \times P', f)$ we must have that $\int_R g = \int_{R \times S} f$.

Similarly we have

$$L(P \times P', f) \leq L(P, g) \leq L(P, h) \leq U(P, h) \leq U(P \times P', f), \quad (11.2.21)$$

and hence

$$U(P, h) - L(P, h) \leq U(P \times P', f) - L(P \times P', f). \quad (11.2.22)$$

So if f is integrable so is h , and as $L(P \times P', f) \leq L(P, h) \leq U(P \times P', f)$ we must have that $\int_R h = \int_{R \times S} f$.

We can also do the iterated integration in opposite order. The proof of this version is almost identical to version A, and we leave it as an exercise to the reader.

[mv:fubiniivB] Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f: R \times S \rightarrow \mathbb{R}$ be integrable. The functions $g: S \rightarrow \mathbb{R}$ and $h: R \rightarrow \mathbb{R}$ defined by

$$g(x) := \int_S f^y \quad \text{and} \quad h(x) := \int_S f^x \quad (11.2.23)$$

are integrable over S and

$$\int_S g = \int_S h = \int_{R \times S} f. \quad (11.2.24)$$

That is we also have

$$\int_{R \times S} f = \int_S \left(\int_R f(x, y) dx \right) dy = \int_S \left(\int_R f(x, y) dx \right) dy. \quad (11.2.25)$$

Next suppose that f_x and f_y are integrable for simplicity. For example, suppose that f is continuous. Then by putting the two versions together we obtain the familiar

$$\int_{R \times S} f = \int_R \int_S f(x, y) dy dx = \int_S \int_R f(x, y) dx dy. \quad (11.2.26)$$

Often the Fubini theorem is stated in two dimensions for a continuous function $f: R \rightarrow \mathbb{R}$ on a rectangle $R = [a, b] \times [c, d]$. Then the Fubini theorem states that

$$\int_R f = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy. \quad (11.2.27)$$

And the Fubini theorem is commonly thought of as the theorem that allows us to swap the order of iterated integrals.

We can also obtain the Repeatedly applying Fubini theorem gets us the following corollary: Let $R := [a^1, b^1] \times [a^2, b^2] \times \cdots \times [a^n, b^n] \subset \mathbb{R}^n$ be a closed rectangle and let $f: R \rightarrow \mathbb{R}$ be continuous. Then

$$\int_R f = \int_{a^1}^{b^1} \int_{a^2}^{b^2} \cdots \int_{a^n}^{b^n} f(x^1, x^2, \dots, x^n) dx^n dx^{n-1} \cdots dx^1. \quad (11.2.28)$$

Clearly we can also switch the order of integration to any order we please. We can also relax the continuity requirement by making sure that all the intermediate functions are integrable, or by using upper or lower integrals.

Exercises

Prove the assertion $U(P \times P', f) \geq U(P, h)$ from the proof of .

Prove .

FIXME

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11.3: Outer measure and null sets

Outer measure

Before we characterize all Riemann integrable functions, we need to make a slight detour. We introduce a way of measuring the size of sets in $\mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be a subset. Define the *outer measure* of S as

$$m^*(S) := \inf \sum_{j=1}^{\infty} V(R_j), \quad (11.3.1)$$

where the infimum is taken over all sequences $\{R_j\}$ of open rectangles such that $S \subset \bigcup_{j=1}^{\infty} R_j$. In particular S is of *measure zero* or a *null set* if $m^*(S) = 0$.

We will only need measure zero sets and so we focus on these. Note that S is of measure zero if for every $\epsilon > 0$ there exist a sequence of open rectangles $\{R_j\}$ such that

$$S \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \epsilon. \quad (11.3.2)$$

Furthermore, if S is measure zero and $S' \subset S$, then S' is of measure zero. We can in fact use the same exact rectangles.

The set $\mathbb{Q}^n \subset \mathbb{R}^n$ of points with rational coordinates is a set of measure zero.

Proof: The set $\mathbb{Q}^n$ is countable and therefore let us write it as a sequence $q_1, q_2, \dots$. For each q_j find an open rectangle R_j with $q_j \in R_j$ and $V(R_j) < \epsilon 2^{-j}$. Then

$$\mathbb{Q}^n \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} \epsilon 2^{-j} = \epsilon. \quad (11.3.3)$$

In fact, the example points to a more general result.

A countable union of measure zero sets is of measure zero.

Suppose

$$S = \bigcup_{j=1}^{\infty} S_j \quad (11.3.4)$$

where S_j are all measure zero sets. Let $\epsilon > 0$ be given. For each j there exists a sequence of open rectangles $\{R_{j,k}\}_{k=1}^{\infty}$ such that

$$S_j \subset \bigcup_{k=1}^{\infty} R_{j,k} \quad (11.3.5)$$

and

$$\sum_{k=1}^{\infty} V(R_{j,k}) < 2^{-j} \epsilon. \quad (11.3.6)$$

Then

$$S \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} R_{j,k}. \quad (11.3.7)$$

As $V(R_{j,k})$ is always positive, the sum over all j and k can be done in any order. In particular, it can be done as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} V(R_{j,k}) < \sum_{j=1}^{\infty} 2^{-j} \epsilon = \epsilon. \quad \text{\textcolor{red}{\texttt{\textbackslash qedhere}}} \quad (11.3.8)$$

The next example is not just interesting, it will be useful later.

[mv:example:planenull] Let $P := \{x \in \mathbb{R}^n : x^k = c\}$ for a fixed $k = 1, 2, \dots, n$ and a fixed constant $c \in \mathbb{R}$. Then P is of measure zero.

Proof: First fix s and let us prove that

$$P_s := \{x \in \mathbb{R}^n : x^k = c, |x^j| \leq s \text{ for all } j \neq k\} \quad (11.3.9)$$

is of measure zero. Given any $\epsilon > 0$ define the open rectangle

$$R := \{x \in \mathbb{R}^n : c - \epsilon < x^k < c + \epsilon, |x^j| < s + 1 \text{ for all } j \neq k\} \quad (11.3.10)$$

It is clear that $P_s \subset R$. Furthermore

$$V(R) = 2\epsilon(2(s+1))^{n-1}. \quad (11.3.11)$$

As s is fixed, we can make $V(R)$ arbitrarily small by picking ϵ small enough.

Next we note that

$$P = \bigcup_{j=1}^{\infty} P_j \quad (11.3.12)$$

and a countable union of measure zero sets is measure zero.

If $a < b$, then $m^*([a, b]) = b - a$.

Proof: In the case of $\mathbb{R}$, open rectangles are open intervals. Since $[a, b] \subset (a - \epsilon, b + \epsilon)$ for all $\epsilon > 0$. Hence, $m^*([a, b]) \leq b - a$.

Let us prove the other inequality. Suppose that $\{(a_j, b_j)\}$ are open intervals such that

$$[a, b] \subset \bigcup_{j=1}^{\infty} (a_j, b_j). \quad (11.3.13)$$

We wish to bound $\sum(b_j - a_j)$ from below. Since $[a, b]$ is compact, then there are only finitely many open intervals that still cover $[a, b]$. As throwing out some of the intervals only makes the sum smaller, we only need to take the finite number of intervals still covering $[a, b]$. If $(a_i, b_i) \subset (a_j, b_j)$, then we can throw out (a_i, b_i) as well. Therefore we have $[a, b] \subset \bigcup_{j=1}^k (a_j, b_j)$ for some k , and we assume that the intervals are sorted such that $a_1 < a_2 < \dots < a_k$. Note that since (a_2, b_2) is not contained in (a_1, b_1) we have that $a_1 < a_2 < b_1 < b_2$. Similarly $a_j < a_{j+1} < b_j < b_{j+1}$. Furthermore, $a_1 < a$ and $b_k > b$. Thus,

$$m^*([a, b]) \geq \sum_{j=1}^k (b_j - a_j) \geq \sum_{j=1}^{k-1} (a_{j+1} - a_j) + (b_k - a_k) = b_k - a_1 > b - a. \quad (11.3.14)$$

[mv:prop:compactnull] Suppose $E \subset \mathbb{R}^n$ is a compact set of measure zero. Then for every $\epsilon > 0$, there exist finitely many open rectangles $R_1, R_2, \dots, R_k$ such that

$$E \subset R_1 \cup R_2 \cup \dots \cup R_k \quad \text{and} \quad \sum_{j=1}^k V(R_j) < \epsilon. \quad (11.3.15)$$

Find a sequence of open rectangles $\{R_j\}$ such that

$$E \subset \bigcup_{j=1}^{\infty} R_j \quad \text{and} \quad \sum_{j=1}^{\infty} V(R_j) < \epsilon. \quad (11.3.16)$$

By compactness, finitely many of these rectangles still contain E . That is, there is some k such that $E \subset R_1 \cup R_2 \cup \dots \cup R_k$. Hence

$$\sum_{j=1}^k V(R_j) \leq \sum_{j=1}^{\infty} V(R_j) < \epsilon. \quad (11.3.17)$$

The image of a measure zero set using a continuous map is not necessarily a measure zero set. However if we assume the mapping is continuously differentiable, then the mapping cannot “stretch” the set too much. The proposition does not require compactness,

and this is left as an exercise.

[prop:imagenull] Suppose $U \subset \mathbb{R}^n$ is an open set and $f: U \rightarrow \mathbb{R}^n$ is a continuously differentiable mapping. If $E \subset U$ is a compact measure zero set, then $f(E)$ is measure zero.

As FIXME: distance to boundary, did we do that? We should!

FIXME: maybe this closed/open rectangle bussiness should be addressed above

Let $\epsilon > 0$ be given.

FIXME: Let $\delta > 0$ be the distance to boundary

Let us “fatten” E a little bit. Using compactness, there exist finitely many open rectangles $T_1, T_2, \dots, T_k$ such that

$$E \subset T_1 \cup T_2 \cup \dots \cup T_k \quad \text{and} \quad V(T_1) + V(T_2) + \dots + V(T_k) < \epsilon. \quad (11.3.18)$$

Since a closed rectangle has the same volume as an open rectangle with the same sides, so we could take R_j to be the closure of T_j . Furthermore a closed rectangle can be written as finitely many small rectangles. Consequently for some ℓ there exist finitely many closed rectangles $R_1, R_2, \dots, R_\ell$ of side at most $\frac{\sqrt{n}\delta}{2}$. such that

$$E \subset R_1 \cup R_2 \cup \dots \cup R_\ell \quad \text{and} \quad V(R_1) + V(R_2) + \dots + V(R_\ell) < \epsilon. \quad (11.3.19)$$

Let

$$E' := R_1 \cup R_2 \cup \dots \cup R_\ell \quad (11.3.20)$$

It is left as an exercise (see Exercise

As f is continuously differentiable, the function that takes x to $\|Df(x)\|$ is continuous, therefore $\|Df(x)\|$ achieves a maximum on E . Thus there exists some $C > 0$ such that $\|Df(x)\| \leq C$ on E .

FIXME

FIXME: may need the fact that the derivative exists AND is continuous on a FATTER E which is still compact and of size ϵ .

FIXME: Then use the whole lipschitz thing we have.

so we can assume that on E

FIXME:

FIXME: Cantor set, fat cantor set, can be done in $\mathbb{R}^n$

FIXME: maybe too much

FIXME

Exercises

FIXME:

If $A \subset B$ then $m^*(A) \leq m^*(B)$.

Show that if $R \subset \mathbb{R}^n$ is a closed rectangle then $m^*(R) = V(R)$.

Prove a version of without using compactness:

a) Mimic the proof to first prove that the proposition holds only if E is *relatively compact*; a set $E \subset U$ is relatively compact if the closure of E in the subspace topology on E is compact, or in other words if there exists a compact set K with $K \subset U$ and $E \subset K$. Hint: The bound on the size of the derivative still holds, but you may need to use countably many rectangles. Be careful as the closure of E need no longer be measure zero.

b) Now prove it for any null set E .

Hint: First show that $\{x \in U : d(x, y) \geq \frac{1}{M} \text{ for all } y \in U \text{ and } d(0, x) \leq M\}$ is a compact set for any $M > 0$.

Let $U \subset \mathbb{R}^n$ be an open set and let $f: U \rightarrow \mathbb{R}$ be a continuously differentiable function. Let $G := \{(x, y) \in U \times \mathbb{R} : y = f(x)\}$ be the graph of f . Show that f is of measure zero.

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11.4: The set of Riemann Integrable Functions

Oscillation and continuity

Let $S \subset \mathbb{R}^n$ be a set and $f: S \rightarrow \mathbb{R}$ a function. Instead of just saying that f is or is not continuous at a point $x \in S$, we need to be able to quantify how discontinuous f is at a function is at x . For any $\delta > 0$ define the oscillation of f on the δ -ball in subset topology that is $B_S(x, \delta) = B_{\mathbb{R}^n}(x, \delta) \cap S$ as

$$o(f, x, \delta) := \sup_{y \in B_S(x, \delta)} f(y) - \inf_{y \in B_S(x, \delta)} f(y) = \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)). \quad (11.4.1)$$

That is, $o(f, x, \delta)$ is the length of the smallest interval that contains the image $f(B_S(x, \delta))$. Clearly $o(f, x, \delta) \geq 0$ and notice $o(f, x, \delta) \leq o(f, x, \delta')$ whenever $\delta < \delta'$. Therefore, the limit as $\delta \rightarrow 0$ from the right exists and we define the *oscillation* of a function f at x as

$$o(f, x) := \lim_{\delta \rightarrow 0^+} o(f, x, \delta) = \inf_{\delta > 0} o(f, x, \delta). \quad (11.4.2)$$

$f: S \rightarrow \mathbb{R}$ is continuous at $x \in S$ if and only if $o(f, x) = 0$.

First suppose that f is continuous at $x \in S$. Then given any $\epsilon > 0$, there exists a $\delta > 0$ such that for $y \in B_S(x, \delta)$ we have $|f(x) - f(y)| < \epsilon$. Therefore if $y_1, y_2 \in B_S(x, \delta)$ then

$$f(y_1) - f(y_2) = f(y_1) - f(x) - (f(y_2) - f(x)) < \epsilon + \epsilon = 2\epsilon. \quad (11.4.3)$$

We take the supremum over y_1 and y_2

$$o(f, x, \delta) = \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) \leq 2\epsilon. \quad (11.4.4)$$

Hence, $o(f, x) = 0$.

On the other hand suppose that $o(f, x) = 0$. Given any $\epsilon > 0$, find a $\delta > 0$ such that $o(f, x, \delta) < \epsilon$. If $y \in B_S(x, \delta)$ then

$$|f(x) - f(y)| \leq \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) = o(f, x, \delta) < \epsilon. \quad (11.4.5)$$

[prop:seclosed] Let $S \subset \mathbb{R}^n$ be closed, $f: S \rightarrow \mathbb{R}$, and $\epsilon > 0$. The set $\{x \in S : o(f, x) \geq \epsilon\}$ is closed.

Equivalently we want to show that $G = \{x \in S : o(f, x) < \epsilon\}$ is open in the subset topology. As $\inf_{\delta > 0} o(f, x, \delta) < \epsilon$, find a $\delta > 0$ such that

$$o(f, x, \delta) < \epsilon \quad (11.4.6)$$

Take any $\xi \in B_S(x, \frac{\delta}{2})$. Notice that $B_S(\xi, \frac{\delta}{2}) \subset B_S(x, \delta)$. Therefore,

$$o(f, \xi, \frac{\delta}{2}) = \sup_{y_1, y_2 \in B_S(\xi, \frac{\delta}{2})} (f(y_1) - f(y_2)) \leq \sup_{y_1, y_2 \in B_S(x, \delta)} (f(y_1) - f(y_2)) = o(f, x, \delta) < \epsilon. \quad (11.4.7)$$

So $o(f, \xi) < \epsilon$ as well. As this is true for all $\xi \in B_S(x, \frac{\delta}{2})$ we get that G is open in the subset topology and $S \setminus G$ is closed as is claimed.

The set of Riemann integrable functions

We have seen that continuous functions are Riemann integrable, but we also know that certain kinds of discontinuities are allowed. It turns out that as long as the discontinuities happen on a set of measure zero, the function is integrable and vice versa.

Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f: R \rightarrow \mathbb{R}$ a bounded function. Then f is Riemann integrable if and only if the set of discontinuities of f is of measure zero (a null set).

Let $S \subset R$ be the set of discontinuities of f . That is $S = \{x \in R : o(f, x) > 0\}$. The trick to this proof is to isolate the bad set into a small set of subrectangles of a partition. There are only finitely many subrectangles of a partition, so we will wish to use compactness. If S is closed, then it would be compact and we could cover it by small rectangles as it is of measure zero. Unfortunately, in general S is not closed so we need to work a little harder.

For every $\epsilon > 0$, define

$$S_\epsilon := \{x \in R : o(f, x) \geq \epsilon\}. \quad (11.4.8)$$

By S_ϵ is closed and as it is a subset of R which is bounded, S_ϵ is compact. Furthermore, $S_\epsilon \subset S$ and S is of measure zero. Via there are finitely many open rectangles $S_1, S_2, \dots, S_k$ that cover S_ϵ and $\sum V(S_j) < \epsilon$.

The set $T = R \setminus (S_1 \cup \dots \cup S_k)$ is closed, bounded, and therefore compact. Furthermore for $x \in T$, we have $o(f, x) < \epsilon$. Hence for each $x \in T$, there exists a small closed rectangle T_x with x in the interior of T_x , such that

$$\sup_{y \in T_x} f(y) - \inf_{y \in T_x} f(y) < 2\epsilon. \quad (11.4.9)$$

The interiors of the rectangles T_x cover T . As T is compact there exist finitely many such rectangles $T_1, T_2, \dots, T_m$ that covers T .

Now take all the rectangles $T_1, T_2, \dots, T_m$ and $S_1, S_2, \dots, S_k$ and construct a partition out of their endpoints. That is construct a partition P with subrectangles $R_1, R_2, \dots, R_p$ such that every R_j is contained in T_ℓ for some ℓ or the closure of S_ℓ for some ℓ . Suppose we order the rectangles so that $R_1, R_2, \dots, R_q$ are those that are contained in some T_ℓ , and $R_{q+1}, R_{q+2}, \dots, R_p$ are the rest. In particular, we have

$$\sum_{j=1}^q V(R_j) \leq V(R) \quad \text{and} \quad \sum_{j=q+1}^p V(R_j) \leq \epsilon. \quad (11.4.10)$$

Let m_j and M_j be the inf and sup over R_j as before. If $R_j \subset T_\ell$ for some ℓ , then $(M_j - m_j) < 2\epsilon$. Let $B \in \mathbb{R}$ be such that $|f(x)| \leq B$ for all $x \in R$, so $(M_j - m_j) < 2B$ over all rectangles. Then

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{j=1}^p (M_j - m_j) V(R_j) \\ &= \left(\sum_{j=1}^q (M_j - m_j) V(R_j) \right) + \left(\sum_{j=q+1}^p (M_j - m_j) V(R_j) \right) \\ &\leq \left(\sum_{j=1}^q 2\epsilon V(R_j) \right) + \left(\sum_{j=q+1}^p 2BV(R_j) \right) \\ &\leq 2\epsilon V(R) + 2B\epsilon = \epsilon(2V(R) + 2B). \end{aligned}$$

Clearly, we can make the right hand side as small as we want and hence f is integrable.

For the other direction, suppose that f is Riemann integrable over R . Let S be the set of discontinuities again and now let

$$S_k := \{x \in R : o(f, x) \geq \frac{1}{k}\}. \quad (11.4.11)$$

Fix a $k \in \mathbb{N}$. Given an $\epsilon > 0$, find a partition P with subrectangles $R_1, R_2, \dots, R_p$ such that

$$U(P, f) - L(P, f) = \sum_{j=1}^p (M_j - m_j) V(R_j) < \epsilon \quad (11.4.12)$$

Suppose that $R_1, R_2, \dots, R_p$ are order so that the interiors of $R_1, R_2, \dots, R_q$ intersect S_k , while the interiors of $R_{q+1}, R_{q+2}, \dots, R_p$ are disjoint from S_k . If $x \in R_j \cap S_k$ and x is in the interior of R_j so sufficiently small balls are completely inside R_j , then by definition of S_k we have $M_j - m_j \geq \frac{1}{k}$. Then

$$\epsilon > \sum_{j=1}^p (M_j - m_j) V(R_j) \geq \sum_{j=1}^q (M_j - m_j) V(R_j) \geq \frac{1}{k} \sum_{j=1}^q V(R_j) \quad (11.4.13)$$

In other words $\sum_{j=1}^q V(R_j) < k\epsilon$. Let G be the set of all boundaries of all the subrectangles of P . The set G is of measure zero (see). Let R_j° denote the interior of R_j , then

$$S_k \subset R_1^\circ \cup R_2^\circ \cup \dots \cup R_q^\circ \cup G. \quad (11.4.14)$$

As G can be covered by open rectangles arbitrarily small volume, S_k must be of measure zero. As

$$S = \bigcup_{k=1}^{\infty} S_k \quad (11.4.15)$$

and a countable union of measure zero sets is of measure zero, S is of measure zero.

Exercises

FIXME:

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11.5: Jordan Measurable Sets

Volume and Jordan measurable sets

Given a bounded set $S \subset \mathbb{R}^n$ its *characteristic function* or *indicator function* is

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases} \quad (11.5.1)$$

A bounded set S is said to be *Jordan measurable* if for some closed rectangle R such that $S \subset R$, the function χ_S is in $\mathcal{R}(R)$. Take two closed rectangles R and R' with $S \subset R$ and $S \subset R'$, then $R \cap R'$ is a closed rectangle also containing S . By and, $\chi_S \in \mathcal{R}(R \cap R')$ and hence $\chi_S \in \mathcal{R}(R')$, and furthermore

$$\int_R \chi_S = \int_{R'} \chi_S = \int_{R \cap R'} \chi_S. \quad (11.5.2)$$

We define the *n-dimensional volume* of the bounded Jordan measurable set S as

$$V(S) := \int_R \chi_S, \quad (11.5.3)$$

where R is any closed rectangle containing S .

A bounded set $S \subset \mathbb{R}^n$ is Jordan measurable if and only if the boundary ∂S is a measure zero set.

Suppose R is a closed rectangle such that S is contained in the interior of R . If $x \in \partial S$, then for every $\delta > 0$, the sets $S \cap B(x, \delta)$ (where χ_S is 1) and the sets $(R \setminus S) \cap B(x, \delta)$ (where χ_S is 0) are both nonempty. So χ_S is not continuous at x . If x is either in the interior of S or in the complement of the closure $\bar{S}$, then χ_S is either identically 1 or identically 0 in a whole neighbourhood of x and hence χ_S is continuous at x . Therefore, the set of discontinuities of χ_S is precisely the boundary ∂S . The proposition then follows.

The proof of the following proposition is left as an exercise.

[prop:jordanmeas] Suppose S and T are bounded Jordan measurable sets. Then

- i. The closure $\bar{S}$ is Jordan measurable.
- ii. The interior S° is Jordan measurable.
- iii. $S \cup T$ is Jordan measurable.
- iv. $S \cap T$ is Jordan measurable.
- v. $S \setminus T$ is Jordan measurable.

FIXME

If $S \subset \mathbb{R}^n$ is Jordan measurable then $V(S) = m^*(S)$.

Given $\epsilon > 0$, let R be a closed rectangle that contains S . Let P be a partition of R such that

$$U(P, \chi_S) \leq \int_R \chi_S + \epsilon = V(S) + \epsilon \quad \text{and} \quad L(P, \chi_S) \geq \int_R \chi_S - \epsilon = V(S) - \epsilon. \quad (11.5.4)$$

Let $R_1, \dots, R_k$ be all the subrectangles of P such that χ_S is not identically zero on each R_j . That is, there is some point $x \in R_j$ such that $x \in S$. Let O_j be an open rectangle such that $R_j \subset O_j$ and $V(O_j) < V(R_j) + \frac{\epsilon}{k}$. Notice that $S \subset \bigcup_j O_j$. Then

$$U(P, \chi_S) = \sum_{j=1}^k V(R_k) > \left(\sum_{j=1}^k V(O_k) \right) - \epsilon \geq m^*(S) - \epsilon. \quad (11.5.5)$$

As $U(P, \chi_S) \leq V(S) + \epsilon$, then $m^*(S) - \epsilon \leq V(S) + \epsilon$, or in other words $m^*(S) \leq V(S)$.

Now let $R'_1, \dots, R'_\ell$ be all the subrectangles of P such that χ_S is identically one on each R'_j . In other words, these are the subrectangles contained in S . The interiors of the subrectangles R_j° are disjoint and $V(R_j^\circ) = V(R'_j)$. It is easy to see from definition that

$$m^*\left(\bigcup_{j=1}^{\ell} R_j^{\circ}\right) = \sum_{j=1}^{\ell} V(R_j^{\circ}). \quad (11.5.6)$$

Hence

$$m^*(S) \geq m^*\left(\bigcup_{j=1}^{\ell} R_j'\right) \geq m^*\left(\bigcup_{j=1}^{\ell} R_j^{\circ}\right) \quad (11.5.7)$$

Therefore $m^*(S) \geq V(S)$ as well.

Integration over Jordan measurable sets

In one variable there is really only one type of reasonable set to integrate over: an interval. In several variables we have many very simple sets we might want to integrate over and these cannot be described so easily.

Let $S \subset \mathbb{R}^n$ be a bounded Jordan measurable set. A bounded function $f: S \rightarrow \mathbb{R}$ is said to be Riemann integrable on S if for a closed rectangle R such that $S \subset R$, the function $\tilde{f}: R \rightarrow \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases} \quad (11.5.8)$$

is in $\mathcal{R}(R)$. In this case we write

$$\int_S f := \int_R \tilde{f}. \quad (11.5.9)$$

When f is defined on a larger set and we wish to integrate over S , then we apply the definition to the restriction $f|_S$. In particular note that if $f: R \rightarrow \mathbb{R}$ for a closed rectangle R , and $S \subset R$ is a Jordan measurable subset then

$$\int_S f = \int_R f \chi_S. \quad (11.5.10)$$

FIXME

Images of Jordan measurable subsets

Let us prove the following FIXME. We will only need this simple

Suppose $S \subset \mathbb{R}^n$ is a closed bounded Jordan measurable set, and $S \subset U$ for an open set $U \subset \mathbb{R}^n$. If $g: U \rightarrow \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that J_g is never zero on S . Then $g(S)$ is Jordan measurable.

Let $T = g(S)$. We claim that the boundary ∂T is contained in the set $g(\partial S)$. Suppose the claim is proved. As S is Jordan measurable, then ∂S is measure zero. Then $g(\partial S)$ is measure zero by . As $\partial T \subset g(\partial S)$, then T is Jordan measurable.

It is therefore left to prove the claim. First, S is closed and bounded and hence compact. By , $T = g(S)$ is also compact and therefore closed. In particular $\partial T \subset T$. Suppose $y \in \partial T$, then there must exist an $x \in S$ such that $g(x) = y$. The Jacobian of g is nonzero at x .

We now use the inverse function theorem . We find a neighbourhood $V \subset U$ of x and an open set W such that the restriction $f|_V$ is a one-to-one and onto function from V to W with a continuously differentiable inverse. In particular $g(x) = y \in W$. As $y \in \partial T$, there exists a sequence $\{y_k\}$ in W with $\lim y_k = y$ and $y_k \notin T$. As $g|_V$ is invertible and in particular has a continuous inverse, there exists a sequence $\{x_k\}$ in V such that $g(x_k) = y_k$ and $\lim x_k = x$. Since $y_k \notin T = g(S)$, clearly $x_k \notin S$. Since $x \in S$, we conclude that $x \in \partial S$. The claim is proved, $\partial T \subset g(\partial S)$.

Exercises

Prove .

Prove that a bounded convex set is Jordan measurable. Hint: induction on dimension.

FIXME

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11.6: Green's Theorem

Change of variables

Note: FIXME4 lectures

In one variable, we have the familiar change of variables

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(x) dx. \quad (11.6.1)$$

It may be surprising that the analogue in higher dimensions is quite a bit more complicated. The first complication is orientation. If we use the definition of integral from this chapter, then we do not have the notion of $\int_a^b$ versus $\int_b^a$. We are simply integrating over an interval $[a, b]$. With this notation then the change of variables becomes

$$\int_{[a,b]} f(g(x)) |g'(x)| dx = \int_{g([a,b])} f(x) dx. \quad (11.6.2)$$

In this section we will try to obtain an analogue in this form.

First we wish to see what plays the role of $|g'(x)|$. If we think about it, the $g'(x)$ is a scaling of dx . The integral measures volumes, so in one dimension it measures length. If our g was linear, that is, $g(x) = Lx$, then $g'(x) = L$. Then the length of the interval $g([a, b])$ is simply $|L|(b - a)$. That is because $g([a, b])$ is either $[La, Lb]$ or $[Lb, La]$. This property holds in higher dimension with $|L|$ replaced by absolute value of the determinant.

[prop:volrectdet] Suppose that $R \subset \mathbb{R}^n$ is a rectangle and $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear. Then $T(R)$ is Jordan measurable and $V(T(R)) = |\det T| V(R)$.

It is enough to prove for elementary matrices. The proof is left as an exercise.

We next notice that this result still holds if g is not necessarily linear, by integrating the absolute value of the Jacobian. That is, we have the following lemma

Suppose $S \subset \mathbb{R}^n$ is a closed bounded Jordan measurable set, and $S \subset U$ for an open set U . If $g: U \rightarrow \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that J_g is never zero on S . Then

$$V(g(S)) = \int_S |J_g(x)| dx. \quad (11.6.3)$$

FIXME

The left hand side is $\int_{R'} \chi_{g(S)}$, where the integral is taken over a large enough rectangle R' that contains $g(S)$. The right hand side is $\int_R |J_g|$ for a large enough rectangle R that contains S . Let $\epsilon > 0$ be given. Divide R into subrectangles, denote by $R_1, R_2, \dots, R_K$ those subrectangles which intersect S . Suppose that the partition is fine enough such that

$$\epsilon + \int_S |J_g(x)| dx \geq \sum_{j=1}^N \left(\sup_{x \in S \cap R_j} |J_g(x)| \right) V(R_j) \quad (11.6.4)$$

...

$$\sum_{j=1}^N \left(\sup_{x \in S \cap R_j} |J_g(x)| \right) V(R_j) \geq \sum_{j=1}^N |J_g(x_j)| V(R_j) = \sum_{j=1}^N V(Dg(x_j)R_j) \quad (11.6.5)$$

... FIXME ... must pick x_j correctly?

Let

FIXME

So $|J_g(x)|$ is the replacement of $|g'(x)|$ for multiple dimensions. Note that the following theorem holds in more generality, but this statement is sufficient for many uses.

Suppose that $S \subset \mathbb{R}^n$ is an open bounded Jordan measurable set, and $g: S \rightarrow \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that $g(S)$ is Jordan measurable and J_g is never zero on S .

Suppose that $f: g(S) \rightarrow \mathbb{R}$ is Riemann integrable, then $f \circ g$ is Riemann integrable on S and

$$\int_{g(S)} f(x) dx = \int_S f(g(x)) |J_g(x)| dx. \quad (11.6.6)$$

FIXME

FIXME: change of variables for functions with compact support

FIXME4

Exercises

Prove .

FIXME

-
1. If you want a funky vector space over a different field, $\mathbb{R}$ is an infinite dimensional vector space over the rational numbers.↩
-

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