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Foundations of Set Theory

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AND

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PREFACE

This book appears exactly half a century after the year (1908) in which set theory, gravely shaken by the antinomies in its original, naive form as created by Cantor, underwent a thorough reconstruction in the hands of Brouwer, Russell, and Zermelo. The present book is mainly dedicated to the description of these widely different and independently developed approaches, in all their ramifications and recent *rapprochement.*

The book is intended to serve as an advanced text for graduate students in mathematics and philosophy who have some knowledge of the elements of set theory and — preferably though not necessarily — of symbolic logic; it may also serve as a book of first reference for those desiring to do research work on the foundations of set theory. We would, however, warn against reading (and quoting) the last, "philosophical" section of the book without having mastered the contents of the whole book before. It is not by accident that this section has been put at the end.

In 1952, the senior of the present authors, in the Preface to *Abstract Set Theory* (1953), announced the planned publication of a companion volume intended to treat in a comprehensive fashion the various aspects of the foundations of set theory, including the logical problems in their entirety. In accordance with this aim, the Bibliography to *Abstract Set Theory* contains many items, published before 1950, pertinent to topics to be discussed in the later volume.

In the meantime, however, there appeared several excellent publications, in English as well as in other languages, some of which present mathematical logic, metamathematics, and the foundations of mathematics in textbook fashion, while others treat selected topics from these disciplines monographically. We mention only WILDER's *Introduction to the foundations of mathematics* (1952), KLEENE's *Introduction to metamathematics* (1952), ROSSER's *Logic for mathematicians*

(1953), and CHURCH's *Introduction to mathematical logic* (1956), among the textbooks; MOSTOWSKI's *Sentences undecidable in formalized arithmetic* (1952), TARSKI's (in collaboration with MOSTOWSKI and R. M. ROBINSON) *Undecidable theories* (1953), HERMES and SCHOLZ's *Mathematische Logik* (1952), MARKOV's *Téoriá algorifmov* (1954), and LADRIÈRE's *Les limitations internes des formalismes* (1957), among the monographs.

This welcome development induced us to restrict considerably the scope of the present book: instead of giving a self-contained and comprehensive account of mathematical logic, metamathematics, and semantics in their entirety, Chapters III and V are now limited to the discussion of a few important topics among those already treated in the mentioned books, as well as to the presentation of approaches not yet covered by them. Chapter IV, in view of HEYTING's *Intuitionism* (1956), is essentially restricted to the discussion of the *fundamentals* of intuitionism. Chapter II, however, still treats the axiomatic foundations of set theory in full detail and is written in a way suitable also for the beginner who knows no more than the elements of Cantor's theory. Chapter I, finally, could now be kept to a minimum just sufficient to impress the reader with the importance of the antinomies as simultaneously annoying and fertilizing factors. Altogether, though the present book has lost, through all these changes, its originally intended character of a self-contained and encyclopedic treatment of all the various aspects of the foundations of set theory, we hope that this loss is balanced by the resulting opportunity to spend part of the space saved for a detailed treatment of certain neglected older views or of quite recent innovations.

The book contains repetitions of more than average frequency. The authors thought that many readers would prefer these repetitions to constant back-references; they have no illusions as to having hit on the right proportions.

We could not very well reprint in this volume all the items contained in the bibliography of the earlier volume and cited here. However, in order to release the reader from the need of constant consultation of that Bibliography, the following compromise solution was employed: the Bibliography attached to the present book contains, in addition to all the pertinent items published after 1950, those more important earlier items which are frequently referred to; for items which are less important for our purposes the reader is referred to the earlier Bibliography, and

these items will, for the benefit of the readers of the present book, be listed also in the forthcoming 2nd edition of the earlier volume.

Though most books and papers cited have been actually read by at least one of the authors, for some items, especially those that appeared in languages with which the authors were not sufficiently familiar (such as Russian, Polish, Hungarian, Finnish, or the Scandinavian languages), they had to rely heavily on the reviews in the *Journal of Symbolic Logic;* the help extended hereby is gratefully acknowledged.

While in the bulk of the book the symbolism of mathematical logic is used but sparely, the more important definitions and axioms, as well as a few of the theorems, are (also) formulated symbolically. Wider use of symbolic formulations is made in Chapters III and V, but even here they are mostly relegated to paragraphs in small print which may be skipped without substantial loss. The rudiments of symbolic logic are given on pp. 25 ff and 272.

The authors have in general refrained from expressing their own opinions regarding controversial points. Only on very few occasions, particularly in connection with certain intuitionistic theses, did they deem it helpful to add their own criticism.

While Chapters I, III, and V were chiefly written by the junior author (Y. B.-H.) and Chapters II and IV by the senior author (A. A. F.), we assume joint responsibility for the book in its entirety.

We wish to thank our Jerusalem colleagues Azriel Lévy and Abraham Robinson who read part of the proofs and gave us valuable advice, and Mr. M. D. Frank, Managing Director of the North-Holland Publishing Company, for the interest he has shown in the publication of the book.

Jerusalem, June 1958

A. A. F.
Y. B.-H.

CHAPTER I

THE ANTINOMIES

§ 1. Historical Introduction

In *Abstract Set Theory* [1]) the elements of the theory of sets were presented in a chiefly *genetic* way: the fundamental concepts were defined and theorems were derived from these definitions by customary deductive methods. To be sure, some quasi-axiomatic ingredients were inserted there in the form of seven Principles, whose main purpose was the delimitation of the notion of set. The precise significance of these Principles will be discussed in detail in Chapter II of the present book.

At a few places in *Theory* [1]) (pp. 18, 132, 282, 301), special precautionary measures had to be taken in order to avoid certain *contradictions* that would have otherwise evolved. These contradictions, arising mainly in connection with a natural unrestricted use of the notions of cardinal number, ordinal, and Aleph, have been called *antinomies*, or *paradoxes*, of set theory. In general, we say that a certain theory contains an antinomy when each of two contradictory statements, or else one single compound statement having the form of an equivalence between two contradictory statements, has been proved within this theory, though the axioms of the theory seem to be true and the rules of inference valid.

Before we go into a detailed systematic investigation of the ways in which antinomies threaten the foundations of set theory, a few historical remarks are appropriate. Cantor's discoveries, starting around 1873 and slowly expanding to an autonomous branch of mathematics, had at first met with distrust and even with open antagonism on the part of most mathematicians and with indifference on the part of almost all philosophers. It was only in the early nineties that set theory became fashionable and began, rather suddenly, to

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[1]) By *Theory*, or by *T*, we refer to *Abstract Set Theory*, by A. A. Fraenkel, Amsterdam, 1953.

be widely applied in analysis and geometry. But at this very moment, when Cantor's daring vision seemed finally to have reached its triumphant climax, when his achievements had just received their final systematic touch, he met the first of those antinomies. This happened in 1895. The antinomy was not published immediately. Two years later, Burali-Forti rediscovered it. Though neither Cantor nor Burali-Forti were able at the time to offer a solution of the antinomy, the matter was not considered to be very serious: this first antinomy emerged in a rather technical region of the theory of well-ordered sets, and it was apparently hoped that some slight revision in the proofs of the theorems belonging to this region would remedy the situation, as had happened so often before in similar circumstances.

This optimism however was radically shattered when Bertrand Russell in 1902 surprised the philosophical and mathematical public with the presentation of an antinomy (see § 2) lying at the very first steps of set theory and indicating that something was rotten in the foundations of this discipline. But not only was the basis of set theory shaken by Russell's antinomy; logic itself was endangered. Only a slight shift in the formulation was required in order to turn Russell's antinomy into a contradiction that could be formulated in terms of most basic logical concepts.

To be sure, Russell's antinomy was not the first one to appear in a basic philosophical discipline. From Zenon of Elea up to Kant and the dialectic philosophy of the 19th century, epistemological contradictions awakened quite a few thinkers from their dogmatic slumber and induced them to refine their theories in order to meet these threats. But never before had an antinomy arisen at such an elementary level, involving so strongly the most fundamental notions of the two most "exact" sciences, logic and mathematics.

Russell's antinomy came as a veritable shock to those few thinkers who occupied themselves with foundational problems at the turn of the century. Dedekind, in his profound essay on the nature and purpose of the numbers [1]), had based number theory on the membership

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[1]) Dedekind 1888.

The literature cited in the present book, except the items preceded by an asterisk, refer to the Bibliography at the end of the book; the initial numerals 19 are dropped (Quine 55 for Quine 1955). The items preceded by an asterisk, e.g. *Dedekind 3, refer to the Bibliography of *Theory*; these items are of secondary importance for the present book.

relation — his method of "chains" may even be taken as a basis for the theory of well-ordered sets (cf. *T*, pp. 103, 319) — and had utilized the notion of a set in its full Cantorian sense for the proof of the existence of an infinite ("reflexive", cf. *T*, p. 40) set. Under the impact of Russell's antinomy, he stopped for some time the publication of his essay, the fundaments of which he regarded as shattered [1]). Still more tragic was Frege's fate: he had just put the final touches on his chief work [2]), after decades of tiresome effort, when Russell wrote him about his discovery. In the first sentence of the appendix, Frege admits that one of the foundations of his edifice had been shaken by Russell [3]).

It is not surprising that many mathematicians who had just begun to accept set theory as a full-fledged member of the community of mathematical disciplines reversed their attitude. This reversal is typically illustrated by the leading mathematician of the time, Poincaré, who himself had contributed to the propagation and application of set theory. For some years after 1902 he met Russell's own proposals for a rehabilitation of set theory (see Chapter III) with an air of mockery [4]).

Cantor himself, to be sure, did not for a moment lose faith in his theory in its full "naive" extent though he was unable to meet the challenge of Russell's antinomy. Other scholars professed not to be especially disturbed by this and other antinomies and, distinguishing between "Cantorism" and "Russellism" [5]), warned against attributing to the "artificially constructed" antinomies any decisive significance. It is however difficult to defend this attitude. Even if Burali-Forti's antinomy does not appear so long as one restricts himself to the ordinals of a few number-classes (*T*, p. 299), this cannot release the serious thinker from the obligation of scrutinizing the theorems that involve the *general* concept of an ordinal; and the contemptuous reference to the "artificial" character of many antinomies should be no more convincing than the claim, say, that every continuous

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[1]) See the preface of the 3rd ed. of Dedekind 1888; cf. *Dedekind **3**, p. 449.

[2]) Frege 1893-03.

[3]) But he set out immediately to repair the damage, though without success; cf. Frege 1893-03$_{II}$, pp. 253 ff, Geach-Black 52 (Preface and pp. 234 ff, especially note o on p. 243), Sobociński 49-50 (pp. 220 ff), Quine 55.

[4]) *Poincaré **5**, book 2.

[5]) See, e.g., *Schoenfliess 1$_{II}$, p. 7; **6**, pp. 250–255.

function has a derivative since continuous functions without derivatives are "artificial". It may be safely stated that, on the contrary, throughout mathematics — and other disciplines — the investigation of the most general notions, in all their unrestricted generality, has often proved to be of extreme value for the advancement of research. To think that difficulties could be overcome simply by disregarding the general case [1]) is somewhat naïve. Finally, to draw a sharp line between mathematics (which is fine) and logic (which a self-conscious mathematician should shun for the benefit of his soul) is less than useless: logic is constantly applied in mathematics, though this use is not often brought into the open and explicitly taken into account, and if one wishes to put restrictions on this application, as some intuitionists do (see Chapter IV), it is better to formulate these restrictions openly and clearly rather than leaving them in the dark.

It is true that the field of mathematical activity proper, both in analysis and in geometry, is not directly affected by the antinomies. They appear chiefly in a region of extreme generalization, beyond the domain in which the concepts of these disciplines are actually used. It is in general not difficult to take precautionary measures in order to avoid the dangerous region. This is the main reason why many mathematicians recoiled so quickly from the initial shock caused by the appearance of the antinomies. The very fact that one continued to speak of *paradoxes*, or *antinomies*, rather than of *contradictions* serves as an indication that deep in their heart most modern mathematicians did not want to be expelled from the paradise into which Cantor's discoveries had led them.

Nevertheless, even today the psychological effect of the antinomies on many mathematicians should not be underestimated. In 1946, almost half a century after the despairing gestures of Dedekind and Frege, one of the outstanding scholars of our times made the following confession:

> We are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world to-day, we have our "crisis". We

[1]) Or by saying that one is going to disregard it; see the witty exposition in *Jourdain **6**, pp. 75 ff. As Russell (08, p. 226) says: "One might as well, in talking to a man with a long nose, say «when I speak of noses, I except such as are inordinately long» which would not be a very successful effort to avoid a painful topic".

> have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and has been a constant drain on the enthusiasm and determination with which I pursued my research work [1]).

Though the present book is officially dedicated to the treatment of the foundations of set theory alone, the fact that set theory is one, and according to some even the only [2]), fundamental discipline of the whole of mathematics on the one hand, as well as part and parcel of logic on the other hand, will force us to interpret our topic very liberally and often go into a discussion of the foundations of logic on the whole and of mathematics on the whole. It is well known that many thinkers are at a loss to delimit the borderline between these disciplines. It is often said that set theory belongs to them simultaneously and forms their common link. We shall be in a better position to discuss this view later on.

Having decided that a treatment of the logico-mathematical antinomies is a task that cannot be dodged, we shall proceed, in the subsequent sections of this chapter, to classify the known antinomies as well as to exhibit some of the most significant ones; we shall also present a preliminary and informal analysis of these specimens and give some references to the abundant literature dealing with the antinomies in general.

§ 2. Logical Antinomies

Since Ramsey [3]) it has become customary to distinguish between *logical* and *semantic* (sometimes also called *syntactic* or *epistemological*) antinomies. The significance of this distinction will become clear in the following section. In this section, we shall present three antinomies of the first kind, viz. the antinomies named respectively after Russell, Cantor, and Burali-Forti.

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[1]) Weyl 46.
[2]) See, e.g., Bourbaki 49, p. 7.
[3]) Ramsey 26.

1. Russell's Antinomy. In 1903, Russell [1]) published the antinomy he had discovered the year before and communicated to Frege by letter. The same antinomy was simultaneously and independently discussed in Göttingen by Zermelo and his circle without however reaching the stage of publication.

It seems to make perfect sense to inquire, for any given set, whether it is a member of itself or not. For certain sets one would hardly hesitate to commit himself to saying that they are not members of themselves: the set of planets, e.g., is certainly not a planet itself, hence not a member of itself. For other sets, one would as little hesitate to regard them as being members of themselves: the set of all sets is an obvious example. Therefore it seems to make perfect sense to ask the same question with regard to *the set of all sets that are not members of themselves*. The answer to this question, however, is alarming: denoting the set under scrutiny by 'S', we see quickly that if S is a member of S, it belongs to the set of all sets that are *not* members of themselves, i.e. it is not a member of itself, but also that if S is not a member of S, it does *not* belong to the set of all sets that are not members of themselves, hence is a member of itself; taken together, we convince ourselves that S is a member of S if and only if S is not a member of S, a glaring contradiction, derived from most plausible assumptions by a chain of seemingly unquestionable inferences.

The careful reader might perhaps have felt that the case was overstated. He might object that the contradiction was derived, among other premisses, also from the assumption that there exists such a thing as the set of all sets that are not members of themselves, the set we called 'S', hence that we are entitled only to derive that *if S exists*, then S is a member of S if and only if S is not a member of S, from which would only follow the falsity of the antecedent, by *reductio ad absurdum*, hence only that S does not exist or, in colloquial terms, that there just ain't no such animal as S.

Though this objection is valid (and will be taken into consideration later in Chapter III), it does but little to reduce the paradoxical character of the result arrived at. That there should not exist the set containing all those objects that satisfy a certain seemingly very delimited condition — viz., that of not containing itself as a member — is probably not less repugnant to common sense than a plain contradiction.

A similar objection against another would-be logical paradox is not only valid but also conclusive. It is worthwhile to deal with this paradox, since it has not been generally recognized that there is indeed a decisive difference

[1]) Russell 03 (in particular § 78 and Chapter X).

between this paradox and Russell's which takes the sting completely out of it. In short one considers the man who, supposedly, shaves all and only those inhabitants of a certain village who do not shave themselves. Abbreviating the expression 'that inhabitant of the village who shaves all and only those inhabitants of the village who do not shave themselves' by '*b*', we arrive, by an argument which is completely analogous to that occurring in Russell's antinomy, at the conclusion that *b* shaves *b* if and only if *b* does not shave *b*. Noticing, however, that we are only entitled to infer that *if b exists*, then *b* shaves *b* if and only if *b* does not shave *b*, we could only derive that *b* does not exist, i.e. that there is no such inhabitant of a village who shaves all and only those inhabitants of that village who do not shave themselves, a result which — though perhaps somewhat surprising to the unaware bystander — is no more paradoxical than, say, the fact that there is no inhabitant of a village who is both more and less than fifty years old [1]).

The condition which the luckless "village barber" was supposed to satisfy simply turned out to be self-contradictory, hence unsatisfiable. (This fact was masked by the circumstance that the insertion of just one inconspicuous word would have made the condition a perfectly satisfiable one: '... all *other* inhabitants ...'.) The condition occurring in Russell's antinomy, on the other hand, does not seem at all to be self-contradictory; the non-existence of a corresponding set is, consequently, a disturbing and unfamiliar result.

The same careful reader who was supposed to make the above-mentioned and partially sustained objection might have also asked himself — recalling the contents of *Theory* — how the emergence of Russell's antinomy can there be obviated. Trying to *prove* the existence of the paradoxical set, he will have noticed that this proof is not forthcoming: the relevant Principle of Subsets (*T*, p. 22) only enables him to prove the existence of a set satisfying a given condition *if this set is a subset of a set already secured.* Russell's paradoxical set, however, cannot be proven to fulfil this additional condition.

Let it be very clearly stated at the outset that there was absolutely nothing in the traditional treatments of logic and mathematics that could serve as a basis for the elimination of this antinomy. We think that all attempts to handle the situation without any departure from traditional, i.e. pre-20th-century, ways of thinking have completely failed so far and are misguided as to their aim [2]). Some departure from the customary ways of thinking is definitely indicated, though it is by no means clearly determined where this departure should take place. Indeed, 20th century research into the foundations of logic and mathematics can be fruitfully classified in terms of the place of departure from the Cantorian approach. This attitude will be adopted in the following chapters.

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[1]) Cf. Helmer 34.

[2]) For one such typical attempt, see Perelman 36.

For the sake of historical completeness it should be admitted that certain misgivings as to the status of "self-referential" concepts, of which being-a-member-of-itself is an obvious specimen, were already voiced in the middle ages [1]). These misgivings, however, never were given the form of a clear proposal for a revision of the customary ways of thinking and expression.

Certain "philosophical" doubts as to the validity of the *tertium non datur*, the logical law of the excluded middle, of which free use was made in the derivation of Russell's antinomy, had already been uttered prior to the intuitionists (see Chapter IV), but again these doubts were nowhere formulated in anything approaching the way they have been expressed in the 20th century, and never until then was a non-Aristotelian logic developed to any tolerable degree of completeness and responsibility.

In order to show that Russell's antinomy is not a specifically mathematical one, depending perhaps on some out-of-the-way peculiarities of the concept of set, we shall briefly reformulate it in purely logical terms: It seems to make perfect sense to inquire of a property whether it applies to itself or not. The property of being red, for instance, does not apply to itself since (the property of being) red is surely not red, whereas (the property of being) abstract, being itself abstract, applies to itself. Calling the property of not applying to itself 'impredicable', we arrive at the paradoxical consequence that impredicable is impredicable if and only if impredicable is not impredicable. The property-theoretical (logical) variant is as paradoxical as the set-theoretical (mathematical) one [2]).

2. Cantor's Antinomy. According to Cantor's theorem (*T*, p. 94), the set $\mathbf{C}s$ of all the subsets of any given set s has a greater cardinal than has s itself. Consider now the set of *all* sets, call it U. Its "power-set" $\mathbf{C}U$, i.e. the set of all subsets of U, has then a greater cardinal than U itself, which is paradoxical in view of the fact that U by definition is the most-inclusive set of sets.

This antinomy was known to Cantor himself in 1899 though — ironically enough — it was published only in 1932 [3]). In June 1901, it came to the attention of Russell who under its stimulation proceeded to construct his own antinomy which is of course much more elementary — at least superficially so — since it makes no allusion to such technical concepts as subset and power-set.

The strong connection between Cantor's antinomy and Russell's

[1]) Cf. Bocheński 56, § 35; Salamucha 37.

[2]) Russell 03, p. 102.

[3]) Cantor 32.

antinomy should be clear to all who recall the proof given for Cantor's theorem in *Theory* (pp. 94–95).

3. Burali-Forti's Antinomy. As the last antinomy of this group — the qualifier 'logical' is in this case rather misleading — we shall mention the historically earliest one: It is named after Burali-Forti who published it in 1897 [1]). Cantor himself, however, discussed it as early as in 1895 and communicated it to Hilbert in 1896.

The formulation of this antinomy is extremely simple: according to the *Corollary to Theorem 2* (*T*, p. 282), the well-ordered set *W* of *all* ordinals has an ordinal which is greater than any member of *W*, hence greater than any ordinal.

Again in the development of set-theory, as presented in *Theory*, neither Cantor's nor Burali-Forti's antinomy are forthcoming since the existence of the relevant sets, viz. the set of all sets and the set of all ordinals, cannot be proven on the basis of the principles laid down there. The reader who guessed that the formulation given there to the Principle of Subsets was intended to obviate the emergence of these and other logical antinomies guessed correctly.

§ 3. Semantical Antinomies

A few years after the appearance of the antinomies mentioned in the previous section, antinomies of a somewhat different kind made their début. Again we shall treat here only a few of the more important ones.

1. Richard's Antinomy. This antinomy, published by Richard in 1905 [2]), is of special significance since it is a sort of caricature of Cantor's diagonal method (*T*, p. 64). Many variants of this antinomy are known; the following is one of the simpler ones.

Let us consider all those real numbers between 0 and 1 that can be uniquely characterized by sequences of English words of any finite (but unbounded) length, e.g. 'point eight', 'the positive square root of point zero seven four', 'the smallest number satisfying the condition that the sum of the square of this number and its product by point one equals point three'. Clearly there are only denumerably many

[1]) Burali-Forti 1897. Cf. *F. Bernstein **2**, *Jourdain **1**. For an exact formulation of Burali-Forti's antinomy, see von Neumann 25, note 24.

[2]) Richard 05, ***2**.

of such numbers. Let R be their set. R can then be enumerated. Consider any such enumeration. We now characterize a real number r as *that real number between 0 and 1 whose n-th digit after the decimal point is the cyclic sequent of the n-th digit of the n-th number in the enumeration under consideration* (where '1' is the cyclic sequent of '0', ..., and '0' the cyclic sequent of '9'). From an argument that is almost entirely analogous to that presented in T, pp. 64–65, it follows that r is different from all the members of R and is therefore not uniquely characterizable by a finite sequence of English words, in plain contradiction to the fact that r has just been characterized in this fashion, viz. by the italicized sequence of English words in the preceding sentence.

Berry's antinomy [1]), essentially only an instructive and ingenious simplification of Richard's antinomy, will not be discussed here since it has no additional theoretical interest and lacks the straightforward connection with the diagonal method that makes Richard's antinomy so especially embarrassing.

2. Grelling's Antinomy. In 1908, Grelling and Nelson [2]) called attention to the following antinomy which they regarded as only a variant of Russell's antinomy but which turned out to be essentially different from, though still remarkably analogous to, the paradox regarding the property of being impredicable.

Grelling's antinomy can be formulated very simply: A few English adjectives, such as 'English' and 'polysyllabic' have the very same property that they denote, e.g. the adjective 'English' is English and the adjective 'polysyllabic' is polysyllabic, while the vast majority, such as 'French', 'monosyllabic', 'blue' and 'hot', do not. Calling the adjectives of the second kind *heterological*, we immediately discover to our dismay that the adjective 'heterological' is heterological if and only if it is not heterological.

3. The Liar. Of this antinomy very many versions are known, among them quite a few that are not truly paradoxical at all. Some of these versions go back to antiquity, to the time when the Megaric philosophers used them to tease the members of Plato's academy [3]). We shall present here one of the more recent versions.

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[1]) Published for the first time in Russell 06.

[2]) Grelling-Nelson 08.

[3]) Cf. Bocheński 56, § 23.

Assume that John Doe utters on December 1st, 1956 the following English sentence and nothing else all day: "The only sentence uttered by John Doe on December 1st, 1956 is false". Since this sentence is a declarative sentence, with nothing elliptical (like "The only sentence uttered by John Doe on December 1st is false") or context-dependent (like "The only sentence uttered by him on December 1st, 1956 is false") about it, one seems entitled to inquire whether this sentence is true or false. However, one realizes before long that the sentence is true if and only if it is false.

Against this antinomy one might raise the objection that it is based on a factual assumption, viz. that John Doe did utter a certain sentence, and nothing else, on a certain day. This is true enough but does little to diminish the paradoxical result. Besides, it has been shown that an analogous antinomy can be constructed which does not rely on any factual assumptions [1]).

§ 4. General Remarks

We have had no intention of presenting an exhaustive description of all the antinomies that have turned up in foundational research during the last sixty years [2]). Among those not treated here so far, the most important is Skolem's paradox because of its basic significance in axiomatic set theory. But just for this reason its exposition and discussion will be postponed to Chapter II, § 6.

We have already remarked that only few mathematicians were seriously disturbed by the appearance of the antinomies. But even among those mathematicians who were alert to the crisis in the foundations of their discipline, brought about by the emergence of the antinomies, the great majority shared Peano's opinion that *Exemplo de Richard non pertine ad mathematica, sed ad linguistica,* from which fact they concluded that qua mathematicians they need not bother about Richard's antinomy and the semantic antinomies in general. Indeed, semantic terms like 'denote', 'characterize', or 'true' are necessary ingredients of these antinomies, and these are not terms about which an ordinary mathematician will feel obliged

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[1]) See Tarski 44, note 11.

[2]) For an enumeration and careful description of some twelve antinomies, including the six treated here, see Beth 55, Livre V.

to think very hard. However in one of the most interesting developments in modern foundational research it became clear that the problem presented by the semantic antinomies was not only a methodological one of at most indirect relevance to mathematics proper, but rather served as the starting point for investigations of immense direct impact on modern mathematics. How this came about will be discussed in Chapter V.

The literature dealing with the antinomies is very extensive. Whereas for the first few years after the publication of Russell's antinomy they were discussed chiefly by mathematicians, they later began to attract the attention of logicians, methodologists, and philosophers at large in an ever increasing measure. Much of this literature is concerned with piecemeal solutions of the various antinomies, exhibiting no general methodological insight and often contradicting each other. Some of them are based on misunderstandings and errors [1]), others lose themselves in epistemological or metaphysical considerations far from the point. On the whole it seems that though a piecemeal solution might occasionally be appropriate with regard to antinomies emerging in the context of ordinary language [2]), insofar as they refer to language systems nothing short of the profound investigations described in the following chapters will do. Exhaustive bibliographical remarks will be found in § 6. Here we shall confine ourselves to a few general comments.

All the antinomies, whether logical or semantic, share a common feature that might be roughly and loosely described as *self-reference.* In all of them the crucial entity is defined, or characterized, with the help of a totality to which it belongs itself. There seems to be involved a kind of circularity in all the argumentations leading up to the antinomies, and it is obvious that attempts have been made to see therein the culprit. However a wholesale exclusion of all reasonings involving any kind of self-reference is certainly too strong a medicine and would throw away the baby with the bath-water. There are innumerably many ordinary ways of expression that are self-referential but still perfectly harmless and useful [3]). To characterize someone

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[1]) Typical is, e.g., *Koyré 1, criticized in Bar-Hillel 47.

[2]) A recent attempt for a solution of Liar-type antinomies in the framework of ordinary language was made by Bar-Hillel 57a.

[3]) For a witty and persuasive defence of self-referential reasoning, see Popper 54.

as the tallest man on a certain team is doubtless utterly innocuous as well as effective, in spite of the fact that the characterization is performed on the basis of a totality to which the man himself belongs. And many a crucial concept in mathematics — as in every other discipline — is formed in a similar fashion. Not all self-reference leads to contradiction, and some self-reference seems to be an indispensable tool in science as in everyday life.

Since the wholesale exclusion of self-referential concept formation is then apparently not feasible, many authors looked for an additional criterium that would separate the sheep from the goats. We shall deal with some of these attempts in Chapter III. Here we shall mention only one such proposal. It amounts, in essence, to disqualifying those would-be concepts whose elimination, on the basis of their definition, would lead to infinite regress [1]); in positive terms, to accepting into the community of scientifically legitimate concepts only those applicants for which finite eliminability can be shown. Without entering here into a detailed discussion, it should be remarked that this proposal, even if effective in overcoming all known antinomies, suffers from the following defect: the proof of finite eliminability, though often extremely tedious, will nevertheless have to be produced from scratch for *every single newly introduced concept.* It is doubtful whether mathematics could stand such a severe imposition. It is therefore understandable that this proposal could not dissuade other authors from looking for more efficient and practicable remedies.

For those mathematicians who believe in the essential soundness of classical mathematics, the task posed by the antinomies is that of constructing a system in which all the notions of classical mathematics can be defined and all (or essentially all) the theorems of mathematics up to and including analysis can be derived but such that its consistency can be proved or, short of this, such that the argumentations leading to the known kinds of antinomies are effectively excluded. It seems that the achievement of this task will require some radical changes in the "naïve" attitude that is still prevalent among most mathematicians. It might not be necessary to abandon the belief in the essential soundness of classical analysis — as the intuitionists would advise us to do — but one might be persuaded to leave the paradise into which Cantor has led the mathematicians and to with-

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[1]) This is "Behmann's solution", proposed in Behmann 31.

draw into a less opulent but more secure habitat. Those unwilling to do this might perhaps prefer to stay in the realm of plenty and build walls around it to keep away the beastly antinomies without, however, being certain that some of these beasts were not walled in themselves. This theme will be developed in a more prosaic way in the following chapters.

§ 5. The Three Crises

The twentieth century is not the first period in which mathematics underwent a foundational crisis. It might add to the perspective in which contemporary antinomies should be looked upon if prior crises are, if only briefly, sketched.

In the fifth century B.C., only a short time after mankind attained one of the most brilliant achievements in its history, viz. the development of geometry as a rigorous deductive science, two discoveries were made that were extremely paradoxical: the first was that not all geometrical entities of the same kind were commensurable with each other, so that, for instance, the diagonal of a given square could not be measured by an aliquot part of its side [1]) (in modern terms, that the square noot of 2 is not a rational number); the other were the paradoxes of the Eleatic school (Zenon and his circle) developing with many variations the theme of the non-constructibility of finite magnitudes out of infinitely small parts [2]).

This crisis shocked the Greek mathematicians into obtaining two more brilliant achievements [3]): the *theory of proportions*, as contained in books 5 and 10 of Euclid's *Elements*, and the *method of exhaustion*, as invented by Archimedes, that was nothing less than a strict, though not sufficiently general, forerunner of modern theories of integration. Their theory of proportions should have enabled the Greeks to define *irrational number* and develop, accordingly, an arithmetical theory of the *continuum*; somehow they did not quite make it.

The Greek theory of proportions was soon forgotten — so much

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[1]) The profound impression made by this discovery may be gathered from Plato's report in *Theaitetos* that Theodoros had proved the irrationality of the square roots of 3, 5, .., 17, a result that was later generalized by his pupil Theaitetos; cf. *A. E. Taylor **1**, *Toeplitz **2**, Reidemeister 49.

[2]) See *T*, p. 11, note 2 and Grünbaum 52, 55 (with further references).

[3]) Cf. *Hasse-Scholz **1**, *Scholz **3**, van der Waerden 54.

so that when rigorous arithmetical theories of irrational numbers were constructed in the second half of the 19th century, one was not at first aware of the fact that these methods were not in principle much different from those already in the possession of the Greek mathematicians two thousand years earlier. Before that, in the 17th and 18th centuries, the great power and fruitfulness of the newly invented calculus led most mathematicians of those times into feverish applications of the new ideas without caring much for the solidity of the basis upon which the calculus was founded [1]). However, the shakiness of this basis became clear at the beginning of the 19th century, constituting *the second crisis in the foundations of mathematics.*

In order to overcome this crisis, Cauchy, in the eighteen thirties, showed how to replace the irresponsible use of infinitesimals by a careful use of limits, whereas Weierstrass and others, in the sixties and seventies, demonstrated how all of analysis and function theory could be "arithmetized". This solidification of the foundations was so successful that Poincaré, in an address delivered in 1900 before the Second International Congress of Mathematicians on the rôle of intuition and logic in mathematics, could proudly claim that mathematics had by then acquired a completely solid and sound basis. In his own words: "Today there remain in analysis only integers and finite or infinite systems of integers ... Mathematics ... has been arithmetized ... We may say today that absolute rigor has been obtained" [2]).

Ironically enough, at the very same time that Poincaré made his proud claim, it had already turned out that the theory of the "infinite systems of integers" — nothing else but a part of set theory — was very far from having obtained absolute security of foundations. More than the mere appearance of antinomies in the basis of set theory, and thereby of analysis, it is the fact that the various attempts to overcome these antinomies, to be dealt with in the subsequent chapters, revealed a fargoing and surprising divergence of opinions and conceptions on the most fundamental mathematical notions, such as set and number themselves, which induces us to speak of the *third foundational crisis* that mathematics is still undergoing.

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[1]) Typical for this attitude is d'Alembert's famous dictum: *Allez en avant, et la foi vous viendra.*

[2]) Poincaré 02.

§ 6. Bibliographical Remarks

Since the date of the publication of Russell's antinomy (1903), the literature dealing with the logical and semantic antinomies has been on an almost constant increase, and there is still no slackening of this interest in sight. Some of these publications dealt with the antinomies in a monographical fashion, occasionally even treating one antinomy at a time, in others the treatment of the antinomies was imbedded into a larger background.

The policy adopted in this section is as follows: Of the early literature — up to 1918 — dealing with the antinomies (when we say 'antinomies', we now mean the logical and semantic antinomies exclusively) only the most important items are listed. For the later literature dealing with this topic in a monographical fashion, a high degree of completeness is aspired to, though short notes of little incisive value, reviews, translations, encyclopedia articles, popular expositions, and dissertations are mostly disregarded. Since every advanced textbook on logic will deal with the antinomies, we shall mention here only those books which — in our opinion — excel in certain respects, e.g. with respect to the degree of clarity of presentation. We will not mention here those publications that deal with the antinomies from points of view that can only be understood after a mastery of the following chapters. Solutions depending, for instance, on the approach through types or semantical categories will be mentioned in Chapter III.

The arrangement will, in general, be temporal, but occasionally different writings of the same author will be joined together. Major contributions will be indicated by bold type. We are heavily indebted to the excellent *Bibliography of Symbolic Logic* compiled by Church and published in *The Journal of Symbolic Logic*, vol. 1, 121–218, vol. 3, 178–192, as well as to the *Index of Subjects*, vol. 3, 193–204, and the various *Indexes of Reviews* (by authors, biennially, published in the December issue of the odd years; by subjects, quinquennially. published in the December issue, starting with 1940) prepared by him,

a. *Antinomies in general*, from a logico-mathematical point of view.

B. Russell 06 (esp. pp. 30–36), 06a, 08, **Hessenberg** 06 (chs. 23 and 24), **Korselt** 06, *2, *Jourdain 3, *Schoenflies 2, 5, *B. Levi 2, **Poincaré** 08, **Whitehead-Russell** 10–13$_{\text{I}}$, *Hagström 1, **J. König** 14, Poli 14, *Korselt 4, *Enriques 2, **Mirimanoff** 17, 17–20, Czeżowski 18,

*F. Bernstein 6, *Grelling 2, *Hölder 3, 4, *Langer 1, **Finsler** 25, 26, 27, **Weyl** 25, 26, Skolem 26, *Brodén 4, *Zaremba 2, *Oikonomou 1, **Ramsey** 26, *3, *5, *Hobson 2_I, *Härlen 1, Fraenkel 27, **Behmann** 27, *5, 31, *Pierpont 2, Hilbert-Ackermann 28/149, **Menger** 28 (pp. 298 ff), *Kamke 3, *Matsumoto 1, Schmeidler 29, *Study 3, Zermelo 30, *Skolem 9, *Fraenkel 13, *Jørgensen 1_{III}, *Cassina 2, **Lewis-Langford** 32, **Church** *4, 34, *6, *Dassen 1, Gonseth 33, *B. Levi 6, 8, **Black** 33, *Menger 9, Skolem 34a, **Helmer** 34, **Carnap** 34, 37, **Curry** 34, 36, **Kamke** 34, *Mania 2, *Andreoli 1, *Albergamo 1, *Fitch 1, **Church** 36, Quine 37, *P. Lévy 5, **Reach** 38, *Burkhardt 1, 2, **Beth** 39, *Fraenkel-Bar-Hillel 1, **Bočvar** 39, Löwenheim 40, *Bar-Hillel 1, *Beth 11, **Curry** 42, *Toranzos 1, *E. P. Northrop 1, Feys 44, Péter 44, *Dumitriu 1, **Gödel** 44, *Bouligand 7, Novikov 47, Zich 47, Geymonat 47, Reichenbach 47, **Mostowski** 48b, Vaccarino 48, Saarnio 49, **Stenius** 49, **Halldén** 49, B. Levi 49, P. Lévy 50, Feys 50, Skolem 50, Schmidt 50, Rosenbloom 50, Curry 51a, Goodstein 51, **Kleene** 52, **Wilder** 52, **Hermes-Scholz** 52, **Rosser** 53, Valpola 53, Specker 54a, Shaw-Kwei 54, Geach 55, **Beth** 55, Moch 56, 56a, **Ladrière** 57.

b. *Antinomies in general,* from a philosophical, informal point of view (the distinction between the two points of view is, of course, vague and subjective):

*Urbach 1, *Samuel 1, *Dingler 2, H. C. Brown 11, **Lipps** 23, *4, *Horák 1, *Betsch 1, *Dieck 1, 2, *Becker 2, **Weiss** 28, *Urbach 1, *Heiss 1, **Kaufmann** 30, *2, *Cassirer 4, *Lafleur 1, *Dingler 1, *Dubislav 2, Gallucci 31, *Burkamp 4, *Zawirski 1, v. Brandenstein 35, *Warrain 3, De Cesare 38, *Vredenduin 4, *van Os 1, Grebe 40, Ushenko 41, *10, *Segelberg 1, *Fitch 11, *Koyré 1, Kattsoff 48, Black 48, Thompson 49, Ladrière 49, Lense 49, Mora 51, Aléksandrov 51, Jørgensen 53, Popper 54, Knaus 54, Wang 55a (pp. 237–238).

c. *Behmann's and Perelman's proposed solutions:*

Behmann 31, *Härlen 6, **Perelman** 36, *3, Grelling 36, 37, *Behmann 8, **Beth** 39, *Fraenkel-Bar-Hillel 1, *Ackermann 11, *Bröcker 1, Vredenduin 40, **Ackermann** 41, *Whittaker 1, **Bočvar** 44, Halldén 49, **Stenius** 49, **Behmann** 55.

d. *(Cantor-)Russell type antimomies:*

Frege $1893–03_{II}$ (Appendix), **Russell** 03, *2, Leśniewski 14, *Guthrie 1, *Eklund 1, *Brodén 1, 5, *Behrens 1, Wittgenstein 22,

Hilbert 25, **Leśniewski** 27–31$_{I}$,29, *Gonseth 4, *Winants 1, Bennecke 34, Vredenduin 40, *Basileioy 1, *Hoensbroech *2*, *Quine *23*, **Curry** 42, *Whittaker 1, **Bočvar** 44, 45, Wedberg 45, **Sobociński** 49–50, **Yuting** 53, Stanley 53, Quine 55, Prior 55 Montague 55.

e. *Burali-Forti's antinomy:*

Burali-Forti 1897, **Russell** 02/06, *Jourdain 1a, *Hobson 1, *Dingler 1, *Hagström *2*, *Cavaillès 3, Quine 41, Rosser **12*.

f. *Grelling's antinomy:*

Grelling-Nelson 08, **Weyl** 18, *Grelling *2*, *J. R. Weinberg *2*, Saarnio **2*, 38, Curry 42, Geach 48, **Copi** 50, Lawrence 50, Ryle 51, Gregory 52, Bowden 52, Landsberg 53, Mackie-Smart 53, **Copi** 54, Meager 56.

g. *Richard-type antinomies:*

Richard 05, **2*,* J. König *2*, Peano 02/06, *Borel 4, **Chwistek** 21, *1, *5, **Weyl** 25, Tarski 31, **Church** 34, **Chwistek** 35, 48, **Kleene-Rosser** 35, Hilbert-Bernays 34–39$_{II}$ (pp. 263–268), Curry 41a, Borel 46, **Mostowski** 46, Zich 47, Bockstaele 49, Mostowski 52, Denjoy 46–54$_{I}$.

h. *Liar-type antinomies:*

*Bolzano 2$_{1}$ (p. 78), *Peirce *2* (II, 370–371; V, 190–222), **Rüstow** 08/10, *Lipps *2*, *H. B. Smith 7, **Tarski** 35/36, *MacIver 1, Ushenko 41, **9*, **10*, *Finsler 7, *Beth **12*, *Koyré 1, Bar-Hillel 47, Langford 47, Ellis 51, Ayer 53, **Quine** 53 (ch. VII), Popper 54, Stroll 54, Evans 54, Geach 55, Löb 55, Yuting 55, Shwayder 56, Meager 56, **Bocheński** 56, Tarski 56, Bar-Hillel 57.

CHAPTER II

AXIOMATIC FOUNDATIONS OF SET THEORY. THE AXIOM OF CHOICE

§ 1. Introduction

The antinomies show that the naive concept of set as appearing in Cantor's "definition" of set (*Theory*, pp. 6 and 15–18) and in the most general conclusions derivable from it cannot form a satisfactory basis of set theory, much less of mathematics as a whole [1]). One may compare this function of the antinomies as controlling and restricting the deductive systems of logic and mathematics to the function of experiment as controlling and modifying the semi-deductive systems of sciences like physics and astronomy. For the purpose of a more solid foundation of set theory it makes no fundamental difference whether we consider the antinomies a disaster whose appearance perforce compels us to look for a different and safer basis, or a (welcome) symptom of an illness which otherwise might have been overlooked; the kind and the degree of the measures to be taken is not necessarily influenced by our position towards the above alternative [2]).

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[1]) Cantor himself recognized this fact after having concluded his work; see his letters to Dedekind of 1899 (Cantor 32, pp. 443-448) where he speaks of *inkonsistenten Mengen*.

[2]) In the beginning of Mostowski 55, one of the most modern and profound expositions of certain problems of the foundations of mathematics, the following illuminating sentences are found regarding the foundations, set theory, and the antinomies. "The present stage of investigations on the foundations of mathematics opened at the time when the theory of sets was introduced. The abstractness of that theory and its departure from the traditional stock of notions which are accessible to experience, as well as the possibility of applying many of its results to concrete classical problems, made it necessary to analyze its epistemological foundations. This necessity became all the more urgent at the moment when antinomies were discovered. However, there is no doubt that the problem of establishing the foundations of the theory of sets would have been formulated and discussed even if no antinomy had appeared in the set theory."

Since the beginning of the present century various ways have been taken to procure a firmer fundament of set theory. Most of them can be classified into three groups each of which divides into several subgroups, viz. the axiomatic, the logicistic, and the intuitionistic attitudes. This order of arrangement may be considered to proceed from more conservative to rather revolutionary attitudes, though the logicistic frame comprises widely different degrees of radicalism. The arrangement is, however, not a historical one; curiously enough, the first and decisive steps in each of these three directions were taken simultaneously and independently during the years 1906–1908. The present and the two following Chapters exhibit some main features of the attitudes mentioned in the above order, while the fifth (last) Chapter deals with more general problems which to a certain extent are common to all of them, though most of these problems have first grown on axiomatic soil.

The axiomatic method in mathematics, emerging with great perfection in Euclid's *Elements* (c. 300 B.C.) and revived only in the course of the 19th century (again in geometry), has developed impetuously since the beginning of the present century; most fields of mathematics and logic and some other scientific theories have since been axiomatized. A general description of the axiomatic method and the problems connected with it is given in Chapter V. The second part of the present section contains a superficial sketch which is sufficient for the understanding of the axiomatic systems described in this Chapter.

The most important directions taken in axiomatic set theory are those of Zermelo and of his early successors, later those of von Neumann, Bernays, and Gödel, of Quine (*New Foundations* and *Mathematical Logic*) and Wang. An exposition of the former directions is given in this Chapter while Quine's and Wang's methods, essentially based upon modern logic, are explained in Chapter III which deals with the logicistic foundations of set theory [1]). On the other hand we shall incorporate in the present Chapter (§ 6) a short exposition of the theorems of Löwenheim-Skolem(-Gödel) which refer not to a specific axiomatization but to a large class of axiomatic systems. A few remarks regarding this point will be useful here.

Cantor's set theory endeavors to explore the "transcendental" domain

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[1]) Chwistek's system (especially 24-25) belongs also to this logicistic direction.

of (finite and) infinite sets (cf. *T*, p. 108, footnote 1) and in this attempt becomes involved in the antinomies. Now an axiomatization of set theory may possibly mean accepting the domain of transcendence as such but restricting it in a way that apparently does not leave room for contradictions. In particular, a set's property of being non-denumerably infinite would then remain an absolute one. On the other hand, a system of axioms may be conceived without any reference to a domain of transcendence; the theorems derived from the axioms are then meant to describe a situation created by the very restrictions of the axioms without an absolute background. In this light, proving a set *S* to be non-denumerably infinite would only mean that the one-to-one mappings which are at our disposal on account of the axioms do not permit to map *S* onto the set of all integers — while in an axiomatic set theory comprising richer means of expression and construction *S* might well turn out to be denumerable. Which of these two views is taken does not depend on the nature of the axioms involved but rather on our attitude towards the axiomatic method in general; it is remarkable that the theorem of Löwenheim-Skolem contains strong arguments in favor of the second, the "constructive" view as against the "transcendental" one. Nevertheless, it is difficult to demonstrate the correctness of either attitude by purely logico-mathematical arguments — analogically as experiments and theories of physics cannot conclusively prove the existence or non-existence of "objects" of an external world to whose phenomena the experiments refer. A pragmatic if not logical argument in favor of the transcendental attitude is the observation that assuming the existence of an absolute non-denumerable continuum makes mathematics much simpler and easier, just as the outlook of physics is presumably simplified by the hypothesis of the existence of physical bodies.

In fact it is the continuum that stands in the foreground of mathematics and not the alephs and general well-ordering or even the second number-class and transfinite induction. The theory of real numbers and real functions upon which the nineteenth century based calculus and analysis in general, deals with sets of real numbers, i.e. sets of sets of integers; in other words with "arbitrary" subsets of the power-set of the set of natural numbers.

We begin the exposition of axiomatic set theory with a detailed development of a *modified form of Zermelo's system,* while the systems of von Neumann and Bernays are treated in their characteristic

features only, with emphasis on those points that distinguish them from Zermelo's. Special attention will be paid to the nature and the implications of the *axiom of choice* which is common to Zermelo's and Bernays' systems and which has throughout half a century formed a focus of discussions. One of our reasons for giving preference to Zermelo's system is that in the preceding book *Abstract Set Theory* the bulk of Cantor's set theory was based on seven Principles which are closely related to Zermelo's axioms. In that exposition a few remarks only hinted at the possibility of deriving the theory from the principles while here (in § 8) the task is actually carried out in its main lines. (It requires a far more intricate and lengthy exposition to achieve this task when the axioms of von Neumann or Bernays are taken as the starting-point.)

Zermelo's system [1]), with certain modifications in various directions, has retained its significance up to this day, though there are strong arguments in favor of other axiom systems developed later. A quasi-Zermelian system has been used to base logic without a theory of types [2]), and Bourbaki's suggestion of a basis sufficient to develop mathematics as a whole [3]) is largely motivated by Zermelo's axioms.

Of other attempts [4]), less elaborate and successful, to base set theory upon a system of axioms three shall be mentioned briefly.

The method of Schoenflies [5]) (1920) takes as the essential primitive concept the part-whole relation ('x is a part, or a proper subset, of y') instead of the membership relation (see below). But a close examination [6]) shows that even if we disregard some technical defects, in

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[1]) Zermelo 08a. Cf. the expositions in Fraenkel 27 and 28, Ackermann 37, Skolem 52.

[2]) Ackermann 37a. A lucid exposition of axiomatic set theory in the frame of logic is given in the second half of Church 40/41.

[3]) Bourbaki 49, particularly pp. 7–8 (cf. Myhill 51), and 54.

[4]) The method of *Ting-Ho 1 has an intention similar to Zermelo. However, his treatment is not strict and clear enough to allow an exact comparison. Cf. *Giorgi 1.

[5]) Schoenflies 21; cf. ***7** and ***10**. The idea of replacing the membership relation by the part-whole relation is also the basis of *Foradori **1, 2, 4**.

[6]) For a careful criticism of the method see Merzbach 25, chs. I–III. Quite recently Schoenflies' idea has been renewed and transformed into an axiomatically serviceable form in Wegel 56 (which paper appeared after the completion of the manuscript).

this way at best a theory of magnitude can be obtained which does not provide — even by an axiom introduced *ad hoc* — for the peculiar properties of "irreducible parts", i.e. members. This also applies to finite sets, whose structure becomes extremely complicated; for instance, a finite set may have infinitely many parts.

The theory of Finsler [1]) is based on three axioms only. From the first two axioms it follows [2]) easily that any consistent model of the Finsler system admits of a further extension, a result which is neither surprising nor peculiar to this axiomatic system [3]). On the other hand, the third axiom postulates the completeness of the system in a sense analogous to Hilbert's completeness axiom for geometry. But in view of the result just mentioned the third axiom entails a contradiction, as would Hilbert's system of axioms without the Archimedean axiom (*T*, p. 165). Therefore, even apart from other arguments — the most serious of which concerns the introduction and wide use of the dubious notion *zirkelfrei* (non-circular) — this way cannot be considered to form a possible basis for set theory.

With respect to the method of avoiding antinomies in the axiomatic system of Gonseth [4]) we refer to what has been said on p. 13 regarding "Behmann's solution". Most characteristic of Gonseth's attitude is his rejecting the assumption that, given a set, it should be definite for any object whether it belongs to the set or not. In consequence the fundamental theorems on the non-equivalence of sets forfeit their validity. As a matter of fact, an actual development of either the classical or a more restricted kind of set theory or even of classical analysis from this axiomatic basis has not been carried out [5]).

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[1]) Finsler 26; cf. ***2**, ***5**, ***6**, *Burkhardt **1,2**, *Locher **1**.

[2]) See Baer 28.

[3]) The same applies, for instance, to Hilbert's axioms for geometry when one drops the axioms of continuity. Cf. Kamke 34.

[4]) Gonseth 33; cf. ***4** and ***5**.

[5]) The criticism of Kamke 34 against an axiomatic foundation of set theory does not seem substantial; cf. Chapter V.

For the attitude of Lorenzen see Chapter III.

Just when the present Chapter went to print, the interesting paper Ackermann 56 appeared that suggests founding set theory upon four principles, which roughly yield a theory based on the axioms of §§ 2 and 3 and VII and VIII of § 5. The decisive (and possibly problematic) point is the third principle which, in a way different from that of §§ 6 and 7, states under which conditions a "collection" of sets is itself a set.

The axiom of choice is omitted by Ackermann as it is supposed to have a

Before presenting Zermelo's system a few explanations regarding the axiomatic method in general are in order; a more extensive treatment will be given in Chapter V [1]).

Every axiomatic theory (with the exception of axiomatic theories of logic itself) is constructed by adding to a certain *basic discipline* — usually some system of logic (with or without a set theory) but sometimes also a system of arithmetic — new terms and axioms, the *specific* undefined terms and axioms of the theory under consideration. However, mathematicians are in general not used to making the underlying basic discipline explicit. They assume that the interpretation of the "logical" words and phrases they employ, such as 'not', 'and', 'if...then', 'all', etc., as well as their performance within deduction, is well known and not in need of special discussion. Yet lately it has become increasingly clear that this happy-go-lucky attitude towards the basic discipline is not quite safe. At any rate, with respect to axiomatic set theory where antinomies are always lurking in the background an explicit taking into account of the basic discipline is now almost universally accepted. This may be done in various degrees of depth and rigor. A complete exposition of the discipline presupposed in our further treatment of axiomatic set theory is out of question if only for reasons of space. We shall employ a somewhat uneasy compromise and describe the basic discipline in general terms only, referring the reader who is interested in details to the ample literature in existence [2]).

In addition to being formalized, the language in which the axiomatic theory is formulated may also be symbolized, i.e. artificial symbols may be used instead of the words of a natural language. A complete symbolization — in contradistinction to a partial symbolization to which every mathematician is accustomed in his daily work — though certainly involving a further increase in rigor and facility of mechanically checking proffered proofs and derivations, is something which for its effective use requires a preliminary training that can be neither presupposed nor required of the average reader. The use of logical symbolism in the main body of this book will therefore be restricted to a minimum,

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logical rather than a mathematical character. This attitude disregards some of the problems raised in § 4, in particular that of the independence of the axiom of choice.

[1]) McNaughton 57 contains specific remarks on the axiomatic method in set theory.

[2]) See for instance Rosser 53 or Church 56. In Church's terminology, we are proceeding according to the informal axiomatic method rather than according to the formal axiomatic method; cf. Church 56, p. 57.

mostly for the formulation of the axioms and of some definitions. (A more extensive use of such symbolism will be made in the remarks and discussions printed in petit, the reading and understanding of which is not necessary for grasping the argument presented in the main body.)

For our purpose of constructing an axiomatic set theory the basic discipline — unless otherwise stated — is assumed to be the so-called *functional calculus of first order*. While we refer the reader, for a complete and rigorous description of this calculus, once for all to Church's excellent recent textbook [1]), we mention here only that this calculus contains a notation sufficient for expressing the logical operations of negation, conjunction, disjunction, (material) implication, (material) equivalence [2]), universal and existential quantification. For these operations we shall use the symbols '$\sim$', '&', 'V', '$\supset$', '$\equiv$', '(A.)...', and '(E.)...', respectively (where in place of the single dot after the "quantifiers" 'A' and 'E' stands some variable and the three dots stand for some statement in which this variable occurs free, i.e. not operated upon by a quantifier; in short, for some statement *free in* this variable).

The language in which one deals with the *expressions* of a given theory (not with the entities denoted by these expressions!) is called the *metalanguage* of this theory. In our case the metalanguage will be ordinary English, supplemented by a few symbols and some rules governing their use. The language in which the theory itself is formulated is called the *object-language* of this theory. In our case the object-language is a certain extremely restricted sub-language of ordinary English, again supplemented by a few symbols and their rules.

As stated above, the object-language of a given theory is sometimes an artifical symbolic language. Only in very rare cases, however, when an extraordinarily high degree of preciseness and rigor is indicated or for certain very special purposes is the metalanguage itself taken to be a symbolic language, in which case its metalanguage, the meta-metalanguage of the theory, is still some ordinary language.

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[1]) Church 56.

[2]) This notion of (material) equivalence has of course nothing to do with the set-theoretical relation of equivalence between sets (cf. *T*, p. 31, and below § 8). The reader should be warned not to confuse this object-linguistic notion with the meta-linguistic notion of logical equivalence which is a relation between statements and for which we use the expression *'equipollent'* (also to avoid a confusion with the set-theoretical relation of equivalence). Cf. Quine 51, p. 30, and Carnap 46, p. 35.

In order to refer in the metalanguage to particular expressions of the theory under discussion, names or other designations of these expressions have to be used. This can be done in various ways [1]). One of the simplest is to employ quotations marks. Often, however, particular signs of the metalanguage are utilized and even more often those expressions themselves are used for this function. This last method in which some expressions are doing double duty, first as normal signs for something different from themselves and second as autonymous signs for themselves, is not without dangers. Since this method is, however, the one favored by almost all mathematicians, we shall use it when the other more exact methods would look pedantic and when no misunderstanding will be likely to arise.

The situation is somewhat more complicated when reference has to be made not to particular expressions of the theory but to classes of such expressions, e.g. to all expressions of a certain kind. In order to do this in a rigorous fashion metalinguistic variables have to be used. (A name of the object-linguistic variable 'x', such as "x" or 'the last-but-two letter of the English alphabet', is of course a metalinguistic constant and not a variable itself.) Various methods are in use and again we decide not to employ in general the most rigorous ones, such as the use of Greek or Gothic signs [2]), of boldtype symbols [3]), or of quasi-quotes [4]), but to rely on the context on the one hand and on the common sense and good-will of the reader on the other. There will be a certain amount of inconsistency in all these matters, but this is probably to be preferred to a usage that would appear overpedantic to most readers.

The use of Greek signs can be illustrated in the following symbolic formulation of a *formation rule* of elementary logic:

For any statements φ and ψ, $\sim\varphi$, $\varphi \,\&\, \psi$, $\varphi \vee \psi$, $\varphi \supset \psi$, and $\varphi \equiv \psi$ are statements. (Notice that in this rule the connectives are used autonymously!)

Sometimes we shall distinguish between *closed* statements that contain no free variables (a variable is called *free* if it does not lie within the scope of a quantifier) and *open* statements that contain at least one free variable.

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[1]) For an extensive treatment of this question see Carnap 37, § 41.

[2]) As used by Carnap 37.

[3]) As used by Church 56.

[4]) As used by Quine 51.

When we deal with a symbolized system we often use '(well-formed) formula' instead of 'statement'.

In the formula '(Ey) $x \in y$' the variable 'y' is bound by the existential quantifier '(Ey)' but the variable 'x' is free; the formula is therefore open. In the formula '(Ax) (Ey) $x \in y$' all variables are bound, 'y' as before and 'x' by the universal quantifier '(Ax)'; the formula is therefore closed.

An open statement free in 'x' will also be called, according to mathematical custom, a *condition on* x [1]) and sometimes rather loosely a *predicate* (of x). In general, however, we shall understand by a predicate rather an expression that is obtained from an open statement by the erasure of its free variables. We shall, then, regard 'x is an even integer' as a condition on x, and 'is an even integer' or 'even integer' as a (*monadic*, singulary, one-place) predicate; similarly 'x is a member of y' as a condition on x and y, and '(is a) member of' as a (*dyadic*, binary, two-place) predicate. (Sometimes 'function' is used for 'predicate'.)

An axiomatic system is in general constructed in order to axiomatize a certain scientific discipline previously given in a pre-systematic, "naive", or "genetic" form. The primitive, undefined terms of the system are meant to denote some of the concepts treated in this discipline while terms denoting the remaining concepts are introduced into the system by definition. The axioms of the system are meant to stand for some of the facts about these concepts while other facts are expressed by the theorems, i.e. the statements that can be derived from the axioms on the basis of the underlying discipline.

If a scientific discipline is axiomatized this discipline forms an *interpretation* or a *model* of the axiom system. In general, however, the axiom system can be interpreted, and even *soundly* interpreted (under a sound interpretation all axioms and theorems turn to true statements), in many additional ways; in that case the original scientific discipline forms the intended or principal interpretation [2]).

§ 2. The Primitive Relation. Equality and Extensionality

As stated above we shall take, for the presentation of a certain variant of Zermelo's set theory, as the underlying discipline the functional calculus of first order, to the exclusion of number theory

[1]) For instance, Rosser 53, p. 200.

[2]) See Church 56, p. 56; also Carnap 39 who uses a somewhat different terminology. Cf. the remarks at the end of § 6 regarding non-standard models.

and, of course, set theory itself. In other words, we are going to present set theory as an *applied functional calculus of first order.* Its only specific primitive symbol — in addition to an infinite list of individual variables w, x, y, z, w', ... (and occasionally other letters), the connectives and quantifiers mentioned above, and such auxiliary signs as commas, parentheses, and brackets — will be the dyadic predicate 'ϵ', meant to denote the *membership relation.* We shall read, then, '$x \epsilon y$' as 'x is a member of y' [1]) and, synonymously, as 'x belongs to y', 'x is contained in y', 'y contains x (as a member)'.

Any formula of the form '$. \epsilon -$', where the dot and the dash are replaced by variables, is well-formed, and these well-formed formulae are the only "atomic" well-formed formulae out of which all other well-formed formulae are obtained by means of the connectives and quantifiers.

In the intended interpretation nothing is supposed about the *range* of the *individual* (or object) *variables* except that it be a well-defined *non-empty* class (of objects), a so-called *universe of discourse.* Especially, nothing is supposed for the time being about the cardinality of this class, not even whether it be finite or infinite. (The terms 'cardinality', 'finite', 'infinite' are used here, of course, only informally.) The *domain* of the membership relation, i.e. the class of those objects that are members of some object (in short, that are *membership-eligible*), will be said to consist of *elements*, the *counterdomain* of this relation, i.e. the class of those objects that contain at least one object as member, will be said to consist of *sets.* For the time being nothing is said about the relation between the domain and the counterdomain; the question whether there are *elements that are non-sets* (sometimes called *individuals* for short, not in the proper logical meaning; Zermelo's *Urelemente*) [2]) or *sets that are non-elements* is so far left open.

As regards the status of the *equality* relation in our system, the following three attitudes may be taken:

a) The equality relation is regarded as *belonging to the underlying logic.* In our case the underlying discipline may, then, be taken as what is known as *the functional calculus of first order with equality* [3]).

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[1]) We avoid the phrase 'x is an *element* of y' since we are using the term 'element' with a somewhat different meaning (especially in § 6).

[2]) Adopted by Ackermann 37a. — As to the null-set, see below.

[3]) Cf. Church 56, p. 48; also Hilbert-Bernays 34. Especially in this case, when equality is treated as belonging to the underlying logic, the term 'identity' is often used to denote it.

This attitude seems to have been adopted in essence by Zermelo [1]). He regards x and y as equal when "they denote the same thing" (exhibiting thereby, incidentally, a confusion between use and mention of signs [2])); eliminating this confusion one winds up with the tautology that x and y are equal when they are the same thing.

b) Equality is regarded as *one of the primitive relations of the system*, on a par with the others. In our case the equality-sign '$=$' could have been taken as a second primitive dyadic predicate. One would, then, have to make sure by appropriate axioms that equality be reflexive, symmetrical, and transitive [3]), i.e. an *equivalence relation*, as well as *substitutive* with respect to the other primitive relation, i.e. that from $x \in y$, $x = x'$ follow $x' \in y$, and from $x \in y$, $y = y'$ follow $x \in y'$.

c) The equality-sign is introduced by *definition* [4]). In our case this may be done in two different ways. *Either* one considers — in line with a tradition that goes back at least as far as Leibniz (*identitas indiscernibilium*) — two objects to be equal if every object (set, class) that contains the one as a member contains also the other [5]), *or* one regards them as equal if they contain the same members. The second way clearly entails that there exists within the universe of discourse *at most* one "individual" (non-set) and seems, therefore, inappropriate for such systems whose universe of discourse under the intended interpretation is to encompass different objects that are not sets, in the ordinary sense of 'set', and *ipso facto* contain no members. Interestingly enough Quine has shown [6]) that this need not be so. By a slight deviation from ordinary usage, viz. by treating individuals as a special kind of sets or rather classes [7]) (classes that contain only

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[1]) Zermelo 08a. Whenever in the present Chapter Zermelo is mentioned without additional reference, this paper is meant; it is not only fundamental for our exposition in general but also contains many details appearing in the following sections.

[2]) See, e.g., Quine 51, § 4.

[3]) For these properties and their interdependence cf. Scholz-Schweitzer 35, Aubert 52.

[4]) Fraenkel 27a; cf. A. Robinso(h)n 39, Hailperin 54.

[5]) Cf., for instance, *Grelling **6.**

[6]) Quine 51, pp. 122–123.

[7]) We are using 'class' in three different senses: first, presystematically, in its everyday sense; second, as a logical term, again in its (more or less) standard sense; third, as a specific term occurring in certain systems, such as those treated in §§ 6, 7. It is hoped that no confusion will result.

themselves as members), he succeeds in introducing equality quite generally after the second manner, without having to renounce individuals in his ontology [1]). As he points out himself the situation can be described alternatively by saying that the ϵ-relation is to be interpreted as 'is a member of, or is equal to', depending on whether the second object involved is a class or not.

The first way seems to be barred by no such obstacles. But this is again illusory. There are systems whose ontology encompasses non-elements, and more than one of them (cf. §§ 6, 7 and Chapter III). For such objects the first way can obviously not be chosen.

In spite of the restrictions involved by attitude c) we adopt it here since it is superior to b) in having less undefined terms and superior to a) in that the underlying discipline is weaker. (Hereby we do not wish to imply that these features are always advantageous.)

Which of the two ways shall we choose now? We already saw that this depends mainly on the ontology we are ready to countenance. Now, at least for mathematical purposes there seems to be no real need to deal with individuals or rather to countenance more than one individual [2]). For such purposes we may therefore treat all objects as sets, with the exception of just one object, the one and only memberless object whose existence is needed for obvious technical reasons. (We would like, for instance, that the intersection of any two sets should belong to the universe even when these sets have no member in common.) We also decide at present not to admit non-elements (though systems encompassing non-elements may be, in a certain important sense, more adequate; cf. §§ 6, 7 and Chapter III). Altogether, we want all objects to be elements and all but one to be sets; in other words, the counterdomain of the ϵ-relation is to contain all but one members of its domain, and its *field* (i.e. the union of the domain and the counterdomain) is to coincide with the universe of discourse. We also want two objects to be treated as equal if and only if they contain the same members and belong to the same sets. Therefore we are free to choose any one of these properties as defining and to stipulate the other by a suitable axiom; in our system both ways of attitude c) are effectively usable.

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[1]) A modification of Zermelo 08a (and herewith, largely, of **Z**) with self-membered individuals in this sense is suggested in Quine 56, No. 5.

[2]) Fraenkel 21/22, p. 234, and 25. This attitude was adopted by von Neumann (see below § 6), and de facto by Bourbaki 49. Cf. Mostowski 39, pp. 201–202.

In view of the decisions taken above and in order to adapt our terminology to the customary one as much as possible we finally decide to use the term 'set' from now on not just for a member of the counterdomain of ϵ, but rather for a member of its field, i.e. for any object belonging to the universe of discourse. In this section and the three following sections we shall therefore not use, in general, the terms 'object' and 'element' nor the terms 'non-element' and 'non-set' (or 'individual'); *the one and only memberless object will also be regarded as a set* and called *'null-set'*, as customary.

After these preliminary and informal remarks we start establishing our system. In general no symbolism beyond the customary set-theoretic symbols 'ϵ', '$\subseteq$' etc. will be used. Only the axioms and some definitions will be fully symbolized, in addition to their ordinary semi-symbolic formulation.

Our system [1]), to be referred to from now on as '**Z**', contains, then, just one specific undefined dyadic predicate, 'ϵ'. All its atomic statements have the form '$x \epsilon y$'. Instead of '$\sim x \epsilon y$' we shall usually write '$x \notin y$'.

DEFINITION I. (Cf. *T*, p. 21.) If, for all x, $x \epsilon y$ implies [2]) $x \epsilon z$ we shall say that y is a *subset* of z (or *comprised* in z, or *included* in z); if, in addition, there is at least one w such that $w \epsilon z$ but $w \notin y$, we say that y is a *proper subset* of z. (Relation of inclusion.) The corresponding symbols are $y \subseteq z$ and $y \subset z$.

From Definition I follows immediately:

THEOREM 1. Every set is a subset of itself ($x \subseteq x$); $x \subseteq y$ and $y \subseteq z$ imply $x \subseteq z$. In other words, the relation $\subseteq$ is reflexive and transitive. The relation $\subset$, on the other hand, is irreflexive, asymmetrical, and transitive. (*T*, p. 176.)

One has to distinguish rigorously between the relations ϵ and $\subseteq$ (of which, in the present exposition, the first is primitive and the

[1]) It rests chiefly on Zermelo 08a; cf. 30. The principal modifications inserted in the following exposition are contained in Fraenkel 21/22, 22, 26, and in Skolem 22/23, 29, 41. Cf. the formalizations in Church 42, Borgers 49, Wang-McNaughton 53 (pp. 15–18), Carnap 54 (pp. 152–156), Thiele 55. A general survey of Zermelo's intention is given in Weyl 46, pp. 10 f.

While the criticism of Poincaré 13 (Chapter IV) is unjustified in many respects, the criticism of Skolem 22/23 is incorporated in the following exposition.

[2]) The term 'implies' is used here as a synonym for 'if ... then---' or 'only if'; the corresponding symbol is '$\supset$'. For the other equally customary sense of 'implies', viz. for 'logically implies', we shall use the term 'entails'.

second derived). The confusion between them, enhanced by equivocalities of Aryan languages — the copula 'is' of Aristotelian fame is used in both these senses (and many others) — had disastrous consequences in the early development of logic. In our terminology, while a set always *comprises* itself and its subsets, it *contains*, in general, neither itself nor its subsets.

Frege seems to have been the first logician to point out the necessity of this distinction; after Peano, only beginners are prone to fall prey to the confusion between ϵ and $\subseteq$. This does not mean, however, that one could not develop the theory of a relation that fuses (not confuses!) the properties of membership and inclusion in a consistent way. As a matter of fact, Leśniewski [1]) did just this.

As stated above we may define equality in either of the two following ways.

DEFINITION IIa. x is called *equal* to y ($x = y$) if and only if, for all z, $x \epsilon z$ implies $y \epsilon z$ and is implied by it, i.e. if every set containing one of the sets x and y contains also the other. If x is not equal to y, it is called *different* from y ($x \neq y$).

In symbols, $x = y =_{\mathrm{Df}}$ [2]) $(Az)\ (x \epsilon z \equiv y \epsilon z)$ [3]).

DEFINITION IIb. x is called *equal* to y if and only if both $x \subseteq y$ and $y \subseteq x$, i.e. if each of the sets x and y is a subset of the other. In other words, going back to a formulation in primitive terms: $x = y$ if every member of one set is also a member of the other. Otherwise x is *different* from y.

In symbols, $x = y =_{\mathrm{Df}} (Az)\ (z \epsilon x \equiv z \epsilon y)$.

On account of either definition the relation of equality is reflexive, symmetrical, transitive, and substitutive with respect to the right-hand argument of ϵ, i.e. $z \epsilon x$, $x = y$ imply $z \epsilon y$ [4]).

However, extensionality — the defining property in IIb — cannot be inferred from IIa, nor can left-hand substitutivity — that $x \epsilon z$, $x = y$ imply $y \epsilon z$ — be inferred from IIb, not even by means of the axioms following below [5]).

[1]) *Leśniewski 3; cf. Sobociński 49-50. See also Chapter III, p. 149.

[2]) Read: is short for, by definition; the symbol '$=_{\mathrm{Df}}$' is of course a metalinguistic symbol that does not belong to **Z** proper.

[3]) Equipollence could be weakened to implication in view of the axioms.

[4]) This is trivial relative to IIb. Relative to IIa, righthand substitutivity can be proved with the help of Axioms IV and V (or II), or even of IV alone, of § 3. See A. Robinso(h)n 39, footnote 4; cf. Thiele 55, pp. 176f.

[5]) Proved by A. Robinson 39; cf. *Vieler 1.

Supplementing each of the Definitions IIa and IIb we therefore introduce an axiom in alternative shapes corresponding to the definitions.

AXIOM Ia. **$x \subseteq y$ and $y \subseteq x$ together imply $x = y$;** in other words, sets containing the same members are equal.

In symbols, $(Ax)\ (Ay)\ [(Az)\ (z \in x \equiv z \in y) \supset x = y]$.

AXIOM Ib. **$x \in z$ and $x = y$ together imply $y \in z$;** in other words, equal sets are contained in the same sets.

In symbols, $(Ax)\ (Ay)\ [x = y \supset (Az)\ (x \in z \supset y \in z)]$.

For what follows it does not matter whether IIa is taken as the definition of equality and Ia as the corresponding axiom, or IIb as the definition and Ib as the axiom. Therefore we shall simply speak of Definition II (*Definition of Equality*) and of Axiom I (AXIOM OF EXTENSIONALITY).

In *Theory* (pp. 18/19) extensionality was introduced in the form of Definition IIb; cf. Principle I, p. 21.

Since a set, according to Definition II and Axiom I, is fully determined by its members we may denote the (finite or infinite) set that contains the members $a, b, c, \ldots$ by

$$\{a, b, c, \ldots\}$$

where the order in which the members are written does not matter.

DEFINITION III. Two sets which contain no common member are called *disjoint*. If the members of a set s are pairwise disjoint, s is called a *disjointed set* [1]).

§ 3. "CONSTRUCTIVE" AXIOMS OF GENERAL SET THEORY

In sections 3 and 4 we introduce five axioms each of which, presuming the existence of (a) certain set(s), ensures the existence of another set. That part of set theory which can be deduced from such axioms may be called *general set theory* [2]). In § 5 further particular axioms will be added, especially "axioms of infinity".

Axioms of such a conditional character are appropriate for the goal of excluding antinomies, for the sets guaranteed by these axioms

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[1]) This includes the trivial cases of a set which contains no member or one only. (In *Theory* 'mutually exclusive' was used for 'disjoint'.)

[2]) Cf. the first footnote of § 5, p. 81.

have an extension which refers to the extension of sets introduced previously and not the absolute comprehensiveness characteristic of the sets which appear in the antinomies.

In this section we shall formulate those axioms of general set theory in which the set to be secured is determined *uniquely* by the set(s) whose existence is presumed (and, in Axiom V, by an additional ingredient). In a loose sense such axioms may be called "constructive". In contradistinction, in § 4 a "non-constructive" axiom is introduced; a given set will then guarantee the existence of another set of a certain type but not the existence of a definite set. In contrast with all these axioms, the axiom of extensionality (§ 2) does not state anything, conditional or absolute, about the existence of sets.

The following axioms roughly correspond to the Principles II–IV, VI, VII of *T* (pp. 22–28, 97, 123). The Principle (V) of Infinity (*T*, p. 42) is here postponed until § 5.

Instead of dogmatically introducing the various axioms we shall precede them by informal remarks apt to point out the purpose of each axiom and to investigate its consequences. In accordance with the character of the axiomatic method in general the axioms have essentially been chosen by means of an *a posteriori* analysis of the methods of Cantor's "genetic" theory, with restrictions meant (at least) to avoid antinomies.

The simple operation of uniting two different sets to a set [1]) is expressed by the

AXIOM (II) OF PAIRING [2]). **For any two different sets *a* and *b*, there exists a set *p* that contains just *a* and *b*** (i.e., *a* and *b* and no different member).

In symbols, $(Aa)\,(Ab)\,\{a \neq b \supset (Ep)\,(Ax)\,[x \in p \equiv (x = a \vee x = b)]\}$.

Since by extensionality all sets containing just *a* and *b* equal each other we may speak of *the* set with the members *a* and *b*. It is also called the *pair* of *a* and *b* and denoted by '$\{a, b\}$' or, synonymously, by '$\{b, a\}$'.

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[1]) Instead of this operation, Kuratowski 25 introduces the operation of forming the union of two sets. Cf. Bernays' axiom II (2), see § 7.

[2]) Zermelo's term is *Axiom der Elementarmengen*; it is appropriate in his exposition as he not only postulates the existence of the pair but, in addition, the existence of the set containing any single object, and also the existence of the null-set. For both we shall be able to prove the existence; hence the axiom can be weakened as done here.

The situation is similar for the following three axioms (III–V); hence we shall in the verbal formulation of these axioms use the definite article ('... there exists *the* set ...').

Incidentally we cannot prove (not even by means of the following axioms) that the pair $\{a, b\}$ is different from a or b.

Dropping the word 'different' in Axiom II would make the statement of the axiom stronger in so far as the existence of p in both cases $a \neq b$, $a = b$ would be postulated. Since the nature of axiomatic method tends to weaken the single axiomatic statements as much as possible it is preferable to restrict Axiom II to the case $a \neq b$, for in the case $a = b$ the corresponding statement can be proved. [1])

If $a, b, c, d, \ldots$ are sets different from each other, repeated application of Axiom II allows to build more complicated sets of various kinds; for instance $\{\{a, b\}, \{c, d\}\}$, $\{\{a, b\}, \{a, c\}\}$, ... However, all sets formed in this way contain just two members.

To obtain sets of a more general character we have to look for another procedure. In *Theory*, as the first and simplest set-theoretical operations we introduced addition and inner multiplication, i.e. the construction of the union and the intersection of given sets, which are also the fundamental operations of Boolean algebra (*T*, pp. 26, 109, 142). For our axiomatic purpose it will suffice to postulate the performability of one of these processes, and we choose union.

According to our program, the sets whose union is to be formed shall not be taken arbitrarily but shall be members of a set the existence of which has already been ensured. Thus we have

AXIOM (III) OF SUM-SET OR UNION. **For any set a which contains at least one member [2]), there exists the set whose members are just the members of the members of a.**

This set is called the *sum-set* of a or the *union* of the members of a; it is denoted by '$\mathsf{U}a$'. Hence $x \in \mathsf{U}a$ holds true if and only if there is a $z \in a$ (at least one z) such that $x \in z$.

In symbols,

$$(\mathrm{A}a)\,\{(\mathrm{E}b)\; b \in a \supset (\mathrm{E}y)\;(\mathrm{A}x)\;[x \in y \equiv (\mathrm{E}z)\;(x \in z \,\&\, z \in a)]\}.$$

[1]) Nevertheless, dropping 'different" has also advantages; for instance, that considerable parts of the theory, e.g. number theory, can then be developed without Axiom IV.

[2]) The case that a contains no member is trivial. One may even assume that a contains two members at least; for if $a = \{b\}$, the axiom only expresses the existence of b.

Roughly speaking, if the set a contains the members $t, u, v, \ldots$ then just the members of $t, u, v, \ldots$ are contained in $\mathsf{U}a$. Sometimes we shall therefore denote the sum-set of a by $t \cup u \cup v \cup \ldots$ [1], where the succession of the terms is insignificant.

If a and b are different sets their union $a \cup b$ certainly exists. For by Axiom II the pair $\{a, b\} = p$ exists, and so does by Axiom III $\mathsf{U}p = a \cup b$.

Axioms II and III together enable us to "construct" sets of various kinds. If, for instance, a, b, c are different given sets, Axiom II yields the sets $\{a, b\} = m$, $\{b, c\} = n$, $\{a, c\} = p$, hence [2], for instance, the sets $\{m, c\} = \{\{a, b\}, c\}$ and $\{a, n\} = \{a, \{b, c\}\}$. Furthermore, by Axiom III the sets $\mathsf{U}m = a \cup b$ and $\mathsf{U}n = b \cup c$ exist. Additional applications of II and III yield the sets

$$\{\mathsf{U}m, c\} = C, \quad \{a, \mathsf{U}n\} = A, \quad \mathsf{U}C = \mathsf{U}m \cup c = (a \cup b) \cup c,$$

$$\mathsf{U}A = a \cup \mathsf{U}n = a \cup (b \cup c).$$

Precisely as in T, p. 115/6, the sets $\mathsf{U}C$ and $\mathsf{U}A$ prove to contain the same members, hence to be equal. In a similar way the general associativity of the union operation can be shown.

$\mathsf{U}A$ contains the members of three different sets a, b, c. Proceeding further in the same or in a similar way — provided we have at our disposal more than three different sets — we may obtain more and more comprehensive sets. But in spite of the considerable strength of Axiom III, a glance at Cantor's theory shows that the axioms of pairing and of sum-set do not give us sufficient liberty in forming new sets, even though fairly strong assumptions about the existence of sets to start with be made. In fact, let us assume that there exist infinite [3] sets of the kind called denumerable, and even denumerably many different such sets. Not even with this assumption would Axioms II and III be strong enough to guarantee the existence of a more-than-denumerable set; for instance, the existence of a continuum.

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[1] In *Theory* (e.g., p. 26/9) $t+u$ was written instead of $t \cup u$, and $\boldsymbol{S}a$ instead of $\mathsf{U}a$.

[2] Here and in the following we should properly add conditions such as $c \neq m$ etc. in order to use Axiom II. But as we shall soon be able to drop these conditions (Theorem 2 on p. 43) it is not worth while emphasizing them.

[3] 'Infinite' and 'denumerable' are not defined until § 5. Here we use only a heuristic argument which is based on notions known from T, §§ 2 and 3, to illustrate the purpose of Axiom IV.

Cantor's (second and principal) tool for reaching sets with a higher cardinality was (transfinite) multiplication, in particular exponentiation. We shall see that for this purpose the power-set (*T*, Theorem 3 on p. 94, the so-called "theorem of Cantor", and Theorem 1 on p. 151) is sufficient; hence we formulate

AXIOM (IV) OF POWER-SET. **For any set *a* there exists the set whose members are just all subsets of *a*.** In symbols,

$$(\mathrm{A}a)\ (\mathrm{E}y)\ (\mathrm{A}x)\ (x \in y \equiv x \subseteq a).$$

The set of all subsets of a is called the *power-set* of a and denoted by '$\boldsymbol{C}a$' [1]). $x \in \boldsymbol{C}a$ holds true if and only if $x \subseteq a$, i.e. if $z \in x \supset z \in a$.

In the axiomatic system as well as in Cantor's theory Axiom IV fulfils a decisive task (cf. § 8), for without it we are not able to form comprehensive enough sets. Yet Axiom IV cannot be utilized for this purpose at the present stage of our axiomatization. For Axiom IV permits only the use of those subsets *whose existence has previously been established.* Now Definition I (p. 31) does not enable us to *form* subsets of a given set s but merely, given a certain set, to ascertain whether it is a subset of s, and Axioms II and III allow the construction of very special subsets only. Hence Axiom IV for itself is not a tool anywise comparable to Cantor's power-set [2]). To use Axiom IV as an instrument for obtaining comprehensive sets another axiom is needed, apt to yield subsets of a given set in a general way.

To realize this more closely we start with a set S which contains at least two members s_1 and s_2; by Axiom II the pair $p = \{s_1, s_2\}$ exists and is, by Definition I, a subset of S. Hence the power-set $\boldsymbol{C}S$ at any rate contains the member p. If S contains more than two members we may do the same with any two members of S, thus obtaining new members of $\boldsymbol{C}S$. By means of Axiom III we obtain more general subsets of S, but only such as are finite in the naive sense. For instance, if s_1, s_2, s_3 are members of S, by Axiom II the pairs $p_1 = \{s_2, s_3\}$ and $p_2 = \{s_1, s_3\}$ exist and are different; hence the pair

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[1]) In *Theory* and, for instance, in Kleene 52 Zermelo's notation '$\mathbf{U}a$' or '$\mathbf{ll}a$' is used. ('$\boldsymbol{C}$' is taken from 'Cantor'.)

[2]) Kaufmann's contrary conception (in 30, p. 176) is due to a misunderstanding of Zermelo's method, caused apparently by the external arrangement of the axioms. (In Zermelo 08a the axiom of subsets (*Axiom der Aussonderung*) precedes the axiom of power-set.)

$p = \{p_1, p_2\}$ exists and also, by Axiom III, $\mathsf{U}p = \{s_1, s_2, s_3\}$, which is a subset of S, i.e. a member of $\boldsymbol{C}S$.

Yet by such methods we fail to obtain infinite subsets of an infinite set S (cf. § 5). If S contains all natural numbers we cannot even guarantee, at the present stage, the existence of the subset of all numbers greater than 1, let alone the subset of all even, or prime, numbers. Therefore as yet we are not able to prove that $\boldsymbol{C}S$ has a greater cardinality than S.

What we want may very loosely be described as follows. The axioms of pairing, of sum-set, and of power-set have an *expansive* function inasmuch as they yield the existence of sets which somehow have a wider range than the sets appearing in the assumptions of the axioms. Now we are in need of a *restrictive* operation in order to obtain sets whose extent is less than that of the given set; viz., subsets of it. Therefore we add the

AXIOM (V) OF SUBSETS. [1] **For any set a and any monadic predicate** [2] $\mathfrak{P}$ which is meaningful ("definite") for all members x of a, **there exists the set that contains just those members x of a which satisfy the predicate $\mathfrak{P}$ (fulfil the condition $\mathfrak{P}(x)$).**

This set, which is clearly a subset of a, is denoted by $a_{\mathfrak{P}}$. In symbols, Axiom V reads

$$(\mathrm{A}a)\ (\mathrm{E}y)\ (\mathrm{A}x)\ [x \in y \equiv (x \in a\ \&\ \mathfrak{P}(x))]$$

where $\mathfrak{P}$ contain 'x' as the only free variable.

Sometimes a stronger form of Axiom V is required where, in addition to x, other free variables $z_1, z_2, \ldots, z_n$ occur in the condition $\mathfrak{P}(x)$; thus we obtain the form

$$(\mathrm{A}z_1)\ (\mathrm{A}z_2)\ \ldots\ (\mathrm{A}z_n)\ (\mathrm{A}a)\ (\mathrm{E}y)\ (\mathrm{A}x)\ [x \bullet y = (x \in a\ \&\ \mathfrak{P}(x, z_1, z_2, \ldots, z_n))]$$

where $\mathfrak{P}$ does not contain the free variable 'y'.

Axiom V is most characteristic of Zermelo's system, in contrast with the pre-axiomatic attitude on the one hand and with that of §§ 6 and 7 on the other — where to any condition on x, $\mathfrak{P}(x)$, a set (or class) s is assigned in the sense of $(\mathrm{A}x)\ (x \in s \equiv \mathfrak{P}(x))$. This attitude earlier led to antinomies, and Zermelo's way out was to apply "comprehension" only to members x of *sets secured previously*.

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[1] Zermelo's term is *Axiom der Aussonderung* (axiom of 'singling out', 'sifting', or 'selecting', viz. selecting those members of a for which $\mathfrak{P}$ comes true).

[2] See above p. 27; the respective condition '$\mathfrak{P}(x)$' is assumed to contain the single free variable x. We may presuppose that $\mathfrak{P}$ is built up of "atomic" formulae of the form $u \in v$.

Prior to a formal analysis of the expression 'definite predicate' used here, a few informal remarks shall illustrate its meaning for readers not interested in a technical logico-mathematical analysis. The idea is that the predicate $\mathfrak{P}$ should be *meaningful* for all $x \in a$ with the implication that, for every single member x of a, *the statement* $\mathfrak{P}(x)$ *is either true or false* [1]; in other words, that any x does, or does not, fulfil the condition $\mathfrak{P}(x)$. This, however, does not presuppose that the decision whether, for a given $x \in a$, $\mathfrak{P}(x)$ is true or false can actually be reached — say, at the present stage of scientific development. A characteristic instance is given in the beginning of *Theory* (p. 14), viz. the condition 'x is transcendental' with respect to real numbers x. Not for every real number (given, for instance, by a rule for its expansion into a decimal or a continued fraction) can we ascertain whether it is algebraic or transcendental; but every number has at most one and, according to classical mathematics and logic, at least one of both properties. Hence 'x is transcendental' is definite for real numbers x, as opposed to 'x is green' or even 'x is finitely definable'.

This explanation of the concept 'definite predicate' cannot satisfy the requirements of a formal deductive theory; making it precise is a principal desideratum within our axiomatic system. On the other hand, a strict definition requires of the reader either some knowledge of modern logic or else certain mathematical technicalities. Only the main features of formalizing 'definite' as used in Axiom V shall be described here. [2]

Zermelo originally (1908) gave the following explanation: A question or statement $\mathfrak{E}$ is called 'definite' if by means of the primitive relations of the system it is settled without ambiguity, in virtue of the axioms and the general laws of logic, whether $\mathfrak{E}$ holds true or not. Likewise a class statement [property, predicate] $\mathfrak{E}(x)$ whose variable x runs over all individuals [members] of a class $\mathfrak{K}$ is called definite if $\mathfrak{E}$ is definite with respect to *each single* individual x of the class $\mathfrak{K}$. For instance, the question whether $a \in b$ ($M \subseteq N$) holds or not, is always [3] definite.

This explanation is certainly unsatisfactory and even seems to leave room for antinomies of the semantic kind (Chapter I, § 3). In 1921/22, independently

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[1] No theory of types (Chapter III) is presupposed here and the logical principle of the excluded middle is freely used (cf. Chapter IV, § 3).

[2] For a definition of subsets by a "quasi-combinatorial" principle see Bernays 35.

[3] The intention is: for any a and b or M and N, $b = a$ not excluded.

[4] Fraenkel 21/22 and 22; Skolem 22/23. The modification proposed in *Vredenduin I is certainly no improvement.

and almost simultaneously, two different methods [4]) were offered for formalizing the concept 'definite' and hereby Zermelo's system as a whole.

The first method formalizes 'definite' by means of a certain concept of function [1]) defined by the operations of Axioms II–V [2]). This function-concept, while more special than Skolem's concept under which it may be subsumed, is sufficient for the purpose of developing general set theory as has been shown by its actual construction [3]).

The second attitude conceives a 'definite' predicate as an elementary well-formed formula [4]), viz. as a well-formed formula of the first-order functional calculus with the free variable x, built up from atomic ϵ-statements [5]). Within this frame, as Quine has shown, the simple theory of types can be derived [6]).

Zermelo, while later admitting the need of formalizing his loose explanation of definiteness, rejected [7]) both methods just described, in particular because they implicitly involve the notion of finite cardinal (integer) which in his view should be based on set theory. Therefore within the frame of his axiomatic system he introduces a special *axiomatization of the concept of definiteness* or, which is essentially the same, of the concept of function. Besides serious doubts whether in this way the use of finite cardinals is avoided, both the inconvenience of one axiomatization overlapping the other and the complexity of the axioms required [8]) render this attitude undesirable. On the other hand, the system then comprises a finite, instead of an infinite, number of (elementary) axioms. (Cf. the following page and § 7.)

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[1]) Cf. also Fraenkel 27 (pp. 103–115) and the important supplement given in von Neumann 28a.

[2]) The inclusion of Axiom V itself automatically leads to a certain hierarchy of orders. Thus the axiom splits up into a sequence of axioms and the analogy to Skolem's concept becomes obvious.

[3]) See Fraenkel 25 and — for the theory of ordered and well-ordered sets, not covered by Zermelo — 26 and 32. Special existence theorems, e.g. those concerning ordinal numbers, want the supplement provided by von Neumann 28a. Cf. Curry 34, p. 590, and 36, p. 375.

[4]) Cf. Skolem 22/23, 29 (§ 2), 30. Skolem uses Schröder's 'algebra of logic', but this is not essential.

The paper Quine 36 (cf. Quine 37 and 53, also *Cartan 1) gives a modification (extension) of the axiom of subsets which renders the other axioms unnecessary, save for extensionality and for the axiom(s) of § 4 and possibly § 5. A further innovation is that the system contains no null-set. A somewhat similar attitude is taken in Lindenbaum-Mostowski 38 and the related researches of these authors.

[5]) Cf. the formulations in Wang 49, Novak 48/51 (p. 90), Mostowski 51a (p. 111).

[6]) See, in particular, the form given to the theory of types by Tarski 33 (pp. 97–103).

[7]) Zermelo 29. Cf. the (justified) criticism of this essay in Skolem 30 and the discussions on Finsler's axiom of completeness (see above p. 23).

[8]) The decisive part is played by a "minimum" axiom (cf. below, § 5). Of course, the number of primitive concepts is increased by this method.

The present authors think the method of Skolem(-Quine) to be preferable, in particular because of its natural adaptation to current logic. No explicit use of this conception of definiteness will be made in what follows, but each instance of applying the axiom of subsets in this chapter can easily be translated into the language of Skolem's method.

The axioms of power-set and of subsets are, as stressed before, closely connected in so far as the former axiom derives its enormous strength from a sufficient liberty in forming subsets. (Here the particular subsets formed by means of the axiom of choice are disregarded; see § 4.) But this connection has the awkward property of being *impredicative.* (A class is called impredicative if it is defined, or definable only, by reference to a totality to which the class itself belongs, and a definition of such kind is impredicative. One may also say that a definition written in symbols is impredicative if it defines an object which is one of the values of a bound variable occurring in the defining expression.) The significance and the riskiness of impredicative definitions and procedures in mathematics, as well as various attempts made since Poincaré and Russell to eliminate them or to render them harmless, will be discussed in Chapter III. Here just the special case of impredicativity involved in the axioms of power-set and of subsets shall be exhibited. (The logical system in which the axioms are imbedded is presumed to include no theory of types.)

Whenever the predicate used in Axiom V to produce a subset of a given set s essentially refers to the power-set $\boldsymbol{C}s$ or to a similar set, a *particular* subset of s is determined by the totality of *all* subsets — which is just the procedure against which Russell's *vicious circle principle* was directed. Naturally a Platonistic attitude would judge this situation quite differently than a constructivistic attitude [1].

No matter whether this situation yields some type distinction in the application of Axiom V or whether — as has now generally been accepted according to the conceptions of Skolem, Ackermann, Quine, and others — one considers the axiom to be an *axiom schema* [2] (or an inference rule) rather than an axiom proper, at any rate V is

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[1] Cf., for instance, Scholz 50; also Bernays 35.

[2] This concept was first introduced in von Neumann 27, p. 13.

distinguished from the preceding axioms and occupies a central position in the system **Z**. (An axiom schema yields infinitely many single axioms; in the present case, corresponding to the infinitely many predicates admitted in Axiom V.)

The situation described here raises several problems, two of which shall be mentioned. The construction of a model for a given axiomatic system whose structure is predicative, can in many cases be achieved in a constructive way, in particular by the iteration of a certain procedure. Yet if there appear axioms with an impredicative character or axiom schemata containing infinitely many single cases, usually the indication of a model causes difficulties of principle [1]). Another problem is whether a given system of (non-logical) axioms, provided it is consistent, is *finitizable*, i.e. can be reduced to finitely many axioms proper. Systems which contain certain impredicative features as an essential ingredient, as does the system of Zermelo according to any of the conceptions of Axiom V described above, are not finitizable [2]). On the other hand, the axiom system of Bernays discussed in § 7 as well as Quine's *New Foundations* [3]) (see Chapter III) and similar systems — in particular, systems containing predicative axioms only — are finitizable or even in the original form contain only finitely many axioms.

Finally we draw a few simple conclusions from Axioms I–V. (The derivation of set theory from our axioms is outlined in § 8.)

DEFINITION. A set n which contains no member (i.e. for which $\sim(\mathrm{E}x)\ x \in n$) shall be called a *null-set*.

THEOREM 1. There exists just one null-set.

Proof. Take in Axiom V for $\mathfrak{P}(x)$ a contradictory condition on x, for instance '$x \neq x$'. Then for any a we obtain a subset $y = a_{\mathfrak{P}}$ which contains no member, i.e. a null-set, and its uniqueness follows from extensionality.

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[1]) See Skolem 29, § 3. Cf. also the exposition of the theory of models in Tarski 54-55.

[2]) See Wang 50d and 52 (cf., however, Wang 55, pp. 82/3), McNaughton 54a, Mostowski 51a and 53. Mostowski 55 (p. 20) gives a modification, due to Tarski of Axiom V which makes the system finitizable; thus an essential weakening of the system is achieved. The modification consists in admitting in Axiom V only predicates in which the quantifiers are restricted in a certain way.

[3]) Quine 37 and 53. See the proof in Hailperin 44 which is rather surprising since Quine's original class axiom schema is impredicative. Cf. below § 7.

The null-set will be denoted by 'O' [1]). According to Definition I on p. 31, O is a subset of every set.

THEOREM 2. For every set b there exists the set $\{b\}$ that contains b and no other member. $\{b\}$ is called the *unit-set* of b.

Proof. Case 1: $b = O$. Since O has no subset besides itself, Axiom IV yields the power-set $\{O\}$.

Case 2: $b \neq O$. By Theorem 1 and Axiom II the pair $a = \{b, O\}$ exists, and if in Axiom V we take this a and for $\mathfrak{P}(x)$ the condition '$x = b$' we obtain $a_{\mathfrak{P}} = \{b\}$.

THEOREM 3. For every two sets a and b there exists the set of the members that belong to both a and b. More generally, for every non-empty set t there exists the set of the members common to *all* members of t.

These sets are called the *intersection* (or meet) *of a and b*, in symbols $a \cap b$ [2]), and the *intersection of the members of t*, in symbols $\bigcap t$.

Proof. $a \cap b$ may be defined as the subset of a which by Axiom V corresponds to the condition $x \in b$. As to $\bigcap t$, by Axiom III there exists the set $S = \bigcup t$; every member of S is contained in at least one member of t. Let $\mathfrak{P}(x)$ be the condition 'x is contained in *each* member of t'. By Axiom V the subset $S_{\mathfrak{P}}$ of S exists and its members are just those common to all members of t, i.e. $S_{\mathfrak{P}} = \bigcap t$ [3]). (If there is no x common to all members of t we have $\bigcap t = O$.)

As to the Cartesian (or outer or cross) product, i.e. Cantor's *Verbindungsmenge* (T, pp. 118, 120), of pairwise disjoint sets, we have

THEOREM 4. For every disjointed set t there exists the set whose members are just the sets [4]) which contain a single member from each member of t.

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[1]) Some authors use Λ, or $\emptyset$. In *Theory* the notation '0' (the same as for the number 0) is used.

[2]) In *Theory* the notation '$a \cdot b$' (inner product), customary in France and Poland, is used.

[3]) One may carry out this proof without using Axiom III. Then instead of S one takes an arbitrary member c of t, obtains its subset $c_{\mathfrak{P}}$ with the $\mathfrak{P}$ used above, and finally proves that $c_{\mathfrak{P}}$ is independent of c, i.e. that for $d \in t$, $d \neq c$, we have $d_{\mathfrak{P}} = c_{\mathfrak{P}} (= S_{\mathfrak{P}})$.

[4]) These subsets of $\bigcup t$ are called 'complexes' in T, p. 120. — Mostly the notion of Cartesian product is extended to the general case where the members of t need not be disjoint; then either a function-concept or ordered sets are utilized. (Cf. p. 47 and § 8; also T, pp. 119 and 121.) For our purpose the restriction to disjointed sets t is preferable.

This set is called the *Cartesian product* of the members of t and denoted by $\boldsymbol{P}t$; one also writes $\boldsymbol{P}t = \tau \times \tau' \times \ldots$ where $\tau, \tau', \ldots$ are the members of t.

If t contains the member O we have $\boldsymbol{P}t = O$.

Proof. Since the members of the desired set are certain subsets of the union $\mathsf{U}t$ we start from the power-set of $\mathsf{U}t$, i.e. from $\boldsymbol{C}\mathsf{U}t = T$, which exists by Axioms III and IV. Let the condition $\mathfrak{P}(x)$ be '$x \in T$, and for *each* $\tau \in t$ the intersection $\tau \cap x$ is a unit-set'; in other words 'the subset of those members of t whose intersection with x contains a single member [1]) equals t itself' [2]). Then the set $T_{\mathfrak{P}} \subset T$ exists by Axiom V; its members are those subsets of $\mathsf{U}t$ which contain just one member from each member of t — i.e. all "complexes" pertaining to t.

The remark regarding $O \in t$ is self-explanatory; indeed, since O contains no member there is no complex having a common member with O.

If t is the null-set or contains one member only, Theorems 3 and 4 become trivial.

Thus of the three operations with sets introduced in § 6 of *Theory* — union, intersection, Cartesian product — the performability of the first has been *postulated* by Axiom III while the performability of the two others within $\boldsymbol{Z}$ has been *proved* (Theorems 3 and 4) by means of Axioms IV and V, which were required anyway. (Theorem 4 also enables us to show the existence of the insertion-set; see *T*, § 7.)

§ 4. The Axiom of Choice

1. Formulation of the Axiom. Its Introduction into Mathematics. After having obtained those subsets of a given set which are determined by a definite predicate we raise the question whether possibly *other subsets*, not obtained in this way, may be conceived and admitted; and if so, how far such subsets are necessary for developing set theory.

The present section deals with an axiom which yields such subsets.

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[1]) Since t is disjointed all these single members are different.

[2]) In a more elaborate form: any member y of t certainly does, or does not, satisfy, with respect to a *given subset* x of $\mathsf{U}t$, the condition that the intersection $x \cap y$ is a unit-set; the subset of those $y \in t$ which do satisfy this condition exists by Axiom V, and its being equal to t is the desired condition on x.

Yet it should be pointed out from the first that within the frame of our system *no ultimate proof has been given showing that such an axiom is actually needed*; i.e. that the axiom of subsets is not sufficient to yield the subsets in question — in which case the following axiom would be redundant in the frame of **Z** as defined by the axioms of sections 2, 3, 5 (see below, II, p. 51).

While no such proof has been obtained and we do not even know in which direction we should look for a proof [1]), there are two indications for the necessity of the axiom in question or of a similar axiom. The first comes from experience. Since 1902 a principle of this kind has been used to guarantee the existence of sets which have not been obtained otherwise; it has also turned out that, long before, Cantor and others had unconsciously applied such a principle as a matter of course. Secondly, since 1922 it has been proved that, provided in the axiomatic system "individuals" in a reasonable multitude are permitted, it is indeed *necessary* to introduce the new axiom; in other words, that in this case Axioms I–V are not sufficient. Yet this does not constitute a full independence proof since individuals are not required for mathematical purposes. (For an alternative to individuals see p. 51.)

The following instance bases this remark on a realistic mathematical foundation. Let t be a disjointed set whose members are *arbitrary non-empty sets of real numbers*. In this case we have no knowledge whatsoever if certain subsets of $\mathsf{U}t$, of the kind described presently, can be obtained by means of Axiom V (in addition, of course, to Axioms I–IV) or if the following Axiom VI is indispensable to guarantee their existence.

We start from a disjointed set t which does not contain the null-set among its members. According to Theorem 4 on p. 43/4 the Cartesian product $\boldsymbol{P}t$ exists and its members, if any, are those subsets of $\mathsf{U}t$ whose intersections with each member of t are unit-sets. The question arises whether in this case $\boldsymbol{P}t$ might be the null-set O. The proof of Theorem 4, while showing that $O \in t$ implies $\boldsymbol{P}t = O$, does not answer our question; though one would expect $\boldsymbol{P}t \neq O$, no valid argument for it has been given so far.

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[1]) Cf., however, Gandy 56. Whether this argumentation will lead to an independence proof has still to be seen. — K. Gödel, in recent lectures in Princeton, outlined a way of arriving at such a proof which he, however, did not yet consider satisfactory. (Communication by P. Bernays.)

The guess $\boldsymbol{P}t \neq O$ relies on the following argument. Since each member of t contains at least one member one might choose *one arbitrary member* in each $y \in t$. If there exists a set c which contains just all those arbitrary members, c is a subset of $\mathsf{U}t$ and satisfies the condition which, according to Theorem 4, is characteristic of the members of $\boldsymbol{P}t$. In this case we therefore have $c \in \boldsymbol{P}t$, i.e. $\boldsymbol{P}t \neq O$, which is the desired result.

But this way of introducing the subset c of $\mathsf{U}t$ is not in accordance with the axiom of subsets — except for the trivial case that every member of t contains one member only, in which case $c = \mathsf{U}t$ satisfies our condition. In the general case the subset c of $\mathsf{U}t$ has not been defined by a definite condition $\mathfrak{B}(x)$ that is characteristic, among all $x \in \mathsf{U}t$, of the $x \in c$ and only of them. On the contrary, suppose $c \subseteq \mathsf{U}t$ is of the desired kind and $\gamma \in c$ belongs to a certain $y \in t$; then, replacing γ by a different member γ' of the same $y \in t$ will yield a new subset $c' \subseteq \mathsf{U}t$ which differs from c, while c' is also a subset of $\mathsf{U}t$ with the desired property. Thus, contrary to the subsets postulated by Axiom V, the subsets of $\mathsf{U}t$ needed for our purpose are not uniquely determined.

Of course, it is quite possible that some subset of $\mathsf{U}t$ with the desired property may be obtained from Axiom V or from other axioms and its existence will then guarantee that $\boldsymbol{P}t \neq 0$. But as long as there is no certainty of obtaining such a subset, one is in need of a special axiom, namely the

AXIOM (VI) OF CHOICE, or Multiplicative Axiom [1]). **If t is a disjointed set which does not contain the null-set, the Cartesian product $\boldsymbol{P}t$ is different from the null-set.** In other words, among the subsets of $\mathsf{U}t$ there is at least one whose intersection with each member of t is a unit-set.

Every such subset u of $\mathsf{U}t$ is called a *selection-set of* t; in general there are various selection-sets of t.

Axiom VI may symbolically be written [2]) in the form

$$(\mathrm{A}t)\,[(\mathrm{A}x)\,(\mathrm{A}y)\,((x \in t\ \&\ y \in t\ \&\ x \neq y) \supset [(\mathrm{E}z)\,z \in x\ \&\ \sim(\mathrm{E}z)\,(z \in x\ \&\ z \in y)]) \supset (\mathrm{E}u)\,(\mathrm{A}x)\,(x \in t \supset (\mathrm{E}w)\,(\mathrm{A}v)\,[v = w \equiv (v \in u\ \&\ v \in x)])].$$

[1]) The former name originates from Zermelo (see below), the latter from B. Russell. (Often 'principle' is used instead of 'axiom'.) In French the axiom is called *axiome du choix* or *axiome de Zermelo*; in German *Auswahlaxiom*, and correspondingly in Italian (*postulato della scelta*) and in other languages.

[2]) Cf. also *van Horn **2**.

Using the notion of *function* which is easily defined in our system (cf. §§ 6 and 8) one may express Axiom VI as follows: For any disjointed set S in which the null-set is not contained, there exists a single-valued function (at least one) $f(s)$ whose domain is S, such that $f(s)$ is a member of s. Each such function determines a selection-set of S.

In view of Theorem 4 on p. 43/4, Axiom VI yields (cf. T, p. 123):

The Cartesian product of the members of a disjointed set t equals O if and only if $O \in t$.

The terms 'choice' and 'selection-set' originate from a psychological consideration which was formulated by Zermelo [1]) as follows: One may express the axiom (VI) also by saying that it is always possible to choose from each member $M, N, R, \ldots$ of t a single member $m, n, r, \ldots$ and to unite all these to a set. (The disjointedness of t guarantees that the "chosen" members are different.) The consequences of this psychologistic formulation, which is liable to misunderstanding, will be described in III and V of this section.

The axiom of choice is probably the most interesting and, in spite of its late appearence, the most discussed [2]) axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago. Previous to a closer examination of its character, its purpose, and its history, we shall glance over its "prehistory".

Presumably the first explicit, if negative, allusion is contained in a paper of G. Peano of 1890 [3]), concerning an existence proof for a system of ordinary differential equations, where he writes: However, since one cannot apply infinitely many times an arbitrary law by which one assigns (*"on fait correspondre"*) to a class an individual of that class, we have formed here a definite law by which, under suitable assumptions, one assigns to every class of a certain system an individual of that class. — In our axiomatic language this would mean: Since one cannot presuppose the existence of a selection-set of t as defined in Axiom VI, we have constructed a predicate furnishing a suitable subset of $\mathsf{U}t$ by means of the axiom of subsets.

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[1]) 08a, p. 266. Cf. 04 and 08.

[2]) See below (V of this section). Cf. the historico-critical exposition in Cassina 36; other expositions of a general and non-technical nature are: Fraenkel 35, Chwistek 42, *Blaquier 1.

[3]) *Peano 2, p. 210.

In 1902 [1]) Beppo Levi, while dealing with the statement that the sum-set of a disjointed set t of non-empty sets has a cardinal greater than, or equal to, the cardinal of t, remarked that its proof depended on the possibility of marking (selecting) a single member in each member of t.

To be sure, Cantor (and others) had applied the principle in question prior to Peano's and Levi's remarks (see below, IV and V). But he did so inadvertently, without being aware of using a procedure which previously had not been applied in classical mathematics or logic.

In 1904, following a suggestion of Erhard Schmidt, Zermelo explicitly formulated the principle of choice and used it as the basis for his first proof [2]) of the well-ordering-theorem (T, pp. 309–315), and in 1908 for his second proof [3]) (cf. T, pp. 319–321). However, he could not then presuppose the set to be disjointed and therefore [illegible] to use looser formulations by means of a functional correspondence (see above), or of a "choice". In 1906 Bertrand Russell [4]) formulated the axiom in its proper "multiplicative" form, restricted to a disjointed set t. In 1908 [5]) Zermelo showed how the general formulation can be obtained from the multiplicative form by means of the other axioms. (Cf. below, § 8 [6]).)

In the present exposition only the fundamental lines regarding the axiom of choice are given; the literature references will enable the interested reader to obtain exhaustive information. The chief points to be discussed here are: *specialized forms* of the axiom; its *existential character*; its *applications* in set theory and in mathematics

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[1]) *B. Levi 1. According to a communication by letter from F. Bernstein, about 1901 G. Cantor and F. Bernstein tried to construct a one-to-one correspondence between the continuum and the set of all denumerable order-types (which has the cardinal of the continuum; T, p. 202). When they met with an insurmountable difficulty B. Levi proposed to solve the difficulty by introducing the principle of choice which he formulated in a general form. (On the other hand, cf. the criticism contained in *B. Levi 7.)

[2]) Zermelo 04.

[3]) Zermelo 08.

[4]) B. Russell 06, pp. 47–52.

[5]) Zermelo 08, p. 110, and 08a, pp. 266, 273 f. Cf. the final chapter of Church 44.

[6]) Cf. the formulation of the axiom in Baer 29, p. 384, and the (stronger) form in Skolem 29, § 1.

on the whole and, in particular, statements equipollent to the axiom; finally the *reaction of mathematicians* to the claim that it is one of the principles underlying mathematical research.

2. Special (weakened) Forms of the Axiom. Its Independence and Consistency. Hitherto we have only supposed the set to be disjointed and not to contain the null-set, while the cardinality of t and of the members of t remained arbitrary. The simplest way of specializing will, then, be *to lay restrictions upon these cardinalities.* Later we shall also consider specializations in form of certain *consequences* of the axiom which seemingly or actually are weaker than the axiom in its general or in specialized forms [1]). In particular, a substitute for the axiom, appropriate for the use in *algebra,* will be discussed in IV.

The most far-reaching specialization is obtained by assuming t to be *finite* [2]). In this case, however, *the axiom becomes redundant* because it can be proved [3]). It is sufficient to consider the case that t contains a single member [4]), for the transition to any finite set t

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[1]) As a specialization of our axiom within the theory of sets of points (because of its using the notion of distance) one may regard the *principio di approssimazione* of B. Levi (see Levi 23 and *7). In *Viola 1–3 and *Scorza Dragoni 1 various theorems about sets of points, usually proved by means of the axiom of choice, are based on Levi's principle.

Another principle referring to sets of points and apparently weaker than our axiom is Knaster's assumption of a function assigning to every linear perfect set of points one of its points (possibly by a one-to-one correspondence). Cf. *Kondô 1 where propositions of an existential character usually based on the axiom of choice — e.g., the existence of totally imperfect sets — are derived from that principle.

[2]) A certain extension of this problem, having a combinatorial character, is raised in *Rado 1; it concerns (a special case of) the problem, under what conditions can more than n sets be "represented" by a selection-set of n members? Cf. *P. Hall 1.

[3]) Cf. *Littlewood 1, p. 14. As has been pointed out by Bernays (e.g., in 30, pp. 359 f; cf. Hilbert-Bernays 34, p. 41), the assertion of the multiplicative axiom for finite sets t is nothing but an application of one of the distributive laws connecting logical conjunction and disjunction. Hence in the general case one may conceive the axiom as a generalization of this elementary logical law to infinite sets; in other words, as a supplement to the logical rules governing general and existential statements. Cf. Collins 54. (Cf. the intuitionistic attitude to the principle of the excluded middle; see Chapter IV, § 3.)

[4]) Cf. Rule C in Rosser 53, pp. 128 ff.

can be achieved by means of ordinary mathematical induction and of the axioms of pairing and of sum-set without involving essential difficulties.

When $t = \{\tau\}$ contains a single member the problem is of a logical rather than of a set-theoretical nature. According to the conditions of our axiom, τ is a non-empty set; accordingly, the task is to "choose" a single member from a non-empty set [1]). But for this purpose the axiom of choice is not required, contrary to an opinion expressed in various publications [2]).

In fact, for $t = \{\tau\}$ the existence of a selection-set follows from the existence of the unit-set $\{x\}$ of any given x — in the present case, $x \in \tau$ — by some simple steps in functional calculus; in particular, by the so-called *theorem of deduction* [3]).

Contrary to the case of a finite set t, *the finiteness of the members of* t *does not trivialize the choice problem.* Already Russell had, in an informal way, hinted at the gap between the use of a definite predicate and the application of the multiplicative axiom by contrasting an infinite set t of pairs of shoes with a (say, equivalent) infinite set of pairs of stockings. In the former case a subset of $\mathsf{U}t$ may be constructively defined as containing all left shoes, and this set is evidently a selection-set of t, obtained without using our axiom. On the other hand, as long as manufacturers adhere to the regrettable custom of producing equal stockings for both feet there is no definite predicate which simultaneously distinguishes one stocking in each of the infinitely many pairs. Hence a set containing just one stocking from each pair exists only by virtue of the axiom of choice. If the set of pairs were, for example, denumerable then we could not without our axiom form a one-to-one mapping between the set t of all pairs and the set $\mathsf{U}t$ of all stockings, proving hereby that the latter set were also denumerable.

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[1]) This case, however, has to be distinguished from that of the formula Dx introduced in *Foster 1 which not only asserts the existence of a member with the respective property but introduces a "representative". For this a special principle is required.

[2]) Notably Kamke 39 (§ 12), Denjoy 46-54$_1$, P. Lévy 50. In these papers it is also erroneously maintained that the general axiom of choice can be inferred, without any further assumption, from the (trivial) case where t contains a single member.

[3]) Cf., for instance, Hilbert-Bernays 34, pp. 151 ff.

From our present point of view which takes into account the cardinalities of the set t and of its members only, the *weakest non-trivial form* of the axiom is obtained by conceiving t as a *denumerable* set [1]) and its members as *finite* sets with cardinals > 1, most simply as pairs [2]).

While in the present section additional (actual or apparent) specializations of the axiom will occur [3]), those mentioned here are sufficient for the following survey of our present knowledge of the *independence of the axiom of choice*, i.e. of the impossibility of proving the axiom within a suitable system of axioms. It makes no great difference what system of axioms serves as basis for constructing a model in which the other axioms are satisfied but not the axiom of choice; for instance, the system **Z** developed in the previous and in the next sections, or *Principia Mathematica* with a simplified theory of types and an axiom of infinity [4]), or the system of Quine 36, or that of Bernays 37–54 or Gödel 40 (see § 7) — each, of course, without the axiom of choice. However, in **Z** as well as in Bernays 37–54 and in similar systems a *fundamental alteration* is required without which the model showing the independence of the axiom of choice cannot be constructed, namely an alteration permitting the introduction of *infinitely many objects which are not sets* ("individuals", *Urelemente*, cf. above p. 28 [5])). For some of the purposes in question even more-than-de-

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[1]) In this case one speaks of the *restricted* axiom of choice (Tarski 48, p. 82), no matter what the cardinality of the members of t may be.

[2]) For an early use of this particular form in the theory of measure, see van Vleck 08.

[3]) For statements equipollent to, or weaker than, the general axiom cf. also *Tajtelbaum-Tarski 3, Tarski ***29**, ***32**, 48, Froda 52. The *Postulat des Abzählens* formulated in von Krbek 55 has no clear meaning and at any rate does not constitute a specialization of the axiom of choice.

[4]) That is to say, the system of Tarski 33.

[5]) In a paper which appeared after this chapter had been sent to print, Mendelson 56a showed that, instead of admitting individuals, alternatively renouncing the axiom of foundation (p. 91) or replacing it by an essentially weaker axiom would enable us to carry out those independence proofs. In fact, if certain *extraordinary sets* (p. 90/1) are used instead of individuals, Mostowski's results remain true. Cf. Bernays $37\text{–}54_{\text{VII}}$ (p. 84), Shoenfield 55, and the *Habilitationsschrift* of Specker (see below p. 135). Of course, this does not alter the fact that within **Z** (or within Bernays' system, etc.) the independence problem remains unsolved — already regarding a set t the members of which are arbitrary sets of real numbers.

numerably many non-sets are required. The existence of non-sets is not compatible with the axioms of **Z** since the axiom of extensionality, in conjunction with other axioms, involves that there exists just one object that contains no member, viz. the null-set (p. 42); the situation is analogous in Bernays' system and in the main variant of von Neumann's system (p. 99). The introduction of such non-sets or of extraordinary sets is an *ad hoc* alteration which is neither useful nor justified in general, and *without this alteration* (or even with only a finite number of non-sets) *the independence of the axiom of choice remains an open problem*: the central problem of independence for which, for the time being, no method of possible solution is in sight [1]).

The most important independence problems related to our axiom are the following. (In b) and e) 'weakest form' does not necessarily mean that the set t is denumerable.)

a) Can the weakest form of the axiom be *proved* (by means of the other axioms)?

b) What interdependence exists between various formulations of the weakest form, in particular with respect to the cardinals of the finite sets which constitute the members of the (infinite) set t?

c) Can every infinite set be ordered? [2]) (The well-ordering theorem stating that every infinite set can be well-ordered is equipollent to the general form of the axiom of choice; see below in IV.)

d) Does the assumption that every set can be ordered imply the well-ordering theorem, hence the axiom of choice?

e) Assuming the validity of the weakest form, can the general form be proved? [3])

f) Can the general form be proved by means of the following *axiom of dependent choices* [4]): If B is a non-empty set and R a dyadic relation such that for every $x \in B$ there is a $y \in B$ with xRy, then

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[1]) Characteristically, the same assumption is required also for Ésénin-Vol'pin's (54) proof that Souslin's problem (T, p. 228) cannot be answered in the affirmative without the axiom of choice.

[2]) The term 'can' has an objective meaning referring to the axioms; see § 8. — A related weaker form is given in Kinna-Wagner 55; cf. below p. 133.

[3]) This problem admits of several specializations, according to the cardinality of t in either the weakest or the general form.

[4]) Bernays $37\text{–}54_{III}$ (Axiom IV* on p. 86); Tarski 48, p. 96. This axiom certainly implies the restricted axiom (p. 51) while in the converse direction the question still seems to be open.

there exists a sequence of members of B, $(x_1, x_2, \ldots, x_k, \ldots)$, such that $x_k R x_{k+1}$ for every k.

g) Does the sum-set of every disjointed set t that does not contain O include a subset equivalent to t?

h) Does every infinite (non-inductive) set include a denumerable subset? (Or else, is every non-inductive set the union of two disjoint non-inductive sets? [1])

i) Is a set which is finite according to a given definition (e.g., non-reflexive) also finite according to certain other definitions of finiteness (e.g., inductive)? [2]

With regard to g), h), i), it is well-known [3] that an affirmative answer follows from the axiom of choice. Hence a negative answer to these questions implies the independence of the axiom.

All questions a), c)–i) *have been answered in the negative* [4] for systems in which the existence of infinitely many "individuals" is compatible with the axioms. This means a multitude, and partly a hierarchy, of independence proofs. The proofs are complicated and mostly use a certain group-theoretic method, analogical to a method of Galois theory in algebra. The result regarding d) shows that the ordering principle is weaker than the well-ordering principle [5].

As to b), several group-theoretic and number-theoretic results

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[1] See Chwistek 35.

[2] A number of definitions of finiteness are arranged, according to the present problem, in the *Annexe* of Tarski 25; see below, p. 63. Thus i) comprises various problems; we refer only to those which are not trivial.

[3] Cf. *T*, pp. 57 f and 41 f, and below, p. 63.

[4] For a) and c) see Fraenkel 22 and 28a; a method regarding e) is developed in Fraenkel 35 and 37. Mostowski 38 (cf. 38a and A. Lévy 58a) and Lindenbaum-Mostowski 38, while adopting Fraenkel's group-theoretic method, based those proofs on a stricter logico-mathematical fundation and used the method of "relativization of quantifiers" (see Tarski 35a and Lindenbaum-Tarski 36). Thus they solved, in addition to a), the problems e), g), h), i). (The detailed exposition announced in Lindenbaum-Mostowski 38 has not appeared.) d) is solved in Mostowski 39 (cf. Doss 45, Shoenfield 55, and the simplification given in G. Schwarz 56), f) in Mostowski 48a (which relies on 39).

[5] For a statement which is (effectively) equipollent to the ordering principle see Łoś 54. Kurepa 53 obtains a statement equipollent to the well-ordering principle by taking the conjunction of the ordering principle and the principle that every partially ordered set has a maximal "anti-chain" (cf. Kurepa 52). Cf. also Kinna-Wagner 55 (see below p. 133).

have been obtained [1]) while a few problems still remain open. Of the number-theoretic results the following two are particularly simple. If Z_n means the statement of the axiom of choice for all sets t whose members have the finite cardinal n, then Z_n implies Z_d for every divisor d of n; the conjunction of the statements Z_{n_1}, Z_{n_2}, ..., Z_{n_k} implies the statement Z_n if for every representation of n in the form $n = m_1p_1 + m_2p_2 + \ldots + m_rp_r$ (p_i prime, m_i positive integer) at least one of the indices $n_1, n_2, \ldots, n_k$ is divisible by at least one of the primes $p_1, p_2, \ldots, p_r$.

Finally, the *consistency of the axiom of choice* was proved in 1938/39 by Gödel [2]). The short note 38 takes as possible bases any of various systems of set-theoretical axioms without the axiom of choice; for instance, a system like **Z** or the *Principia Mathematica* in the modification of Tarski 33 or the (modified) system of von Neumann 29. In the frame of any such system a special model can be constructed which satisfies the axiom of choice.

Gödel's classical booklet 40 adopts as its basis a modification Σ of Bernays' system of axioms which is described below in § 7. He proves the existence of the class of all "constructible" sets and from it obtains a model Δ satisfying Σ in which there is a well-ordering of all sets. (Here "constructible" has to be taken *cum grano salis*, for the non-constructive theory of ordinals is utilized and every ordinal proves constructible.) Thus the validity of the axiom of choice in Δ, hence its compatibility with Σ, is easily obtained.

3. The Existential Character of the Axiom. Save for the properly intuitionistic attitudes (Chapter IV) which are justified from their own point of view, the majority of the attacks on the axiom of choice [3]) derived from not sufficiently appreciating its *purely existential character*. In fact the axiom does not assert the possibility (with

[1]) Mostowski 45, Szmielew 47, Sierpiński 55. In the first paper the group-theoretic method mentioned above is used.

[2]) See Gödel 38 and 40 (cf. 39).

[3]) In addition to the literature quoted on p. 77, for instance *J. König 5 (pp. 170 f), *Dingler 5 (pp. 88 f of the first ed.), *Richard 4. Also the hypothetical part assigned to the multiplicative axiom in *Principia Mathematica* (cf. Chapter III) is apparently influenced by a quasi-constructive conception. As Ramsey 26 (p. 355) convincingly suggests, the existential conception, not implying that a selection-set can be specified, would be compatible with a "tautological" character of the statement. Cf. its introduction in von Neumann's axiomatic system (§ 6).

scientific resources available at present or in any future) of *constructing* a selection-set; that is to say, of providing a rule by which in each member τ of t a certain member of τ can be named. On the contrary, providing such a rule would mean obtaining the respective subset of $\mathsf{U}t$ by the axiom of subsets, without involving the axiom of choice. The latter just maintains the *existence* in $\boldsymbol{Z}$ of a selection-set, i.e. the non-emptiness of the Cartesian product $\boldsymbol{P}t$ (whose existence is guaranteed without our axiom). In other words, the axiom maintains that, its assumptions fulfilled, among the subsets of $\mathsf{U}t$ such subsets as contain a single common member with each member of t *will not be absent.*

Too little attention was paid to this fundamental point during the first decades of the present century and hereby many sterile discussions were caused. The difference in question will become clear in the light of the following examples, which are treated informally, i.e. without reference to the system $\boldsymbol{Z}$.

1. Let $t = \{\tau\}$ contain a single non-empty element τ. As pointed out on p. 50, in this case a selection-set exists independently of our axiom. However, even in this simplest case a member of τ cannot always be named; for instance, if τ is a suitable set of transcendental numbers. Before Liouville first (in 1851) constructed transcendental numbers, hereby proving the existence of such numbers, one still might have conditionally asserted the existence of a unit-set containing a transcendental number, using the principle of the excluded middle but not the axiom of choice. (In a similar way the principle of the excluded middle is used in each single step of a procedure dependent on a sequence of bisections, for instance in the proof of Bolzano-Weierstrass' theorem. Regarding their existential character such proofs resemble those in which an application of the axiom of choice is involved.)

2. Let t be an infinite (for instance, a denumerable) *disjointed* set $\{t_1, t_2, \ldots, t_k, \ldots\}$ whose members t_k are non-empty sets of *natural numbers*. Then the axiom of choice is not required for the purpose of forming a selection-set, the axiom of subsets being sufficient. We may, for example, define the condition $\mathfrak{P}(x)$ as 'there is a $y \in t$ such that x is the least number in the set y'. Hereby to each $y = t_k$ a definite natural number $x = n_k$ corresponds and $n_{k_1} \neq n_{k_2}$ if $k_1 \neq k_2$; hence we obtain a subset of $\mathsf{U}t$ which is a selection-set of t. The fact that in every non-empty set of natural numbers there is a least

number makes the use of the axiom of choice unnecessary [1]).

3. The situation is entirely different when t is an infinite set whose members are arbitrary sets of *real numbers*. Then, in general, we do not know a rule which simultaneously assigns to each member of t one of its members — except for the case that the sets have a special quality which enables us to form a constructive rule; for instance, when each member of t (or each except for a finite number) contains algebraic numbers, in which case an effective enumeration of the set of all algebraic numbers (T, p. 56) makes the use of our axiom superfluous.

This becomes yet more obvious when we take Zermelo's general axiom of choice without assuming t to be disjointed. Denoting by C a continuum (e.g., the set of all real numbers) let us take for t the set $\mathbf{C}C - \{0\}$, i.e. the set of all non-empty subsets of C. To define a function assigning to every member x of t a single member of x, we ought to form a rule which, in every non-empty set of real numbers, marks one number contained in the set; that is to say, we ought to name a property characteristic of the numbers to be marked, analogous to the property of being the least natural number in the preceding example. One cannot expect to name such a property and one can prove that, in the present and in similar cases, no function "analytically representable" in a well-defined sense exists which expresses such a property [2]).

Lebesgue also pointed out an example [3]) which shows that distinguishing between construction and existence in the sense described is not just a matter of logic and principles but affects the significance of concrete mathematical statements. A well-known theorem of Hilbert's [4]) states that a geometrical construction achievable with ruler and compasses can also be achieved by means of a ruler and a protractor of segments, even of one segment only, provided the problem has the property that all its solutions (for any values of the

[1]) From an intuitionistic point of view one may remark that the same applies to any "enumerable" (effectively denumerable) disjointed set of non-empty sets. For an effective enumeration of the sets t_k does not seem possible but by naming a definite member in each t_k, whereby the construction of a selection-set evolves, Cf. Lusin 27, p. 81. The same applies to any effectively well-ordered disjointed set of non-empty sets.

[2]) See Lebesgue 07.

[3]) Lebesgue 41, p. 117.

[4]) See Hilbert 1899/1930, ch. VII (theorem 67 of the 7th edition).

parameters concerned) are real. This theorem was proved in a constructive way on condition that the number of parameters does not surpass 3. In the general case, however, the resources of the theory of real fields (due to Artin and Schreier) which are required for the proof use a *basis* which is secured by the axiom of choice. Thus even in the theory of geometrical constructions the existential character of our axiom puts in its appearance.

It proves at least psychologically useful to express the existential character of our axiom *in a negative way*, viz. by denying the possibility that, t fulfilling the conditions of our axiom, among the subsets of Ut there are no subsets containing a single common member with each member of t — even when we fail to construct such a subset by the axiom of subsets. In fact, the modern views taken in the foundations of mathematics by intuitionism (Chapter IV) or by metamathematics (Chapter V) seem to find the most acceptable approach to our axiom in maintaining that proving a mathematical theorem by the axiom of choice means showing that any attempt of proving its negation is hopeless.

However, the relation between our axiom and the antithesis between construction and existential proof is more intricate than it appears in view of the preceding remarks. To illustrate this we use the notion of *effective example* [1]).

To give proper value to a definition, no matter whether within mathematics and logic or without, the existence of objects (at least one object) satisfying the definition should be shown. Normally this is done by providing a particular object that satisfies the definition, i.e., by giving an effective example. Not always need the example be given in a constructive way; its formation may make use of a non-predicative procedure (pp. 174 ff) or be based upon joining an existential proof which shows that *there are* objects satisfying the definition, to a demonstration that *no more than one* such object can exist. One might maintain that also in this way an effective example was given.

The possible combinations regarding the use of the axiom of choice are far more diverse than might be expected at first sight. In par-

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[1]) For the significance of this notion (introduced for sets of points by Borel and Lebesgue; cf., e.g., *Lebesgue 4 and 5 [p. 238]) see, besides *F. Bernstein 4 (§ 4), especially Sierpiński 19, 21, 28, *7, 33, *15; furthermore *Knaster-Kuratowski 1, Lusin 25 and 27. The notion of *effective correspondence* with its specialization to effective denumerability (*enumerability*) has a particular importance.

ticular, its use for proving the existence of a certain set and the possibility of specifying a member of the set thus procured are independent.

It may be possible, for instance, to prove by our axiom the existence of objects with a certain property whereas no way of determining a particular such object is known. This is the case in the original problem leading to the axiom of choice (p. 46) where the existence of a certain subset of Ut is maintained. Another famous instance is that of a set of points (real numbers) with the cardinal $\aleph_1$. (Incidentally, the problem of giving an effective example of such a set of points is an important weakening of, and therefore approach to, the continuum problem [see § 7], whose solution in the affirmative would mean constructing an effective one-to-one mapping between the linear continuum and the second number-class.) On account of the comparability of cardinals (i.e. of the axiom of choice) the continuum has subsets of the cardinal $\aleph_1$, yet all attempts [1]) to provide an effective example of such a subset have failed so far [2]).

There are other instances where the existence of objects with a certain property may be shown without the axiom of choice while no effective example can be provided. But it even happens that *an effective example can only be constructed by means of the axiom of choice*, in spite of its existential character. More precisely, a certain object is formed by a constructive procedure, yet the proof for its being an example for the purpose intended essentially uses our axiom. This case occurs in a remarkable proof [3]) of the well-ordering theorem which is based upon the assumed comparability of cardinals without using the axiom of choice.

Let M be the set whose members m are all well-ordered sets of real numbers, regardless of the succession of numbers by magnitude. (The existence of the set M within our system $\boldsymbol{Z}$ is easily shown; cf. § 8.) Distribute the members of M into pairwise disjoint classes

[1]) See *Hardy 1 (cf. *2, and *Hobson 1); *Hausdorff 2_{II} (p. 156); *Lusin 7 and 8, where some important related problems are raised; Sierpiński 54; Fraïssé 55 (§ 31).

[2]) Another example is a one-to-one mapping between the continuum and the set of its denumerable subsets (cf. p. 48). Producing such mappings involves the axiom of choice and no effective example is at our disposal (Sierpiński 19, p. 145; 21, p. 113).

[3]) Hartogs 15; cf. Sierpiński 21, p. 117.

c by allotting to the same class all sets m with equal ordinal numbers; hence every class is a subset of M. ($\{O\}$ is one of the classes.) Let C be the set of all classes, ordered by the following rule: c_1 and c_2 being different members of C, $c_1 \prec c_2$ shall hold if the members of c_1 have an ordinal less than the ordinal of the members of c_2. C is evidently well-ordered and the section of C determined by a certain c (T, p. 245) is similar to the members of the class c.

The cardinal (Aleph) of C is neither less than, nor equal to, the cardinal $\aleph$ of the continuum. In fact, any subset C_0 of C with a cardinal $\leqslant \aleph$ is equivalent to some set of real numbers, which can be well-ordered so that it becomes similar to C_0; this well-ordered set, therefore, is a member of a certain $c \in C$, hence similar to the section of C determined by c. From this we infer that C_0 is a proper subset of C since a well-ordered set is not similar to a section of itself (T, p. 266).

The set C has been formed constructively. Assuming now the comparability of cardinals, i.e. the axiom of choice, we infer from the last paragraph that C is an effective example of a well-ordered set with a cardinal surpassing $\aleph$.

Furthermore, since the well-ordered set C has a cardinal greater than $\aleph$ there is among its sections a first section $\bar{C}$ with the cardinal $\aleph$. *$\bar{C}$ is an example of a well-ordered set with the cardinal of the continuum* and every member of the class c which in C determines the section $\bar{C}$ is a well-ordered set of points (numbers) with the cardinal $\aleph$. Yet whereas in the case of C not the definition but the proof of its characteristic property requires the axiom of choice, the very definition of $\bar{C}$ relies on this axiom [1]).

Naturally, also Cantor's continuumhypothesis and its generalization are apt to produce notions for which effective examples are not available.

4. Some Typical Applications of the Axiom [2]). A comparison of the axiom of choice with Axioms II–V may cause the reader to

[1]) By a slight modification (cf. Sierpiński 21) one may without the axiom define an effective example of a well-ordered set which in view of the axiom proves to have the cardinal $\aleph$; namely a well-ordered set which without the axiom proves to have a cardinal neither less nor greater than $\aleph$.

[2]) For applications of the axiom in analysis see in particular Sierpiński 19, *Littlewood 1 and 2, *Lusin 10, *Egyed 1, and many other papers, especially within the theory of measure; most of them appeared in *Fundamenta Mathematicae*.

wonder why we stress so much its significance. It might appear as if its statement, excluding the non-existence of a certain kind of subsets of Ut, applied to special problems and methods only and meant but little for the general theory. This supposition seems to be supported by the fact that the axiom was introduced only at the beginning of the present century; that is to say, at a time when the bulk of both the theory of abstract sets and the theory of sets of points, including the nucleus of the modern theory of real functions, had already been developed.

Yet this supposition does not to conform to the actual situation. On the contrary, *fundamental and general theorems and methods* in the theory of sets as well as in analysis, algebra, and topology *are based on the axiom of choice,* in the sense that we do not know a way of avoiding its use. A remarkable number of the theorems in question even prove *equipollent* to our axiom. True, the axiom was introduced only at the beginning of the 20th century, but it had been utilized long before while only much later was it observed that in the respective proofs an argumentation not used and recognized in earlier mathematics is involved.

Therefore the axiom of choice, provided it is independent (cf. p. 52), must be admitted among the other acknowledged principles of mathematics. According to Hilbert [1]) it rests on "a general logical principle which is necessary and indispensable already for the first elements of mathematical inference".

To enable the reader to form his own opinion in this matter we shall now present a few characteristic applications of our axiom. Five examples will be given, selected not only in view of their fundamental character and of a minimum of technicality entering but so as to cover a maximum variety of domains: two examples from the general theory of sets and one from each arithmetic, analysis, and algebra [2]).

The first example, taken from the elements of *abstract set theory* [3]), concerns the operations on cardinals (addition, multiplication, exponentiation; see §§ 6 and 7 of *Theory*) and partly on order-types

[1]) Hilbert 23, p. 152.

[2]) No example of topology is given here, to avoid technicalities; see, for instance, *Tukey 1 (cf. below, p. 68); also Kelley 50 for a characteristic topological theorem which is equipollent to the axiom of choice.

[3]) Of course, the non-vanishing of a product of cardinals $\neq 0$ is also an example, but this is the multiplicative principle itself (p. 46).

and ordinal numbers (§§ 8 and 10). Since the point is the same in all these cases it will be sufficient to take the simplest case, viz. the *addition of cardinals* [1]). To obtain the sum of infinitely many [2]) (finite or infinite) cardinals we assign to each cardinal as its representative a set with that cardinal [3]) on condition that the representatives be pairwise disjoint; then the cardinal of the union of the representatives is the sum of the cardinals. Accordingly the sum would depend on the arbitrarily chosen representatives, yet the independence is guaranteed by a theorem (*T*, p. 112) stating that different ways of choosing the representatives necessarily yield equivalent unions, hence the same sum-cardinal.

Now the proof of this theorem is based on simultaneous one-to-one mappings between the representatives attached to the same cardinal by different choices. More precisely, if the cardinal $f(t) = \mathfrak{c}_t$, where t runs over a certain set T, is represented once by a set a_t and again by b_t (hence $b_t \sim a_t$), let $\psi^{(t)}$ be a certain mapping between a_t and b_t; by combining the mappings $\psi^{(t)}$ for all $t \in T$ we easily obtain a mapping between the union of the sets a_t and the union of the sets b_t. However, $\psi^{(t)}$ is not uniquely determined by the equivalent sets a_t and b_t; save for trivial cases, there are various mappings between these sets and infinitely many when a_t (hence b_t) is infinite. As will be shown in § 8, the set $\Psi^{(t)}$ of all mappings between a_t and b_t exists on account of our axioms I–V, and so does the (disjointed) set I' whose members are all sets $\Psi^{(t)}$ when t runs over T. But what we actually need is a set containing *a single member* from each member

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[1]) For the use of the axiom in the arithmetic of cardinals and ordinals in general cf., e.g., Sierpiński 28 (Chapters VI and XII) and Tarski 49 (pp. 239–243).

[2]) If the number of terms is finite the procedure is the same but our axiom is not required.

[3]) Apparently here arises the question how to "obtain" such representatives. This depends on the axiomatic system used and on the way of deriving the theory from it. According to the direction taken in § 8 to derive set theory in **Z**, no cardinals appear at all and representatives are used from the first. Within the systems of §§ 6 and 7 a cardinal may be taken as its own representative. Of course, if a cardinal appears as the set (class) of all sets "with this cardinal", the axiom of choice is already required to obtain a set of representatives.

Accordingly, our example refers rather to the theorem of *T*, p. 112, than to cardinals proper; in fact, this theorem and its analogues are the key to the arithmetic of cardinals.

$\Psi^{(t)}$ of Γ; to obtain such a set the axiom of choice is required, Γ taking the place of the set t on p. 46.

Hence the addition of cardinals depends on our axiom, provided the number of terms $\mathfrak{c}_t$ is infinite (even if the terms themselves are finite cardinals > 1). The same applies to the other operations with cardinals and with order-types.

As a second example we take *the distinction between finite and transfinite cardinals* or sets, bearing upon the concept of finite number (integer), the very fundament of arithmetic. This matter is closely connected with the theorem stating that every infinite set includes a denumerable subset (T, pp. 57/8), i.e. that $\aleph_0$ is the least transfinite cardinal. Of the two proofs A and B given *l.c.*, A (but not B) essentially uses the axiom of choice because in proof A 'infinite' means 'non-inductive'.

As in T, p. 37, a set s shall be called *inductive* if either $s = O$ or there exists a positive integer n such that s contains exactly n memmers [1]). A set is called *reflexive* [2]) if it is equivalent to a proper subset of itself [3]). Our purpose is to investigate whether inductive sets and non-reflexive sets are the same (finite sets) [4]).

First one easily proves by mathematical induction that an inductive set cannot be reflexive (T, p. 38); hence a reflexive set is not inductive and it only remains to see whether sets exist which are neither inductive nor reflexive — a question raised by Lebesgue as

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[1]) For the present purpose this formulation which presupposes the notion of positive integer is sufficient. But we may as well define with Russell as follows: a set of cardinals is called *hereditary* if its containing n implies that it contains $n + 1$; a cardinal is called *inductive* if it is contained in every hereditary set that contains the cardinal 0; a set is called *inductive* if its cardinal is inductive. ($n + 1$ is here defined by means of the union.) Cf. the attitude of Bernays; see below, pp. 117 f.

[2]) The reflexivity of sets has, of course, to be distinguished from the reflexivity of relations (p. 31).

[3]) 'Non-reflexive' is, then, Dedekind's so-called first definition of finiteness (T, p. 40). It may also be formulated in a positive way as follows: for every one-to-one mapping of the finite set F upon a subset $F_0 \subseteq F$ one has $F_0 = F$.

[4]) Even without going into matters of principle we perceive the fundamental difference between the two concepts when we attempt to prove, on the basis of 'non-reflexive', a theorem like the following: the power-set of a finite set is finite. On the basis of 'inductive' this is an elementary theorem of combinatorics, easily proved by mathematical induction.

early as 1904 [1]). The cardinals of such sets, sometimes called 'mediate cardinals' [2]), would therefore constitute an intermedium between finite and transfinite cardinals.

To show that mediate cardinals do not exist we have to prove that every non-inductive set is reflexive, hence every non-reflexive set — inductive. But for this purpose we rest upon the theorem, proved by means of the axiom of choice, that every non-inductive set includes a denumerable subset; the latter set is reflexive and transfers this property to the original non-inductive set (cf. *T*, pp. 41/2). Thus we obtain: The proof that every inductive set is non-reflexive has a constructive character, yet *for the converse statement the axiom of choice is required.*

'Inductive' and 'non-reflexive' are two of the various meanings of 'finite' (finite set, finite cardinal). Tarski [3]) gave several definitions of finiteness, arranging them in such a way that a set which is finite on account of a certain definition proves finite on account of every succeeding definition by elementary methods, while the transition

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[1]) See Borel 14, p. 156.

[2]) Cf. *Wrinch **5**, *Chwistek **3**. The paper *von Seckendorff **1** is erroneous because of an unnoticed use of the axiom of choice. See also Doss 45.

[3]) Tarski 25, pp. 93–95. The bulk of this important essay (cf. *Zermelo **4**, *Vredenduin **1**) is devoted to the exhibition of certain properties which without the axiom of choice prove equipollent to 'inductive' and of other properties of an inductive set which do not yield inductivity without the axiom of choice. Inductivity and equipollent properties appear there singled out by the fact that all other properties mentioned can be derived from them without the axiom. (The view expressed in Denjoy 46–54$_1$ that Tarski uses the axiom of choice is due to misunderstanding; cf. p. 50, footnote 2.)

A classification of definitions of finiteness on a different and more general basis by means of Gödel's method of arithmetization is given in Mostowski 38; in a parallel way also axioms of infinity (cf. § 5) are classified there. More far-reaching results are obtained in Trahténbrot 50 where, in particular, it is shown (with regard to suitable axiomatic systems) that among the essentially different definitions of finiteness neither a "strongest" nor a "weakest" exists. Cf. Church 56, pp. 342 ff.

Dedekind's so-called second definition of finiteness (Dedekind 1888, p. XI of the 2nd ed.) is equipollent to inductivity (contrary to Dedekind's conjecture), as shown in Tarski 25. Incidentally, some of Tarski's methods in basing the theory of finite sets and numbers without the axiom of choice had been anticipated by Dedekind in a posthumous essay published in 1932 (see *Dedekind **3** III, pp. 450 ff); cf. E. Noether's remarks, *ibid.*, and *Cavaillès **1**.

in the converse direction requires the axiom of choice [1]). Take, for instance, the following two definitions:

(*) A set f is finite if its power-set $\boldsymbol{C}f$ is not equivalent to any proper subset of $\boldsymbol{C}f$ (i.e. if $\boldsymbol{C}f$ is non-reflexive) [2]).

(**) A set f is finite if it is not the union of two disjoint sets each of which is equivalent to f [3]). (Cf. p. 53.)

According to the arrangement of definitions described above, the property (*) intervenes between inductivity and non-reflexiveness while (**) succeeds non-reflexiveness.

More than in any other field of mathematics the axiom of choice is utilized in analysis; in particular in the theories of sets of points and of real functions. Most of these applications involve technical notions of the theories concerned. Here we shall give an example from the very first elements of analysis with which all readers are familiar.

One might expect the most common instance to be the following. After having proved that for each point x of a given set there exists at least one neighborhood of x — i.e. an open interval containing x — with a certain property, one chooses for each given x a definite such neighborhood. Apparently here our axiom is used inasmuch as for each x simultaneously an arbitrary neighborhood is chosen. However, in general the axiom can be dispensed with through a restriction to neighborhoods with rational ends; then only an (effectively) denumerable set of possibilities is left for every x and it is easy to mark a definite one among them by a general rule. (Cf. exercise 9) in T, p. 61.)

Yet with respect to concepts of an even more fundamental character we do depend on the axiom of choice. As usual (cf. T, p. 232) a point p shall be called an *accumulation point* of a (linear) set K of points if in every neighborhood of p there is a point of K different from p. On the other hand one may base the elements of analysis upon the notion of *limit point*, defining p as a limit point of K if there exists

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[1]) For more recent characterizations of finite sets cf., among others, Popadić 51, Kurepa 52, A. Lévy 58. Cf. also Rabin 54.

[2]) On the other hand, the property that $\boldsymbol{C}(\boldsymbol{C}f)$ is non-reflexive is equipollent to inductivity.

[3]) That infinite sets have the opposite property, i.e. that every transfinite cardinal $\boldsymbol{t}$ satisfies the equation $\boldsymbol{t} = \boldsymbol{t} + \boldsymbol{t}$, is a consequence of the well-ordering theorem, hence of the axiom of choice (T, pp. 303 and 315).

a sequence (k_ν) of different points of K ($\nu = 1, 2, \ldots$) such that the sequence has the limit p.

Without the axiom of choice one easily proves that if p is a limit point of K, p is also an accumulation point of K. On the other hand, let us suppose that also every accumulation point of K is a limit point of K. By this supposition one proves the following lemma $\mathfrak{L}$ [1]: If $S = (S_1, S_2, S_3, \ldots)$ is a sequence of pairwise disjoint non-empty sets of points, there exists a sequence of points $(p_1, p_2, p_3, \ldots)$ such that p_k with different indices k belong to different S_n. Conversely it is easy to infer from $\mathfrak{L}$ without the axiom of choice that every accumulation point of K is as well a limit point of K. Hence, without involving our axiom, the equipollence between the notions 'accumulation point of K' and 'limit point of K' is a necessary and sufficient condition for the validity of the lemma $\mathfrak{L}$.

It is evident that $\mathfrak{L}$ constitutes the statement of the axiom of choice for an effectively denumerable set t of sets of points. Thus the connection becomes clear between our axiom and the equipollence between two fundamental and elementary notions of analysis which usually are identified without further ado.

This equipollence implies relations of equipollence between other fundamental concepts of analysis which can be defined by means either of accumulation point or of limit point: not only those of 'derived set' and of 'closed [2]), dense-in-itself, perfect set' (T, pp. 232 f) but also the concept of *continuous function*. In fact, in the elements of analysis one uses either of the following two definitions of continuity:

a) $f(x)$, defined for $a < x < b$, is called continuous at the point x_0 of that interval if to every positive ε corresponds a positive δ such that

$$|x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \varepsilon.$$

[1]) Sierpiński 19, p. 120.

[2]) One may define a set K of points as *closed* either (as in T) on condition that every accumulation point of K belongs to K, or on condition that every limit point of K belongs to K; analogically for the following concepts.

Thus one can also conceive the theorem of Bolzano-Weierstrass in two different meanings, as asserting that a bounded infinite set of points has either an accumulation point or a limit point. To be sure, this makes no difference for the proof if one conceives 'infinite' as 'reflexive'; for 'non-inductive', however, the proof of the theorem with the meaning of 'limit point' requires the axiom of choice.

b) $f(x)$ is called continuous at x_0 if

$$\lim_{k\to\infty} x_k = x_0 \text{ implies } \lim_{k\to\infty} f(x_k) = f(x_0).$$

While one easily proves that a function which is continuous at x_0 in the sense a) is also continuous there in the sense b), the converse assertion rests on the axiom of choice; more precisely, the proposition $\mathfrak{L}$ of p. 65 is necessary and sufficient for proving the equipollence of the definitions a) and b) without the axiom [1]).

Naturally, the same alternative exists for the definition of the derivative of a real function, where the situation is analogous to that regarding continuity.

It was a complete surprise for the mathematical world when in 1910 Steinitz [2]) showed the important task performed by the axiom of choice in algebra, both for certain problems of classical algebra and still more for abstract algebra (which became an important branch of mathematics through the influence of that very essay of Steinitz).

To render a typical problem of this branch intelligible also to readers not familiar with modern algebra we proceed from a starting-point known to everybody, if not properly algebraic.

The so-called *fundamental theorem of algebra* [3]) (cf. below Chapter IV, pp. 256 f) can be expressed as follows. A polynomial

$$p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \qquad (a_0 \neq 0)$$

with integral rational coefficients a_k and the positive integral degree n has at least one zero $x = x_1$ within the field of all complex numbers; hence it has n, not necessarily different, zeros.

However, the field of all complex numbers — which, as well as the field of all real numbers, is a concept of analysis and not of alge-

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[1]) Yet the transition from b) to a) can be accomplished without the axiom if b) is presumed *for the entire interval*, i.e. for every convergent sequence of points from the interval. See Sierpiński 19, pp. 131 ff.

[2]) Steinitz 10, §§ 19–24. Cf. the comments of R. Baer and H. Hasse (pp. 18–26) of the 1930 edition, and *van der Waerden **21** (chapters VIII–X of the first, or the third [or fourth] edition); also *von Neumann **5**.

[3]) This name has historical reasons only. From a modern point of view the name should be attributed to the theorem stated below regarding an algebraically-closed extension.

bra — quite incidentally enters this theorem, viz. because it is a well-known and would-be "elementary" concept. Returning to algebra, by *algebraic number* we mean any (complex) number that is a zero of $p(x)$, i.e. a root of an algebraic equation $p(x) = 0$ with integral rational coefficients (cf. *T*, p. 14). The domain of all algebraic numbers — which is denumerable, contrary to the domain of all complex or real numbers — has the following two properties: 1) it constitutes a field with respect to addition and multiplication; 2) also if the coefficients a_k are any *algebraic* numbers the polynomial $p(x)$ has a zero, hence n zeros, in the field of all algebraic numbers.

A field F (not necessarily a field of numbers) is called *algebraically-closed* if it does not admit of an algebraic extension; that is to say, if every polynomial $p(x)$ "in F" (i.e., with coefficients from F) has a decomposition into linear factors $x - x_k$, x_k belonging to F. In particular, F is called an *algebraic* algebraically-closed extension of a field F_0 if F_0 is a subfield of F, F is algebraically-closed, and every member of F is algebraic with respect to F_0, i.e. a zero of a polynomial in F_0. Therefore the field of all complex numbers is algebraically-closed, but not an algebraic extension of the field R of all rational numbers since the transcendental numbers are not algebraic with respect to R. On the other hand, the field A of all algebraic numbers is an algebraic algebraically-closed extension of R.

The construction of the field A from R does not involve the axiom of choice (nor transfinite induction), owing to the denumerability of the field However, the well-ordering theorem, hence our axiom, is required to prove by transfinite induction that there is essentially *only one* such extension of R; 'essentially' refers to isomorphic extensions.

Now we may generalize this train of ideas by starting not just with the field R of the rationals but with *any field* F_0 whose members need not even be numbers [1]). In this case, as proved by Steinitz, the above theorem still holds true, i.e. there exists one, and essentially only one, algebraic algebraically-closed extension F of F_0. Then, however, both parts of the theorem rest upon the well-ordering theorem

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[1]) As shown in Steinitz 10, it is important to distinguish between fields of the characteristic 0 (comprising R as a subfield) and fields of the characteristic p in which, by repeatedly (a prime number p of times) adding the unity of the field, one obtains the zero of the field.

since transfinite induction is required for the proof of the first part also.

This is a typical and important use of the axiom of choice in algebra. For other algebraic problems requiring the axiom a few references are given below [1]).

Furthermore, the use of our axiom in algebra has yielded an important by-product: a principle of an apparently different character which nevertheless proves equipollent to the axiom of choice and which is considered by algebraists a welcome substitute for the axiom, for our axiom and transfinite induction are rather foreign to algebra in general and to abstract algebra in particular. In fact, in modern algebra the tendency has increased of avoiding a direct application of the axiom of choice or the well-ordering theorem.

As proved by Hausdorff [2]), every partially ordered set (T, p. 179) includes at least one *maximal* completely (or totally) ordered subset, i.e. an ordered subset which is not a proper subset of any completely ordered subset. The proof uses the axiom of choice and follows the method of Zermelo's first proof of the well-ordering theorem (see below, p. 70). Independently but also by means of the axiom of choice a similar theorem was later proved by Kuratowski [3]). Again independently of these results and of each other, Zorn [4]) and Teichmüller [5]) introduced, for the use in abstract algebra instead of the well-ordering theorem [6]), two principles which, with slight modifications due to Bernays [7]), can be formulated as follows.

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[1]) We mention the following of the earlier researches of this kind (up to 1930): *Artin-Schreier **1**, *Baer **1**, *Burstin **1**, *Kamke **2**, *Krull **1** (pp. 110 f), *E. Noether **1** and **2**, *Ostrowski **1**, *Prüfer **1**, *Souslin **1**, *Tambs Lyche **1**, *Zermelo **6**. For the use made of the axiom in the theory of representations for Boolean algebras, see *Stone **4**.

[2]) Hausdorff 14, pp. 140 f.

[3]) Kuratowski 22, p. 89.

[4]) Zorn 35. Cf. *R. L. Moore **1**, p. 84 (1932).

[5]) Teichmüller 39. Various formulations are given there and some specializations of the axiom of choice, fit for applications in algebra, are discussed.

[6]) In certain cases, especially in proofs of general set theory, it is convenient to use both a maximum principle and the axiom of choice; cf., e.g., Hönig 54.

[7]) Bernays 37-54$_V$, pp. 91-93. Here various forms of the maximum principle are deduced in the frame of the axiom system of Bernays or of part of it. (See below § 7.) Cf. also the exposition in Bernays 58, chapter VI. (This booklet could no more be used in the present book.)

(Z) Every non-empty closed set A of subsets of a given set B [1]) has a maximal member. (A set A of sets is called 'closed' if for every chain c — i.e. every set that is completely ordered by subsumption: $\sigma_1 \prec \sigma_2$ if $\sigma_1 \subset \sigma_2$ — which is a subset of A, the sum-set of c is a member of A.)

(T) A set A of subsets of a given set has a maximal member if A has the following property: a set s belongs to A if and only if every finite subset of s belongs to A.

The principle in one of these or of related forms [2]) is now generally called *Zorn's lemma.*

G. Birkhoff [3]) and others [4]) proved the equipollence of Zorn's lemma to other maximum principles (among them those state above) and the well-ordering theorem, subject to a suitable axiomatic basis.

We shall sketch a proof of the well-ordering theorem on the basis of Zorn's lemma (Z). [5])

Let s be an arbitrary set. By $\langle t, O_t \rangle$ we denote the ordered pair of a subset t of s and of a well-ordering O_t of t. [6]) Let B be the set of all such pairs. Two pairs

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[1]) In Zorn's original formulation 'set of sets' stands in place of 'set of subsets of a given set'. Within the present, or Bernays', axiom system these formulations are equipollent.

[2]) Cf. also the forms given in Beth 53.

[3]) Birkhoff 48, pp. 42–44. The axiom formulated there as "the original Axiom of Choice of Zermelo" is not Zermelo's general axiom (Zermelo 08a, p. 274) as opposed to Russell's Multiplicative Axiom but the particular case of the former used in Zermelo 04 and 08 to prove the well-ordering theorem. — Also in Gottschalk 52 various formulations of the maximum principle are given and their totality is called "the extremum law".

[4]) Wallace 44, Fort 48 (specialization of Zorn's lemma), Klimovsky 49, Obreanu 49, **Bourbaki** 50 and 56, **Kneser** 50, Szele 50, **Witt** 51 (cf. Inagaki 52), **Vaughan** 52, Banaschewski 53, Seki 55 (here the basis is Gödel's axiom system, cf. § 7). (The more important papers are given in bold type.) Equipollence proofs are also given in some modern textbooks; for instance in Rosser 53 (pp. 493 ff), Kelley 55 (pp. 32–36), A. Robinson 55 (pp. 15 f), Hermes 55 (pp. 136 ff). (For Bernays see § 7.) The maximum principle considered in Büchi 53 is not properly Zorn's lemma since it is stated in the frame of the logical theory of types (not of set theory) as developed by Church 40; on the other hand, the type-theoretic counterpart of the axiom of choice involved corresponds not to Zermelo's axiom but to Hilbert's ε-axiom (Chapter III).

[5]) Cf. the proof in *T*, pp. 319–321, which is closely connected with Zermelo's second proof of the well-ordering theorem. For details we are indebted to R. Brauer and A. Robinson.

[6]) Ordered pairs are easily reduced to pairs $\{a, b\}$, and a (well-)ordering of t is a certain set of ordered pairs of members of t; furthermore, the set of all $\langle t, O_t \rangle$ with $t \subseteq s$ exists. See § 8.

$\langle t_1, O_{t_1}\rangle$ and $\langle t_2, O_{t_2}\rangle$ are called *comparable* if $t_1 \subset t_2$, O_{t_2} orders t_1 in the same way as O_{t_1}, and t_1 is an initial (cf. p. 128) of t_2 according to that ordering. Let A be the set of all subsets B' of B such that any two members of B' are comparable; A is easily seen to be closed and not empty. By (Z) A contains a non-empty maximal member C. The union t_0 of all sets t which appear in the first place of the pairs $\langle t, O_t\rangle \varepsilon C$, is well-ordered in an obvious way by the second elements of those pairs. Denoting this ordering by O_{t_0}, we find that $C \cup \{\langle t_0, O_{t_0}\rangle\}$ belongs to A; hence, by the maximum property of C, $\langle t_0, O_{t_0}\rangle$ belongs to C. Moreover, t_0 *coincides with* s; for if $s - t_0$ contained a member, it would be possible by a well-known argument to extend C (cf. T, p. 314 or 321). This shows that O_{t_0} defines a well-ordering of $s = t_0$, which completes the proof.

A last example is again taken from abstract set theory; it is the very case for which our axiom was originally introduced, viz. the well-ordering theorem and the comparability of cardinal numbers (T, pp. 309–315).

In both proofs of the well-ordering theorem given by Zermelo [1]) the fundament is a simultaneous choice of particular ("distinguished") members in every non-empty subset of the set s to be well-ordered; in other words, the supposition that a single-valued function f exists whose domain is the set of the non-empty subsets x of s, such that always $f(x) \epsilon x$.

The comparability theorem may either be inferred from the well-ordering theorem in view of the comparability of well-ordered sets or be proved directly through the axiom of choice (T, pp. 319–321).

But the axiom of choice, the well-ordering theorem, and the comparability theorem are even *equipollent* in the following sense. It is evident that the well-ordering theorem implies the multiplicative principle in its formulation of p. 46, i.e. for a disjointed set t, and therefore is equipollent to our axiom [2]). For an arbitrary well-ordering $\mathfrak{W}$ of the sum-set $\mathsf{U}t$ simultaneously well-orders every subset of $\mathsf{U}t$, hence every member of t; therefore a selection-set of t exists by the axiom of subsets; for instance, the subset of $\mathsf{U}t$ that contains the first member of each member of t on account of the well-ordering $\mathfrak{W}$.

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[1]) Zermelo 04 and 08. For explicitly embedding the proof of Zermelo 08 in the present axiomatic system, cf. *Vieler I, pp. 4–7. Bernays $37\text{-}54_{IV}$, pp. 143–145, compares the axiomatic simplicity of both proofs and gives an intermediate form of proof which has certain advantages (see below § 7).

[2]) Lebesgue (41, p. 118), in a somewhat sceptical comparison with Poncelet's continuity principle, straightforwardly expresses our axiom as follows: any property applying to every well-ordered set applies to every set, provided the property does not deal with order.

Incidentally, any well-ordering of an arbitrary set s assigns to every non-empty subset of s a "distinguished" member of the subset; thus one directly obtains the above form of the axiom where the disjointedness is not assumed.

The transition to the comparability theorem from either the axiom of choice or the well-ordering theorem has just been mentioned. The converse direction is taken in Hartogs' proof [1]) which is not restricted to sets of points, i.e. to subsets of the continuum, but applies to any abstract set as well. (This proof, as far as the theory of equivalence is concerned, uses the present axiomatic system; regarding the ingredients of order and well-order it is easily adapted in view of § 8.) Starting with an arbitrary non-empty (ordered or unordered) set s we obtain, by the method of p. 58/9 and without using the axiom of choice, a certain well-ordered set C — on p. 59 called the set of all "classes" — each of whose members contains similar well-ordered subsets of s; just as in the particular case of the linear continuum, C proves to be neither equivalent to s nor to any subset of s [2]). (This, by the way, means that no set exists whose cardinal surpasses the cardinal of every well-ordered set.)

Hence, *if we accept the comparability of sets as a principle,* s is equivalent to a subset of the well-ordered set C and has a cardinal less than C. s can, therefore, be well-ordered on account of a one-to-one mapping on a certain subset of C. Thus comparability implies the well-ordering theorem, hence the axiom of choice. Accordingly, *these three statements are equipollent principles*; taking one of them as an axiom we obtain the others as provable theorems. That the axiom of choice is favored in the foundations of set theory and of mathematics in general has its reason in the general logical character of this axiom and not in a superior evidence. Even in those cases where a function which provides a particular member in every non-empty subset of a given set can be *constructed,* its construction is in general not easier than directly well-ordering the set.

The purely existential character of our axiom becomes most con-

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[1]) Hartogs 15.

[2]) By the same additional step as in the case of p. 59 we thus obtain the result: To every set s one can assign in an effective way a well-ordered set $w(s)$ for which *it can be proved without the axiom of choice that the cardinal (Aleph) of* $w(s)$ *is neither less nor greater than the cardinal of* s while every section of $w(s)$ has a cardinal less than s.

spicuous (and lamentable) in its application to proving the well-ordering theorem. For the purpose of producing a particular well-ordering from which one might draw conclusions regarding the Aleph of the set concerned, the proof is of no use; especially not in the most important case where the continuum is the set to be well-ordered. This explains the inaccessibility of the continuum problem in both its special and its generalized forms, which to a great extent is responsible for the distrust our axiom has met with.

Regarding *the connection between the axiom of choice and the generalized continuum hypothesis* (T, p. 318, and below §§ 5 and 7), let us formulate the latter in the following three different expressions:

(H_1) If $\boldsymbol{c}$ is a transfinite cardinal (power) there is no cardinal $\boldsymbol{d}$ which satisfies $\boldsymbol{c} < \boldsymbol{d} < 2^{\boldsymbol{c}}$ (in other words, $\boldsymbol{c} \leqslant \boldsymbol{d} \leqslant 2^{\boldsymbol{c}}$ implies either $\boldsymbol{d} = \boldsymbol{c}$ or $\boldsymbol{d} = 2^{\boldsymbol{c}}$).

(H_2) If $\boldsymbol{a}$ is an Aleph there is no cardinal $\boldsymbol{d}$ which satisfies $\boldsymbol{a} < \boldsymbol{d} < 2^{\boldsymbol{a}}$.

(H_3) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α.

Lindenbaum and Tarski [1]) asserted in 1926 that the assumption (H_1) implies the axiom of choice, which assertion was proved twenty years later by Sierpiński [2]).

Naturally (H_3) — already in the particular case $\alpha = 0$ — is sufficient for well-ordering the continuum.

Lindenbaum-Tarski 26 also asserted that (H_2) and (H_3) are equipollent and that either of these weaker expressions of the hypothesis requires the addition of the axiom of choice to obtain a statement equipollent to the stronger expression (H_1).

Furthermore, if (H_1) is true for the particular values $\boldsymbol{c}$, $2^{\boldsymbol{c}}$ (and $2^{2^{\boldsymbol{c}}}$), then these cardinals are Alephs [3]).

Similar distinctions between seemingly equipollent assertions apply in more general cases and yield heterogeneous situations

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[1]) Lindenbaum-Tarski 26, p. 314; cf. Tarski 29 where, apparently for the first time, the opinion is expressed that the continuum hypothesis is independent of the usual axioms of set theory, an opinion shared nowadays by most mathematicians (cf. also Baer 29).

[2]) Sierpiński 47. Hence, according to Specker 53, (H_1) does not hold in Quine's *New Foundations* (37).

[3]) Asserted in Lindenbaum-Tarski 26, p. 314, with respect to the three above cardinals; proved in Sierpiński 47 for $\boldsymbol{c}$ under the assumption (H_1) for $\boldsymbol{c}$, $2^{\boldsymbol{c}}$, and $2^{2^{\boldsymbol{c}}}$, in Specker 54 for $2^{\boldsymbol{c}}$ under the assumption (H_1) for $\boldsymbol{c}$ and $2^{\boldsymbol{c}}$.

regarding the axiom of choice. Take, for instance, the following statements about "successive" cardinals $\mathfrak{m}$ and $\mathfrak{n}$:

(P_1) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that
  a) $\mathfrak{m} < \mathfrak{n}$
  b) $\mathfrak{m} < \mathfrak{p} < \mathfrak{n}$ does not hold for any cardinal $\mathfrak{p}$.

(P_2) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that
  a) $\mathfrak{m} < \mathfrak{n}$
  b) for every cardinal $\mathfrak{p}$, $\mathfrak{m} < \mathfrak{p}$ implies $\mathfrak{n} \leqslant \mathfrak{p}$.

(P_3) For every cardinal $\mathfrak{m}$ there is a cardinal $\mathfrak{n}$ such that
  a) $\mathfrak{m} < \mathfrak{n}$
  b) for every cardinal $\mathfrak{p}$, $\mathfrak{p} < \mathfrak{n}$ implies $\mathfrak{p} \leqslant \mathfrak{m}$.

As Tarski [1]) has shown, (P_1) can be proved without the axiom of choice while (P_2) is equipollent to the axiom; whether (P_3) can be proved without the axiom is not yet known. [2])

It is not surprising that the comparability of cardinals proves equipollent to the axiom of choice. With respect to most theorems of the arithmetic of cardinals the alternative holds that either the theorem is provable *without* the axiom or — if the proof requires the axiom — that the theorem is *equipollent* to the axiom, i.e. implies the axiom as well. Even for those mathematicians who reject the axiom because of its existential character, such theorems of equipollence retain a definite meaning [3]).

While for the multitude of results known in this direction we refer to the literature [4]) a few statements equipollent to the axiom shall be stated here:

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[1]) Tarski 54; cf. already Tarski 48, pp. 80–84 and 93.

[2]) In a forthcoming paper of A. Lévy, which uses the model of Mostowski 39, it is shown that the answer is in the negative — even if the ordering principle (p. 52f) is added to the axioms. (Addition in the proof.)

[3]) The same applies to the (less far-reaching) result that certain statements can be proved by means of the axiom but not without it — subject to the independence of the axiom altogether.

An analogous situation subsists with respect to consequences of the continuum hypothesis or statements equipollent to it — no matter what attitude one takes towards this hypothesis.

[4]) In particular Lindenbaum-Tarski 26 (pp. 311 ff), *Tarski 3 and 32, Lindenbaum-Mostowski 38, and the exhaustive list in H. Bachmann 55, § 31; cf. also *Tarski 29. As pointed out above, many theorems not pertaining to the arithmetic of cardinals, within or without set theory, are also equipollent to our axiom.

A) $\boldsymbol{c}^2 = \boldsymbol{c}$ for every non-inductive cardinal $\boldsymbol{c}$ [1]).

B) For any two cardinals $\boldsymbol{c}$ and $\boldsymbol{d}$, the inequality $\boldsymbol{c} + \boldsymbol{c} < \boldsymbol{c} + \boldsymbol{d}$ implies $\boldsymbol{c} < \boldsymbol{d}$ (while without our axiom $\boldsymbol{c} + \boldsymbol{c} > \boldsymbol{c} + \boldsymbol{d}$ implies $\boldsymbol{c} > \boldsymbol{d}$, and $\boldsymbol{c} < \boldsymbol{d}$ implies $\boldsymbol{c} + \boldsymbol{c} < \boldsymbol{c} + \boldsymbol{d}$).

C) Every non-inductive cardinal is a prime number, i.e. is not the product of two smaller cardinals [2]).

5. Mathematicians' Attitude towards the Axiom. We conclude our survey of the axiom of choice by a glance over the attitudes taken by mathematicians towards the axiom since its explicit formulation in the beginning of the present century. (The "prehistory" of the axiom was described on pp. 47 f.)

The negative attitude of most intuitionists, because of the existential character of the axiom, will be stressed in Chapter IV. To be sure, there are a few exceptions, for the equipollence of the axiom to the well-ordering theorem (which is rejected by all intuitionists) depends, *inter alia*, on procedures of a supposedly impredicative character; hence the possibility subsists of accepting the axiom but rejecting well-ordering as it involves impredicative procedures. This was the attitude of Poincaré.

Save for the intuitionistic point of view, both the positive and the negative attitude towards our axiom are *far more strongly influenced by emotional or practical reasons than by arguments of principle.*

For those accepting and using the axiom, the chief reason is its — or the well-ordering theorem's — indispensability for proving important theorems of analysis and set theory; this argument proved so strong that even scholars who in principle rejected existential procedures did not refrain from using the axiom to a certain extent in their analytic researches. As to algebra, the historical development can be hardly illustrated more strikingly than by the following quotation (translated) from Steinitz' pioneer work [3]). "As yet many mathematicians take a negative attitude towards the principle of choice. With the increasing recognition that there are mathematical questions which cannot be decided without the principle, the opposition

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[1]) Cf. *T*, p. 303, (2). A direct proof of this equality through Zorn's lemma (p. 69) without the detour via well-ordering theorem and ordinals is given in *Zorn 2.

[2]) See *Sierpiński **26.**

[3]) Steinitz 10, *Einleitung.*

against it will presumably fade more and more. On the other hand, for the sake of purity of method it seems suitable to avoid the principle as far as its application is not required by the nature of the problem concerned. I endeavored to make this boundary clearly visible."

Yet the opponents have also been considerably influenced by psychological rather than logical reasons — in spite of the opinion of as authoritative an observer as Lebesgue who maintained that no discussion between the two parties had been possible "because they had no common logic" so that they could do no better than to insult each other. (Cf., however, the quotations below on p. 77/8.) As a matter of fact, as long as the (implicit and unconscious) use of the axiom by Cantor and others involved just arithmetical operations with cardinals and order-types or guaranteed the non-vanishing of a product whose factors differ from zero — in short, involved only generalized arithmetical concepts and properties well-known from finite numbers — nobody took offence. The same applies to its use in some elementary proofs of analysis. Yet at the moment (1904) when the axiom, explicitly enounced, was used by Zermelo to prove and confirm one of the earliest assertions of Cantor, viz. the well-ordering theorem, mathematical journals [1]) were flooded with critical notes rejecting the proof, mostly arguing that our axiom was either illegitimate or meaningless.

True, various and widely different reasons for the rejection were given; but one cannot help feeling that the common denominator from which the various reasons derived was the unwillingness to accept its consequence, namely the theorem in which the opponents did not believe. This unwillingness deepened enormously when it became obvious that the main purpose connected with well-ordering since 1880, viz. to ascertain the place of the power of the continuum in the series of Alephs, had not become furthered a bit by the well-ordering theorem. The tremendous difficulties of this continuum problem, more clearly perceived to-day than 70 years ago, had by

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[1]) See in particular vol. 60 of *Mathematische Annalen* (Zermelo's first proof had appeared in vol. 59), Hadamard 05, and note IV in Borel 14. As becomes apparent especially in this note there was an early bifurcation among the opponents with regard to the case that the set t is denumerable; in this case Borel, as well as Denjoy 46-54, deems rather acceptable the assumption that a selection-set exists while Lebesgue sees none but possibly psychological reasons for distinguishing between different transfinite cardinalities of t.

the turn of the century induced many mathematicians to believe that the (linear) continuum and more complicated sets could not be well-ordered at all; in other words that $\aleph = 2^{\aleph_0}$, $2^{\aleph}$, etc. were no Alephs. Cantor's contrary conviction, dramatically coming forward at the Third International Congress of Mathematicens (1904; cf. *T*, p. 307, footnote 3), had no suggestive effect even on many set-theoreticians, let alone mathematicians in general. Now when Zermelo in his short notes 04 and 08, simple in technique in spite of their sagacity, proved just the contrary and confirmed Cantor's conviction yet without providing a method of ascertaining the Alephs concerned, one was inclined to believe that Zermelo's proofs yielded too much and were incorrect. On the other hand, the majority of critics did not contrive to find mistakes in the proofs and therefore distrusted their basis, the axiom of choice — not so much for itself as for its consequences. Of course, this scepticism is based upon not appreciating the existential character of the well-ordering theorem which a priori sentences it to uselessness for the decision of constructive questions like the continuum problem.

To be sure, not all opposition directed against Zermelo's proofs derived from antagonism to our axiom. There is the attitude of Poincaré mentioned above, rejecting an (actual or seeming) use of an impredicative procedure in the proof. Poincaré, in spite of his intuitionistic inclination (see Chapter IV), was ready to accept the axiom of choice, admitting the possible existence of rules which cannot be constructively, or anywise completely, formulated; in the present case the existence of subsets of $\mathsf{U}t$ which cannot be obtained by means of the axiom of subsets. This attitude is in accordance with his considering non-contradiction as the decisive criterion of mathematical existence.

Incidentally, the proofs of the well-ordering theorem have also been attacked with actually erroneous arguments while on the other hand unsufficient proofs of the theorem have been proposed [1]; in both cases there was a connection with the series of all ordinals (antinomy of Burali-Forti), either by wrongly assuming sound proofs to depend on this series or by inadvertently just using the series [2].

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[1] Notably *Jourdain 10; *Schoenflies 8, pp. 172–173.

[2] In a sarcastic form many objections against the first proof are refuted in Zermelo 08, where further references to the literature (up to 1908) are given.

Apart from the well-ordering theorem there are statements of a quite different character — in particular geometrical statements — proved by means of the axiom of choice, which because of their paradoxical character induced many mathematicians to reject the axiom [1]). Presumably the earliest statement (1914) of this kind is Hausdorff's discovery that half of a sphere's surface is congruent to a third of it [2]); other paradoxical consequences were found later [3]).

It may surprise scholars working in the field of abstract or applied set theory that even to-day, after half a century of utilizing the axiom of choice and the well-ordering theorem, a number of first-rate mathematicians (especially French) have not essentially changed their distrustful attitude; not even such as have been working most successfully in the domain of point sets and of real functions. Some lectures and discussions delivered at an international conference on foundations of mathematics in Zurich 1938 [4]) are most characteristic of what may be called *a stagnation of controversy during several decades*, in spite of an enormous actual development of research in the field under discussion. It is remarkable what Lebesgue says there of his attitude toward the axiom of choice: *En énonçant ainsi mes exigences, je n'ai nullement voulu decider en faveur de la position purement négative que j'ai adoptée. Je ne la regarde nullement comme définitive; ce n'est qu'une position d'attente; je l'ai qualifiée de position de prudence et même peut-être de routine, il y a de bien longues années déjà, dès les premiers jours de la controverse* [5]).

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[1]) For instance, Borel *7 and 47, also Bouligand 47 (cf. Bockstaele 49); cf. the reaction of P. Lévy 50 who maintains that the axiom has a logical character. Denjoy (46-54, part V) justly remarks that certain complications in analysis caused by the rejection of the axiom are no less paradoxical than Banach-Tarski's theorem (footnote 3). A remarkable illustration is given by the result of *Sierpiński 29 that the following apparently evident statement is in some way equipollent to our axiom: if by a single-valued mapping (with the domain s) the members of a set s are related to those of a set t then the cardinal of t cannot be greater than the cardinal of s.

[2]) Hausdorff 14, p. 469.

[3]) In particular, Banach-Tarski 24, von Neumann 29a, R. M. Robinson 47.

[4]) Notably Lebesgue 41, Sierpiński 41, and the respective discussions.

[5]) *l.c.*, p. 118. It is worth while to reproduce (in translation) some of Lebesgue's subsequent remarks; he says with regard to our axiom: In the past, audacity and caution have been collaborating at each important progress. Why not do it once again? ... I desired to make it clear ... that the discussion has neither been nor is purely logical .... Each of us takes pains to understand,

It has been maintained that, after in 1904 "the powder keg had been exploded through the match lighted by Zermelo" in his first proof of the well-ordering theorem, the strange situation emerged that those who up to that moment had utilized set theory were opposed to the axiom of choice while other mathematicians were ready to accept the axiom [1]). Whether this is correct or not, at any rate the situation has since thoroughly changed owing to various reasons: to the applications of the axiom to problems of almost every domain of mathematics; to the penetrating and many-sided investigations on statements equipollent to the axiom (from 1915 on) whose import is independent of the admission of the axiom; to the researches on the independence of the axiom (from 1922 on) in its weaker and stronger forms, confronted with our inability to prove many important results without the axiom; to the fact that none of the conclusions drawn from the axiom has led to a contradiction, a fact finally crowned by Gödel's proof of the consistency of the axiom, i.e. its compatibility with a reasonable system of axioms; to the use by Hilbert of an enormously generalized "logical" axiom of choice in his attempts to prove the consistency of mathematics as a whole (see Chapter V). One should also not overlook that there are axiomatic systems of set theory, notably that of von Neumann (see § 6), where the axiom of choice does not appear explicitly but is a consequence of other axioms which seemingly lack the odium incurred by the axiom of choice.

It is true that the character of our axiom does not conform to the realm of pure arithmetic or to geometrical rigor in the Greek sense. But this does not justify its being made a scapegoat, considering that

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and to be certain of, a substratum underlying the words used. To this purpose we utilize comparisons, instances taken from History of Science, we proceed audaciously or timidly ... perhaps according to our age or to our race. Thus it is a question of trying to elaborate a new chapter of mathematics. What, then, is the use of logic? Certainly not to convince, to create confidence. ... At the historical moment of the paradoxes of set theory, when we emerged from discussions where none of us saw a way of repairing a logic that appeared ruined, nevertheless we continued applying that very logic to the problems we were studying; for an attitude of philosophical doubt taken in a discussion does not at all prevent the full certainty. ... Logic does not create confidence ... The researches on the foundations and the method of mathematics should give plenty of space to psychology and even to aesthetics.

[1]) See Lebesgue and Sierpiński, *l.c.*

modern "classical" analysis, and therefore geometry, in general — in contrast with intuitionistic mathematics (Chapter IV) — has not that character either [1]). Considering the fact that no possibility of construction is asserted there is no reason why the non-vanishing of the Cartesian product of non-vanishing sets or the idea of a well-ordering of the continuum should be regarded as hazardous hypotheses.

True, the axiom of choice was explicitly formulated only in the twentieth century and apparently was not implicitly used earlier than two decades before. But, at that, every mathematical principle was once expressed for the first time, mostly long after it had been used implicitly and unconsciously. The development of mathematics through the centuries has been achieved in two directions: extensively by drawing *new conclusions* from previously admitted premisses, in a less conspicuous way by adding *new premisses* or principles to those admitted before, in accordance with the needs of science.

In fact one has arrived at the axiom of choice just as at other mathematical principles, viz. by *a posteriori* examining and logically analysing concepts, methods, and proofs actually found in mathematics whose original development in an intuitive manner rests on psychological rather than on logical foundations. This way of analysing then yielded the principle in question, and a reference to the intuitive or logical evidence of the principle was at best secondary [2]). Thus the Greek mathematicians were induced to include the axiom of parallels among the principles of geometry — an achievement whose ingenuity was fully appreciated only more than two thousand years later. When the independence of the axiom of parallels was ultimately proved by the middle of the 19th century nobody would have proposed to renounce the parts of geometry (metric and affine geometry) which depend on the axiom. In an analogous way we should admit the argumentations of algebra, analysis, and set theory that use the axiom of choice but at the same time examine what results can

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[1]) Cf. Hadamard 05.

[2]) In the philosophical systems of Kant and Fries this idea plays an essential part. In particular in logic many principles are chiefly justified by the evidence of their consequences (cf. *Schiller 1) and in geometry one may regard the existence of similar but non-congruent figures as much more evident than the axiom of parallels. Thus it seems hardly possible to accept the contrary attitude of *Collingwood 1.

be obtained without the axiom [1]) and avoid it whenever possible. Hereby one learns to distinguish the domains of mathematics which are independent of the existential principles of choice and well-ordering from those where they are indispensable.

The analogy with geometry, where Absolute Geometry would correspond to the former domains, suggests the question, what shape will analysis and set theory assume by accepting a principle *contradicting* the axiom of choice? Such a "non-Zermelian" theory in some sense corresponds to non-Euclidean geometry — provided that the independence of the axiom of choice is ultimately proved [2]).

The last remarks seem to suggest the attitude taken in *Principia Mathematica*, namely not to ask whether the axiom of choice is "true", unavoidable, or admissible, but what additional parts of mathematics can be obtained by the admission of the axiom. This attitude appears even more legitimate today after the consistency of the axiom has been proved. However, just according to Gödel's quasi-platonic conception of mathematical truth [3]) it becomes doubtful whether this relativistic position is tenable. For according to that conception, whether taken in a naive sense or on account of a suitable system of axioms (possibly including axioms of an unforeseen character), the continuum hypothesis — even in the strong form (H_1) of p. 72 — is a statement which should be *either true or false*, and in the former case, according to the results of Lindenbaum-Tarski-Sierpiński (l.c.), the statement *implies the truth of the axiom of choice*. Thus at least for certain conceptions of mathematical truth the statement of the axiom has a categorical rather than a hypothetical character.

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[1]) It is remarkable that this is possible even with respect to many problems of abstract spaces; cf. *Faedo 1. Regarding analysis, in particular the Lebesgue integral, cf. Cipolla 13, and *Tonelli 1_I (chapters III and IV); regarding set theory, Lindenbaum-Tarski 26 and *Tarski 7, in addition to other papers cited above.

[2]) Church 27 investigates what shape the theory of ordinal numbers, in particular of the second number-class, assumes on account of certain forms of negating the axiom of choice. For contradicting principles (alternatives to the axiom of choice) in general, see H. Bachmann 55, § 39, and 56, pp. 113 f.

[3]) See Gödel 47; cf. below pp. 93 f.

§ 5. Axioms of Infinity and of Restriction

General set theory [1]), as derived from Axioms I–VI (or from corresponding axioms in the systems of §§ 6 and 7), *does not enable us to prove the existence of an infinite set.* In fact, Theorems 1 and 2 (p. 42/3) and the axiom of pairing (II) only yield the null-set and, for any given sets s and t, the sets $\{s\}$ and $\{s, t\}$; the sum-set of a finite set of finite sets and the power-set of a finite set are finite, and the same holds for the subsets of a finite set and for the intersection and the Cartesian product (Theorems 3 and 4 on p. 43/4) of finitely many finite sets. In short, with the null-set as starting-point Axioms I–VI ensure the existence of finite sets only, though infinitely many sets will exist.

Incidentally, as long as we restrict ourselves to finite sets, not all those axioms are needed. With respect to the axiom of choice this was pointed out on p. 49; it is easy to see that the situation is similar regarding the axioms of power-set and of subsets.

These questions, however, are of no importance when our interest is directed towards set theory or analysis and not towards certain domains of arithmetic alone. In the former cases the *infinite* sets with their cardinals and order-types are indispensable and for this purpose Axioms I–VI are not sufficient [2]).

Hence we have to add an existential axiom that guarantees an infinite (reflexive) set [3]); most simply, a denumerable set (T, p. 44). The definition of 'reflexive' (cf. p. 62) uses the concept of equivalence; yet at this juncture we prefer to avoid the use of derived notions such as 'equivalent', 'denumerable', 'order' etc. which will be formally introduced in § 8.

Dedekind [4]), and in a similar way Bolzano [5]) four decades before, believed that they had proved the existence of infinite sets. However, not only are their methods incompatible with the restrictions of

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[1]) This term is used here in a sense different from that of Bernays (37-54$_{IV\text{-}V}$ and below § 7).

[2]) For more elaborate independence arguments cf. Bernays 37-54$_{VI}$ and Mendelson 56. Cf. Rosser 55, pp. 57 ff.

[3]) For axioms of infinity in *Principia Mathematica* and in related systems with or without a theory of types, see Chapter III.

[4]) Dedekind 1888, § 5. Cf. the more elaborate examples given in *O. Becker **2** (pp. 98 ff of the book edition) and *Scholz **2**.

[5]) *Bolzano **3**, § 13.

our axiomatic system but they are just those that lead to the logical antinomies (Chapter I). Dedekind, for instance, contemplated the set T whose members are all "objects of thinking" and proves its reflexiveness as follows. If t is any member of T, the thought 't is an object of thinking' is also a member of T and clearly differs from t. Hence the totality of all thoughts with the particular form 't is an object of thinking' defines a proper subset T_0 of T, and a one-to-one mapping between T and T_0 is formed by relating to every $t \in T$ the member of T_0 that expresses 't is an object of thinking'; therefore T is a reflexive set. Yet the set T is of the type that involves antinomies. In fact after the publication of Russell's antinomy Dedekind withdrew for some time his work — whose eminent importance, however, is independent of the argument mentioned.

From the axiomatic viewpoint there is no alternative for securing infinite sets but *postulating* them, and we shall express an appropriate axiom in two (not essentially different) forms. While the first corresponds to Zermelo's original axiom of infinity the second implicitly refers to von Neumann's method of introducing ordinal numbers [1]).

AXIOM OF INFINITY VII. **There exists at least one set Z with the following properties** [2])

a) $O \in Z$

b) **if** $x \in Z$, **also** $\{x\} \in Z$ [3]).

AXIOM OF INFINITY VII*. **There exists at least one set Z^* with the following properties** [4])

a*) $O \in Z^*$

b*) **if** $x \in Z^*$, **also** $(x \cup \{x\}) \in Z^*$ [5]).

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[1]) Zermelo 08a, von Neumann 23 (see below § 6); a more general, though equipollent, formulation of VII is given in Wang 49, p. 151.

[2]) Of course, such sets are incompatible with a theory of types (Chapter III). Cf. Whitehead-Russell 10-13$_{II}$, pp. XI f and 198 f.

[3]) A stronger form than Zermelo's VII is introduced in Fraenkel 27 (p. 114, axiom VIIc) and later employed by R. M. Robinson 37 (axiom 8.3) and Bernays 37-54 (parts III and VI, axiom VI*). In the stronger axiom instead of $\{x\}$ more general functions of x are admitted; the new axiom, like Axiom VIII (p. 85), is independent of the other axioms including VII (see Rosser-Wang 50, p. 128).

[4]) $\{x\}$ exists on account of Theorem 2 on p. 43, and $x \cup \{x\}$ on account of Axioms II and III.

[5]) For a slightly different (more general) formulation, cf. Gödel 40.

In symbols,

VII. $(\mathrm{E}z)\,[O \in z \;\&\; (\mathrm{A}x)\,(x \in z \supset \{x\} \in z)]$.

VII*. $(\mathrm{E}z)\,[O \in z \;\&\; (\mathrm{A}x)\,(x \in z \supset (x \cup \{x\}) \in z)]$.

The "first" members of any Z satisfying VII are evidently O, $\{O\}$, $\{\{O\}\}$, $\{\{\{O\}\}\}$ etc. and they are different from each other; for instance, $\{O\} \neq O$ because the former contains a member, viz. O, and the latter none.

As to Z^*, $O \in Z^*$ implies $\{O\} \in Z^*$. Hence also $\{O\} \cup \{\{O\}\} = \{O, \{O\}\}$, and $\{O, \{O\}\} \cup \{\{O, \{O\}\}\} = \{O, \{O\}, \{O, \{O\}\}\}$ etc. are members of any Z^* that satisfies VII*.

In drawing the next conclusions from the axiom of infinity [1]) we content ourselves with the form VII; *mutatis mutandis* the results hold true for the form VII* as well.

Let Z be an arbitrary set [2]) with the properties named in VII, $U = \boldsymbol{C}Z$ the power-set of Z, and $\overline{U}$ the set of those subsets u of Z which contain among their members the null-set and, together with any $x \in u$, also the member $\{x\}$. $\overline{U}$ is a subset of U and exists on account of the axiom of subsets. Clearly the intersection $Z_0 = \bigcap \overline{U}$ of the members of $\overline{U}$ has both properties characteristic of the members of $\overline{U}$; since $Z_0 \subseteq Z$, also $Z_0 \in \overline{U}$.

Let now Z' be another arbitrary set with the properties a) and b) [3]) and Z_0' the intersection formed from Z' in the same way as Z_0 is formed from Z. The intersection $Z_0'' = Z_0 \cap Z_0'$ has again the two properties named above; since $Z_0'' \subseteq Z$ and $Z_0'' \subseteq Z'$, we have $Z_0 \subseteq Z_0''$ and $Z_0' \subseteq Z_0''$. Hence $Z_0'' = Z_0 = Z_0'$.

Thus the set Z_0, though constructed by means of an arbitrary set Z which satisfies the conditions of VII, proves independent of Z. Z_0 is a subset of every such set Z, hence the least set with the properties of VII; it contains O, hence $\{O\}$, hence $\{\{O\}\}$ etc.

Z_0 can be conceived as the set of all non-negative integers, for we may denote the null-set O by the numeral '0', $\{O\}$ by '1', and generally $\{x\}$ by 'the successor of x' in the terminology of Peano's axioms for natural numbers [4]).

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[1]) Cf. Zermelo 08a, pp. 267 and 280 f; Fraenkel 25, § 4.

[2]) Neither VII nor VII* yields a uniquely determined set, in contrast with Axioms II–V. A modification which entails uniqueness is given in Skolem (52 and) 57a, pp. 40–43.

[3]) Our axioms do not enable us to construct the set of *all* sets with the properties a) and b).

[4]) Peano's axioms, roughly speaking, postulate that each number have a "successor", that there be a particular number which is not successor of any number, that each number, except for the particular, be a successor of (at

Starting from Axiom VII* instead of VII we obtain, instead of the set

$$Z_0 = \{O, \{O\}, \{\{O\}\}, \{\{\{O\}\}\}, \ldots\},$$

the set

$$Z_0^* = \{O, \{O\}, \{O, \{O\}\}, \{O, \{O\}, \{O, \{O\}\}\}, \ldots\}$$

which may as well be conceived as the set of all non-negative integers. In fact the members of Z_0^* are the finite ordinals in von Neumann's theory (p. 101). Of any two members of Z_0^*, one is both a member and a subset of the other, whereby a natural "order" is established in Z_0^*.

Within the axiomatic theory of equivalence (below, § 8) Z_0 proves to be an infinite (reflexive) set since a one-to-one mapping of Z_0 onto its proper subset $\bar{Z}_0 = Z_0 - \{O\}$ is produced by relating $x \in Z_0$ to $\{x\} \in \bar{Z}_0$. As Z_0 is a subset of every set Z satisfying VII, every such Z is reflexive; for instance, a suitable mapping of Z onto $Z - \{O\} \subset Z$ is formed by using the above mapping with respect to $Z_0 \subseteq Z$ while every member of $Z - Z_0$ is related to itself. The same holds true for Z^*. Hence our axiom of infinity only requires a starting member O and a primitive function whereas the mapping needed is not postulated but constructed [1]).

True, our axiom of infinity only guarantees the existence of such infinite sets as axiomatically correspond to the denumerable sets of Cantor's naive theory. However, the axioms of § 3 enable us to obtain more extensive sets (see § 8).

Nevertheless, *Axioms I–VII are not sufficient to guarantee the existence of certain kinds of sets* [2]) whose counterparts in Cantor's

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most) one number, and that (*principle of mathematical induction*) there be no numbers in addition to those required by the conditions stated. For weaker formulations see, e.g., Devidé 55. Cf. footnotes 1) and 2) in *T*, p. 250, and for profounder questions involved, Hilbert-Bernays $34\text{-}39_I$.

The use of 'etc.' (p. 83) and of dots (above) is quite informal. For a stricter treatment see the end of § 6 (relativization) and Chapter V (completeness, categoricalness).

[1]) This is essentially the method of Dedekind 1888. Cf. Bernays 30, pp. 363 f.

[2]) See Fraenkel 21/22 where an axiom of substitution is first suggested; Skolem 22/23 (No. 4) and 29; cf. Fraenkel 27, pp. 114 f; also the previous hint in Mirimanoff 17. The fundamental importance of Axiom VIII for the theory of ordinal numbers was first pointed out in von Neumann 23; cf. von Neumann 28a and Lindenbaum-Tarski 26.

theory have never been questioned. Presumably the simplest instance of a set the existence of which cannot be proved by means of Axioms I–VII is the denumerable set

$$A = \{Z_0, Z_1, Z_2, \ldots\}$$

where Z_0 has the above meaning and Z_{k+1} is the power-set of Z_k. The significance of the set A is emphasized by the fact that the sum-set of A has a cardinal greater than the cardinal of any member of A (hence a cardinal $\geqslant \aleph_\omega$), while our previous axioms are not strong enough to yield a set of this cardinal.

In addition to the insufficiency of our axioms with respect to particular (extensive) sets, the general theory of ordinal numbers (§§ 6 and 8) and the general method of transfinite induction (*T*, pp. 251 f) cannot be developed on the axiomatic basis laid as yet. The supplement required shall be formulated as

AXIOM (VIII) OF SUBSTITUTION or Replacement. **For any set s and any single-valued function** [1] **f with a free variable x, there exists the set that contains just the members $f(x)$ with $x \in s$.**

In other words, if the domain of a single-valued function is a set, its range (counter-domain) is also a set.

An alternative formulation [2] reads: If s is a set and φ a dyadic predicate (of sets) such that for every $x \in s$ there is at most one y satisfying $\varphi(x, y)$, then there is a set t such that $y \in t$ is equipollent to the existence of a member x of s that satisfies $\varphi(x, y)$. (Naturally it is assumed that φ contains neither s nor t as free variables.)

In symbols

$$(As)\,\{(Ax)\,(Ay)\,(Ay')\,[x \in s\ \&\ \varphi(x, y)\ \&\ \varphi(x, y') \equiv y = y']\} \supset$$
$$(Et)\,(Ay)\,[y \in t \equiv (Ex)\,(x \in s\ \&\ \varphi(x, y))].$$

VIII is, like V, not an axiom proper but an axiom schema (p. 41); it includes the stronger form of the axiom of infinity (p. 82, footnote 3) if VII on p. 82 is granted.

Axiom VIII, in a stronger sense than VII, does not belong to the axioms of "general set theory" as developed in §§ 2–4; in a loose sense

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[1] Or 'predicate', cf. above p. 38, also regarding additional free variables.

[2] Cf. Skolem 29, p. 3, and Church 42. See also the "schéma de sélection et réunion" in Bourbaki 54, p. 64. (The alternative form is preferable for logical reasons.) A weaker form of the axiom is formulated in Tarski 48, p. 80.

it may be included among axioms of infinity since together with the axioms of § 3 and VII it guarantees not just infinite but extremely comprehensive sets. But while the facts of the general theory which can be proved by means of Axioms I–VI would remain insignificant if infinite sets, as yielded by Axiom VII, did not exist, the bulk of Cantor's theory can be deduced from Axioms I–VII [1]). Still, Axiom VIII is required for various important purposes.

The meaning of *'function'* in Axiom VIII is not different from that of *'predicate'* appearing in the axiom of subsets (V); however, for the first formalization of p. 40 a more comprehensive conception is needed here [2]).

While Axioms VII and VIII aim at *extending* the domain $\boldsymbol{Z}$ by postulating the existence of certain comprehensive sets, we now turn to arguments which suggest a certain *limitation* of the domain. There is, however, a fundamental difference between both directions. Axioms VII and VIII were introduced to make possible procedures which in Cantor's naive theory are admitted as a matter of course; in the following, however, problems are raised which for Cantor's theory involve no less difficulty than for axiomatic set theory, in the present shape or in any other. These problems are (1) inaccessible numbers, (2) extraordinary sets, (3) the (simple or generalized) continuum problem.

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[1]) Therefore it would not be suitable to drop the axiom of subsets although it is easily proved by means of the axiom of substitution; for this purpose one has only, in the formulation of both axioms (VIII in its alternative form), to take $y = x \;\&\; x \in s \;\&\; \mathfrak{P}(x)$ for $\varphi(x, y)$, with restriction to the x for which $\mathfrak{P}(x)$ holds true.

[2]) See von Neumann 28a; cf. *Tarski **29** and **32** where important applications of Axiom VIII are given. Cf. below §§ 6 and 7 for further comments on the axiom of substitution and the connection between the concepts of function and set. — It can be shown, on account of Gödel 31 and Rosser-Wang 50, that the axiom system I–VIII is essentially stronger than I–VII (cf. Wang-McNaughton 53, p. 18; Mendelson 56). Montague 56 expresses this "additional strength" in the following form: if 'Zermelo set theory $\boldsymbol{Z}$' means the system of Axioms I–V, VII, 'Zermelo-Fraenkel set theory $\boldsymbol{Z}^*$' the system I–V, VII–IX, then $\boldsymbol{Z}^*$ is not a finite extension of $\boldsymbol{Z}$, i.e. cannot be obtained by adding finitely many axioms to $\boldsymbol{Z}$. (VIII is an axiom schema.) — Cf. also Rosser 56.

From the viewpoint of powers (cardinals) one might say that Axiom IV permits us to advance by single steps while Axiom VIII, in conjunction with III and IV, allows to progress to the limit.

Since very few final results have been attained in these directions the following exposition will be somewhat sketchy and informal.

The first problem chiefly concerns regular initial (ordinal) numbers ω_α whose index α is a limit-number, and the corresponding cardinals [1]), now generally named *inaccessible* (or unattainable) *numbers* [2]) (cf. *T*, p. 323). From the present axiomatic viewpoint it is preferable to deal with the question whether (well-ordered) *sets* with such ordinals or cardinals exist. As early as 1914 Hausdorff wrote: If there are regular initial numbers with a limit-index (and as yet one has not succeeded in discovering a contradiction in this hypothesis) the least among them has so exorbitant a magnitude that it will hardly ever come into consideration for the usual purposes of set theory [3]).

In the meantime the situation has considerably changed. First, we no more expect to prove that such numbers cannot exist. On the other hand it has been proved [4]) that, for instance, the axiom system of Gödel [5]) (§ 7), provided it is consistent, *remains consistent* after the addition of an axiom stating that all cardinals are accessible. Thus the existence of inaccessible numbers proves *independent* of the usual axioms or assumptions of both axiomatic and naive set theory. Furthermore, contrary to Hausdorff's supposition, the inaccessible numbers proved since 1925 to have significance not only for the foundations of set theory but also for certain applications [6]).

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[1]) Independently of ordinals, these cardinals $\boldsymbol{i}$ may be characterized as those ($\neq 0$) that satisfy the following conditions: a) if, to each member x of any set X with a cardinal $< \boldsymbol{i}$, a cardinal $\boldsymbol{c}_x < \boldsymbol{i}$ is assigned, we have $\sum_{x \in X} \boldsymbol{c}_x < \boldsymbol{i}$; b) for every cardinal $\boldsymbol{h} < \boldsymbol{i}$ there is a cardinal $\boldsymbol{g}$ such that $\boldsymbol{h} < \boldsymbol{g} < \boldsymbol{i}$.

[2]) This term has also been used in a narrower sense since 1930 (Sierpiński-Tarski 30, and in particular Tarski 38; the least ordinals corresponding to inaccessible numbers in the narrow sense are called *Grenzzahlen* in Zermelo 30). The new condition, imposed instead of b), reads: if $\boldsymbol{a} < \boldsymbol{i}$ and $\boldsymbol{b} < \boldsymbol{i}$, also $\boldsymbol{a}^{\boldsymbol{b}} < \boldsymbol{i}$. (As to the cardinals 2 and $\aleph_0$, the terminology is varying but without significance.)

[3]) Hausdorff 14, p. 131; cf. already *3 (1908), p. 443. Also in the penetrating papers *Mahlo 1 and 2 (1911/13) the problem is raised.

[4]) Firestone-Rosser 49 (cf. already Kuratowski 25 and Baer 29). Cf. the independence proofs (referring to inaccessible ordinals) of Mostowski 49a, Shepherdson 51-53, Mendelson 56.

[5]) As to the system *Z*, see Kuratowski 25.

[6]) See Tarski *5, 29, and particularly 38, Kuratowski 25, Lindenbaum-Tarski 26 (§ 4), furthermore *Koźniewski-Lindenbaum 1 and *Erdös-Tarski 1. For imbedding the question in general axiomatic problems, see Baer 29,

Finally a special axiom, postulating the existence of inaccessible sets, has been proposed by Tarski [1]) in the following general form. For every set t there exists a set s with the properties I)–III):

I) t is equivalent to a subset of s;

II) the set of all those subsets of s which are not equivalent to s, is equivalent to s;

III) there is no set whose power-set is equivalent to s [2]).

One easily perceives that this axiom makes some of our axioms redundant; in particular it allows to prove the axioms of power-set and of choice. Yet such a reduction is hardly desirable, considering the extreme strength of this axiom which is not needed for ordinary purposes.

Postulating thus the existence of inaccessible numbers as large as desired (cf. Tarski's axiom, I)) [3]) is one possible attitude towards the problem: an attitude strengthening Axioms VII and VIII. However, the converse attitude can be taken as well. In his attempt to base set theory constructively and not axiomatically Kamke [4]) proposed, as an alternative to stronger limitations (for instance, to the second number-class), a constructive procedure which should yield just all ordinals less than the least inaccessible ordinal number. While Kamke's constructive method appears somewhat problematic, a similar attitude from the axiomatic viewpoint is rather natural and has since 1921 yielded attempts to introduce an *axiom of restriction* (limitation) [5]). Such an axiom would be analogous to Peano's last (inductive) axiom,

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Zermelo 30, Kamke 34, Neumer 51; for applications to analytic problems, notably in the theory of measure, *Banach 2 and *Ulam 2 (also *Sierpiński 13 and 17, *Ulam 3). Particularly remarkable is the application in Mostowski 49a where an undecidable statement about real numbers is constructed, the truth of which cannot be established without presupposing the existence of inaccessible ordinals.

For further details, including a comparison of different definitions of inaccessible ordinals and cardinals, see the comprehensive exposition in H. Bachmann 55, §§ 40–42; cf. Bachmann 56.

[1]) Tarski *32 (1939), p. 183; cf. 38, pp. 87 f. The numbers treated there are inaccessible in the narrow sense.

[2]) Property III) is well-known also for comparably "small" sets, e.g. those of the cardinals $\aleph_0$ and $\aleph_\omega$; see T, p. 305.

[3]) A much more far-reaching axiom is proposed in the forthcoming Thesis of A. Lévy.

[4]) Kamke 34.

[5]) Fraenkel 21/22, pp. 233 f (*Axiom der Beschränktheit*).

and conversely analogous to Hilbert's *Vollständigkeitsaxiom* [1]) (axiom of completeness) in geometry and analysis. While the latter, in order to reach categoricalness (see below), postulates that the domain be as *comprehensive* as compatible with the axioms [2]), in the case of set theory an axiom of restriction would demand the domain to be as *narrow* as compatible with the axioms; this may mean that the domain be the intersection of all "realizations" (minimal model) of the system of axioms, provided that such an intersection can be shown to exist and to fulfil the axioms. Then the non-existence of inaccessible numbers or of the corresponding sets, as well as of extraordinary sets as described below, could be proved.

A suitable axiom of restriction should enable us to prove that all models of the axiom system are isomorphic [3]), i.e. admit a one-to-one mapping which preserves the ϵ-statements. (Cf. also Gödel's postulate of constructibility, § 7.)

However, there are strong arguments against the possibility of formulating an axiom of restriction such that a consistent realization can be shown to exist [4]). And even if such an axiom could be admitted it seems that it would not guarantee the categoricalness of the system. It may, however, possibly yield a solution of the continuum problem (see below). This situation might be compared to a geometry with an axiom of completeness but without the axiom of parallels, where non-isomorphic Euclidean and hyperbolic realizations are compatible with the axioms [5]).

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[1]) Hilbert 1899/30 (from the second ed. on, axiom V2) and *3. For axioms of this type, cf. Bernays 55 and the literature cited there.

[2]) This, of course, presupposes the existence of a "realization" (model) of the axiom system which does not admit of a further extension. In (three-dimensional) Euclidean geometry this proves to be fulfilled after the introduction of Archimedes' axiom.

[3]) Such a system of axioms is called *categorical* according to *Veblen 1, p. 346, or *monomorphic* according to Carnap. The notion of categoricalness has, however, meanwhile turned out to be far more involved than was originally expected; see the remarkable definition in Mostowski 55, p. 13; also Wang 53b, the end of § 6 below, and Chapter V.

[4]) See, in particular, the arguments given by von Neumann 25, pp. 230 f and 238 ff. Cf. Shepherdson $51\text{–}53_{III}$, p. 161.

[5]) Cf. the situation in arithmetic (see Chapter V); for instance, Kleene 52, p. 427, theorem 44.

True, recently Suszko [1]) and Carnap [2]) proposed a vindication of this axiom of restriction, and Carnap formulated it symbolically, as an axiom of a minimal model [3]); to achieve this more easily he admitted individuals in addition to sets while otherwise adopting the axioms of §§ 2–4 and Axioms VII and VIII of the present section. (Also for independence proofs regarding the axiom of choice the existence of individuals was presumed, see above p. 51.) But it still appears doubtful whether hereby the problem of a general axiom of restriction is settled.

Yet there is another type of restrictive postulate which at any rate should be introduced in the system. To understand its purpose we consider certain paradoxical sets to which attention was first called by Mirimanoff.

If the members of a set are again sets, including the null-set, and not individuals — as presumed in our system — it stands to reason that the primary constituents (called *Urelemente* by Zermelo) of any set would be confined to the null-set alone.

If we had admitted as the members of a set also individuals besides the null-set, as did Zermelo (p. 28), we should similarly obtain a finite or infinite number of primary constituents serving as the original bricks from which all sets are "built up" in accordance with the axioms. In both cases a set would not contain itself as one of its members, let alone as its sole member.

However, all these conjectures are not sufficiently justified. Since 1917 one has become aware that another type of sets (classes) with a strange structure, called *extraordinary sets* [4]), is compatible with the

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[1]) Suszko 51. He calls the axiom 'axiom of canonicity' and maintains that it "constitutes a precise formulation of the so-called Beschränktheitsaxiom of Fraenkel". The connection between the axiom and the Skolem paradox (§ 6) is there discussed in detail.

[2]) Carnap 54, p. 154. Cf. Myhill 52 and the (not quite perspicuous) remark in Wang 53, p. 389.

[3]) The pith of the axiom is then the demand that no "partial relation" $\tilde{\varepsilon}$ should fulfil the conditions expressed by the other axioms. Cf. *Carnap-Bachmann 1 (1936) and Specker 54a, p. 243.

[4]) See Mirimanoff 17 and 17-20, *Eklund 1 (the objections of *Brodén 1 are not justified). Cf., besides Zermelo 30, also *Sierpiński 4, Skolem 22/23, Finsler 26, Fraenkel 28 (pp. 354 f), Yuting 53 (who calls a class s with the property (M) a groundless class). As Gödel 44 points out, the exclusion of extraordinary sets bears upon the problem of reconciling axiomatic set theory with a simplified theory of types. Cf. Quine 51, §§ 24 and 28, Fitch 53, McNaughton 53.

axioms. An extraordinary set s may, for instance, have the property

(M) $$\ldots \in s_{k+1} \in s_k \in \ldots \in s_2 \in s_1 \in s,$$

thus yielding an infinite descending sequence of members of members. In particular, s_1 may be the only member of s, and generally s_{k+1} the only member of s_k. Furthermore, not necessarily is $s_k \neq s_n$ for $k \neq n$; a set may even contain itself, the extreme case being that s is the only member of s. Particular cases are those of two sets each of which has the other as its only member, or of a finite sequence of sets $s_1, s_2, \ldots, s_n$ with $s_2 \in s_1, s_3 \in s_2, \ldots, s_1 \in s_n$ (n-circular set). In these cases there are no primary constituents at all.

In fact our system of axioms, while guaranteeing the existence of certain sets and excluding the existence of others, leaves a large scope between these two domains, a scope in which sets may exist or not. Our axiomatic system thus lacks the property of categoricalness (p. 89), or even of completeness in a weaker sense (Chapter V).

von Neumann, while raising objections against a general axiom of restriction, proposed a special axiom to eliminate at least part of the extraordinary sets [1]). This axiom is useful and has meanwhile been widely accepted. It may be formulated as follows:

AXIOM (IX) OF FOUNDATION or of Grounding [2]). **Every non-empty set a contains a member b such that a and b have no common member.**

In other words, there exists no set $a \neq O$ such that *every* member of a has a member which is also contained in a.

In symbols,

$$(Aa)\ \{(Ex)\ (x \in a) \supset (Eb)\ [(b \in a)\ \&\ (Ay) \sim (y \in a\ \&\ y \in b)]\}.$$

Hence no sets s, t, u can exist with the properties $s \in s$ (for $\{s\}$ would then contradict Axiom IX), or $s \in t$ and $t \in s$, or $s \in t$, $t \in u$, $u \in s$, etc. The strength of Axiom IX still wants further examination (cf. the literature mentioned above, and § 7).

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[1]) von Neumann 25, p. 239, and 29, p. 231. Zermelo 30 also introduced this postulate and called it *Axiom der Fundierung*, because according to the axiom a descending sequence of members should terminate (reach its bottom or "foundation") after a finite number of steps. Bernays (37-54$_{II}$, p. 6) uses the term 'Restrictive Axiom'. (von Neumann and Bernays formulate the axiom with respect to classes a; see §§ 6 and 7, cf. Wang-McNaughton 53, p. 17.)

[2]) See Yuting 53; cf. Montague 55.

The set of all individuals of the domain (including the null-set) is called by Zermelo [1]) its "basis", and the sets are classified in layers, the lowest layer coinciding with the basis and each layer comprising the sets whose members belong to preceding layers; thus a progression is formed which corresponds to that of finite and infinite ordinals. The progression may be stopped with any regular initial number α that fulfils a certain condition; e.g. with ω. The cardinal of the basis and the ordinal α together are an invariant characteristic of the intended domain of sets. The first step leads to the domain of finite sets, the second to the domain of sets up to the first inaccessible number.

However, Zermelo's proof that this invariant guarantees the monomorphism (categoricalness) of the domain can hardly be considered stringent, and even the concepts used, e.g. "cardinal of the basis", are objectionable. Also the exclusion of the axiom of choice from this system of axioms — although "it is, as a general logical principle, taken for granted during the whole investigation" — seems to weaken the logico-axiomatic value of Zermelo 30, in contrast with 08a.

Finally, the problem of a restriction, or extension, of the domain in order to approach categoricalness, naturally bears upon the *continuum problem*. In *Theory*, pp. 93 f and 316 ff the history of Cantor's, and of the generalized, continuum hypothesis is sketched and literature given regarding equipollent statements and the use of the hypothesis for proving various theorems [2]). For different formulations of the hypothesis itself and for its connection with the axiom of choice, cf. above p. 72.

Here our interest concentrates upon the question whether the continuum hypothesis, in its simple or its general form, can be proved or disproved by means of the present, or of another, system of axioms of set theory. Cantor's efforts in the beginning of the 1880's were tragic and unsuccessful, and during the next fifty years no actual progress was made, notwithstanding Hilbert's attempt [3]). A decisive

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[1]) Zermelo 30; cf. *10 (1935).

[2]) In Fraïssé 55 (§§ 31–35) the problem is embedded in his theory of n-ary relations.

[3]) Hilbert 25.

step forward is only due to Gödel's discoveries of the late 30's [1]).

Gödel's main result is that *the continuum hypothesis cannot be disproved* since it turns out to be compatible with a certain system of axioms for set theory; that system is described in § 7 (pp. 119 f) and does not essentially differ from Bernays' system. The compatibility is proved by the construction, within that axiom system, of a model of set theory in which the generalized continuum hypothesis is satisfied. (Within ***Z***, the situation is principally the same.)

Gödel's result is all the more important because certain paradoxical consequences of the continuum hypothesis [2]) seemed to suggest that it could be refuted — a chance excluded by Gödel's proof. However, far from solving the problem, the result raises a number of new penetrating questions. First, is perhaps the *negation* of the continuum hypothesis as well compatible with the same system of axioms? If so, the continuum hypothesis would be *independent* of the other axioms [3]); set theory and analysis would bifurcate (or multifurcate) at the continuum hypothesis as plane geometry bifurcates (to Euclidean and hyperbolic geometries) at the axiom of parallels, on account of its independence of the other axioms of absolute geometry. Yet this result might hardly be considered to be a definitive settlement of the problem, at any rate not by those who believe that the concepts and axioms of set theory, as Gödel puts it, "describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its indecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality."

Hence it seems that axioms of a different kind must be added to those mentioned in the present section in order to bring about a veritable solution of the problem. There may exist "other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts".

Gödel also stresses a peculiar difficulty which touches the very

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[1]) Gödel 38, 39, 40; cf. his expository paper 47 from which we quote in the following.

[2]) Cf. a similar situation regarding the axiom of choice, as described on p. 77.

[3]) For the interdependence between the independence problems regarding relations of the form $2^{\aleph_\mu} = \aleph_{\mu+1}$ for *different* values of μ, see Mostowski 55 and Hajnal 56.

gist of axiomatization of set theory. Since the continuum can be conceived as the set of all subsets of a certain denumerable set we should, before trying to solve the problem, exactly know *what we mean* by those subsets: sets definable in some manner, say by the axioms of subsets and of choice, or arbitrary multitudes of members of our denumerable set. One cannot soundly expect to determine the number of subsets before it is clearly settled what they are.

Thus the task of adding further axioms, appropriate for better clarifying the concept of set, becomes rather imperative. But, alas, we have no idea in what direction such axioms might be sought and whether they would settle a situation which is well-defined within present mathematics, or else belongs to a mathematical reality not yet conceived.

In the present section a number of axioms, in addition to those of general set theory (§§ 2–4), have been introduced to guarantee the existence of particular types of sets. In §§ 7 and 8 it will be shown that our axioms, in the form considered so far or in that of the following sections, enable us to develop set theory to a similar extent as actually done in Cantor's theory. Yet we still have to examine in what sense the original goal of axiomatization, namely *avoiding antinomies*, has been reached.

To be sure, a proof of the consistency of axiomatic set theory can hardly be given [1]). This is not surprising or particularly disappointing, for we do not know a satisfactory way of proving the consistency of classical analysis [2]) either and even the consistency of the ordinary arithmetic of integers cannot be proved within this arithmetic. The reasons will be discussed in Chapter V. On the other hand we shall see that on the assumption of the consistency of the axiom system **Z**, or even of part of it, also stronger systems as considered in the next sections can be proved to be consistent (§ 7).

Therefore our purpose regarding the non-existence of contradictions in axiomatic set theory must be far more modest; we may only expect to show that antinomies of the types considered in Chapter I, §§ 2 and 3, cannot be obtained in the familar way. Still this is of

[1]) For certain serious steps in this direction cf., in particular, Mostowski 53. — The consistency proof of Ackermann 37 refers to a set theory which has no axiom of infinity (though it includes infinitely many sets).

[2]) For attempts in this direction (cf. Lorenzen 51, Schütte 52 and 54a) see Chapter V. See also Wang 50a.

considerable importance, for the impossibility of arriving at antinomies is the crucial test which both urges and justifies replacing Cantor's theory by an axiomatic system.

When we considered the antinomies of the logical type (Chapter I, § 2), e.g. those of Burali-Forti and of Russell, we perceived as a common characteristic trait the unlimited extent of the sets concerned: the set of all ordinals, of all sets having a certain property, etc. Such sets do not occur in our domain **Z**, since there is a definite limited starting-point and well-defined restricted procedures only are admitted to lead from the start to any object (set) of **Z**. In fact, the null-set and infinite sets as secured by the axiom of infinity (p. 82) — essentially the set of all integers — serve as the starting-point, and the "operations" admitted to form from them new sets are those given by the axioms of § 3, together with the axiom of choice and, in addition, the axiom of substitution. Some of these axioms — the axioms of subsets and of choice — have no "extending" power at all, others — such as the axiom of power-set when combined with the axiom of subsets, or the axiom of substitution when combined with the axiom of sum-set — do enable us to proceed from given sets to new and more extensive ones; this transition, however, remains in all stages strictly relative to the given sets. In particular, predicates like 'is an ordinal' or 'does not contain itself as a member' cannot be used to determine new sets in an independent way but only as subsets of sets previously secured.

There seems, then, to be no straight way of proving the existence of boundlessly comprehensive sets (such as the set of all ordinals, or the set of all sets that do not contain themselves as members, or the set of all sets) in our system, in contradistinction to Cantor's system where their existence follows from his "definition" of the set-concept. Since the assumption of their existence and the use made of them led us to contradiction we may now regard the "paradoxical" arguments (Chapter I) as just a proof, by *reductio ad absurdum*, of the non-existence of such sets [1]). (Cf., however, § 6.) This does not exclude the possibility of proving by some as yet unknown means the existence of over-comprehensive sets which might show the inconsistency of our system.

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[1]) In particular, by substituting $x \notin x$ for $\mathfrak{P}(x)$ in Axiom V we then infer, instead of Russell's antinomy, that no universal set exists (Thiele 55, p. 176).

The impossibility of reproducing, within our system, the arguments leading to the semantic antinomies follows from the simple fact that predicates like 'denotes', 'definable', or 'true', which are characteristic of these arguments, are not definite in the sense of the axiom of subsets; cf. p. 40. See, however, Chapter V, § 6.

§ 6. The Axiom System of von Neumann. The Set-Theoretical Relativism (Paradox of Skolem)

Sections 6 and 7 deal with axiom systems which, in a definite sense, are stronger than the system **Z** described thus far [1]). Systems of another character, notably those introduced by Quine and Wang, are discussed in Chapter III.

In the decisive point, namely the introduction of a more extensive domain of which the domain **Z** of §§ 2–5 is a partial domain only, the systems of von Neumann (from 1925 on) and of Bernays (from 1937 on) are similar. The former is outlined in the present section with regard to its historical significance and its fundamental features while Bernays' system is described in § 7 in more detail.

A common feature of both systems is the elimination of the predicate variables that appear in **Z** in the axioms (axiom schemata) V of § 3 and VIII of § 5. Hereby we get rid of *axiom schemata* altogether and the resulting axiom systems become "finite".

Von Neumann presented his axiomatization in two extensive essays [2]), the first of which stresses the fundamental points and the general axiomatic and metamathematical questions connected with the system while the second performs in detail the difficult technical development, including the derivation from the axioms of the set-theoretical operations, of (well-)ordering, similarity and ordinals [3]), equivalence and cardinals, and transfinite induction. A third paper [4])

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[1]) Cf., however, the end of § 7.

[2]) von Neumann 25 and 28. The essential points of 28 had already appeared in the Thesis of the author (in Hungarian, Budapest, 1925).

Symbolical expositions of some of the main features of the system are given in Skolem 41, pp. 32 ff, and in Wang 49 where (bottom of p. 151) an interesting strengthening of the system is proposed.

[3]) For the introduction of ordinals, cf. already von Neumann 23 (and 28a); see below pp. 101 f.

[4]) von Neumann 29. – In Klaua 57 von Neumann's method is somehow amalgamated with the theory of types. (Addition in the proof.)

discusses certain general features of the system and shows the compatibility of its strongest axiom, which is peculiar to von Neumann's system, with the ordinary axioms of set theory.

Finally, the so-called Skolem paradox, though equally referring to other axiomatizations described in Chapters II and III, is exhibited here, for von Neumann was the first (in 25) to discuss the paradox in detail after its discovery by Skolem (22/23).

The axioms of **Z** guarantee the existence of a part only of those sets which exist according to Cantor's naive conception. True, the antinomies show the danger inherent in too comprehensive sets or domains, and the domain **Z**, in spite of its limited bounds, turns out to be sufficient for all legitimate purposes. However, many of the sets whose existence cannot be inferred from the axioms of **Z** seem to be quite harmless and the demarcation line between the sets admitted and those not admitted is drawn in **Z** in an arbitrary manner.

It was von Neumann's daring idea that it is not the existence of the overcomprehensive sets as such which leads to contradictions but *their being taken as members of other sets* (elementhood). This attitude suggests to drop, in the axiom of subsets (§ 3), the requirement that the set corresponding to a given predicate should be a subset of a set secured previously. In particular the universal class V, corresponding to the condition (predicate) '$x = x$', and the complement $V - s$ of any given set s become thus admissible. Hence the appearance of antinomies which evolve from using overcomprehensive sets may be excluded through disallowing, by suitable axioms, the elementhood of the "dangerous" sets. Those objects which cannot serve as members of other objects shall be called *classes*; von Neumann justifies an axiom (see below) which excludes from elementhood just those objects that are "of the same extent" as the universal class [1]). On the other hand, those objects which can become members of a set or a class shall now be called *elements*; all sets existing in **Z**

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[1]) The necessity of distinguishing between sets and classes was already felt by Cantor in the 90's; in his letters to Dedekind of 1899 he speaks of "inconsistent systems", having approximately in mind the classes (Cantor 32, pp. 443–448, cf. p. 470). On the other hand, classes may be conceived as "ideal elements" in the sense of Hilbert 25, in contradistinction to sets which are mathematical objects.

are certainly elements. The axiom of subsets can, then, be replaced by an axiom guaranteeing the absolute existence of classes, subject to the restriction that quantifiers entering the condition which defines the class shall be restricted to element-variables.

The terms 'class' and 'element' are not von Neumann's; his terms, which are less convenient, are mentioned below. The term 'class' corresponds to the same term in § 7, while the term 'element', largely corresponding to Bernays' term 'set', is used particularly by Quine.

While these are innovations characteristic of all systems described in this and in the next section, von Neumann also introduced some peculiar and rather accidental features. In the axioms of subsets (p. 38) and of substitution (p. 88) appear predicates or conditions, i.e. expressions denoting functions. But from the mathematical point of view there is no essential difference between set and function. A set means a division of the universal class into two categories of objects (arguments), those which are members of the set and may be allotted the function-value 1 (or B) and those which do not belong to the set and are given the value 0 (or A) [1]. Conversely, a (single-valued) function f, or $y = f(x)$, is nothing but a set of ordered pairs (y, x) satisfying certain conditions (cf. § 8). Therefore, to avoid the use of both concepts simultaneously, von Neumann restricts himself to *functions only* [2], using the notation $[f, x]$ for the value assumed by the function f at the argument x; hereby the variable character of both arguments and functions is more clearly expressed than by the usual notation $f(x)$ which overemphasizes the variability of arguments. To distinguish arguments from functions, the arguments (members) are called by von Neumann 'I-things', the functions 'II-things', and those functions which may also serve as arguments 'I II-things'; the latter, then, are the sets (elements) which may serve as members of classes. The functions are defined everywhere, i.e. for all argu-

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[1]) Cf. T, pp. 151 f; in particular, de la Vallée Poussin's *characteristic functions*. Then, a function f may be called a *domain* if, for each x, either $[f, x] = A$ or $[f, x] = B$, and the *members of the domain* are those x for which $[f, x] = B$. (A domain is a set if it is also an argument.) For $[f, x] = B$ one may write $x \in f$. A convenient way is taking $A = O$ and $B = \{O\}$.

[2]) Hereby, incidentally, the connection is facilitated between axiomatic set theory and combinatory logic (see, e.g., *Schönfinkel 1 and *Curry 2); cf. Cogan 55. See Chapter III, p. 195.

ments. While the operation $[f, x]$ produces the value of f at x, which again is an argument, a special provision or axiom is wanted which states what *functions* may also serve as arguments. The characteristic property of a function, namely being determined by the course of its values, is expressed by an appropriate axiom, von Neumann's I4 (in our terminology, an axiom of extensionality); it states that functions with the same course of values are equal — but not that to each course of values a function is assigned, which would mean a return to Cantor's concept of an arbitrary set.

The axioms of von Neumann are divided into five groups I–V. There is no need to go into details of the groups I–III (cf. the following section). Group I contains general axioms regarding arguments and functions; for I4 see above. II secures simple "arithmetical" constructions, e.g. a function f with $[f, x] = x$, and also functions of any finite number of variables. III provides "logical" constructions, including a certain normal form for functions of several variables. In these groups two primitive concepts are introduced in addition to those already mentioned, namely the concept of *ordered pair* (cf. §§ 7 and 8) from which ordered n-ads are easily derived, and a couple of arbitrary arguments A and B (see above).

Most characteristic of von Neumann's axiomatization are the two axioms of group IV, particularly IV2. Formally IV1 and IV2 are analogous in so far as they specify when either an argument or a function is an element. They state:

IV1. There is a function f such that an argument x is an element (function-argument, I II-thing) if and only if $[f, x] \neq A$.

IV2. A function f is *not* an element if and only if there is a function g such that to every argument x exists an argument y which fulfils $[f, y] \neq A$ and $[g, y] = x$.

The purpose of IV1 is simple. Properly the axiom might have postulated, in the terminology of § 2 (p. 30), that all objects be sets. Instead IV1 leaves open the existence of "individuals" in the domain, yet this possibility is not utilized in the system and the author explicitly gives the alternative just mentioned instead of IV1 [1]).

In contrast herewith, IV2 is the strongest axiom not only of the present system but of any axiomatic set theory in existence. Its

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[1]) von Neumann 25, p. 238, axiom VI1. This axiom does not belong to the axiom system proper; it would render IV1 superfluous.

purpose is, as mentioned above, to settle when a function is also permitted to serve as an argument. By means of the arbitrarily chosen argument A, the property of being an argument is restricted to those functions f whose values $[f, x]$ are not "too frequently" different from A, namely *for not too many arguments* x. More strictly, IV2 says that the function f is not an argument if and only if the totality of those x for which $[f, x] \neq A$ can be represented on the totality of all arguments — it being understood that only representations through our functions are admitted, viz. of the form $y = [g, x]$, where g is single-valued. Hence all functions f which are $\neq A$ "more rarely" or "not more frequently" than a certain element are also elements.

In the usual language this approximately means that a class c is also a set if and only if c admits no mapping onto the universal class V.

One easily perceives that IV2 guarantees the existence not only of those sets which in **Z** originate by means of forming subsets, i.e. by the axioms of subsets and of choice, but also, for instance, of those guaranteed by the axiom of substitution (p. 85). In accordance herewith the well-ordering theorem is easily proved by means of IV2 [1]) without an explicit intervention of the axiom of choice or a similar principle. (Also the equivalence theorem does not directly enter the theory.) Therefore, besides the question of the desirability of so strong an axiom (which is a question of axiomatic "taste"), the problem of the consistency of IV2 arises (see below p. 104).

A last group V contains three axioms which are largely analogous to the axioms of infinity, of sum-set, and of power-set in **Z**. However, their importance for the general theory is inferior to that of their counterparts in **Z** inasmuch as here already the previous axioms, particularly IV2, perform substantial tasks which in **Z** are fulfilled by the latter two axioms. The new axioms additionally state that the objects obtained are elements; they serve the theory of cardinals rather than the general theory of sets.

For the derivation of the first beginnings of set theory from the

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[1]) The argumentation is roughly as follows. The assumption that the class of all ordinals (which can easily be introduced, see below p. 101) is an element leads to the Burali-Forti antinomy. Hence this class is not an element and is, by Axiom IV2, equivalent to the class of all arguments. Thus the latter class proves equivalent to a set which is easily well-ordered; accordingly the class of all arguments can also be well-ordered, hence any class or set.

axioms we refer to the following two sections. Here we content ourselves with an outline of von Neumann's earliest achievement in this field, his *theory of ordinals* [1]), which can be imbedded in every suitable axiomatization of set theory; it is the first formalized theory of ordinal (and cardinal) numbers. True, in the following we use the terminology of (well-)ordered sets, which will be derived from an axiomatic basis only in § 8. However, the theory of order (i.e., of the relation $\prec$) is almost trivial within von Neumann's theory of ordinals because, for any two different members a, b of the sets appearing here, there always holds either $a \subset b$ or $b \subset a$; hence by writing $a \prec b$ for $a \subset b$ we obtain (well-)ordered sets, and the simple facts of the classical theory of well-ordered sets (T, §§ 10 and 11) used in the following are easily translated to axiomatic language. Thus von Neumann established his theory of ordinals without the general concept of order-type (T, § 8), as done originally by Cantor in a different sense.

Denoting the section (*Abschnitt*) of an ordered set M which is determined by $x \in M$ (T, p. 245) by '$A(x; M)$', von Neumann defines as *ordinals* those well-ordered sets M which, for every $x \in M$, satisfy the condition '$x = A(x; M)$'; in other words, in an ordinal M every $x \in M$ equals the section of M determined by x (cf. T, p. 276, Theorem 2). It follows that every ordinal is well-ordered by the relation $\subset$.

One easily sees that to a given well-ordered set there exists *at most one* ordinal similar to the set; for two similar ordinals prove equal by transfinite induction [2]).

It is far more difficult and requires the use of the axiom of substitution to prove that there exists *at least one* ordinal which is similar to an arbitrarily given well-ordered set M. This is shown by transfinite induction with regard to the sections of M, in a way similar to the usual proofs in the theory of well-ordered sets [3]). Hence we may speak of *the ordinal P of a given well-ordered set M.*

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[1]) von Neumann 23 and 28a. Cf. *Tarski **29**, p. 200.

[2]) *Proving* by transfinite induction causes no difficulty. As to *defining* ("recursion theorem"), besides the derivation within his own axiomatic system, von Neumann also gave (in part III of 28a) a derivation on the basis of **Z**; for a well-ordered *set* the axiom of substitution proves unnecessary while it is required for the domain of all ordinals. Cf. Tarski 55.

[3]) T, pp. 251–257. The proof is given in detail in von Neumann 28a, pp. 379–381. The axiom of substitution is used to form the set P of all $f(x)$ with the domain M where $f(x)$ is the ordinal similar to the section of M determined by x (hence the set of all $f(y)$ satisfying $f(y) \subset f(x)$). P is then an (the) ordinal similar to M.

The main properties of the ordinals defined in this way are: a) P is the set of all ordinals Q with $Q \subset P$, i.e. '$Q \in P$' and '$Q \subset P$ and Q is an ordinal' are equipollent; b) P and Q being different ordinals, we have $P \subset Q$ or $Q \subset P$, whence we obtain the comparability of well-ordered sets; c) every set of ordinals, ordered according to $\subset$ (see b)), is well-ordered; d) Π is an ordinal if and only if $P \in \Pi$ implies that P is an ordinal and a subset of Π; hence, in particular, O is an ordinal.

To construct ordinals we start with O and make use of the almost evident fact that, P being an ordinal and M a set of ordinals, also $P \cup \{P\}$ and $\mathsf{U}M$ are ordinals. In analogy to the addition of ordinals (T, pp. 193 and 261) we may denote $P \cup \{P\}$ by '$P + 1$'. O is the first ordinal, $P + 1$ the (immediate) succesor of P. One easily proves that an ordinal P for which an ordinal R with $P = R + 1$ exists satisfies $\mathsf{U}P = R$ while otherwise (i.e. for limit-numbers P) we have $\mathsf{U}P = P$.

This concept of ordinal enables us to develop the theory of non-negative integers without an axiom of infinity [1]). The first finite numbers were named on p. 84; the numbers corresponding to ω, $\omega + 1$ etc. are obtained according to the preceding paragraph. Finally, the transfinite *cardinals* are defined as the initial numbers (T, p. 299), i.e. those transfinite ordinals which are not equivalent to any member of themselves. One proves that for every well-ordered set M there is a single cardinal equivalent to M, "the cardinal of M"; the proof is not as simple as it might appear because of the restrictions enjoined on the use of classes. For the general theory of cardinals, to be sure, the well-ordering theorem is required; in von Neumann's system this means, Axiom IV2.

An important simplification of von Neumann's system was given much later by R. M. Robinson [2]). (A far more fundamental modification of the system, also going back to 1937, is exhibited in § 7.) We shall not touch upon the simpler and more lucid form which Robinson gives to von Neumann's axioms in general nor upon his *definition* of the ordered pair (see the following two sections); the main advantage

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[1]) Done in detail in von Neumann 28, pp. 743 ff. This theory is analogous to that of Dedekind 1888 (see, in particular, *Satz* 126). An interesting exposition of the historical development, particularly of Dedekind's work as the source of Peano's axioms, is given in Wang 57.

[2]) R. M. Robinson 37. — Neubauer 50 simplifies some of von Neumann's proofs.

of the latter modification is that a "pair" is introduced as a certain I II-thing and not conceived as primitive. Also the theory of ordinals is considerably simplified and hereby a more natural proof of the well-ordering theorem obtained (cf. footnote on p. 100); we return to this simplification in § 7.

Yet the decisive progress — not only technically but, so to speak, psychologically — is that *functions need not be defined for all arguments.* This is done by replacing '$= A$' with 'is undefined'. Hereby a more "constructive" interpretation of the system is made possible; most of the sets (elements) concerned, namely all those which are actually required for mathematics, appear in a form rather similar to their appearance in **Z**. One may start with the function 0 which is not defined for any argument and therefore needs no value; in other words, $[0, x]$ is not defined whatever x may be. Then a function 1 is constructed such that $[1, 0] = 0$ and $[1, x]$ is undefined for $x \neq 0$. By using matrices the upper line of which contains the arguments of the domain while the lower line contains the corresponding values of the respective function, the next steps yield the functions

$$\binom{0}{1} \quad \binom{1}{0} \quad \binom{1}{1} \quad \binom{0,1}{0,0} \quad \binom{0,1}{0,1} \quad \binom{0,1}{1,0} \quad \binom{0,1}{1,1} \text{ etc.}$$

Still, the most characteristic technical feature of von Neumann, viz. Axiom IV2, is retained by Robinson. Significantly enough, this axiom is the center of von Neumann's last publication on the subject [1]).

First, this paper explains that Axiom IV2 easily yields the axioms of substitution and of choice, hence (p. 86, footnote 1) also the axiom of subsets. This suggests the idea of replacing IV2 by the former two axioms which are more plausible. By this change a characteristic feature of the system would disappear and one would obtain a domain which is similar to that of **Z** of §§ 1–5. Thus the question arises: what is contained in IV2 in addition to the axioms of substitution and choice and how can this additional ingredient be justified?

Before answering this question let us recall that the problem of the *consistency of the axioms of set theory* (in some sense even of classical analysis) does not seem solvable with the present resources of mathematics; cf. p. 94. However, one may raise and solve more restricted ("relative") consistency problems; for instance, assuming the con-

[1]) von Neumann 29.

sistency of a system containing the "general" axioms of § 3 and one or two axioms of infinity (VII, and possibly VIII, of § 5), *does this system remain consistent after the adjunction of other axioms?* If the adjoined axiom is that of choice or even the continuum hypothesis, the answer is in the affirmative as shown by Gödel.

von Neumann (*l.c.*) gives a partial but profound relative consistency proof [1]) in the case that the original system contains all his axioms save IV2, yet with the addition of the axioms of substitution and choice in a suitable formulation, while the adjoined (problematic) axiom is the remaining ingredient of Axiom IV2. The consistency proof is rather complicated; we shall touch upon analogous proofs of relative consistency at the end of § 7. (On the other hand, the consistency of von Neumann's system clearly implies the consistency of ***Z***, since to every theorem on *sets* in ***Z*** there corresponds a theorem on *elements* in von Neumann's system [2]).) Yet hereby the problem of the consistency of von Neumann's system relatively to ***Z*** is not solved; an additional important step is due to Wang [3]) who shapes the former system as a straightforward extension of ***Z***.

The proofs of (von Neumann and) Wang have an additional interest with regard to the systems of Quine which are dealt with in Chapter III, namely ***N***(*ew*) ***F***(*oundations*) *and* ***M***(*athematical*) ***L***(*ogic*). As Quine remarks in a survey given much later [4]), ***ML*** stands in a similar relation to ***NF*** as von Neumann's system stands to ***Z***. In this respect, therefore, Wang's proof that ***ML*** is consistent if ***NF*** is consistent has a tendency analogous to that of the consistency proofs mentioned above.

We now turn to what is called by Skolem and von Neumann *relativism of set theory*, in particular of cardinals.

[1]) To accomplish his consistency proof von Neumann introduces additional postulates, namely the one mentioned on p. 99 (that all arguments be elements), a fixation of A and B (footnote 1 on p. 98), the usual interpretation of an ordered pair, and some form of an axiom of foundation; in fact, the latter essentially constitutes the "remaining ingredient". In this light, von Neumann 29 approaches a consistency proof of the axiom of foundation.

[2]) Cf., in a more general frame, Wang 50a.

[3]) Wang 49. In addition to the mathematico-technical side of his results he remarks that, if we regard ***Z*** as an applied functional calculus of the first order, von Neumann's system turns out to be an applied functional calculus of the second order. — A model of von Neumann's set theory constructed within the theory of ordinals is given in Takeuti 54a by means of the methods of Gödel (§ 7).

[4]) Quine 53, pp. 94 ff. ***NF*** is Quine 37 (improved in 53) and ***ML*** is Quine 51.

In 1915 Löwenheim, a mathematical logician of Schröder's school who was (known then to a narrow circle only), proved, in the frame of investigations of logical calculi, a theorem which may be expressed as follows [1]).

> Any well-formed formula of the first-order functional calculus [2]) — hence also a finite system of such formulae — which is satisfiable in an infinite domain, is satisfiable in a *denumerable* domain, for instance in the domain of all integers; in short, if it has an infinite model it has a denumerable model [3]).

In 1920 Skolem [4]) simplified Löwenheim's proof and generalized the theorem to the case of a system of *denumerably many formulae*, provided the system (or each subsystem of finitely many formulae) is satisfiable in an infinite domain (or in respective infinite domains). To build such a denumerable model one has to connect the objects by the primitive relation(s) in a suitable manner.

These theorems [5]) are intrinsically connected with Gödel's fundamental theorem [6]) on the completeness of the first-order func-

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[1]) Löwenheim 15, *Satz* 2. Cf. *Kalmár 5 and *Skolem 15.

[2]) Löwenheim and Skolem use Schröder's term '*Zählaussage*', Gödel the term 'formula of the restricted functional calculus'. Each axiom of von Neumann (or Bernays) constitutes such a formula; so do the axioms of **Z**, save for the axioms of subsets and of substitution which are schemata of such formulae.

[3]) A very lucid non-technical account of the meaning of this theorem and of its generalization by Skolem is given in Myhill 53.

[4]) Skolem 20; cf. 29, § 4, and the important paper Malcev 36. A particularly simple way of proving Skolem's theorem constructively, by building a numerical interpretation of the system, is given in Quine 54; cf. Church 56, pp. 238 ff, and Takeuti 57.

[5]) Including the theorem of Tarski, stating that if no finite model exists there exists — in addition to a denumerable model — also a non-denumerable model; hence the system of formulae *is not categorical*. See Skolem 34, p. 161, Tarski 49b, Vaught 54; cf. A. Robinson 51, p. 73, and 56, furthermore (for the connection with the axiom of choice) Vaught 56. In Vaught 54 and Łoś 54a (independently) the notion of *categoricalness in a given power* (cardinal) is introduced which has particular interest in connection with the results of Löwenheim-Skolem-Tarski. Cf. Łoś 55; also Asser 56.

[6]) Gödel 30. Cf. the stronger results of Bernays in Hilbert-Bernays $34\text{-}39_{II}$ (in particular, pp. 178–189 and 252 f); also A. Robinson 51 and 56, Church 56 (pp. 233 ff). A simplification of Gödel's proof, together with a somewhat problematic generalization of the result, is given in Henkin 47 and 49 (cf. Hasenjaeger 53); cf. the topological proofs of Beth 51 and 53, Hasenjaeger 52, Rasiowa-Sikorski 51 and 52, Reichbach 55, also Mostowski 49a. (Beth 56, ch. II of which

tional calculus. A modern and comprehensive exposition with complete proofs of these and many other theorems was given by Kleene [1]). The Löwenheim-Skolem theorem [2]) which interests us in connection with the following was originally proved (in Skolem 20) by means of the axiom of choice, but later proofs of Skolem and others dispense with this axiom while they are not fully constructive [3]). The theorem may also be conceived as an undecidability theorem in the sense of Gödel [4]) and has been dubbed "the most beautiful theorem in the domain of the decision problem".

Clearly the axiom system of von Neumann can be considered as a finite system of formulae in the sense of Löwenheim's theorem (hence, by taking their conjunction, as a single formula) [5]); according to this theorem, since the system is satisfied by ordinary set theory [6]), *the system can be satisfied within a denumerable domain* — for set theory is supposed to be consistent although no actual consistency proof of von Neumann's set theory, or of any other, is at our disposal. The axiom system **Z** of §§ 3–5, to be sure, is a set of denumerably many formulae owing to the axiom schemata V and VIII; yet in view of Löwenheim-Skolem's theorem this system can also be satisfied within a denumerable domain [7]). Thus we obtain what is nowadays called a

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contains a lucid exposition of the subject, appeared after the completion of the manuscript.) See also Chapter V.

[1]) Kleene 52, § 72 (–74). Cf., for instance, Beth 55, pp. 83 ff, and Hermes-Scholz 52, pp. 23 f.

[2]) With regard to the (possibly misleading) name *Skolem-Löwenheim theorem* see the historical remarks in Feferman-Tarski 53: while above we assumed the condition of a *satisfiable* system of formulae (according to Löwenheim-Skolem), the theorem proven in Rasiowa-Sikorski 52 is based on the more general condition of a *consistent* system and should rather be named *Skolem-Gödel theorem*. For a Platonistic viewpoint, to be sure, this distinction is irrelevant. — Cf. furthermore Kreisel-Wang 55.

[3]) Save for the interpretation of the theorem in Herbrand 30 (Thèse, p. 118) for which an intuitionistically valid proof is given. But this interpretation is not sufficient for the conclusions drawn in the following. Cf., however, Łoś-Mostowski-Rasiowa 56 (with regard to Herbrand).

[4]) Gödel 31; cf. Kreisel 50. A characteristic application of Löwenheim-Skolem's idea to the construction of an undecidable statement in the theory of real numbers is given in Mostowski 49a, pp. 153 ff and 162.

[5]) The same applies to Bernays' system (§ 7). Cf. Myhill 52 and Beneš 55.

[6]) Or 'is consistent' according to Skolem-Gödel.

[7]) Cf. the model Σ' constructed by von Neumann (25, pp. 232 ff) in another way; Σ' satisfies von Neumann's system and is nevertheless denumerable —

"non-standard model" of set theory [1]; in fact the relation representing 'ϵ' in the model is not the class-membership relation. On the other hand, Cantor's theorem holds true in each of these systems and yields, in view of the axiom of infinity (VII in § 5 or its counterpart in § 6), non-denumerably infinite sets. This apparent contradiction is known as the *Skolem paradox* [2].

While the intuitionists (Chapter IV) welcome this paradox as an additional argument against the existence of sets of transfinite cardinality [3] and while Skolem himself and a few more conservative scholars [4], too, have sometimes drawn rather defeatist conclusions for analysis and set theory from Skolem's paradox, the majority of mathematicians take a more cautious attitude [5]. Encouragement may be derived from the very close relatedness of the paradox to Richard's antinomy (Chapter I) and from parallel features in modern undecidability theorems and theories. In fact a certain set, for instance the continuum (say, the set of all sets of integers), may prove non-denumerable within the axiomatic system considered because there exists in the system no one-to-one mapping (i.e., no suitable set of pairs; see § 8) assigning different integers to different members of the

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not within the system but in a "higher" system, or in the absolute sense of Cantor's theory. Also in § 2 of Skolem 29 and in Suszko 51 the paradox is reached without the Löwenheim-Skolem theorem.

[1] Rosser-Wang 50, Myhill 53; cf. already Henkin 47 and 49. With regard to ϵ, cf. the more precise statement in Shepherdson 51-53$_{III}$.

[2] Pointed out first in Skolem 22/23, in more detail in 29. Cf. Wang 55, pp. 69–72.

[3] With respect to transfinite ordinals, also rejected by intuitionists, the rejection can much less rely on our paradox.

[4] Even von Neumann in 25 (in particular pp. 235 and 239). The conclusions drawn in Beth 51a are radical in a different direction as they stress the character of *immediacy* peculiar to mathematical knowledge; cf. the well-founded attitude of Myhill 53.

[5] Cf. the vigorous discussion (of 1938) recorded in Skolem 41, pp. 47–52; also, for instance, Kleene 52, pp. 426 f, and Wang 53a. The profound and convincing arguments of Kreisel 50 cannot be discussed here; they confirm the opinion that no contradiction and no crisis of cardinals is involved.

Incidentally, the Skolem paradox has also its bearing on the concepts of *model* and *categoricalness* (above, p. 89), and the notions of *absolute* and *non-absolute* models are connected with the paradox. See Skolem 33 and 34, Kemeny 48, Mostowski 47a, 53, 55 (pp. 12 ff), Henkin 50, Rosser-Wang 50, Ryll-Nardzewski 53.

In particular it has been pointed out that the only conditions that can be

continuum; Cantor's theorem states the non-existence of such a mapping. Yet at the same time the predicates used in Axiom V of **Z** (p. 38) are available only in a denumerably infinite multitude and yield no more than $\aleph_0$ subsets of the set of all integers, i.e. *an enumeration of the continuum from outside the axiomatic system* [1]; additional subsets that might be obtained by the axiom of choice will not change this picture. More generally, to a given axiomatic set theory Σ there always exists a more comprehensive theory in which all infinite sets of Σ prove equivalent, namely denumerable; the enumeration from outside utilizes the structure of the system as a whole which cannot be reached by operations within the system. This is but a particular case of a general feature which arises whenever we try to consider formal objects apart from the formalism in which they were introduced. The situation is in a way parallel to that of Gödel's incompletability theorem (Chapter V) which shows that in "normal" deductive systems certain statements are true in view of the structure of the system in its entirety but cannot be proved by means of the formal instruments available in the system.

Still we find ourselves in a dilemma similar to that which applies to the continuum problem[2]). Either the concept of subset of a given set is taken as a Platonic reality, hence non-denumerability is conceived as something absolute — then axiomatization seems impracticable in view of Skolem's paradox and we retreat to naive set theory with its menace of antinomies. Or else we accept axiomatization in whatever shape, then the notions 'denumerable' and 'non-denumerable' turn out to be relative and we have to accept *the relativization of cardinals* [3]) first pointed out by Skolem. In a given

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imposed (in systems formalized within the first-order functional calculus) upon the domain of a relation with respect to its cardinality, are a) to be empty, b) to have *at least n* members (for any finite *n*), c) to be infinite, d) to be universal. Thus the "paradox" becomes natural: a lower bound $\leq \aleph_0$ can be given but no upper bound. See Hermes-Scholz 52, p. 24; Myhill 53.

[1]) This contrast which "solves" the paradox of Skolem is described in detail by Carnap (34, and in the English edition 37). According to him infinite sets may have different cardinal numbers and be nevertheless syntactically of the same cardinal number.

[2]) Cf. pp. 93 f. — A philosophical discussion of the ontological significance of our dilemma is contained in (Berry 53 and) Myhill 53.

[3]) The relativization does not apply to the ordinals to the same extent. Yet a purely ordinal conception of cardinals (as initial numbers) in a sense which

axiomatic set theory we may find all features of Cantor's theory including infinitely many different transfinite cardinals and even inaccessible numbers, but with the introduction of subtler criteria as furnished by a higher system all distinctions dissolve and $\aleph_0$ alone survives. This means that Cantor's set theory is so copious as to admit absolutely non-denumerable sets while axiomatic set theory is so limited that every non-denumerable set becomes denumerable in a higher system or in an absolute sense.

This relativism, incidentally, concerns not only more-than-denumerable sets but *finite sets as well.* von Neumann pointed out [1]) that the properties of finiteness and of being well-ordered are subject to a similar relativization. In fact, if finiteness is taken, for instance, in the sense of Dedekind ('not equivalent to a proper subset') or of Tarski [2]) (which is equipollent to Russell's inductiveness) the finiteness of a set S depends on certain mappings or on the totality of subsets of S; it is well conceivable that in a given system there is an insufficient wealth of mappings or subsets causing S to appear finite or well-ordered, while the superior wealth of a higher system renders S (denumerably) infinite or no more well-ordered. The latter case would mean that in the higher system there is a subset without a first member, which subset does not exist in the original system.

This double relativism should not be exploited to strengthen any of the existent philosophies of mathematics. If, on the one hand, it seems to support the intuitionistic slogan that by formalistic (axiomatic) methods no reality can be reached and everything remains vague and relative, on the other hand the supposedly absolute original intuition of positive integer (Chapter IV, § 5) forfeits as well its pretended evidence owing to indistinct passages from finiteness to transfinite denumerability [3]). Thus not only is categoricalness or monomorphism of set theory called in question but also the absolute authority and distinctness of the notion of integral number.

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would transfer some invariant character from ordinals to cardinals, independently of the relativistic concept of well-ordering, seems hardly tenable.

[1]) von Neumann 25, pp. 239 f. Cf. already Skolem 22/23, pp. 223 f. For 'well-ordered' cf. the example in § 2 of *Skolem 10.

[2]) Tarski 25.

[3]) See also Wang 50, in particular Theorems 9 and 10.

§ 7. The Axiom Systems of Bernays and Gödel. Relative Consistency Proofs

The fundamental idea of von Neumann, namely of admitting "most comprehensive" aggregates (classes) for themselves but not as arguments (elements), is preserved by Bernays; but his axiomatization retains much more of the features of ***Z*** and hereby assumes a simpler and more lucid structure. Additional steps in the same direction were taken by Gödel; see pp. 119 ff.

(The discovery that admitting comprehensive classes does not necessarily lead to contradictions has its significance also outside set theory; we may, for instance, speak freely of the class of all groups or of all fields.)

In the system of Bernays [1]), henceforth denoted by '***B***', two different sorts of objects appear, called *classes* and *sets*; his classes roughly correspond to von Neumann's functions, the sets to the arguments [2]). There are, in addition to equality (identity) and to truth-functions and quantification, *two primitive relations* to be interpreted as membership in a *set* and in a *class*, respectively. Following Bernays we use small italics when referring to sets, capital italics when referring to classes; the primitive relations shall be denoted by 'ϵ' and 'η', respectively, viz.

$x \epsilon a$ (x is a member of the set a)

$x \eta A$ (x is a member of the class A) [3]).

The left-hand argument *x is restricted to sets*. This has a twofold significance; first that a class cannot be a left-hand argument of ϵ or η, second that ***B*** does not contain "individuals" in the sense of p. 28. In the latter respect the system ***B*** resembles ***Z*** (p. 31) and

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[1]) Bernays 37-54. This sequence of seven articles is the most comprehensive and profound research in existence about axiomatic set theory. Cf. also Bernays 58. As most quotations of Bernays in this section refer to this sequence, each article is cited by the *Roman letter only* of the respective part of 37-54.

[2]) In Bernays' words, "the distinction may be thought of in this way that a set is a multitude forming a proper thing whereas a class is a predicate regarded only with respect to its extension" (I, p. 66). There is no need of distinguishing between 'arguments' and 'elements' because Bernays' attitude is from the first that of the footnote on p. 99 (that is to say that all arguments are sets; cf. § 2, p. 31).

[3]) Bernays has 'element' instead of 'member'.

von Neumann's system 29 and contrasts with Zermelo's attitude [1]).

As to *equality* (or, as Bernays expresses it, 'being the same individual') ***B*** regards it not as a primitive relation of the system but as "a logical concept immediately connected with the idea of an individual" [2]). This attitude differs from that taken in § 2; in fact Bernays introduces the axiom of extensionality in the form Ia of p. 33 but doubled for the equality of *sets* and of *classes* (the latter being of secondary importance only) [3]). The relations $\subseteq$ and $\subset$ are defined as on p. 31 both for sets and for classes and also between a set and a class (subset of a class, subclass of a set). A new concept naturally involved herewith is "the set *a representing* the class A" or vice versa [4]); it is defined by '$a \subseteq A$ and $A \subseteq a$'.

In addition to the axioms of extensionality (I) there appear two elementary axioms II postulating the existence of the null-set (empty set) O [II(1)] and of the set $s \cup \{c\}$ for any given sets s, c with $c \notin s$ [II(2)]. II(2) replaces the axiom of pairing of ***Z*** (p. 34) and guarantees, in view of II(1), the set $\{c\}$ for every c.

The ordered pair or dyad $\langle a, b\rangle$ is defined as the set $\{\{a\}, \{a, b\}\}$ (cf. below on p. 131); a is called its *first*, b its *second element* [5]). Clearly $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$. $\langle b, a\rangle$ is called *converse* to $\langle a, b\rangle$.

Finally the ordered triad $\langle a, b, c\rangle$ is defined as $\langle a, \langle b, c\rangle\rangle$. We shall henceforth drop 'ordered' and simply speak of *dyads* $\langle a, b\rangle$ and *triads* $\langle a, b, c\rangle$.

These preliminaries are followed by the corner-stone of Bernays' system, namely by the *axioms for the construction of classes* (III) through which axiom schemata such as the axiom of subsets in § 3 and the axiom of substitution in § 5 are be reduced to axioms. The final purpose pursued by the axioms III is to reach the following fundamental principle, called the *class axiom.*

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[1]) Both in Zermelo 08a and 30. (*Urelemente* are there admitted though not required.)

[2]) Henceforth 'individual' is understood in its logical sense.

[3]) Part II of Gandy 56 will prove that if Bernays(-Gödel)'s system *without* the axiom of extensionality is consistent it remains so after adjoining the axiom.

[4]) A related idea appears in Löwenheim 40, as in general the method of restriction used in this paper (in order to avoid antinomies) is analogous to axiomatic set theory.

[5]) The term 'element' (of a dyad or k-ad) should be distinguished from the term 'member' (of a class or set).

(B) For any predicate 'F' which contains no bound class-variables there exists the class Y that contains just those x which satisfy 'F', or in symbols

$$(\mathrm{E}Y)\ (\mathrm{A}x)\ [x\ \eta\ Y \equiv F(x)].$$

'Y' is here a class-variable not appearing in '$F(x)$'. (Cf. p. 113.)

The main difference between the principle (B) and the axiom of subsets of **Z** (§ 3, p. 38) lies in that *there* the existence of a set y, determined by a given predicate, depends on the pre-existence of a set of which y is a subset while *here* the existence of the class Y is stated unconditionally, i.e. independently of any assumption save for the defining predicate 'F'.

The principle (B) is again a *schema* because of the occurrence of the predicate-variable 'F'. However, Bernays (in part I) gives the rather complicated proof that the principle (B) can be derived from a number of single *axioms* which are united to an axiom group III. To be sure, this derivation is not carried out within the axiomatic system but in a metamathematical way. Changing slightly Bernays' arrangement with regard to the axioms III and to some immediate conclusions [1], we formulate the following seven axioms.

III (1) Every set is represented (p. 111) by a class, i.e.
$(\mathrm{A}z)\ (\mathrm{E}Y)\ (\mathrm{A}x)\ [x\ \eta\ Y \equiv x \in z]$ [2].

III (2) There exists the class of all dyads $\langle x, y\rangle$ which satisfy $x \in y$, i.e. [3]
$(\mathrm{E}Z)\ (\mathrm{A}x)\ (\mathrm{A}y)\ (\langle x, y\rangle\ \eta\ Z \equiv x \in y)$.

III (3) For every class X its complementary class $\mathrm{V} - X$ exists, i.e.
$(\mathrm{A}X)\ (\mathrm{E}Y)\ (\mathrm{A}z)\ [z\ \eta\ Y \equiv \sim(z\ \eta\ X)]$.

III (4) The intersection $X \cap Y$ of any two classes X, Y exists, i.e.
$(\mathrm{A}X)\ (\mathrm{A}Y)\ (\mathrm{E}Z)\ (\mathrm{A}u)\ [u\ \eta\ Z \equiv (u\ \eta\ X\ \&\ u\ \eta\ Y)]$.

III (5) For every class X of dyads there exists the class the members of which are the first elements of the members of X (the *domain* of X); i.e.
$(\mathrm{A}X)\ (\mathrm{E}Y)\ (\mathrm{A}z)\ (\mathrm{A}u)\ [z\ \eta\ Y \equiv (\mathrm{E}u)\ (\langle z, u\rangle\ \eta\ X)]$.

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[1] We here follow the arrangement of (Gödel 40 and) Wang-McNaughton 53, distinguishing, however, between the relations 'ϵ' and 'η'.

[2] It will turn out that the converse does not hold in **B**, i.e. that not every class is represented by a set. This is to be expected from the first, according to the very purpose of the distinction between classes and sets.

[3] In the *verbal* expressions of Axioms (2), (5), (6), (7) some matter-of-course simplifications have been inserted; in (2), for instance, Z might contain also members other than dyads, and the same holds for X in (5), etc.

III (6) For every class X there exists the class the members of which are the dyads whose first element is a member of X; i.e.
$(AX)\,(EY)\,(Az)\,(Au)\,[\langle z, u\rangle\ \eta\ Y \equiv z\ \eta\ X]$.

III (7) For every class X of triads $\langle z, u, v\rangle$ there exists the class of the corresponding triads $\langle v, z, u\rangle$ and the class of the corresponding triads $\langle u, z, v\rangle$; i.e.
$(AX)\,(EY)\,(Az)\,(Au)\,(Av)\,[\langle v, z, u\rangle\ \eta\ Y \equiv \langle z, u, v\rangle\ \eta\ X]$,
$(AX)\,(EY)\,(Az)\,(Au)\,(Av)\,[\langle u, z, v\rangle\ \eta\ Y \equiv \langle z, u, v\rangle\ \eta\ X]$.

Among the immediate consequences of these seven axioms it is sufficient to mention that there exists

a) the class the members of which are all sets containing *a single member* [1];
b) for every class C of dyads, the class of the converse dyads, in short the *converse class* of C;
c) the empty class, the class of *all sets* (by III (3)), and the class of *all dyads*;
d) the *union* of two given classes (by III (3) and (4));
e) for every class C of dyads, the class of their second elements (the *converse domain* of C), and also the class of all elements of the members of C (the *field* of C).

In analogy to the transition from dyads to triads we can define k-ads [2], and certain types among them as *normal k-ads*; for $k = 4$, $\langle a_1, \langle a_2, \langle a_3, a_4\rangle\rangle\rangle$ is a normal tetrad.

By means of these axioms Bernays derives the class axiom (B); more precisely and generally, he proves the existence of a class whose members are just the k-ads that satisfy the predicate 'F' which contains k free set-variables. While the proof is rather difficult (see part I, pp. 70–77) its gist is lucid enough. In fact, for certain very simple predicates 'F' (B) is already expressed by the axioms III, and from the conditions '$F(x)$', '$G(x)$', '$H(y, z)$' we may pass, in view of the axioms III, to conditions of the form '$\sim F(x)$', '$F(x)\ \&\ G(x)$', '$(Ey)H(y, z)$'. In particular one proves the existence of the classes which are

1) the intersection of the members of a given class (or set),
2) the union of the members of a given class (or set),
3) the power-class [3] of a given class (or set).

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[1] In part I this appears as an axiom. For its proof see VII, pp. 94 f.

[2] Cf. Quine 34, Chapter I. For a direct introduction of ordered k-ads as sets, cf. Skolem 57a, pp. 43–46 (also Rosser 53, p. 281).

[3] I.e., the class of all subsets.

Bernays' part II opens with a further group of (six) axioms, namely the counterparts of axioms known to us from **Z** (§§ 3–5): *the axioms of choice* (Bernays' IV), *of subsets* (Va), *of substitution* (Vb), *of sum-set* (Vc), *of power-set* (Vd), *of infinity* (VI), and *of foundation* (VII). These axioms require here additional precision because of the distinction between classes and sets.

First, a (single-valued) *function* is defined as a class F of dyads $\langle a, b\rangle$ such that different dyads of F have different first elements; in other words, for every member a of the domain of a function its "value" b is uniquely defined. A one-to-one mapping is, then, a class M of dyads such that both M and its converse class are functions.

Using this concept, Axioms IV-VII of the system **B** read:

IV. Every class C of dyads has a subclass which is a function and has the same domain as C. (Axiom of choice.)

Va. If a class is represented by a set, every subclass is also represented by a set. (Axiom of subclasses.)

Vb. If the domain of a one-to-one mapping is represented by a set, the converse domain is also represented by a set. (Axiom of substitution.) [1])

Vc. The sum-class (union of the members) of a set is a set.

Vd. The power-class of a set is a set.

VI. There exists a set s such that there is a one-to-one mapping of s onto a proper subclass of s. (Axiom of infinity.)

VII. The axiom of foundation (or restriction); it can approximately be taken in the form of p. 91. In accordance with Bernays' terminology we shall call it *the axiom of restriction* in the present section.

The axioms V will be used separately for different purposes as described in the following. In particular, Vc and Vd will be avoided as far as possible.

Of the immediate consequences of Axioms IV–VII only a few need explicit mentioning. From IV and the axioms III the general axiom

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[1]) To be sure, in the assumption of the axiom on p. 85 only a function appears and not a one-to-one mapping (biunique function). In view of the axioms IV and Va, however, this makes no difference. Conversely, from the original form of the axiom of substitution, adapted to the present terminology and denoted in the following by V*, Va and Vb can be deduced as theorems. V* is frequently used by Bernays instead of the conjunction of Va and Vb, especially when the axiom of choice is not presupposed.

of choice follows, viz. the statement that for every class C of non-empty sets there exists a function with the domain C which assigns to each member of C one of its members. On the other hand, Russell's Multiplicative Axiom (p. 46) follows from III (1), IV, Va or b. From VI we infer the existence of a set which has a one-to-one mapping onto a proper subset, i.e. of a set which represents a "reflexive" class.

It is not intented to outline here the development of set theory on the basis of the axioms of Bernays, except for some topics as ordinals and number theory. The derivation of the fundamental points of the theory from the axiomatic basis of §§ 2–5 is given in § 8, and a few remarks in the following paragraph will illustrate the derivation by Bernays (mostly given in part III). Furthermore it shall be pointed out what axioms of ***B*** are required for the foundation of various mathematical branches; this is one of the most important achievements of Bernays, who also obtains profound results about the independence of the axioms.

In addition to the axiom groups I–III, which will always be presupposed, the a(xiom) of restriction is sufficient to develop the general theory of *ordinals* (see below), number theory, and the theory of finite sets (and classes); with a slight modification of the definition of ordinals the a. of subclasses may be taken instead of the a. of restriction. For *analysis*, I–III are supplemented by the a. of subclasses, a stronger form of the a. of infinity (namely, that every enumerable class is represented by a set), and the restricted a. of choice [1]). For *general set theory* [2]), the axioms of choice, of subclasses, and of substitution are added to I–III. Regarding analysis, real numbers are defined as Dedekind cuts (sections, cf. *Theory*, § 9) by means of *fraction triplets* of non-negative integers (a, b, c) with $c \neq 0$, to be interpreted as $\frac{a - b}{c}$. While the least upper bound and even a definition of the Lebesgue measure may be obtained along custom-

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[1]) See above, p. 51, note 1. As Bernays shows, the restricted axiom is sufficient for the theory of the second number class.

[2]) As mentioned before, Bernays uses this phrase in a sense different from that used in §§ 3–5. Bernays' conception is adapted to the tendency, recognizable already in von Neumann's system, "of separating the assumptions which are required for the conceptual formations from those which lead to the Cantor hierarchy of powers" (part IV, p. 133). Cf. p. 116.

ary lines, a sequence of real-valued functions of a real variable can in general not be represented by a set, save for the case of continuous functions (which are already determined by their values at rational arguments; cf. *T*, p. 160).

The interdependence of various forms of an a. of infinity is discussed, including the axioms VII and VII* on p. 82, the far stronger axiom mentioned on p. 82 footnote 3, and the above a. of enumerable sets.

Part IV derives those parts of general set theory that are available without the axioms of sum-set, power-set, and infinity — which axioms are relevant for the construction of the series of cardinals rather than for the general theory. As customary, the a. of choice is only used when unavoidable. While the algebraic equivalence laws of union and intersection of classes (i.e. elementary Boolean algebra) and of Cartesian products can be deduced from Axioms I–III alone, the laws concerning the exponentiation of sets (*T*, § 7) also require a. V* and a *pair class axiom* which postulates that the product $A \times B$ is represented by a set whenever the same holds for the classes A and B. (Instead of these two axioms the original axioms IV, Va, Vb may be used.) To obtain the equivalence theorem (*T*, pp. 98 f) for classes and Cantor's theorem (*T*, p. 94) for sets, the addition of a. Va is sufficient. It is remarkable that of the two fundamentally different methods of proving the equivalence theorem (see *T*, p. 103) the one which dispenses with the use of a denumerable set proves to be axiomatically *less elementary* and can be used in the present system only when the original class (*S* in the formulation of *T*, p. 99, fourth paragraph) is represented by a set. Herewith the discussion, continued for many decades, of the question which method if preferable finds an answer in a well-defined sense. As to Cantor's theorem, the proof by means of a. Va yields only that the *class* of the subsets has a higher cardinal [1]) (power); the proof that this class is represented by a set requires the use of a. Vd which is beyond general set theory in the above sense. Obviously Cantor's theorem cannot be formulated for classes in general since, for instance, the class of all sets coincides with the class of its subsets; nevertheless Bernays succeeds in adapting Cantor's argument, if not his result, even to the subclasses of any

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[1]) The concept of cardinal is not used here; for its explicit introduction see below, p. 118.

class. At this juncture it becomes particularly lucid how the antinomies are avoided through the distinction between classes and sets.

By means of the axioms IV, Va, and Vb the inequality of König-Jourdain-Zermelo (*T*, p. 132) is obtained.

While the proof of the comparability of *classes* transcends the general theory, as it requires Vc, Vd, VII, the comparability of *sets* is obtained by an adaptation of Zermelo's first proof (*T*, pp. 310 ff) of the well-ordering theorem, for which Axioms IV, Va, and Vb are sufficient. The concept of order is introduced by either of the two methods explained in the next section (pp. 129 ff; cf. *T*, p. 192, c)). Zermelo's second proof (cf. *T*, pp. 319 f), as well as Hartogs' proof (see above p. 71), require a. Vd and appear less elementary in this light; yet Bernays gives an intermediate course between the first and the second proof which contents itself with the above means of general set theory.

Prior to these results regarding classes and sets in general, the *theory of ordinals* is developed in part II. Bernays follows R. M. Robinson's modification of von Neumann's method (above p. 102) where the theory of ordinals, in contrast with von Neumann, is not based on well-ordering but is developed independently; he also uses certain ideas expressed by Gödel in 1937. A set t is called *transitive* if every member of any member of t is also a member of t; ordinals are defined as transitive sets c such that $a \in c$ and $b \in c$ with $a \neq b$ imply either $a \in b$ or $b \in a$. By (*B*) on p. 112 *the class of all ordinals exists*. By means of the a. of restriction it is proved that every transitive proper subclass of an ordinal is represented by a certain member of that ordinal and that the intersection of the members of a non-empty class of ordinals is represented by a member of that class. The first property makes the use of a. Va superfluous in this theory while the second yields their well-ordering, provided that the ordinals are ordered by the relation C; in fact the second property then means that in every non-empty class of ordinals there is a first. Now it easily follows that every member of an ordinal is an ordinal and that every ordinal represents the class of all preceding ordinals (see § 6). From hence the usual theory of ordinals, say in the sense of von Neumann, is easily developed and it follows that *the class of all ordinals cannot be represented by a set*, in view of a. VII [1]).

[1]) As remarked in R. M. Robinson 37, the theory of ordinals might be modified in a way avoiding the a. of restriction. In this case, however, we need a. Va to obtain the above result.

Finally a *cardinal* [1]) is defined as an ordinal n such that there exists no one-to-one mapping of n onto any member of n (cf. p. 102). For every set there exists one and only one corresponding cardinal.

The theory of finite ordinals and of *finite classes and sets* is now easily obtained by specialization.

An ordinal is called finite if it and each of its members satisfy the condition of either being empty or having a last member. The class of all finite ordinals exists by (B). Peano's axioms, including the principle of mathematical induction, easily follow from the definition; the theorem of finite recursion (definition by induction) can be derived either analogically to Dedekind [2]) or by a modification due to Bernays which relies on the special case of an iteration. Addition, multiplication, and exponentiation of non-negative integers are defined in essentially the usual way; more generally, numerical functions using the concept of "the least number such that —" can be defined [3]).

A class or a set is called finite if there exists a one-to-one mapping of it onto a finite ordinal. Accordingly an ordinal proves to be a finite set if and only if it is a finite ordinal (see above). The further development of the theory of finite sets and numbers meets with no difficulty provided the a. of restriction (or Va) is granted.

The final parts V–VII deal on the one hand with procedures and results of classical set theory and certain selections of axioms sufficient for deriving them, on the other hand with properties of the axiom system itself (see Chapter V), in particular with independence proofs.

In the first direction a general recursion theorem requiring only Axioms Va and Vb is proven, from which by means of the a. of choice various maximum principles (see p. 69) are deduced. The above results regarding finite recursion are extended to transfinite recursion; this can be done either by means of a. Vc or by a weaker form in which case the notions of sum and product of transfinite ordinals are required (part V). Then the axioms of sum-set and of power-set which so far were mostly avoided are used to prove that for every cardinal there exists a greater one; moreover, the following result

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[1]) Cf. also the definition in Specker 54, p. 332.

[2]) See p. 102. Dedekind's procedure, however, requires the a. of infinity in addition to Va or VII. Zermelo (*4, cf. *Grelling 1) was the first to show that the theory of finite sets and numbers can be derived from set-theoretic axioms without assuming the existence of an infinite set. Cf. Wang 53 (and 57).

[3]) Cf. Hilbert-Bernays $34\text{-}39_1$, pp. 401 ff.

is obtained which characterizes the central point of von Neumann's system (see his axiom IV2, p. 99): a given class A of ordinals is either represented by a set or else there is a one-to-one mapping of A onto the class of all ordinals. The main tool used here is a class Π used by von Neumann in a somewhat different form for his consistency proof (p. 104). (Π turns out to be the class of all sets.) For the members of Π the important concept of degree (or rank) is introduced [1]). Furthermore, the task of the axiom of restriction and the possibility of weakening it are discussed (part VI).

The rest of part VI and the bulk of the (last) part VII are devoted to the construction of models serving the purpose of *independence proofs* (see Chapter V) by means of a function Ψ and the class Π introduced by von Neumann [2]). The main result is that in a system containing Axioms I–III, Va–d, and the axioms of infinity VI and of restriction VII, each of the four axioms Vb, Vc, Vd (of substitution, of sum-set, and of power-set), and VI is independent, i.e. not provable in the subsystem created by deleting the respective axiom. In part VII further independence proofs are given regarding the axioms of subclasses and of restriction, both without and with the a. of infinity; here a method of Ackermann's is used [3]). The a. of choice is partly included. Finally the possibility of constructing the models in a number-theoretic frame, instead of the set- and class-theoretic frame, is discussed.

Gödel's system Σ of axioms [4]) follows pretty close Bernays' system (which was known to Gödel by letter while only part I had been published by then). It is essentially a simplification and this quality makes it not only fit for its immediate purpose, viz. the proof of the consistency of the axiom of choice and of the continuum hypothesis, but also a convenient instrument for consistency (or independence) proofs in general.

[1]) This concept of rank is defined in a simple way in Tarski 55a; here some of its important applications are stated and the development of the theory of ordinals within the theory of ranks is sketched.

[2]) von Neumann 29, pp. 236 f; here the function Ψ, as the *Bereich* Π, is used for the consistency proof.

[3]) Ackermann 37.

[4]) Gödel 40; cf. 38 and 39. Cogan 55 shows how Gödel's system can be formulated within a suitable system of combinatory logic (cf. above, p. 98, footnote 2).

Gödel's main formal innovation is that *every set is identified with the class represented by that set*; accordingly, of the relations ϵ and η only one (ϵ) is retained (which makes the system more similar to **Z**, see above § 2) [1]). While '$s \epsilon t$' is *meaningful* for any classes s, t, i.e. for sets and "proper classes", it is *true* only when s is a *set* which is a member of t. Gödel divides his system into five groups A–E [2]). A contains, in addition to extensionality and III (1), the axiom of pairing as on p. 34 (not in the sense of dyads). Group B essentially coincides with Bernays' axioms III [3]) and yields the result (cf. (*B*) on p. 112) that, given a predicate in which no bound class-variable appears [4]), there exists a class which contains the sets satisfying that predicate. Group C contains suitable axioms of infinity, sum-set, power-set, and substitution, the latter taken in the original form of p. 85, while D and E are the axiom of restriction (Bernays' axiom VII) and a general form of the axiom of choice (for classes of non-empty sets). Thus a closer approach to the system **Z** is obtained, save for the distinction between sets and classes and for Bernays' conception of the *Aussonderungsaxiom* by means of the axioms III or (*B*); for the latter purpose, Gödel's derivation is simpler than Bernays'.

The theory of ordinals is then developed, essentially along the lines of von Neumann and R. M. Robinson (§ 6). [5])

On this basis a certain *model* Δ for Σ is defined whose classes and sets satisfy the axioms of Gödel. Its very ingenious definition cannot be given here and it must suffice to say that the sets in Δ are the members of the class L of values assumed by a certain function whose domain is the class of all ordinals. The function is created by a transfinitely

[1]) Cf. the elegant method proposed in Quine 56, p. 269. As to a further reduction of the primitive concepts, cf. the remarks of Mostowski and Tarski in Mostowski 39, p. 208, and 49a, p. 144.

[2]) For a symbolic formulation of the axioms (with a slight modification) see Mostowski 49a, p. 144.

[3]) For a simplification which shows the axiom B6 to be redundant, see Markov 48. On the other hand, the axiom B8 turns out to be redundant in view of a combinatorial theorem proven in Hajnal-Kalmár 56 (but not both axioms).

[4]) The latter condition is dropped in *202 of Quine 51. This modification seems to make Quine's system more powerful; see Wang 49, note 5. On the other hand, Quine's form is apparently not reducible to finitely many axioms. Cf. below Chapter III.

[5]) Sprinkle 56 obtains a theory of cardinals based upon Gödel's axioms (without the axiom of choice) by defining cardinals of *ordinals*, and not of arbitrary sets.

continued periodic alternation of nine particular set-theoretic operations. Δ satisfies von Neumann's axiom IV2 (p. 99). The members (sets) of L are "constructible" in a certain loose sense, meaning that the application of the operations (starting with the null-set) may be iterated any transfinite number of times. In particular, every ordinal proves constructible.

Now the compatibility of the generalized continuum hypothesis (and of the axiom of choice) with Gödel's axioms is proven by showing that the conjunction of these axioms and of the assumption that every set is constructible (in short, the *postulate of constructibility*) implies the validity of the continuum hypothesis; hence this hypothesis holds true in Δ. (This idea is related to a much earlier program of Hilbert's [1]) which has never been carried out in its original form.) The proof of that implication uses a one-to-one and (with respect to magnitude) similar mapping of the class of ordinals onto that of transfinite cardinals, which yields a one-to-one correspondence between the ordinal $\omega_{\alpha+1}$ and the set of subsets of ω_α.

It turns out that the result is not essentially dependent on the particular axioms used by Gödel but rather on their property of rendering precise the concept of predicate used in the axiom of subsets of § 3; in other words, on outlining the *concept of class* according to Bernays' or Gödel's axiomatization (see (*B*) on p. 112). This essential improvement, in comparison with **Z**, enables Gödel to prove that the adjunction of the postulate of constructibility to the other axioms does not alter the consistency of the system, provided it is consistent before the adjunction. To be sure, Gödel is clearly opposed to the idea of accepting the postulate of constructibility as one of the axioms of set theory [2]). Nevertheless it is remarkable that through this postulate a scholar as "Cantorian" and "Platonistic" as Gödel introduced, and efficiently utilized, a principle of a constructive character which reminds us of the methods of recursive functions and even of intuitionism (Chapter IV).

In the present and the preceding section we have seen that the axiom systems of von Neumann, Bernays, and Gödel are superior to the system **Z** of §§ 2–5 in many respects, an important feature being

[1]) Hilbert 25.

[2]) See particularly Gödel 47. For the problem of the postulate's independence of other set-theoretical axioms, cf. Mostowski 55, p. 24, and A. Lévy 57.

that the concept of set (class) is explicitly conceived *as comprehensive as possible* in view of the antinomies. Hence it is natural to ask whether we have not paid too dear for these achievements, viz. at the risk of increasing the danger of contradictions arising within the system. (Regarding this problem in **Z**, see pp. 94–96.)

It is therefore a great and somewhat unexpected [1]) achievement that recently *the systems of von Neumann-Bernays-Gödel have been shown to be consistent, provided* **Z** *is consistent.* Gödel's form is most convenient for such a proof. Three different proofs have been given for this relative consistency theorem: by I. Novak, by Rosser and Wang, and in a stronger (more finitistic) way by Shoenfield [2]).

Miss Novak's proof relies on a particular method of constructing models whereas the proof of Rosser-Wang is based on general considerations regarding "non-standard models" [3]) for formal logics. While none of the proofs can be given here *in extenso*, the argumentation of the second proof shall be sketched. A decisive moment is that in the systems of Bernays and Gödel classes are defined in a predicative way through the axioms III or (*B*); the chief instrument of the proof is Löwenheim-Skolem's theorem (p. 105).

Let us assume that the system **Z** is consistent. By Löwenheim-Skolem, **Z** has a model in the domain of positive integers; to every set of **Z**, then, a certain integer is assigned and the set-theoretical relation ϵ is replaced by a certain number-theoretical relation ζ such that we have $m \zeta n$ if and only if $s \epsilon t$ (the integers m, n being respectively assigned to the sets s, t). Now the axioms III of Bernays (p. 112 f), or B of Gödel, contain seven statements [4]) which may be regarded as operations assigning a class to classes (or sets) which

[1]) See Wang 49, p. 152.

[2]) Novak 48/51 (cf. Mostowski 51a and McNaughton 53, pp. 141 f); Rosser-Wang 50 (cf. Wang-McNaughton 53, p. 21). Also Mostowski, in an unpublished address at Princeton 1949, gave a proof based upon an argument similar to Rosser-Wang and showed that every theorem of Gödel's system which involves *set*-variables only is also a theorem of **Z**; see Rosser-Wang 50, pp. 125 and 128, and Mostowski 51a. While those proofs start from far-reaching consistency premisses, the proof of Shoenfield 54 is "finitary" in the sense of Hilbert-Bernays $34\text{-}39_{II}$, p. 340, being achievable in their "recursive number theory". — Other relative consistency proofs for Zermelo's or Gödel's theory (relative to simpler systems) are given in Orey 56a; cf. Firestone 47/49. For relative consistency problems regarding **Z** and Quine's *New Foundations*, see Rosser 56.

[3]) See above p. 107, footnote 1 (particularly Henkin).

[4]) In the case of Gödel, one axiom of A has to be added.

appear in the premiss of the respective statement, or else as forming a class unconditionally (if there is no premiss, as in III(2)). Leaving aside the axiom III(1) which is *sui generis*, the axioms III belong to three different types: either there is no premiss, or the premiss contains a single class as does III(3), or it contains two classes as III(4).

We now assign integers of the form $2^l \cdot 3^m \cdot 5^n$ to the classes postulated by the axioms III, l assuming the values 1, 2, 3, ..., 8 in the succession of the axioms [1]). As to III(1), if to the set z the integer m is assigned, $2^1 \cdot 3^m \cdot 5^n$ with an arbitrary n shall be assigned to the class representing z. In the other cases

a) if there is no premiss at all, $2^l \cdot 3^m \cdot 5^n$ with arbitrary values m, n shall be assigned to the class postulated in III(l); e.g., $2^2 \cdot 3^m \cdot 5^n$ to the class of all dyads;

b) if the premiss of III(l) contains one class X to which the integer m is assigned, then $2^l \cdot 3^m \cdot 5^n$ with an arbitrary n shall be assigned to the class postulated; e.g., $2^3 \cdot 3^m \cdot 5^n$ to the complement of the class X to which m is assigned;

c) if the premiss of III(l) contains two classes X, Y to which the integers m, n respectively are assigned, then $2^l \cdot 3^m \cdot 5^n$ shall be assigned to the class postulated; e.g., $2^4 \cdot 3^m \cdot 5^n$ to the intersection of X and Y if m is assigned to X and n to Y.

Hereby to every class postulated by the axioms III at least one positive integer is assigned, and the same integer is never assigned to different classes. To obtain a model of Bernays' system in the domain of integers we may now, according to the operations expressed by the axioms III, introduce two new relations θ and κ between integers such that we have $m \,\theta\, n$ or $\bar{m} \,\kappa\, \bar{n}$ if and only if respectively $s \,\eta\, T$ (in Gödel's symbolism, $s \in T$) or $\bar{s} \in \bar{t}$, the integer $m(\bar{m})$ being assigned to the set $s(\bar{s})$, the integer n to the class T, and the integer $\bar{n}$ to the set $\bar{t}$.

Since the remaining axioms of Gödel are not essentially different from the corresponding axioms of $\boldsymbol{Z}$ we thus obtain a model of Gödel's system in the domain of positive integers, as an extension of the arithmetical model of $\boldsymbol{Z}$ from which we started; thus the consistency of Gödel's system relatively to the consistency of $\boldsymbol{Z}$ is proven.

We may contemplate these consistency proofs also in the following light. In § 5 it was pointed out that if we denote the system of the axioms I–VII of $\boldsymbol{Z}$ (§§ 2–5) by $\bar{\boldsymbol{Z}}$, then the axiom of substitution VIII —

[1]) For III (7) two values, 7 and 8, are required.

or even the stronger form of VII, see p. 82, footnote 3 — is independent of $\bar{\boldsymbol{Z}}$. In other words, $\bar{\boldsymbol{Z}}$ *is weaker than the system* $\boldsymbol{Z}$ [1]). On the other hand, in view of the consistency proofs just mentioned $\boldsymbol{Z}$ and the systems of von Neumann and of Bernays-Gödel may be considered to possess similar strength in a certain sense. This is remarkable since the "proper classes" (non-sets) which appear in the systems of §§ 6 and 7 cannot be secured in $\boldsymbol{Z}$.

§ 8. Derivation of Set Theory from the Axioms

In addition to $\boldsymbol{Z}$ and the axiom systems of §§ 6 and 7, exhibited in the present Chapter, some "logicistic" systems, in particular those of Russell and Whitehead, of Quine, and of Wang, are discussed in Chapter III, and intuitionistic set theory in Chapter IV. Of others, the systems T_n ($n = 1, 2, 3$) of Wang and McNaughton [2]) may be mentioned; essentially they are weaker subsystems of $\boldsymbol{Z}$. T_1 drops the axiom of infinity, thus yielding the theory of finite cardinals or ordinals; by the method of type theory and a suitable class axiom (analogous to (*B*), p. 112) T_1 is successively extended to T_2, yielding the continuum, i.e. a set theory for real numbers, and to T_3, yielding a set theory for functions of real variables.

Yet to accomplish the task of the present Chapter it is not sufficient to exhibit various axiom systems, to compare them, and to characterize the strength of each — as done so far — but we have *actually to derive the principal facts of classical set theory from the axioms.* Various hints about methods of achieving this from the axioms of von Neumann and Bernays were given in §§ 6 and 7 but it would require too much space to carry this out in detail as done in the original papers of von Neumann and Bernays. For the system $\boldsymbol{Z}$, however, the actual derivation shall now be outlined in its main features [3]).

In accordance with the substance of historical set theory (cf. *Theory* or other textbooks) the following exposition divides into two parts, the *theory of equivalence* and the *theory of order.* After laying the foundations of these theories the further development ensues

[1]) Cf. Wang-McNaughton 53, p. 18, also for the terms 'weaker' and 'of similar strength' used here.

[2]) Wang-McNaughton 53, ch. VI.

[3]) Only after the manuscript had gone to print did the paper Thiele 55 become accessible to the authors. This paper, too, has the purpose of deriving set theory from the axioms of $\boldsymbol{Z}$, maintaining "that the axiom system of Zermelo and Fraenkel is sufficient to construct all branches of set theory; in

more or less parallel to traditional lines. Therefore the main task is to define the basic concepts of both theories, by reducing them to the single primitive concept of **Z**, viz. the relation ϵ. We shall in general only use Axioms I–V of the "general theory" (§§ 2/3); whenever the axiom of choice (VI) is used this is pointed out. The axioms of infinity and of substitution are utilized in particular cases only.

The following idea will suggest a suitable definition of *equivalence*. S and T denoting equivalent disjoint sets, a one-to-one correspondence between the members of S and of T — i.e. a single-valued and uniquely invertible function with the domain S and the counterdomain T or conversely — can clearly be conceived as a set R whose members are pairs $\{s, t\}$ satisfying the following conditions:

a) $s \in S$, $t \in T$;

b) $\{s_1, t_1\} \neq \{s_2, t_2\}$ implies $s_1 \neq s_2$ and $t_1 \neq t_2$ [1]);

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particular, the range of this theory is no less than that of von Neumann's theory".

While in general the same direction — though with a higher degree of formalization — is taken by Thiele as in Zermelo 08a, Fraenkel 25, 26, 32, and in the present § 8, there is a difference of principle regarding the introduction of cardinal and ordinal numbers: whereas in von Neumann 23 and 28a these numbers are, within **Z**, based on the axiom schema VIII of substitution (§ 5), Thiele avoids VIII by introducing the concepts of 'cardinal (ordinal) of a set m (well-ordered set m) *relativized with regard to a set* p'. (These concepts are void if the cardinal [ordinal] of p is inferior to that of m.) The rôle of the set p, with regard to which these concepts are defined, corresponds to that of the universal class (in the systems of von Neumann, Bernays, etc.) which is not available in **Z**. Thiele proves that the "magnitude" of a cardinal or ordinal is independent of the particular set p used for its definition. — Nevertheless, the axiom schema of substitution still proves necessary for the purpose for which it was originally introduced, namely for guaranteeing the existence of comprehensive enough sets, at least up to, and with the exclusion of, the first inaccessible cardinal succeeding $\aleph_0$.

Minor differences or additions in Thiele's paper are: a proof of the comparability of cardinals, prior to the theory of order, by means of Zorn's principle and along the lines of *T*, pp. 319–321; the introduction of order according to Hausdorff instead of Kuratowski (below p. 131); the procedure of defining by transfinite induction within **Z**, given similarly to von Neumann 28a (by means of the axiom of substitution; cf. also *T*, pp. 251–253); elaborate proofs of classical theorems of the theory of ordinals.

[1]) This is a *special* condition, in accordance with the biuniqueness of the correspondence — in contrast with the definition on p. 111 where we dealt with *ordered* pairs (dyads). According to the definition of equality between sets we have in general $\{s_1, t_1\} \neq \{s_2, t_1\}$ if $s_1 \neq s_2$.

c) every member of the union $S \cup T$ appears in at least one pair $\{s, t\}$ — hence, in view of b), in just one pair.

Now we reverse this train of ideas. Considering that *all* pairs $\{s, t\}$ — not only the members of a set as characterized by a) to c) — belong to the Cartesian product $S \times T$ (theorem 4 on p. 43/4), we start with the following

Definition of Equivalence. S and T being disjoint sets, S is called *equivalent* to T ($S \sim T$) if the Cartesian product $S \times T$ has at least one subset ϕ such that every member of $S \cup T$ belongs to just one member of ϕ. Every such set ϕ is called a (one-to-one) *mapping* of S onto T.

The null-set is called equivalent to itself. Clearly equivalence is a symmetrical relation, hence we may speak of mappings *between* S and T.

The Cartesian product $P = S \times T$ of the disjoint sets S and T exists (p. 43), hence also its power-set $\mathfrak{C}P = U$ (Axiom IV, p. 37). If and only if $S \sim T$, U contains at least one member ϕ (a subset of P) for which the condition stated in the definition is satisfied [1]). Hence if U_0 denotes the subset of U defined by this condition (Axiom V) the inequality $U_0 \neq O$ [2]) means $S \sim T$ and every member of U_0 is a mapping between S and T.

In view of this definition which enables us to ascertain whether two disjoint sets are equivalent or not, it is easy to develop the traditional theory of equivalence by means of Axioms I–VI [3]). In particular we obtain the concept 'image in T of a given $s \in S$ in view of a given mapping between the equivalent disjoint sets S and T'. To get rid of the restriction to *disjoint* sets, a restriction which involves intolerable complications, we may prove the following general theorem. S being a set which is not neccessarily disjointed, one can by means of our axioms (without the axiom of choice) form a *disjointed* set S' which is equivalent to S and whose members are equivalent to the members of S in the following sense: if Ψ denotes a certain mapping of S onto S' and s' denotes the image in S' of $s \in S$ in view of Ψ, we have $s \sim s'$. (In particular, one can arrange for the sum-sets of S and S' to be disjoint.) Hereby [4]) the definition of equivalence can be generalized

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[1]) The condition expresses that the subset of $S \cup T$, each member of which belongs to just one member of ϕ, coincides with $S \cup T$ itself.

[2]) U_0 exists as well if S and T are *not* equivalent; then U_0 is the null-set.

[3]) For the details see Zermelo 08a, § 2; Fraenkel 25, § 3.

[4]) For the present purpose a weaker theorem is sufficient, but the above general theorem will be required presently.

so as to include sets S and T which are not disjoint, namely by the condition that either set be equivalent to a third which is disjoint to both. This equivalence is clearly a transitive relation.

Beside the usual theorems of equivalence theory, including the theorems of Cantor (T, p. 94) and of König-Jourdain-Zermelo (T, p. 132) — the latter requiring the axiom of choice — we thus obtain the generalization of Axiom VI (Multiplicative Axiom) that constitutes Zermelo's general principle of choice (p. 48). For starting from a set S of non-empty sets, in the way just outlined we obtain an equivalent *disjointed* set S' of sets which respectively are equivalent to the members of S. Axiom VI then applies to S'; hence, passing on from any selection-set of S' to corresponding members of the respective members of S by means of simultaneous mappings between the members of S and those of S', we obtain a single-valued function which assigns to each member x of S a single member of x.

For the purpose of arranging sets "according to magnitude" the definition of T, p. 90, can be used without the concept of cardinal, and the equivalence theorem (T, pp. 98 f) may be proved in one of the usual ways, even without relying on the sequence of integers. Yet the comparability theorem cannot be proved at this stage, save for the method of T, pp. 319–321, which would cause considerable difficulty. The lack of cardinals for themselves certainly somewhat complicates these developments but does not actually affect them. (The method of introducing cardinals by the detour via order theory and ordinal numbers, in accordance with von Neumann, can be used as hinted on p. 132 [1]).

Proceeding now to the theory of *ordered sets* we start from the traditional concept of order to reach a method of reducing this concept to the primitive relation ϵ of **Z**. The following method [2] essentially

[1]) There exist also independent axiomatizations of the cardinal concept, notably Baer 29 (cf. *Fraenkel 4), and Ackermann 37a, § 7. Baer's paper uses the generalized continuum hypothesis and the assumption that there is no regular initial number with a limit-index above ω (p. 87). Among other interesting results, the continuum hypothesis proves independent of the formal laws of cardinal arithmetic. This, to be sure, does not amount to proving its independence of the axioms of set theory. — Cf. also Tarski 49.

[2]) It was initiated by Hessenberg 06 (ch. 28; also *10, p. 74; cf., however, Fraenkel 26 [*Fund. Math.*]) and completed by Kuratowski 21. Cf. *Combébiac 1, Mirimanoff 17/20, Sierpiński 21a, Quine 34 (ch. 1).

rests on the fact that the relation $x \subset y$ (x is a proper subset of y) which derives from ϵ, has the characteristic properties of (partial) order, viz. asymmetry and transitivity.

A being an ordered set, a subset of A (conceived in the order of A) is called *an initial of* A if the subset, together with any of its members, also contains all the preceding members of A (T, p. 182). Hence A itself as well as the null-set O are initials of A. The set I of *all initials* of A obviously has the following four properties:

A) Of two different members of I one is a proper subset of the other.

B) a and a' being different members of A, there is at least one member of I that contains *just one* of them. (If $a \prec a'$ in A then, for instance, the initial of A which contains a and its predecessors, and the initial which contains all predecessors of a' but not a', have the property in question.)

C) I_0 being any subset of I, $\cup I_0$ and $\cap I_0$ are also initials, hence members of I.

D) $A \epsilon I$, $O \epsilon I$.

Conversely, these properties together — but not part of them — prove *characteristic* of the set of all initials of A, in the sense that a set of subsets of the ordered set A having the properties A)–D) coincides either with I or else with the set of *all remainders* of A (T, p. 182) [1].

Property A) expresses that the initials can be arranged in a continually increasing (or decreasing) series. A set of sets x — in particular a set of subsets x of given set — such that for any two different x, x' we have either $x \subset x'$ or $x' \subset x$, shall be called a *chain.* (Some authors use the term 'monotone set'.)

It is remarkable that, as Kuratowski has proven, the conjunction of the properties B)–D) can be replaced by the following completeness (or "saturation") property: I is a *maximal* chain of subsets of the ordered set A, i.e., a chain which is not a proper subset of another chain; every maximal chain, then, has also the properties B), C), D). The proof will be sketched below.

Inverting this train of ideas, we now start with any (unordered) set S, and define as follows.

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[1] In fact initial and remainder are correlative in this sense and we may choose either. The papers quoted above mostly prefer remainders. Yet for practical reasons initials are preferable in view of the application to well-ordered sets, whose definition favors initials.

Definition of Order. A set S is called *orderable* if *a maximal chain M of subsets of S exists*; that is to say, if the power-set $\mathbf{C}S$ has at least one subset M with the following two properties:

a) if $m \in M$, $m' \in M$, $m \neq m'$, either $m \subset m'$ or $m' \subset m$;

b) if $\bar{M}$ is any subset of $\mathbf{C}S$ with the property a) and if $M \subseteq \bar{M}$, then $M = \bar{M}$.

Every such set M is called a set which orders S, or in short *an order of S*.

The questions now arise whether to every S there corresponds at least one order M and how a given order M of S defines an order in S in the sense of classical set theory (T, § 8).

The first task seems simple. Let x be a set of subsets of S, i.e. a subset of $\mathbf{C}S$ or a member of $\mathbf{CC}S = K$. We then take the condition on x 'x is a maximal chain' and abbreviate it by '$\mathfrak{P}(x)$'. By the axioms of power-set and of subsets, $K_{\mathfrak{P}}$ exists; every member of $K_{\mathfrak{P}}$ is an order of S in the sense defined. So far the axiom of choice has not been used, but it is required to show that $K_{\mathfrak{P}} \neq O$; for instance, by means of the well-ordering theorem, whose second proof by Zermelo can be imbedded in our axiomatic system without difficulty, or of Szpilrajn's theorem cited in T, p. 212, which also uses the axiom of choice [1].

As to the second task, the crucial point is showing that the properties a) and b) of the Definition of Order imply the property B) of p. 128 [2].

Let s and s' be two different members of S. We shall prove that there are members of M (i.e. of any subset of $\mathbf{C}S$ with the properties a) and b)) which *contain just either s or s'*, and we shall even construct a particular member of M with this property. For this purpose we may assume, without impairing the generality, that every member of M which contains s' contains s too.

Let D be the intersection of those $x \in M$ for which $s' \in x$; hence $s' \in D$ and by our assumption also $s \in D$. *D is a member of M.* To show this we prove that D has the property a) with respect to *every* $m \in M$, i.e. that $D \subset m$ or $m \subseteq D$; then $D \in M$ follows from the maximum property b). We distinguish between the case where, m being an arbitrarily fixed member of M, m is a subset of every $x \in M$ with the property $s' \in x$ and the case where there is at least one $x_0 \in M$ with

[1] *Vieler 1; Szpilrajn 30. — If S is finite, the axiom is not needed.

[2] That a) and b) imply D) is evident. As to C), a way of deducing it from a) and b) is easily gathered from the following proof; for this very purpose the proof is extended somewhat beyond its proper goal.

$s' \in x_0$ such that m is not a subset of x_0. In the first case m is also a subset of the intersection D; in the second case a) implies $x_0 \subset m$ and *a fortiori* $D \subset m$. Hence $D \in M$ is true.

Now put $D - \{s'\} = D^*$; then $s \in D^*$, $s' \notin D^*$. It remains to prove that $D^* \in M$. This follows, in view of the maximum property b), from the fact that for every $m \in M$, $m \neq D^*$, either $D^* \subset m$ or $m \subset D^*$. To show this we distinguish between the cases $s' \in m$ and $s' \notin m$. In the first, according to the definition of D we have $D \subseteq m$, hence $D^* \subset m$. In the second case, in view of a) m is a proper subset of every $x \in M$ with $s' \in x$, hence of D and even of $D^* = D - \{s'\}$, because of $s' \notin m$. Herewith the proof is completed.

The Definition of Order thus involves that for a given pair of different members s, s' of S every order M of S has at least one member x which contains a single member of the pair. *If $s \in x$ and $s' \notin x$ we write* $s \prec s'$ ("s precedes s' in S on account of M"). Obviously $s' \prec s$ is incompatible with $s \prec s'$, for in view of the property a) there is no $y \in M$ with $s' \in y$ and $s \notin y$. The transitivity of $\prec$ follows immediately. M, then, introduces into S an order relation in the ordinary sense.

The definition $s \prec s'$ is of course arbitrary, viz. adapted to the conception of an order of S as a set of *initials* of S. Exchanging initials for remainders leads to the definition $s' \prec s$.

Clearly to a set S with more than one member [1]) belong *different* orders M; to an infinite set, infinitely many orders (if any).

According to the Definition of Order an *ordered set* may be considered to be the pair of S and of an order M of S; yet M is sufficient since $S = \cup M$. We shall use the expression "x is a member of an ordered set" in the ordinary sense.

One easily obtains examples for this conception of order by using initials. For instance, a closed one-dimensional interra C, conceived as an ordered set of points in one of the two ways of natural order, is ordered according to our definition in the following way. Denote by e one of the ends, by x any point of C, by X the (unordered) set of points between e and x, including e and excluding x (for $x = e$, X is the null-set), and by $\bar{X}$ the union $X \cup \{x\}$. Then the set which contains all sets X and $\bar{X}$ is an order of C. (The set D^* which contains s but not s', corresponds to X with $x = s'$.)

[1]) If S is empty or contains one member only, the respective orders are uniquely determined, viz. $M = O$ or $M = \{O, S\}$. For $S = \{s_1, s_2\}$ we have the different orders $M_1 = \{O, \{s_1\}, S\}$ and $M_2 = \{O, \{s_2\}, S\}$.

In the present case C might be "ordered" in the usual sense by the set of all X only. Yet this set, while having the properties A) and B) of p. 128, satisfies neither C) (or D)) nor the maximum condition b) of p. 129, hence it is not an order in our sense. The second fact is evident; the first follows from that the intersection of infinitely many X may be an $\overline{X}$.

Another method of reducing 'ordered set' to 'ϵ' originates from Hausdorff [1]; it is based upon conceiving an ordered set as the *set of the ordered pairs* $\langle a, b\rangle$ which correspond to the statements $a \prec b$ holding in the set. Inverting this conception one may consider a set Σ of ordered pairs $\langle a, b\rangle$ with $a \epsilon S$, $b \epsilon S$ to be *an ordered set which contains just the members of* S if Σ fulfils certain conditions, notably those corresponding to the asymmetry and the transitivity of the relation $\prec$; these conditions are

a) for $a \epsilon S$, $b \epsilon S$, $a \neq b$, $\langle a, b\rangle \epsilon \Sigma$ or $\langle b, a\rangle \epsilon \Sigma$ holds true, but not both;

b) $a \epsilon S$ implies $\langle a, a\rangle \notin \Sigma$;

c) $\langle a, b\rangle \epsilon \Sigma$ and $\langle b, c\rangle \epsilon \Sigma$ imply $\langle a, c\rangle \epsilon \Sigma$.

In view of b) and c) one may in a) drop the condition 'but not both'.

This method requires a reduction of the concept of *ordered pair* (dyad) to that of set. The simplest method of doing this is that of Kuratowski, which conceives the ordered pair $\langle a, b\rangle$ with $a \neq b$ as the pair (p. 35)

$$\{\{a, b\}, \{a\}\}\ ^{2}).$$

(The pair $\{\{a, b\}, a\}$ would not do since possibly a is a pair one of whose members is $\{a, b\}$, as long as the axiom of foundation is not presupposed.)

An analogous procedure can be applied to any dyadic relation defined within a set S, thus *reducing the general concept of relation to that of set.* If a dyadic relation ρ has S (or a subset of S) as domains of both its arguments, then ρ may be conceived as the set of all those ordered pairs $\langle a, b\rangle$ with $a \epsilon S$, $b \epsilon S$ for which $a \rho b$ holds true. Conversely, every set of ordered pairs whose members belong to S furnishes a dyadic relation defined in S. Analogous remarks apply to relations with any finite number of arguments.

On the basis of the Definition of Order the theory of ordered and well-ordered sets can be developed similarly to the classical theories [3]). Analogically to the mappings defined on p. 216, in the present case

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[1]) Hausdorff 14, pp. 70 f; 27, p. 42.

[2]) Kuratowski 21, p. 171; cf. already *N. Wiener 1. — From the viewpoint of type theory, the increase of type involved by this definition of an ordered pair (and the analogous one for a finite ordered set) is undesirable. It can be avoided by more complicated definitions; cf. *Quine **29**, Goodman 41, Schwabhäuser 54. See, furthermore, Skolem 57a.

The only difference between this definition of ordered pair and the definition according to the general concept of order as described before is that the null-set O is dropped from the members of the order $\{O, \{a\}, \{a, b\}\}$.

[3]) See Fraenkel 26 and (for the theory of well-order) 32.

we obtain, for any two disjoint ordered sets, the set of all *similar mappings* between them (without using the axiom of choice). Whenever this set is not empty the ordered sets are called *similar.*

To establish the theory of *well-ordered* sets we need not start with the general order-concept (specializing it) but may use a short cut by modifying the order definition in the light of the following remark. (Cf. *T*, pp. 242/3.) If M_0 is a subset of an order M of S we have $\cap M_0 \in M$ by property C) on p. 128. Let us now demand that $\cap M_0$ be a member (not only of M but even) of the subset M_0, i.e. that every $M_0 \subseteq M$ contain a "minimal" member, i.e. a minimal initial of the ordered set. This easily proves *characteristic* of well-ordered sets. — As above in the theory of equivalence we thus renounce the use of ordinals (above cardinals) and instead content ourselves with *sets* [1].

On the other hand, in the framework of our axioms the ordinal numbers can be defined [2] as *sets* with certain properties, provided that the axiom(-schema) of substitution (VIII on p. 85) is added to the axioms I–VII [3]. This was done by von Neumann (p. 101). Then also cardinals are definable, viz. as initial numbers.

Now we can axiomatically formulate a problem raised on p. 52. The well-ordering theorem maintains that, given any set s, there exists a special order w of s which *well-orders* s. This includes the more general *ordering theorem* (OT) maintaining that there exists an order of any given set s; in other words, that the set of all orders of s is not empty. OT not only formally maintains less than the existence of a well-ordering of s — since OT does not include the property $\cap M_0 \in M_0$ (see above) — but also intuitively appears weaker; the linear continuum, for instance, can trivially be ordered while the possibility of its well-ordering depends on the axiom of choice.

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[1] Kuratowski 22 develops a general method of avoiding the use of ordinals, which is closely connected with Zermelo's second proof of the well-ordering theorem.

[2] An axiomatic foundation of ordinal numbers independently of set theory in general is given in *Tarski 5; cf. Lindenbaum-Tarski 26, § 4.

[3] Without the axiom of substitution (or VII*, or a stronger form of VII, cf. p. 82) not even the existence of the set of all *finite* ordinals in the sense of von Neumann-Bernays can be proved. — An essentially weaker theory of "relative" ordinals and cardinals is developed in Thiele 55 (see above p. 125) without the axiom of substitution. As is proven in the same paper, the adjunction of the axiom of substitution guarantees, for every ordinal α, the existence of a set with the cardinality $\aleph_\alpha$.

Therefore the question arises whether OT *is actually weaker than the well-ordering theorem* and can be proved by some specialization of the axiom of choice. We may also ask what part of general set theory — perhaps the whole of it? — can be deduced from the axiomatic basis of **Z** if the axiom of choice (VI) is replaced by OT. Or else, can OT be proved by means of the axioms I–V only? (Cf. problems c) and d) on p. 52.)

The latter question is easily answered in the negative [1]). Let T be an infinite disjointed set the members of which are non-empty *finite* sets; for instance, pairs (as in the weakest form of the axiom of choice, p. 51). By OT there is at least one order of the sum-set $\mathsf{U}T$; let R be an arbitrary such order. Since every $t \in T$ is a finite subset of $\mathsf{U}T$, t has a first member in view of R. Hence the subset of $\mathsf{U}T$ defined by the condition 'x is the first member of some $t \in T$ in view of R' contains a single member from each $t \in T$, i.e. it is a selection-set of T.

Therefore, if the ordering theorem was provable by means of Axioms I–V, the same would apply to the weakest form of the axiom of choice which, however, is independent of Axioms I–V and VII (see p. 53) — at least for axiom systems which permit the existence of infinitely many non-sets or of certain extraordinary sets. Hence the ordering theorem is independent in the same sense.

Whereas the ordering theorem implies the weakest form of the axiom of choice the converse question is still open [2]). On the other hand, Mostowski succeeded in showing [3]) that the ordering theorem *does not imply* the well-ordering theorem; in other words, by replacing the axiom of choice with the ordering theorem we do not obtain a sufficient basis for general set theory. Hence *the general form of the axiom of choice is stronger than its weakest form.*

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[1]) See Fraenkel 28a.

[2]) This applies to the formulation of the weakest form which regards *any* (disjointed) set of non-empty finite sets. If upon the cardinalities of the latter certain restrictions are imposed, a negative answer to the converse question follows from Mostowski 45.

[3]) Mostowski 39. This essay is based on the axiom system of Bernays while using the "model" method of Fraenkel 22; cf. p. 53. A related weaker form of the axiom of choice is given in Kinna-Wagner 55 by means of the following property of a set S: there exists a single-valued mapping φ of $\boldsymbol{C}S$ into itself such that, for every $S_0 \subseteq S$ with at least two members, $\varphi(S_0)$ is a non-empty proper subset of S_0. By replacing here 'subset' with 'initial' one obtains a "cleavable ordering" of S.

The theory of (simply) ordered sets has in the above been incorporated in the theory of plain sets which rests upon the single primitive relation ϵ. On the other hand, there exists an independent axiomatization of the theory of *multiply ordered sets* (*T*, pp. 178 f) [1]. The multiple order in question cannot be constructed by iterating the relation of simple order but rests on two different primitive relations, each with $n + 1$ variables, when the multiplicity of order intended is n. The number of axioms is independent of n save for a differentiation between $n = 1$ and $n > 1$, and mathematical induction is not presupposed except for the very notion of a relation with n variables.

Whether this order concept can be reduced to the concept of set, has not been clarified so far.

We conclude with a few remarks on special sets, namely finite and denumerable sets.

For the theory of *finite sets* one might *a priori* assume it unnecessary to use axioms of infinity or even the axiom of choice, which turned out to be provable in the case of a finite set (p. 49). In the first direction we refer to the remark on p. 118 (Zermelo), and in a more general sense to Chapter III where Russell's Axiom of Infinity is contemplated in the light of the theory of types.

The rôle played by the axiom of choice in the theory of finite sets was explained on pp. 62–64. Among the "elementary" definitions of finiteness, i.e. those from which the entire theory can be deduced without using the axiom of choice, the definition of a finite set as a doubly-well-ordered set should be mentioned (*T*, p. 270). In view of our axiomatic treatment of well-ordering this definition is easily based upon our axioms [2]) and it is proven without the axiom of choice that every set which is finite in the elementary sense can be well-ordered.

Regarding *denumerable sets*, it was shown on p. 83 that the axiom of infinity implies the existence of a denumerable set, e.g. the set of all

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[1]) Hudekoff 30. This theory may, under appropriate assumptions, be interpreted in the space of n dimensions. Cf. G. Schwarz 54.

[2]) Cf. Tarski 25, see above p. 63. An independent axiomatization of the domain of the subsets of a finite set, which admits a further generalization, is given in Menger 28 (pp. 309 ff); cf. *G. Bergmann 1.

*Whitehead 5 conceives the notion of finite number in a way which fundamentally differs from its conception in *Principia Mathematica*.

finite numbers. The theory of denumerable sets in general is easily based upon the axioms of **Z** [1]). The main theorem, stating that every infinite set has a denumerable subset (T, pp. 57 ff) can be proved in **Z** as follows, provided 'infinite' is conceived in the sense of 'reflexive'. (If 'infinite' is taken as 'non-inductive', the axiom of choice is required; see above p. 63.)

On account of p. 126 we may start with a reflexive set R which has no common member with the denumerable set $Z_0 = \{O, \{O\}, \{\{O\}\}, \ldots\}$ (see p. 83). Let R be equivalent to its proper subset $R' = R - Q$ with $Q \neq O$ and let q be an arbitrary member of Q. In view of the axioms of power-set and of subsets there exists the set P of all those subsets ρ of the Cartesian product $R \times Z_0$ which contain the pair $\{q, O\}$. A one-to-one mapping between the equivalent sets R and R' such that to every $r \in R$ a certain $r' \in R'$ is related, enables us to define, corresponding to every $\rho \in \mathsf{P}$ (hence $\rho \subseteq R \times Z_0$), the subset of the members $\{r, z\} \in \rho$ for which also $\{r', \{z\}\} \in \rho$. Hence there is a certain subset P^* of P which contains those $\bar{\rho} \in \mathsf{P}$ that satisfy the condition that, together with $\{r, z\} \in \bar{\rho}$, also $\{r', \{z\}\} \in \bar{\rho}$. The intersection I of all members of P^* has the same property, in addition to $\{q, O\} \in I$.

Clearly I is a disjointed set and the intersection of $\mathsf{U}I$ and R is equivalent to Z_0. Hence R has a subset equivalent to Z_0, which completes the proof [2]).

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[1]) Zermelo 08a; Fraenkel 25, pp. 272 f; *Vieler I, pp. 31 ff. For *order* in denumerable sets cf. Fraenkel 26, pp. 154 ff.

[2]) Only after the final proofs of Chapter II had been printed, did Specker's important Habilitationsschrift „*Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom)*" become accessible to us; this paper, written in 1951, appeared in the *Zeitschr. f. math. Logik u. Grundlagen d. Math.* (*3*, 173–210) by the end of 1957.

From the viewpoint of the present exposition, its most interesting achievement is an ingenious proof of the *independence of the axiom of foundation*, different from the proof in Bernays $37–54_{VII}$. The independence of the axiom of choice is proved (cf. p. 51) by the use of extraordinary sets, i.e. by renouncing (or essentially weakening) the axiom of foundation. Finally, in addition to the direction of Church 27, other alternatives to the axiom of choice and their surprising implications are developed, and the connection with the existence of inaccessible cardinals is pointed out.

CHAPTER III

TYPE-THEORETICAL APPROACHES

§ 1. The Ideal Calculus

We do not believe that there exists at this moment a single classification of the various approaches to the foundations of set theory which is decisively simpler and more "natural" than its competitors. We therefore make no such claim for the classification adopted here. The various approaches presented together in one chapter will, of course, have many common features, but the degree of communality will differ from chapter to chapter and the reader will notice that, occasionally, an approach dealt with in one chapter will have less in common with another approach treated in the same chapter than with one of the approaches mentioned in another chapter. As we see it, there exists a multi-dimensional continuum of possible attitudes out of which those that were historically chosen form a sometimes rather arbitrary-looking selection; of these systems those described in this book form an additional, perhaps not quite so arbitrary, selection.

In order to get a good preview of the features that distinguish the approaches to be treated in this chapter from those described in the preceding chapter, let us start with the presentation of a certain calculus that might be regarded as an adequate formalization of one "naive" approach: According to this approach, all the entities, if not within the whole universe at least within the "universe of discourse", are essentially alike in their status, and it therefore makes always sense to claim that one of these entities is a member of any other entity, even of itself, though such a claim may be wrong, sometimes perhaps even absurdly wrong. All the variables of the calculus in which this view is to be formalized would then be of the same kind, and one formation rule of this calculus would state, for instance, that any expression of the form '$\ldots \in ---$', where the dots and dashes are replaced by occurrences of variables (different or identical), is a well-formed formula — a wff, for short — and that for certain versions of

such a calculus, such expressions are the only atomic wffs, from which the other wffs are formed with the help of the customary connectives, quantifiers, and parentheses.

In addition, according to the same approach, all those entities which fulfil a given condition always constitute another entity, a class, and only one class, since any two classes which have all their members in common are identical. This means that the rules of transformation of the corresponding calculus will contain, first, an *axiom(-schema) of comprehension* that might be symbolized as

$$(\ ) \qquad (\mathrm{E}y)\ (\mathrm{A}x)\ [x \in y \equiv F(x)],$$

where '$F(x)$' represents any wff free in 'x' and the initial pair of parentheses is meant to indicate the *universal closure* [1]) of the following formula, i.e. to stand for the string of the universal quantifiers binding all the remaining free variables of '$F(x)$', if any; and, second, an *axiom of extensionality* that might be symbolized as

$$(\mathrm{A}x)\ (\mathrm{A}y)\ [(\mathrm{A}z)\ (z \in x \equiv z \in y) \supset x = y],$$

where '$x = y$' is defined as short for '$(\mathrm{A}w)\ (x \in w \equiv y \in w)$'. (For a discussion of the problems raised by this definition, see Chapter II, pp. 29 ff).

The calculus, whose outline has been given here, will be called — following Hermes-Scholz [2]) — the *ideal calculus* and denoted by '***K***'. This calculus may be extended by adding any or all of the axioms of infinity, choice, and substitution (see Chapter II).

K and its extensions, though indeed constituting an ideal formalization of the naive approach in many aspects, have unfortunately one drawback: *they are inconsistent.* It is very easy to derive in them a formalized counterpart of Russell's antinomy. Let us do this in some detail.

Taking '$x \notin x$' for '$F(x)$' in the axiom of comprehension, we get

$$(1) \qquad (\mathrm{E}y)\ (\mathrm{A}x)\ (x \in y \equiv x \notin x).$$

According to a standard theorem of the first-order functional calculus (which is again taken as the logic underlying the ideal calculus)

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[1]) For the term, see Church 56, p. 228; for the symbol, see Carnap 37, p. 94.
[2]) Hermes-Scholz 52, pp. 57 ff.

we have

(2) $$(Ax)\ (x \in y \equiv x \notin x) \supset (y \in y \equiv y \notin y).$$

According to a standard rule of the functional calculus,

(3) $$(Ey)\ (Ax)\ (x \in y \equiv x \notin x) \supset (Ey)\ (y \in y \equiv y \notin y)$$

is derivable from (2). (1) and (3) yield together, by *modus ponens*,

(4) $$(Ey)\ (y \in y \equiv y \notin y),$$

from which follows, according to another standard theorem,

(5) $$(Ey) \sim (y \in y \equiv y \in y).$$

On the other hand,

(6) $$y \in y \equiv y \in y$$

is a theorem of the functional calculus, from which, according to another standard rule of this calculus,

(7) $$\sim(Ey) \sim (y \in y \equiv y \in y)$$

can be deduced. (5) and (7), however, are obviously a pair of contradictory statements. A calculus, in which both are provable, is inconsistent.

We may look upon the approaches discussed in the preceding chapter and to be discussed in the present chapter as so many different attempts to modify the ideal calculus in order to overcome its shortcomings. The "axiomatic" attitudes of the preceding chapter do not essentially change the formation rules: neither the two kinds of variables in Gödel's system, small letters for sets and capital letters for classes, nor the two kinds of constants denoting the membership relation in Bernays' system, 'ϵ' for membership in a set, 'η' for membership in a class, are indispensable, and it is easy to give these systems otherwise equivalent formulations in which none of these deviations from the ideal calculus is retained [1]). The decisive changes are made in the axiom of comprehension: in Zermelo's system it is replaced, on the one hand, by the much weaker *axiom(-schema) of subsets* ("Aussonderungsaxiom"), whose symbolization, adapted to our present purposes, is

( ) $$(Az)\ (Ey)\ (Ax)\ [x \in y \equiv (x \in z\ \&\ F(x))],$$

and, on the other hand, by a certain number of specific cases of the original unrestricted axiom of comprehension, i.e. by a certain number

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[1]) Cf. Chapter II, p. 120, footnote 1.

of axioms in which the original '$F(x)$' is replaced by certain specific wffs; in one case, for instance, '$F(x)$' is replaced by '$x \subseteq z$', i.e. by '$(Aw)\ (w \in x \supset w \in z)$', and we get the *axiom of power-set*,

$$(Az)\ (Ey)\ (Ax)\ [x \in y \equiv (Aw)\ (w \in x \supset w \in z)].$$

A less far-reaching weakening of the axiom of comprehension consists in introducing a different conjunctive component into the right side of the equivalence, viz. the wff '$(Ez)\ (x \in z)$' — instead of Zermelo's '$x \in z$' —, getting thereby

$$(\ )\qquad (Ey)\ (Ax)\ \{x \in y \equiv [(Ez)\ x \in z\ \&\ F(x)]\}.$$

This axiom-schema, originating with von Neumann, at least in essence, assures the existence of a class comprising all and only those entities that fulfil a given condition and are members of *some* class or other, in short, of a class comprising all and only those *elements* that fulfil the given condition. (Zermelo's *Aussonderungsaxiom* guaranteed the existence of a class answering a given condition only if its prospective members were already members of a *given* class.) It is still weak enough not to allow the reproduction of the argument leading to contradiction in the ideal calculus. The reader would do well to check this by himself. He will find that instead of arriving at a contradiction he will be able to prove only the harmless theorem that the class of those entities that are not members of themselves is a non-element.

The common attitude characteristic for the approaches treated in the preceding chapter can then be summarized by saying that its deviation from the naive approach consists in repudiating the assumption that to any given condition there always corresponds a (membership-eligible) class, viz. the class of all and only those entities that fulfil this condition.

There exists a different attitude, going back in essence to Russell, that attempts to overcome the antinomies by tampering not with the axioms and rules of transformation but rather with the rules of formation of the ideal calculus, viz., by repudiating the assumption that formulae which correspond to ordinary language clauses naively regarded to be conditions are necessarily well-formed. By stipulating, e.g., that the formula, '$x \notin x$', corresponding to the ordinary language clause occurring in Russell's antinomy, 'x is not a member of x', be not well-formed, the antinomy is overcome in the rather trivial sense that its counterpart can no more be formulated in the calculus.

However, simply stipulating that no formula of the form '... ϵ ---', where the dots and dashes are replaced by (occurrences of) the same variable, be well-formed, would not do. It can easily be shown that a Russell-type antinomy can be generated from the formula '$x \epsilon y$ & $y \epsilon x$' that would not be disqualified by the mentioned stipulation. In order to do away with all antinomies of this type, a more systematic and more radical deviation from the rules of formation of the ideal calculus seems to be required.

§ 2. The general theory of classes

Instead of having only one kind of variables, as ***K*** does, the calculus we are now going to outline — we shall call it the *general theory of classes* [1]) and denote by '***T***', in honor of its relationship to Russell's type theory (see § 8) — contains a denumerable hierarchy of *levels* of variables. Each variable belongs to one level and one only. The *level-numbers*, 1, 2, 3, ..., will be indicated by superscripts. The only atomic formulae of ***T*** are again formulae of the form '... ϵ ---', but now the level-number of the left-hand variable must be exactly lower by one than the level-number of the right-hand variable, in other words,

$$x^i \epsilon y^j$$

is a wff if and only if $j = i + 1$.

Since ***T*** contains infinitely many levels of variables, the single axiom of extensionality of ***K*** has to be replaced by an axiom-schema of extensionality

$$(\mathrm{A}x^i)\ (\mathrm{A}y^i)\ [(\mathrm{A}z^{i-1})\ (z^{i-1} \epsilon x^i \equiv z^{i-1} \epsilon y^i) \supset x^i = y^i]$$

and the axiom-schema of comprehension

$$(\ )\qquad (\mathrm{E}y^{i+1})\ (\mathrm{A}x^i)\ [x^i \epsilon y^{i+1} \equiv F(x^i)]$$

has become even more schematic in its new version.

We have already seen that Russell's antinomy is not reproducible in ***T***. Let us check now that Cantor's antinomy cannot be reproduced

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[1]) This is the name used by Tarski 56 (p. 241) to denote a calculus that differs from the one outlined here only in containing an axiom of infinity, in addition to certain inessential points of notation. Tarski's general theory of classes lies between ***T*** and the system ***T**** introduced below.

in it either. Indeed, there is no (well-formed) term in ***T*** corresponding to the locution 'the class of all entities' [1]) but only to 'the class of all entities of level i', hence no formula that can be interpreted as asserting the existence of an absolute all-inclusive universal class. But it is only the assumption of the existence of such a class which, in view of Cantor's theorem, leads to paradox.

It can finally be shown that none of the customary arguments leading up to the other known logical antinomies are reproducible within ***T***. The situation is different with regard to the semantic antinomies. We shall, however, waive the treatment of these antinomies, for the time being, and take it up only later on.

How good is ***T***? This question is meant in two senses: (a) How much of naive set theory, and how much of classical mathematics in general, can be developed on its basis; (b) how sure can we be of its consistency?

With regard to question (a), the following can be said. Let us supplement ***T*** by an axiom of infinity to the effect that the number of different objects of level 1 is $\aleph_0$ — where an object is regarded as belonging to a certain level if it belongs to the range of a variable of that level — and by axioms of choice for the various levels and denote the resulting system by '***T****'. ***T**** is essentially equipollent to (a simplified version of) the system presented in *PM* (*Principia Mathematica*) in which counterparts of all basic theorems of classical set theory, arithmetic and analysis have been rigorously proved and which is, therefore, almost generally considered to have provided a sufficient foundation for these disciplines [2]). However, the avoidance

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[1]) The reason for this is not that ***T*** contains no notation for the definite article; such a notation could have easily be supplemented, and in many different ways. One could, e.g., introduce a notation for definite description through a contextual definition of the so-called iota-operator along lines well-known from *PM*, and then define the circumflex operator of abstraction. See, e.g., Quine 53, pp. 85 ff, and Schröter 56.

The reason is rather that the symbol '$\hat{x}$' for 'the class of all entities x such that' just does not occur in ***T***, since all variables have to carry superscripts, so that we have only '$\hat{x}^i$'.

[2]) This essential equipollence is rather surprising in view of the fact that ***T*** is notationally so much inferior to ***PM***, as it contains neither propositional nor relational variables. However, propositional variables can be replaced by schemata, a procedure introduced by von Neumann. Two-termed relations

of the logical antinomies, which inspired the transition from ***K*** to ***T***, has been achieved only at the price of certain drawbacks, some of which will be mentioned here.

Cardinal numbers can no longer be simply and uniquely defined as classes of equivalent classes (see *T*, p. 77), in the tradition created by Frege, but, according to the level of the equivalent classes, we get an infinite series of numbers on all levels (from two upwards). Each level has its own (quasi-)universal class, its own null-class, and the complement of a class becomes a quasi-complement, containing not all non-members of that class but only those non-members that belong to the same level as the members of that class.

Many mathematicians will find this reduplication repugnant, not only for intuitive reasons but also, perhaps mainly, because it restricts severely their accustomed freedom of expression and notation and often requires complicated technical operations in order to overcome the unwanted notational restrictions.

Some of these technical drawbacks could be overcome by utilizing a principle of *typical ambiguity*, in adaptation of a procedure of *PM*. Instead of proving each theorem with the variables (and constants introduced by definition) carrying with them general superscripts and provided with occasionally long-winded clauses stating the relationships between these superscripts [1]), one could decide not to use those superscripts from the beginning but assume that each theorem is preceded by the clause 'provided that all terms belong to the appropriate levels'. The resulting formalism [2]) would look once

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can be replaced by classes of ordered pairs, and ordered pairs in their turn by certain classes. If the ordered pair is homogenous, i.e. of the form $\langle x^i, y^i \rangle$, it can be replaced by $\{\{x^i\}, \{x^i, y^i\}\}$, according to the well-known method of Wiener-Kuratowski (cf. p. 131). Otherwise, i.e. if of the form $\langle x^i, y^j \rangle$, with $i \neq j$, this pair has first to be "homogenized", in a self-explanatory fashion, prior to the application of the Wiener-Kuratowski transformation. Many-termed relations, finally, can be replaced by classes of ordered n-tuples, which in their turn are reducible to ordered pairs in a way exemplified by the replacement of $\langle a, b, c \rangle$ by $\langle a, \langle b, c \rangle \rangle$, after homogenizing, if required.

[1]) Another procedure to the same effect would be to prove the theorem with smallest possible superscripts attached to all occurring terms and proving once for all a meta-theorem stating that each theorem remains valid when all superscripts have been raised by the same amount.

[2]) This formalism is called PM4 in Quine 51a. Quine regards it as a certain version of ***PM***. In his review, *J.S.L.* **17**, p. 212 (1952), Beneš points out that PM4 differs from ***PM*** — or rather from ***T*** — in containing a universal class

more like that of ***K*** and one could think that it would be notationally as easily manageable.

In practice, however, it would sometimes be quite difficult to keep track of all the tacit provisions [1]). In addition, special care would have to be taken in the formulation of the rules of formation so that formulae leading to antinomies should not inadvertently be reestablished as well-formed. Though, for instance, in a calculus embodying a principle of typical ambiguity, each of the two formulae '$x \in y$' and '$y \in x$' will separately be regarded as well-formed, their conjunction '$x \in y \ \& \ y \in x$' must most definitely be denied this character, on pain of reestablishing a Russell-type antinomy.

The required rules of formation are best stated in terms of *stratification*, following Quine's usage. An atomic formula of the form '$\ldots \in ---$' is called *stratified* if its left-hand and right-hand variables can be assigned to consecutive levels (which is, of course, always the case except when they are (occurrences of) the same variable). A formula, in general, is called stratified if all the variables occurring in its expanded form, i.e. the form it assumes when expanded into primitive notation exclusively, can be assigned to levels in such a way that all atomic formulae, forming part of it, are simultaneously stratified (all occurrences of the same variable being assigned to the same level). A formula will then, finally, be regarded as well-formed if and only if it is stratified.

Though the careful specification of the rules of formation of this calculus would keep the logical antinomies out, working with it would require additional care in view, for instance, of the fact mentioned above that no longer can a conjunction of two wffs be automatically taken to be a wff itself.

In addition, adoption of typical ambiguity does not help to over-

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and in completely abolishing the segregation of entities into types, the last deviation making PM4 "a set theory rather than a type theory". Without trying to diminish the extent of the deviations, the issue seems to be a verbal one. Russell's original type theory differed from Zermelo's original set theory in countenancing an infinite hierarchy of universes of discourse as against a single universe with Zermelo, as well as in declaring certain formulae as meaningless regarded as meaningful by Zermelo (in addition, of course, to many other differences). In PM4, the first difference is indeed abolished, but the second subsists. Cf. also below, pp. 172 ff.

[1]) For a case where the masters themselves, Russell and Whitehead, failed, see Gödel 44, pp. 145–146.

come other intuitively repugnant features of ***T****, such as the reduplication of the universal and null classes, of the cardinal numbers, etc.

With regard to question (b), i.e. the consistency of ***T****, let us be satisfied for the moment with stating that ***T**** is consistent if $\bar{\boldsymbol{Z}}$ — the system of the axioms I–VII (cf. p. 123) — is; more will be said about this topic in Chapter V.

§ 3. Quine's New Foundations

The mentioned shortcomings of ***T**** induced quite a few logicians to look for ways and means by which these could be avoided without — if and as much as possible — endangering its relative security. One of the most interesting attempts in this direction was made by Quine who tried to combine Zermelo's approach that regarded the unrestricted use of the axiom of comprehension, with the resulting admittance of "too large" sets, as guilty of producing logical antinomies with Russell's approach that saw the culprit rather in the use of "meaningless" phrases. Would it not suffice to leave the formation rules of the ideal calculus alone, hence not to discard unstratified formulae as offending against a revised set of such rules, but rather to sterilize their harmful effects by restricting the '$F(x)$' of the axiom of comprehension to stratified formulae?

The resulting system was described in a paper *New Foundations for Mathematical Logic* (1937) and became subsequently known simply as *New Foundations*, abbreviated to ***NF***. Since the deviation of ***NF*** from ***K*** is minimized, it is in many respects very convenient to handle, and many of the shortcomings of ***T**** are indeed overcome. There are unique universal and null classes, each class has a complement, the numbers constructed in this calculus are unique, etc.

What about the antinomies? Since the decisive formulae in the derivation of the customary logical antinomies are unstratified, the existence of the corresponding antinomic classes cannot be proved directly on the basis of the axiom of comprehension, in its ***NF*** version, though occasionally the existence of a class corresponding to an unstratified formula can be proved by devious means.

But it turns out that ***NF***, though it is perhaps not downright inconsistent, exhibits many strange and counter-intuitive features. One can, for instance, prove in ***NF*** that not every class is equivalent to the class of its unit subclasses; this is not the case, e.g., for the

universal class. On the other hand, there is a class which is equivalent to the class of all its subclasses, viz. the universal class. Does this not prove that Cantor's antinomy can be reestablished in ***NF***? No, it does not; Cantor's theorem (*T*, p. 94) does not hold here. In its standard proof, there occurs an unstratified formula, as the reader will easily verify for himself. Serious difficulties arise in connection with the fact that mathematical induction holds only for stratified formulae, whereas the standard proofs of important arithmetical theorems often involve unstratified conditions [1]). The axiom of choice is incompatible with ***NF*** and the universal class cannot be well-ordered [2]). Finally, there exists no standard model for ***NF***, in the sense of Henkin-Rosser-Wang [3]).

For many logicians, these traits disqualify ***NF*** from being a serious participant in the contest for the calculus most adequate for the foundations of mathematics [4]). But it seems that at least some of the drawbacks can be easily overcome: by inserting, for instance, the condition 'x is Cantorian' (where a class is called Cantorian if it is equivalent to the class of its unit-subclasses) at certain crucial points in the formulation of theorems and their proofs, the counterparts of most classical theorems — including those that require the axiom of choice for their proofs — would still be forthcoming, the axiom of choice might still safely be assumed to hold for Cantorian classes, and the class of these classes could still be well-ordered [5]); the classes in which the mathematician is ordinarily interested are surely all Cantorian. The significance of the absence of a standard model will be discussed later; for the moment we shall only remark that ***NF*** is by no means the only set theory that suffers from this defect — if a defect it is [6]).

With regard to consistency, let us once more be satisfied for the moment with noticing that ***NF*** is consistent if and only if ***PM4*** (see footnote 2 on p. 142) is consistent [7]).

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[1]) Cf. Quine 53, p. 99.

[2]) This at the time rather surprising result was found by Specker 53.

[3]) See Rosser-Wang 50, pp. 115 ff.

[4]) Curry, e.g., in his review of Rosser 53 in the *Bull. A.M.S.* **60**, pp. 266–272 (1954), strongly questions the wisdom of choosing an ***NF***-type system as a basis for a textbook of logic for mathematicians.

[5]) See Rosser 56, p. 292 and Quine 56, p. 270n.

[6]) See Rosser-Wang 50.

[7]) See Quine 51. Additional material on ***NF*** is contained in the following papers: Rosser 39, Hailperin 44, Rosser 52, Beneš 54. Hailperin's formulation

§ 4. Quine's Mathematical Logic

In addition to being an interesting system in its own right, ***NF*** has also served as the starting point for the construction of other, in certain respects more attractive, systems [1]). The best-known among these is Quine's own system developed in his book *Mathematical Logic*, whose first edition appeared in 1940, while the third edition, containing one major revision, was published in 1951. This system became again known under the name of the book often abbreviated to ***ML***. Whereas ***NF*** is roughly characterizable as a hybrid between ***T*** and ***Z***, the systems of Russell and Zermelo, ***ML*** employs ideas borrowed rather from von Neumann, in addition to a core taken from ***T***. Though ***ML*** is then in many respects stronger and more convenient than ***NF*** — just as von Neumann's system, especially in the version given it by Bernays, is stronger and more convenient than ***Z*** —, it is interestingly enough as secure as ***NF***, i.e. it is consistent if ***NF*** is; this is perhaps not too surprising, in view of the fact that once more the same relationship has been shown to hold between von Neumann's system and ***Z*** (cf. p. 122). More will be said about this topic in Chapter V.

To be now more specific, ***ML*** (in the revised version of 1951) differs from ***NF*** in that the axiom of comprehension of ***NF***, C_{NF} for short, which we might formulate as

C_{NF}: If '$F(x)$' is stratified and does not contain 'y', then

$$(\ ) \qquad (Ey)\ (Ax)\ [x \in y \equiv F(x)],$$

is replaced by two rules: the one, C_{ML}, resulting from C_{NF} by dropping the clause, 'if '$F(x)$' is stratified' and adding '$(Ez)\ x \in z$' — read: 'x is membership-eligible' or 'x is an element' — as a conjunctive component to the right hand of the equivalence sign, yielding altogether

C_{ML}: If '$F(x)$' does not contain 'y', then

$$(\ ) \qquad (Ey)\ (Ax)\ \{x \in y \equiv [(Ez)\ x \in z\ \&\ F(x)]\};$$

the other asserting in effect that all those classes which exist at all for ***NF*** are elements in ***ML***. This second rule, and this rule only,

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of ***NF*** is of special interest since in it the axiom-schema of comprehension is replaced by a finite list of axioms.

[1]) One such interesting system is presented in Stanley 55.

contains then once more the characteristic stratification condition. In this form, incidentally, C_{ML} looks very much like the axiom of subsets of ***Z***, which we could indeed as well regard as the Zermelian version of the axiom of comprehension, abbreviate by 'C_Z' and formulate as

C_Z: If '$F(x)$' does not contain 'y', then

$$(\ ) \qquad (\mathrm{E}y)\,(\mathrm{A}x)\,\{x \in y \equiv [x \in z\ \&\ F(x)]\}.$$

The reader will again easily verify for himself that Russell's antinomy, for instance, is not reproducible in ***ML*** and that the counterpart of the standard argumentation leads only to the double result that, on the one hand, there just is no such thing as the class of all *entities* that are not members of themselves and that, on the other hand, the class of all *elements* that are not members of themselves is a non-element [1]).

It turned out, interestingly enough, that the original (1940) version of ***ML*** was nevertheless inconsistent, a Burali-Forti-type antinomy being derivable in it. But this was, so to speak, only a accident; certain minor revisions, suggested by Wang, in the formulation of one of the axiom-schemata were sufficient to obviate that inconsistency [2]).

Whereas in set theories of the Zermelian tradition either the existence or, at least, the elementhood of "very large" classes can be disproved, the universe of classes in ***NF*** and the universe of elements in ***ML*** are symmetrical. In ***ML***, e.g., it can be proved that every finite class is an element as well as every class that contains all elements except a finite number of them, so that non-elements differ from both the null-class and the universal class in point of infinitely many members.

ML is certainly one of the most elegant and manageable systems that have been proposed so far. The freedom of manipulation is restricted only to the degree that occasionally the elementhood of a class has to be proved. Its architectonics is superb. The two basic disciplines of elementary logic, the propositional calculus and the functional calculus, are built upon exactly one primitive notion each, in addition, of course, to an infinity of term-variables — reducible,

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[1]) Cf. Quine 51, pp. 130–131.

[2]) Cf. *Rosser 12 and Wang 50b.

in principle, to one variable and an accent — and a left and right parenthesis; the only primitive of the propositional calculus is the symbol of joint denial '$\downarrow$' [1]), the down-arrow, on the basis of which — as is well-known — all other sentence connectives can be defined, and the only primitive of the functional calculus is the symbol for universal quantification '(A)'. 'ϵ', the symbol of class-membership — we shall see immediately that this reading is not exact —, is the only additional symbol, on which the theory of classes and thereby all of higher logic, arithmetic and analysis, is based. Each of these three parts has, of course, its own set of axiom-schemata.

There is an interesting by-product of the extreme notational economy of ***ML*** (as well as of ***NF***) [2]). In ***ML***, as in Zermelo's original system, it is left open whether all objects are classes or whether there exist also individuals, i.e. memberless objects (in addition to the null-class). So long, then, as the existence of individuals is not excluded, '$x \epsilon y$' cannot be interpreted as 'x is a member of y', in case that y is not a class. Quine is therefore, in this case, free to interpret this expression as he likes and, by interpreting it as 'x is identical with y', manages not only to save the principle that *every* expression of the form '$\ldots \epsilon ---$' is significant but also to allow for the definition of '$=$' on the basis of 'ϵ' through

$$x = y =_{Df} (Az)\ (z \epsilon x \equiv z \epsilon y),$$

which was impossible in Zermelo's original system and was possible in ***Z*** only through the fact that all its objects were sets (p. 32). One result of this definition is, however, that every non-class, if there exist such, is identical with its unit-class. This is something which many mathematicians will find hard to swallow; it will look to them as a return to the times before Frege and Peano made the greatest

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[1]) Church 56 calls this truth-function 'non-disjunction' and symbolizes it by '$\bar{\vee}$'. The only other binary connective that could serve as a basis is '$|$', Sheffer's stroke, the symbol for alternative denial (or non-conjunction).

[2]) This economy could still be improved upon. Quine himself showed how to replace the threefold primitive notation by one containing two symbols only, those for inclusion and abstraction (see Quine *17 and 53, pp. 94 ff). For a different reduction of the primitive notation see Henkin 53 and Stanley 55. These notations, however, are much less perspicuous than the threefold one and tend to fuse the three compartments of logic which some logicians would like to keep as much apart as possible: the propositional calculus, the functional calculus, and the class-calculus.

efforts to drive home the distinction between the class-membership and the class-inclusion relations. The damage caused by this feature to the intuitiveness of ***ML*** should, however, not be exaggerated; the identification of certain objects with their unit-classes is restricted to non-classes, and no actual harm is done thereby nor can palpable paradoxes be derived from it.

This peculiarity of ***ML*** can also be formulated in an alternative way: ***ML*** countenances two kinds of classes, ordinary ones that are not identical with their unit-classes, and extra-ordinary ones that are identical with their unit-classes (i.e. contain themselves as their sole members). This formulation has the technical advantage of restoring the simple reading 'x is a member of y' for '$x \in y$' which, according to the first formulation, would have to be read as, say, 'x is a member of y if y is a class, or x is identical with y if y is a non-class'. [1])

ML is the culmination of one trend in the foundations of set theory that was started off by the theory of types, the trend, namely, which saw in this theory only a device that, though efficient enough to stop the antinomies, had to pay for this achievement too highly in terms of all kinds of restrictions on the customarily desired maneuverability of the mathematical symbolism, and therefore was ready, in order to restore this notational freedom to the highest possible degree, to sacrifice whatever intuitive plausibility was connected with the type hierarchy and to retain only those technical features which seemed to suffice for keeping the antinomies out. This sacrifice might well turn out to be a very serious one. ***ML***, just as ***NF***, has no standard model. With regard to consistency, however, ***ML*** is no worse off than ***NF***, in spite of its greater power: Wang [2]) has shown that ***ML*** is consistent if ***NF*** is consistent. There is even a possibility that ***ML*** might in a certain respect be better off than ***NF***; though ***NF*** might conceivably suffer from the annoying property of *ω-inconsistency* (see Chapter V, § 4), this would not necessarily endow ***ML*** with the same defect [3]).

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[1]) Cf. pp. 29–30.
[2]) Wang 50b.
[3]) Cf. Quine 53a.

§ 5. The Hierarchy of Languages and the Ramified Class Calculus

Before we turn to the presentation and discussion of the other trend noticeable in the type-theoretical approach, and as a partial preparation for this discussion, let us deal with a question which must have by long arisen in the mind of the careful reader: type hierarchies and stratification seem to work well enough for the avoidance of the logical antinomies, but what about the semantic antinomies? Indeed, it seems on first sight as if a Richard-type antinomy can be derived within a calculus embodying a type rule. For the purpose of descriptional simplicity, let us choose for illustration not the calculus ***T*** treated before, but a certain variant that, for other reasons (see Chapter V), has achieved great fame: this variant, to be called calculus ***P***, has natural numbers as individuals (rather than as certain classes of classes) and a few more axioms corresponding to the well-known Peano axioms for the arithmetic of natural numbers (p. 83, footnote 4); the reader will easily verify that the transition to this variant is not essential for the validity of the following argument but only serves to simplify it considerably. There are in ***P*** denumerably many well-formed expressions determining classes of natural numbers. Let these classes be enumerated as, say, r_1^1, r_2^1, r_3^1, ...; then the axiom of comprehension seems to ensure the existence of a class, say r_0^1, that contains all and only those natural numbers n that do not belong to the class r_n^1. r_0^1, being a class of natural numbers determined by a well-formed expression, must be identical with some r_n^1, say r_m^1. What about the relation between m and r_m^1? If m belongs to r_m^1, then from the identity of r_0^1 and r_m^1 and the definition of r_0^1 it follows that m does not belong to r_m^1; if m does not belong to r_m^1, it belongs by definition to r_0^1, hence to r_m^1. The responsibility for this contradiction cannot be charged to a violation of the type rule.

It can however be argued, and has been argued, that the condition '$n \in r_n^1$' playing a decisive role in the above argument is not really well-formed. 'r_n^1' is short for 'the n-th class of natural numbers determined by well-formed expressions of ***P*** (according to some enumeration)', hence is not (an abbreviation for) an expression of ***P*** itself but rather of the metalanguage of ***P***. Strict adherence to the distinction between object-language and metalanguage is enough to remove the ground from under the argumentation leading up to

Richard's antinomy (and the other semantic antinomies). By implementing the hierarchy of levels within one language by a *hierarchy of languages* [1]), the last threats presented by the classical antinomies seem to have been successfully repulsed.

Once more, there is a price to be paid for this success. The strict distinction between the language layers looks counter-intuitive to many thinkers, to about the same degree as the strict distinction between the levels. In addition, the question must be raised whether the layering of languages excludes the detrimental possibility that a given object-language might contain expressions which — though not identical with metalinguistic expressions — would still stand in some kind of isomorphic correspondence to them, thereby allowing for the reestablishment of the semantic antinomies, if only in a roundabout way. This is a very serious possibility indeed, and in Chapter V the subject will be taken up again.

Yet, the semantic antinomies can also be overcome by a different implementation of the type hierarchy considered so far. We recall that the "Richardian" class of natural numbers was determined through reference to a totality, i.e. the totality of all such classes determined by well-formed expressions of **P**, of which it turned out to be a member itself. There have been, and still are, quite a few thinkers who regard such a type of determination, if "essential", i.e. not replaceable by a determination not exhibiting this feature [2]), as illicit, since viciously circular. No object, they would say, should be regarded as belonging to a certain totality if the very existence of this object can be shown only by the use of an expression referring to that totality. By outlawing expressions purporting to refer to such over-inclusive totalities as the totality of all well-formed expressions, an illicit application of the axiom of comprehension can be avoided and the existence of the "Richardian" class and other antinomic classes no more proved. This aim can be achieved, e.g., by changing the rules of formation of **T**, to which calculus we return now after **P** has served its purpose, such that each variable will have to carry

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[1]) This insight, prepared by Ramsey's distinction between the logical and the "epistemological" antinomies and Hilbert's distinction between mathematics and metamathematics, is due mainly to Tarski; see Tarski 56, VIII. Cf. also Carnap 37, pp. 211 ff.

[2]) This is the type of determination that is known as "impredicative". Cf. Chapter II, p. 41 and below, § 10.

along with it not one but two indices, one, say a right superscript, indicating its level as before, the other, say a left superscript, indicating its "order". The formulation of the axiom of comprehension is now changed to

> If the highest order-index of any bound variable of level $i + 1$ occurring in '$F({}^{k}x^{i})$' is j then
>
> $$(\ ) \ (\mathrm{E}^{j+1}y^{i+1}) \ (\mathrm{A}^{k}x^{i}) \ [{}^{k}x^{i} \in {}^{j+1}y^{i+1} \equiv F({}^{k}x^{i})]$$
>
> (the order-index of 'y^{i+1}' being 1, if no such bound variable occurs in '$F(x)$' altogether).

In the resulting *ramified class-calculus*, ***RT***, the semantical antinomies are indeed eliminated. Trying to reproduce, e.g., the argumentation leading up to Richard's antinomy we are immediately impressed by the fact that no longer does there exist in ***RT*** a counterpart for 'for all classes of natural numbers of level i' but only for 'for all classes of natural numbers of level i and order k'. Forming the Richardian (class) corresponding to an enumeration of the classes of natural numbers of order k determined by the well-formed expressions of ***RT***, we notice that its order is $k + 1$, so that we have no longer any right to claim that it is identical with one of the classes of order k.

Not only can the semantical antinomies not be reproduced in ***RT***, this calculus is even demonstrably consistent [1]). However, as might be expected, this very desirable trait has its price. On the basis of ***RT*** only a fraction of classical mathematics can be reconstructed. This will be easily understood, without our having to delve into details, as soon as we notice, e.g., that the least upper bound of a class of real numbers, constructed in ***RT*** as certain classes of classes of ..., is of an order higher than these real numbers [2]).

In order to counterbalance the deplorable loss of strength incurred by the ramification, one could stipulate an *axiom of reducibility* whose effect would be to ensure, corresponding to each class of a certain level and any order, the existence of a class of the same level and order 1 containing the same members as that class. These two classes would nevertheless not be regarded as identical since otherwise the

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[1]) Such a demonstration has been given many times, both model-theoretically, as in *Fitch **4**, and proof-theoretically, as in Lorenzen 51 and Schütte 52. For these two methods of consistency proofs, see Chapter V, § 4.

[2]) This difficulty was discussed already by Weyl 18, p. 23.

effect of the introduction of the axiom of reducibility would be nothing but the exact cancellation of the ramification; but any two classes containing the same members *are* identical, according to the axiom of extensionality. This axiom would then have to be sacrificed. This, however, would now have repercussions of its own, which we shall not pursue here any further [1]).

§ 6. WANG'S SYSTEM Σ

Reinforcing ramified type theory through an axiom of reducibility never enjoyed much popularity, and is now generally regarded to have led into a blind alley. Ramified type theory as such, however, seems to be undergoing a process of rejuvenation at the hands of such able logicians as Wang and Lorenzen. It is not so much its ability of eliminating the semantic antinomies which makes it so attractive — this aim can be attained more simply through the method of language hierarchies outlined above — as its connection with a "constructivist" philosophy of mathematics. Postponing the philosophical discussion

[1]) Cf. below, p. 190. The situation concerning the axiom of reducibility is still quite confusing. The objection raised by Ramsey, Waismann and others, as if this axiom were of an empirical character rather than of a logical one, hence if true only factually so and, so to speak, by lucky coincidence — recalling a similar criticism against the axiom of infinity (see below, p. 165) — seems to have little justification. As a matter of fact the axiom of reducibility is incomparably weaker than the unrestricted axiom of comprehension in the ideal calculus; this last axiom, however, is rejected by intuitionists and other constructivist authors not because it is regarded as empirically false or doubtful but because its acceptance does not square with a constructivist attitude. For a Platonist, the axiom of reducibility is no less logical than the axiom of comprehension, and if he rejects either or both, it is done because the logical system incorporating them is either contradictory or in some other sense inefficient. In an appropriate metalanguage, the logical character ("analyticity") of the axiom of reducibility is easily demonstrable. See especially Quine 36a, Copi 50, Church 51, Copi 54 (Appendix B) and Church 56, pp. 354–356.

Altogether, the real choice, as many have observed, seems to be between the simple theory of types, appealing to the Platonist, and the ramified theory without the axiom of reducibility, appealing to the constructivist. The latter has then the choice between the "heroic", or rather, "quixotic" course taken in the twenties by Chwistek, for instance, of countenancing finite types only and being therefore obliged to sacrifice large parts of classical analysis, or working with types of transfinite order, in the line of Wang and Lorenzen to be discussed in the following sections.

let us turn to the presentation of a system that aims at preserving the desirable features of a ramified type theory but avoids its undesirable traits without recourse to such dubious means as an axiom of reducibility. The major point of this system is that it restores to a sufficiently high degree the freedom of use of such phrases as 'for all real numbers' instead of the awkward 'for all real numbers of order i'. The system we are going to present, Hao Wang's system Σ, exists so far only in outline [1] — a more complete version being promised for the near future — but this outline looks so interesting that at least a short discussion is indicated.

The hierarchy of objects in Σ differs in many respects from that of ***RT***. First, the messy two-dimensional array of ***RT*** is replaced by a one-dimensional array that manages to combine the level and order distinctions through allowing for "mixing of types". The lowest (or 0-th) *layer* — we shall use this term instead of Wang's own 'order' so as to keep it apart from the "orders" of ***RT*** — consists of some denumerable totality of objects (which may be taken to be, for instance, the positive integers or all the finite sets built up from the empty set — see p. 84). The first layer contains all the objects of the 0-th layer and, in addition, all those sets of these objects which correspond to conditions that contain no bound variables ranging over objects of the first or higher layers; in general, the $n+1$-st layer contains all the objects of the n-th layer together with all such sets of these objects as are determined by conditions whose bound variables range over objects of the n-th layer at most. We have here a combination of a conception of *cumulative* layers [2] according to which all objects of a certain layer belong also to all higher layers — a rather serious deviation from the more customary conception of *exclusive layers* — with a strict embodiment of the *vicious-circle principle* [3] according to which no entity determined by a condition that refers to a certain totality should belong to this totality. For any two objects, there exists therefore a layer (hence, infinitely many ones) to which they both belong; the lowest layers, on the other hand, to which they belong may of course be different.

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[1] Wang 54.

[2] The conception of cumulative layers (or orders) is not original with Wang. Cf., e.g., Quine 53, p. 123.

[3] For an especially penetrating discussion of this principle, see Gödel 44, pp. 133 ff.

Secondly, the hierarchy of layers is continued beyond the finite ordinals [1]). Layer ω, for instance, is the sum of all finite layers, and layer $\omega + 1$ contains, in addition, also such sets as are determined by conditions whose bound variables range over the objects of layer ω at most. Attempts to extend the type hierarchy beyond the finite ordinals were made before [2]), but in Wang's system this device is exploited to its utmost. Though the naive freedom of expression is not yet fully restored by it and expressions corresponding to 'for all real numbers' are still not reestablished, we are already entitled to use 'for all real numbers of finite layers', though only from the partial system $\Sigma_{\omega+1}$ onwards, that deals with the entities of layer ω, and not in any system with lower index. That this is indeed a far-going improvement can be realized from the fact, for instance, that the least upper bound of a set of real numbers that belong to layer n is, in general, an entity that belongs to layer $n + 1$, hence different from all these numbers, according to the vicious-circle principle and the exclusive-layer conception; in Wang's system, however, the original real numbers and the upper bound of their set do belong to a common layer, indeed to all layers from $n + 1$ upwards. This clears the ground for a proof of the least upper bound theorem pretty much along the traditional lines. And the situation is similar with respect to other basic theorems in analysis, such as the Bolzano-Weierstrass and the Heine-Borel theorems that proved to be stubborn obstacles in the way of any constructivist reconstruction of analysis.

Wang also forms the union, Σ, of all the partial systems Σ_α, where α is any "constructive" ordinal of the second number class, but this procedure raises many moot questions with regard to the exact characterization of the term 'constructive' in this context as well as with regard to the legitimacy of this procedure in general; these questions will not be discussed here. It is clear, however, and admitted by Wang, that Σ itself is no more a formal theory in the sense in which its partial systems Σ_α are.

With this proviso in mind, we shall now describe the structure of Σ in outline. Σ is based on standard functional calculus, meant to

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[1]) Whitehead and Russell explicitly reject this procedure though their reasons are none too convincing. See PM_{I}, p. 53.

[2]) See, e.g., L'Abbé 53, especially footnotes 2, 3, 4 and 15 in which the relevant literature is referred to.

hold for entities of any layer. It seems that Wang intends to regard all entities as sets — as in the axiomatization of Chapter II — since he defines identity in terms of equal extension. (But he might also have in mind an adaptation of Quine's treatment described above.) Leibniz' principle is then taken as an axiom, in an appropriate form. Wang claims that all the sets of layer α are enumerable by a function E_α (i.e. a certain relation, hence a certain set — cf. p. 131) of layer $\alpha + 2$, that a truth definition for Σ_α can be formulated in $\Sigma_{\alpha+2}$, and that the consistency of Σ_α can be proved in $\Sigma_{\alpha+2}$. (The transition to $\alpha + 2$ comes about through the fact that in these definitions and proofs sets have to be used that correspond to formulae which contain bound variables ranging over entities of layer $\alpha + 1$.) All sets of any subsystem Σ_α of Σ are therefore enumerable in some other subsystem of Σ, and the consistency of each subsystem provable within some other subsystem, hence — in a sense which does not yet seem to be quite clear — all the sets of Σ are enumerable in Σ and the consistency of Σ provable in Σ itself; the Gödel method of constructing undecidable statements (see Chapter V) is not directly applicable to Σ. All this is achieved with the help of powerful *axioms of limitation* whose effect is to ensure that there are no entities in Σ except those enumerated by the functions E_α. Since the functions E_α, by enumerating all the entities of layer α, automatically well-order them, certain theorems corresponding to the axiom of choice are provable in Σ. The continuum hypothesis, at any rate, is not independent of the other axioms but becomes either provable or refutable, according to whether the equivalence between sets of layer α is defined by the existence of a mapping relation or one-to-one correspondence (cf. *T*, p. 31) between these sets within $\Sigma_{\alpha+2}$ (or higher layers) or within Σ_α itself.

It is probably advisable to postpone judgment on the system Σ until a more detailed and rigorous exposition is available. But the partial system $\Sigma_{\omega+1}$ is interesting on its own account and looks indeed promising enough. True, many of the attractive (but also perhaps treachorous) features of Σ, such as the inclusion of its own consistency proof, the immunity towards Gödel's incompletability arguments, and — for some thinkers at least — the enumerability of all its sets, do not hold for $\Sigma_{\omega+1}$, but $\Sigma_{\omega+1}$ seems, on the other hand, to be quite sufficient as a foundation for large parts of analysis, with additional parts becoming reconstructible in, say, Σ_{ω^2+1} or $\Sigma_{\omega^\omega+1}$.

Whereas Σ_ω is roughly equivalent to a ramified class-calculus (with-

out an axiom of reducibility) [1] — neither containing full-fledged counterparts of such decisive phrases as 'for all real numbers of finite order' —, $\Sigma_{\omega+1}$, by containing bound variables ranging over all entities of layer ω, i.e. over all entities of any finite layer, allows for the formulation of just such counterparts and therefore takes care of all those procedures for the smooth operation of which the axiom of reducibility was introduced. Whereas, however, the axiom of reducibility all but destroys the constructive character of the ramified class-calculus, this character is completely preserved in $\Sigma_{\omega+1}$, thereby enabling Wang to present the outlines of a model-theoretic consistency proof similar to that given by Fitch [2] for the ramified class-calculus. Incidentally, Wang also claims to be able to give a proof-theoretic consistency proof for $\Sigma_{\omega+1}$ (as well as for any partial Σ_α) analogous to those given by Lorenzen and Schütte [2] for the ramified class-calculus.

And here we come to the one immediately apparent drawback of $\Sigma_{\omega+1}$ (and of Σ), if indeed a drawback it is: Cantor's theorem (*T*, p. 94), according to which — among other things — the power-set of a denumerable set contains *absolutely* more members than this set, is not valid, and we saw that, on the contrary, any infinite sets can be enumerated in an appropriate partial system of Σ. That part of Cantor's theory which countenances indenumerable sets is thrown overboard, as a consequence of the repudiation of certain formulae which play a decisive role in the proof of this theorem, though his theory of transfinite ordinals (at least within the second number class and perhaps only insofar as they are "constructive") is kept intact. However, the number of mathematicians and logicians who would not regard this consequence as a serious loss and would even welcome it as materializing their constructivist intuitions is now quite considerable and apparently on the increase. If the price for providing mathematics with sound foundations is no more than renouncing a certain part of the paradise into which Cantor had led the mathematicians at the end of the 19th century, then many thinkers would not regard this price as too high. [3]

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[1] As is the essentially equipollent system L_ω of Quine 53, pp. 124 ff.

[2] See footnote 1 on p. 152.

[3] A good semi-formal description of Wang's system is given in Stegmüller 56–57, pp. 66 ff.

§ 7. Lorenzen's Operationist System

Wang's approach is admittedly strongly influenced by that of Lorenzen who in a series of articles [1]) developed his own version of constructivism in the foundations of mathematics. Lorenzen's work culminated so far in a book *Einführung in die operative Logik und Mathematik* [2]), published in 1955. As evident from the title, Lorenzen prefers now the adjective 'operationist' — probably the most adequate English rendering of the German term '*operativ*' — to the adjective 'constructivist' he had used before. This is more than a mere verbal matter. According to Lorenzen, 'constructivist' in mathematics should be reserved for those methodological attitudes that insist on restricting their investigations to the effectively calculable or describable, whereas 'operationist' indicates rather that the entities investigated are schematic operations. (There exists therefore a connection with the "operational" methodology of Bridgman [3]) and an ever closer contact with Dingler's [4]) views.)

We presented Wang's views before those of Lorenzen because we felt that the fundamental philosophical outlook of Lorenzen is still somewhat obscure, in any case obscurer than that of Wang. We nevertheless hope that the following outline will render the gist of some of Lorenzen's most important tenets.

For Lorenzen, the main (though not the only) subject of mathematics is the treatment of calculi — this should by no means be misunderstood as a claim that mathematics *is* a calculus, which Lorenzen would very definitely reject — where a calculus is understood to be a system of rules for schematic operations with figures, which may but need not be marks on paper; they might as well be pebbles (*calculi*) or any other physical objects. In addition to this precisification of the subject-matter of mathematics — and only slightly connected with it — Lorenzen stipulates that the methodical frame be as wide as compatible with the *conditio sine qua non* that all mathematical statements be *definite*. Due to the fact that Lorenzen — who in this respect shows a certain affinity to intuitionistic thinking (Chapter IV) — cares very little for exact formalization, insisting that precisification

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[1]) Lorenzen 50, 51, 51a, 51b, 53, 54.

[2]) Lorenzen 55; cf. also 55a, 56, 57 and Hermes 56, pp. 67 ff.

[3]) *Bridgman 1.

[4]) *Dingler 2, 7.

is by no means identical with axiomatization or formalization, the exact meaning of this central term is not altogether clear. It certainly has nothing to do with the term 'definite' as used by Zermelo (p. 39) and is clearly wider than the term 'definite' as used by Carnap [1]). In Carnap's usage, a statement containing a regular (unlimited, or unrestricted) quantifier, such as '$(Ax)F(x)$', is indefinite — and so is a language whose rules of formation allow for such sign-sequences as well-formed formulas — whereas it is definite in Lorenzen's usage, more precisely refutation-definite (*widerlegungsdefinit*), on condition that '$F(x)$' is definite, and this because '$(Ax)F(x)$' is refutable through a derivation of an instance of '$F(x)$'; being a derivation is a definite characteristic, since it is decidable through the performance of schematic operations on the figures that make up the lines of the derivation.

Lorenzen's system is then much more liberal than constructivist systems of the most rigorous brand that either contain no quantifiers at all [2]) or, at most, only *limited* quantifiers [3]) in which quantified statements with a constant limit are equipollent to conjunctions or disjunctions, respectively, of finite length. Nevertheless' it still excludes impredicative concept formations. The limitations thereby imposed are overcome, just as with Wang, by a utilization of the transfinite layers. Lorenzen, however, is able to go into much greater details in the justification of his hope that almost all of classical analysis will remain essentially intact, though certain reformulations will be required. Absolutely indenumerable sets don't exist any more, of course; in those mathematical fields, such as topology or the theory of integration (Lebesgue measure theory is an immediate example), where the contrast: denumerable — indenumerable seems to play a decisive role, the contrast: primary — secondary — where a set is called *primary* if it belongs to a layer whose index is less than some arbitrary limit-number θ_1 and *secondary* if the index of the smallest layer to which it belongs is greater than θ_1 but less than some other fixed (non-problematic, i.e. "constructive") limit-number θ_2 — can in general serve as a satisfactory substitute. However, should it turn out that certain classical theorems will not be reproducible on the operationist basis, there remains the possibility

[1]) For details, see Carnap 37, p. 45 and pp. 160 ff.

[2]) Such systems were especially advocated by Skolem; cf. Chapter V.

[3]) I.e., quantifiers of the form 'for all [some] x up to y'. For a theory of the generalized notion of restricted quantification, see Hailperin 57.

of paying serious heed to the words of Skolem: "We ought not to regard all that's written in the traditional textbooks as something sacred".

The logic of dealing with schematic operations turns out to be the intuitionistic logic (Chapter IV) in which the *tertium non datur* is not generally valid. By adding this law to the effective functional calculus, Lorenzen gets the "fictional" functional calculus, corresponding to classical functional logic. For the justification of this transition, Lorenzen uses an argumentation which is strongly related to the ideas developed by van Dantzig in connection with the so-called *stable* statements (Chapter IV, p. 238).

The fact that Wang and Lorenzen, starting from quite different backgrounds, the one from a Russell-Zermelo tradition, the other from a constructivist Hilbertism (Chapter V), converge to systems which have so much in common cannot but bring their streamlined version of the ramified type theory back into the race. The old animadversions against this theory do not hold any more.

§ 8. The Logicistic Thesis

Though we might have thought that metaphysical considerations should be of no relevance to the problem of providing a secure foundation for set theory (or mathematics in general), so long as one judges by what logicians and mathematicians *say* about this problem, this is definitely not so. Let us pause for a while to see this at some detail.

Metaphysical convictions certainly make no difference — with the possible exception of some very tough-minded intuitionists — for the application of mathematics to science and technology. No director of research in some industrial outfit would inquire into the metaphysical beliefs of the mathematician he is about to hire. There seems to exist no correlation between these beliefs and the performances in which the director of research is interested. In the attempt to solve some set of differential equations, all mathematicians will peacefully cooperate though they might thereafter, during a lunch hour conversation, disagree violently about "the nature of mathematics". We make these commonplace observations, not in order to disparage discussions about the nature of mathematics — this whole book is dedicated to the discussion of one aspect of this problem — but to put them in their right place.

Most present-day thinkers — though by no means all [1]) — would agree that there are at least two different kinds of true scientific statements: the empirical truths on the one hand and the logical and mathematical truths on the other. We shall take here this approach for granted and investigate only into the relation between the sets of the logical and the mathematical truths among themselves. Any two sets stand in one and only one of five fundamental relations. As to the relationship between the two sets of statements, five views are therefore theoretically possible: they are identical, the set of mathematical truths is a proper subset of the set of logical truths, the set of logical truths is a proper subset of the set of mathematical truths, the two sets are mutually exclusive, the two sets are overlapping. In addition, one could of course also be less committal and subscribe to any disjunction of at most four of these theses.

Looking at the literature, one could indeed find expositions of each of the five fundamental views as well as of many of their disjunctions, if one were to take these expositions at their face value. This would, however, probably be a mistake since very often the terms 'logical' and 'mathematical' are used in different, sometimes in very different, senses by different authors (and occasionally by the same author in different publications), so that it is often very hard to distinguish between purely verbal and real disagreements, or to decide that a verbal agreement is an indication of a substantial agreement.

We mention all this because the type-theoretical approach to the foundations of set theory, to the various variants of which this chapter is devoted, has often been identified with the *logicistic* approach to the foundations of mathematics, i.e. with the thesis that mathematical truths form a proper subset of the logical truths. This identification, however, is definitely misleading and due to nothing more but the fact that there is a union between the inventor of type theory and the foremost logicist of our time in the person of Bertrand Russell. Logicism, however, was created by Frege long before type theory was invented, and there are now many formulations of type-theoretical systems that do not embody the logicistic thesis. Let us turn now to an exposition of this thesis.

The last quarter of the 19th century saw the triumph of the so-called

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[1]) For a forceful presentation of the minority view, see Quine 53, especially essay II. Cf. also the remarks on significs in Chapter IV, pp. 206-207.

arithmetization of mathematics. By explicit, though occasionally quite complicated, definitions, the various kinds of numbers, up to complex numbers and beyond, and the operations upon them were introduced on the basis of the natural numbers and the operations upon them. Almost all mathematicians were satisfied with this achievement and regarded with suspicion any further inquiry into the reducibility of natural numbers to other entities or into the deducibility of the arithmetical theorems from other theorems [1]). Frege, however, had no great difficulties in showing[2]) that all the interpretations given in his time to the calculus of natural numbers were gravely deficient and that the last refuge of the mathematicians in this respect, i.e. leaving the calculus without interpretation, could not be seriously upheld since in certain contexts numerical signs must be regarded as interpreted, e.g. in "every quadratic equation has exactly two roots". By his own interpretation, the natural numbers were the cardinal numbers [*Anzahlen*] of certain concepts, with 'the cardinal number of the concept *F*' defined as short for 'the extension of the concept equinumerous-with-the-concept-*F*'; finally, the statement 'the concept *G* is equinumerous with the concept *F*' was regarded as short for 'there exists a one-one-correlation between the objects falling under the concept *F* and the objects falling under the concept *G*'. This last expression, Frege was able to show, could be reduced to purely logical expressions. With infinite care and meticulousness, using a strange geometrical symbolism, Frege continued to define one basic arithmetical term after the other, proving, as he went along, the fundamental arithmetical theorems governing these terms. For this purpose, he almost recreated formal logic, gave the first complete axiom system for the propositional calculus, and greatly expanded the functional calculus.

Even if Frege had been entirely successful in reducing arithmetic to logic — which he was not in view of the emergence of antinomies in his system [3]) —, it is not always sufficiently realized today and has not been sufficiently realized by Frege himself that this would still not have meant that all of analysis were reducible to logic, in

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[1]) Let us recall Kronecker's much-quoted dictum: "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*". (The good Lord made the integers, everything else is human creation.) Cf. p. 245.

[2]) In his masterfully written Frege 1884. However, at least up to the turn of the century he met with the antagonism of most mathematicians.

[3]) See above, p. 3.

spite of the already accomplished arithmetization of analysis. This is so because the arithmetization means reduction to "integers and finite or infinite systems of integers", in Poincaré's terms quoted above [1]), hence not only to integers but also to what we would now call *sets of integers*. The reduction could therefore be regarded as completed only if, in addition to arithmetic, also the general theory of sets — or at least the theory of sets of integers and, probably, of sets of sets of integers etc. — were "reduced" to logic [2]). This, however, was never done nor even attempted by Frege. This really final step was attempted only by Russell (and Whitehead). It was when dealing with Cantor's set theory in order to reduce it to logic that Russell ran afoul of his antinomy. This accident forced him to a revision of what logic amounted to and made him give it the specific shape of the Ramified Type Theory. Historically speaking, type theory is then an accidental by-product of an attempt to implement the logicistic thesis, but systematically their connection is very slight. In many current variants of type theory, given by Gödel, Tarski, Carnap and others, the logicistic thesis is abandoned and logic and arithmetic are simultaneously developed.

We already mentioned above that the usual formulations of the logicistic thesis are far from being precise and universally accepted. The following formulation, e.g., seems to be precise enough: All specific mathematical terms are definable on the basis of the logical vocabulary, and for the proof of all mathematical theorems no axioms beyond the axioms of logic and no rules of inference beyond those accepted in logic are needed. The precision of this thesis, however, is not as great as it appears to be on first sight. There exists no universal agreement on the meaning of most of the terms occurring in it, such as 'logic', 'axioms of logic', 'rules of inference' or 'logical vocabulary'. Depending on the specific interpretation of these terms as well as on a precisification of the term 'mathematics' itself, e.g. whether geometry is or is not to be regarded as mathematics (a decision that depends of course in its turn on the interpretation given to 'geometry'), the logicistic thesis runs the whole gamut between an uninteresting truism through a pious hope, whose basis is not quite clear, to an almost obvious falsity.

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[1]) p. 15.

[2]) This point is made in Wang 54, p. 242.

We have no intention of dealing here with the problem of the relationship between logic and mathematics in all its generality. (Cf. also Chapter IV.) At the moment, we are concerned only with set theory. Unfortunately, this does not reduce by much the vastness of the problem, nor is it thereby appreciably simplified. The situation is by no means such that there exists a clear conception of what logic on the one hand and set theory on the other hand are, allowing a dispassionate, though perhaps still difficult, investigation into their relationship; on the contrary, different apriori conceptions of this relationship color the conceptions of these disciplines themselves, forming one complex of problems whose disentanglement is a formidable task indeed.

The axiomatic approach to set theory, treated in Chapter II, is inherently neutral with regard to our problem. To construct set theory, or any theory, for that matter, as an uninterpreted axiom system based upon some smaller or greater fragment of logic — in Chapter II, this fragment consisted of the first-order functional calculus (without equality) — means, by definition, that the question of interpretation of this axiom system is so far not taken up. It is by no means excluded that one possible interpretation should be a logical one, in the sense that the primitive terms of this axiom system — for most systems discussed in Chapter II just 'ϵ' — is defined on the basis of the vocabulary of either the underlying fragment of logic or of some more extensive fragment in such a way that the axioms become logical theorems. Of course, he who strongly believes in the existence of such a logical interpretation and moreover does not believe in the existence of other useful interpretations may regard the erection of an independent axiom system as a waste of time and prefer to present his theory from the beginning as a part of logic. This is what Frege and Russell, and many others after them, have done. By interpreting 'ϵ' as denoting class-membership — which they take to be a logical notion though perhaps one that belongs to a "higher" part of logic — they identified from the very start the (mathematical) theory of sets with the (logical) theory of classes.

This identification, however, meets with certain difficulties. They arise only when the realm of finite sets is left and therefore were of greater concern to Russell who intended to reconstruct all of set theory to its complete Cantorian extent as part of logic than to Frege who, a contemporary of Cantor, was interested in the theory

of sets mainly insofar as it was needed for the arithmetic of the natural numbers. Whereas most axioms of ***Z*** are easily provable in logic after this identification, the counterparts of the axioms of infinity and of choice could not be proved in ***PM***. For the counterpart of the two versions of the axiom of infinity of ***Z*** this is immediately obvious since their formulation violates the rule of types adopted in ***PM***, according to which the membership of any class must be homogeneous, whereas both the classes whose existence is asserted by these axioms have a heterogeneous membership. But neither can any statement to the effect that the number of individuals, i.e. of the entities belonging to the lowest level, is denumerably infinite be derived from the axioms of ***PM***. For each statement provable on the basis of logic plus this axiom, a theorem in the form of an implication whose antecedent is the axiom and whose consequent is the statement is derivable from logic alone. Whitehead and Russell were therefore able to prove, corresponding to each mathematical theorem T for whose derivation the axiom of infinity AxInf was needed, not T itself but rather $\text{AxInf} \supset T$. But this meant, of course, that any such mathematical theorem T by itself could not be shown to be a theorem of logic unless AxInf was taken to be an axiom of logic. The authors of ***PM***, however, were very reluctant about taking this step since the content of this axiom, i.e. the existence of an infinity of individuals, had a definite factual look, indeed so much so that not only its logicality but even its truth was in doubt: whether the universe was composed of a finite or infinite number of ultimate particles — taking these particles to be the "individuals" of the system of ***PM***, which was indeed one of the explicitly admitted interpretations — seemed to be a question which could be answered, if at all, only by physics, and this in spite of the fact that InfAx was formulated in logical terms exclusively. This situation was disturbing even for the authors of ***PM*** themselves and caused others to reject this "reduction" of mathematics to logic.

The situation is different, but no less disturbing, with regard to the Axiom of Choice, or the Multiplicative Axiom — MultAx, for short — as Russell called one of its variants (see p. 48). Again there were large parts of higher classical mathematics whose derivation required the use of MultAx. But the authors of ***PM*** could not persuade themselves to treat it as a logical truth on a par with the other logical axioms. They had an intuitive notion of logical truth, from which

they were unable to derive, to their own satisfaction, either the truth or the falsehood of MultAx. Afraid that MultAx might eventually be shown to be false, they had to content themselves once more, with regard to those classical mathematical theorems T in whose proof MultAx was commonly used, to prove MultAx $\supset T$. Though they liken the situation to that with which the geometrician is confronted by the Axiom of Parallels, it is doubtful whether this analogy is to be taken seriously: had someone been able to show that MultAx is completely independent of the other logical axioms, i.e. that neither it nor its negation are derivable from them, it rather seems that they would have been ready to stretch their intuitive notion of logical truth — this notion was admittedly never quite definite [1]) — so as to include MultAx but not its contradictory. At any rate, no attempt was made by them to develop mathematics on the assumption of the truth of the contradictory of MultAx — in analogy with the development of non-Euclidean geometries.

But with regard to MultAx, the situation should probably be looked upon not so much as throwing doubt on the reducibility of mathematics to logic as on the adequacy, reliability, and determinateness of our logical intuitions. There is certainly nothing specifically mathematical in its formulation. Whether it should, might, or should not be accepted as an axiom, provided that it is demonstrably, or at least presumably, independent of the other axioms, is a moot question within the philosophy of logic, into which to delve at any length and depth it is not feasible without a detailed discussion of this discipline, something that is beyond our reach here.

It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx, according to the interpretation intended by them. But is this interpretation under which the individuals are certain factual entities like particles, events, etc., an integral part of the thesis? This is another moot question, in the discussion of which it is very difficult not to fall into verbal traps. But, whether essential deviation from this thesis or inessential reinterpretation of it, there exist logical systems which embody a unified simultaneous development of logic and arithmetic — the expression 'reduction of mathematics to logic' could be applied to this develop-

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[1]) See, e.g., *Russell 5, p. 205.

ment only in a rather stretched sense — thereby managing to sidestep the disturbing problematic status of InfAx within ***PM***.

These systems, of which Gödel's system ***P*** mentioned above (p. 150) is one of the better known, are couched within what might be called, following Carnap, *coordinate-languages* [1]). In such languages, in contradistinction to the more customary *name-languages*, the objects of the fundamental domain are not designated directly by proper names but indirectly by systematic positional coordinates, i.e. by symbols that show the place of the objects in the system and thereby their positions in relation to each other. Instead of saying, then, that entity *a*, i.e. the entity whose proper name is '*a*', is blue, as customary in name-languages, one says in a coordinate-language that the entity occupying such and such a position is blue. In some of the languages specifically constructed and treated by Carnap, the basic domain of positions is taken to be a one-dimensional series with a definite direction whose initial position is designated by '0', the succeeding position by '0'', etc.; in other words, the natural numbers are treated as the coordinates of these languages. Without going into any more detailed description of these languages, let us only notice that through adopting some Peano-type axioms the infiniteness of the basic domain becomes provable. But — and this is the decisive point — the fact that the infiniteness of the domain is part of the logic of the system is by no means as disturbing as within ***PM***. No longer does the statement of infinity assert the existence of infinitely many different particles or other physical entities but rather the fact that the one-dimensional series of positions has no last member, leaving the answer to the question how many of these positions are occupied by physical entities entirely to extra-logical science.

Those authors who are strongly objected against having existence assumptions in logic, whether of an infinity of "entities" or even of at least one "entity", seem to have had name-languages in mind; these objections look rather pointless when directed against coordinate-languages, where these "entities" are nothing but positions that might well be empty. It appears that the strengthening of the logicistic attitude towards the foundations of mathematics entailed by the shift from name to coordinate-languages has not yet been sufficiently taken into account by most authors in this field.

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[1]) See Carnap 37, p. 12.

Let us notice in this connection that Carnap [1]) has succeeded in proving that the axiom of choice was analytic — where 'analytic' was understood to embody a certain refinement of the pre-systematic concept of logical validity — in his Language II, one of the mentioned coordinate-languages; for his proof, he assumed that an axiom of choice held in the metalanguage of Language II. This assumption does not make the proof circular in any formal sense but, on the other hand, it does not contribute to the solution of the problem whether such an axiom is to be regarded as a logical principle. Carnap himself, at least in the thirties, regarded this problem not as one referring to the material truth of the principle of choice but rather to its expediency. Since, within Language II, an axiom of choice is expedient and allows for the reconstruction of a mathematical calculus adequate for science and since it is unlikely that this language will turn out to be inconsistent, there is very little to be said against the admission of the axiom of choice as a logical principle, if one accepts Carnap's ontology-free standpoint.

§ 9. Types, categories, and sorts

The idea that not all objects are of one kind, in the sense that there are properties which can be meaningfully predicated of certain objects but not of others, is an old one and had originally no connection with the problem of antinomies. This conception can be traced back to Aristotle [2]), Schröder drew level distinctions in 1890 [3]), and indications to this effect are to be found in Frege's writings [4]). The philosopher E. Husserl dealt at length with categories of meaning [*Bedeutungskategorien*] [5]) and exerted much influence on the Polish school of logicians.

Russell himself was never quite satisfied with the ontological validity of his own type distinctions though he remained convinced that some sort of hierarchy is necessary [6]). In one of his later publi-

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[1]) Carnap 37, pp. 122 ff. Cf. already Ramsey 26.

[2]) *Categoriae*, ch. 3.

[3]) See *Church **12**.

[4]) But no more than indications. Frege's hierarchy of *Stufen* was a consequence of certain formal considerations. He almost explicitly rejected a theory of types. See *Church **12**.

[5]) *Husserl $I_{II,1}$, pp. 317 ff. Cf. also Bar-Hillel 47a, 57.

[6]) Reply to Criticisms (Schilpp **44**), p. 692.

cations, he was even ready to admit — perhaps somewhat rashly — that his definition of types was wrong, insofar as he had originally distinguished different types of *entities* whereas he should have made these distinctions rather with regard to *symbols* [1]).

Among Russell's followers, we can distinguish between those for whom type distinctions were a matter of intuitive coercion, a philosophical necessity, and those who regarded them as a necessary evil, something to be grudgingly admitted so long as no better way of avoiding the antinomies was in view. The foremost proponent of the first conception is probably the Polish logician Stefan Leśniewski [2]) for whom, as a matter of linguistic intuition, (almost) all well-formed expressions of any language, ordinary or artificial, belonged to exactly one *semantic category* out of a potentially infinite and highly ramified hierarchy of semantic categories; this hierarchy consisted of two *fundamental* categories, that of *names* and *statements* (always including open names such as 'the father of x' and open statements, respectively), and an infinity of *functor categories*, distinguished according to the category and number of the argument-expressions of the functors belonging to them and to the category of the expression resulting from the application of these functors to their arguments.

Among the simpler members of this hierarchy, we have the category of functors that out of a name form a statement, that out of two names form a statement, ..., that out of a statement form a statement, that out of two statements form a statement, ..., that out of a name form a name, that out of two names form a name, ..., that out a statement form a name, ... Using a convenient quasi-arithmetical notation originating with Ajdukiewicz [3]), these categories could be denoted by

$$s/n,\ s/nn,\ \ldots,\ s/s,\ s/ss,\ \ldots,\ n/n,\ n/nn,\ \ldots,\ n/s,\ \ldots,$$

respectively. This symbolism also indicates how the more complex members of this hierarchy could be characterized and denoted: the category of a functor, e.g., that out of a functor (as argument) that out of a name forms a statement

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[1]) *Ibid.*, p. 691.

[2]) Leśniewski 29. See also Słupecki 53, Chapter 1 (but cf. review by Lejewski, *J.S.L.* **21** (1956), pp. 188–191), 55, p. 9 (and references given there in footnote 8) and Grzegorczyk 55, pp. 85 ff.

[3]) *Ajdukiewicz 3, further elaborated in Bocheński 49, Bar-Hillel 50, 53, Wundheiler 55. This notation should be compared with those developed in Carnap 37, § 27 (who uses 'functor' in a sense which is roughly that of the expression 'name-forming functor' used here), and Church 40 for types of expressions.

forms a functor (as value) that out of a name forms a statement would be denoted by '*s/n//s/n*'. A few illustrations taken from ordinary English (but to be evaluated *cum grano salis*) and the set-theoretical symbolism adopted in this book will help:

Category	English	Set theory
n	John, poet	x, y
s	John is a poet	$F(x)$, $x \in y$
s/n	... is-a-poet	$F(\ )$, $\ldots \in y$
s/nn	... is-a ---	$\ldots \in ---$, $\ldots \subset ---$
s/s	It-is-not-the-case-that ...	$\sim(\ )$, $(Ax)(\ )$
s/ss	... if-and-only-if ---	$\ldots \equiv ---$
n/n	The-father-of ...	$\mathbf{C}(\ )$
n/nn	The-product-of ... and ---	$\ldots \cap ---$
n/s	whether ...	$\hat{x}(\ )$
s/ns	... doubts-whether ---	$(A\ )(\ )$
s/n//s/n	gracefully (in 'John gracefully retired')	
s/nn//s/nn		/(in 'ϕ')

For Leśniewski, transgressing the limits set by the *Theory of Semantic Categories* is a sin against logic, occasionally though not necessarily punished by the arisal of antinomies. The validity he claimed for his theory was quite apart from the fact that by adherence to it some of the logical antinomies could not be formulated [1]) and was based rather, as stated above, upon an intuitive insight into the conditions of meaningfulness of ordinary language expressions, in elaboration of Husserl's views. Leśniewski, in his turn, was followed by a host of Polish philosophers and logicians, the most important among whom were Kotarbiński, Ajdukiewicz, and Tarski. How strong the influence of Leśniewski among his pupils was, can be judged, e.g., from a passage written by Tarski in the early thirties where he claims that

> the theory of semantical categories penetrates so deeply into the fundamental intuitions regarding the meaningfulness of ex-

[1]) Not all logical antinomies are due, according to Leśniewski, to a confusion of categories. Some result from a violation of certain rules of definition. In addition, Leśniewski claims that the current conception of the term 'class' confuses two quite different notions, namely that of "distributive class" and that of "collective class" (roughly the same distinction already made by Russell 03, pp. 68 ff, in terms of "class-as-many" and "class-as-one") and traces in particular the origin of Russell's antinomy to this confusion. Cf. Sobociński 49-50.

> pressions that it is scarcely possible to imagine a scientific language in which the sentences have a clear intuitive meaning but the structure of which cannot be brought into harmony with the above theory [1]).

Later on, however, Tarski lost faith in the intuitive necessity of Leśniewski's theory of semantic categories, probably when realizing the grave strictures imposed by it upon the structure of languages, and started investigating language systems in which this theory was not obeyed [2]). It seems that Wang's recent studies (cf. above, § 6) form a direct continuation of this line of thinking.

Whatever the importance of the Theory of Semantical Categories as a logico-philosophical doctrine, there can be little doubt as to the importance for linguistics of its syntactical counterpart, the *Theory of Syntactical Categories* [3]), especially in the form of the notational calculus it was given by Ajdukiewicz [4]).

Typical for the second, opportunistic conception of the theory of types is Reichenbach [5]) who regarded the fact that this theory made language consistent as its best possible justification.

We shall not deal here with the various philosophical objections that have been launched against the theory of types, its intuitive plausibility, its applicability to ordinary language, or its consistent formulability. Whatever the force of these objections with regard to ordinary languages, there can be no doubt that calculi embodying type distinctions can be constructed according to the highest standards of rigor and that it can be even be proved of some of these calculi that they are consistent. Many mathematicians have nevertheless found the theory of types repugnant as a foundation for mathematics. There are probably many reasons behind this reaction. It might often be due to no more than an unanalyzed idiosyncratic aversion, leaving us nothing more to say about it besides acknowledging the fact, but the type division has also certain technical drawbacks some of which we have discussed above.

Let us deal here with what is probably the most serious disadvantage

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[1]) Tarski 56, p. 215.
[2]) *Ibid.*, VIII, Appendix.
[3]) Bar-Hillel 47a, 50.
[4]) See note 3 on p. 169.
[5]) Bertrand Russell's logic (Schilpp 44), p. 38.

of the Theory of Simple Types. Developing set theory on the basis of a logic incorporating a type stratification of its variables means that this logic is no more the functional calculus of the first order but rather the functional calculus of order ω, i.e. one that contains quantifiable variables of any finite level whatsoever. Set theory as developed in *PM*, in contradistinction to **Z**, is therefore no more an *elementary* (or *standard*) theory and no longer enjoys the desirable features of such a theory, the most important of which is that the *proof procedures of its underlying logic are complete* (see Chapter V, § 4). In an elementary theory, all variables have the same range and there exists therefore just one single universe of discourse. The set theory of *PM* is not the only non-elementary one; so is the system **B** of von Neumann-Bernays in which there are two disjoint universes, that of the classes and that of the sets.

Now, there exists a technique of converting any *many-sorted* theory, i.e. any theory containing more than one universe of discourse, into an equipollent *one-sorted* theory [1]). Applying this technique to the set theory of *PM* we arrive at a system that was called by Quine [2]) the *Standardized Theory of Types* and represents an interesting transition between the simple theory of types, or rather its still more simplified version **T*** of § 2, and Zermelo's original system. It is still a type theory in the sense that its ontology comprises individuals, classes of individuals, classes of classes of individuals etc., and that these classes remain homogeneous. But it differs from **T*** in that its variables are not merely typically ambiguous, i.e. ranging in a given context over a definite, though unspecified type, but fully general, i.e. always ranging over all types.

Without going into the details of the highly elegant treatment given by Quine to this transformation, let us briefly describe one resulting system of axiom-schemata for the standardized theory of types. Let 'T_0' be the predicate that holds for all and only the individuals, 'T_1' the predicate true of all and only the classes of individuals, etc. (Quine shows that all these predicates can be defined in terms of 'ϵ' alone and need therefore not be added to the primitive notation.) The axiom-schemata of comprehension and extensionality

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[1]) See A. Schmidt ***2**, 51, Wang 52a, Hintikka 55.

[2]) Quine 56.

receive now the following form:

(1) ( ) $(\mathrm{E}y)\{T_{n+1}(y) \ \& \ (\mathrm{A}x)[T_n(x) \supset (x \in y \equiv F(x))]\}$

(2) ( ) $\{T_{n+2}(z) \ \& \ T_{n+1}(x) \ \& \ T_{n+1}(y) \ \&$
$(\mathrm{A}w)\,[T_n(w) \supset (w \in x \equiv w \in y)] \ \& \ x \in z\} \supset y \in z.$

To these axioms a new set of axioms has now to be added through the axiom schema:

(3) ( ) $x \in y \supset [T_n(x) \equiv T_{n+1}(y)].$

(The function of the T-clauses in (2) is to prevent the identification of the individuals with each other and with the null-sets of the different types as well as the identification of the various null-sets among themselves.)

Notice that according to (3) any formula of the form '$x \in y$' where x and y do not belong to consecutive ascending types, hence especially the formula '$x \in x$', is *false* and not meaningless as in the original system. This is a straightforward effect of the standardization. That this deviation from the original approach has no technically pernicious consequences should have some impact on the evaluation of those views that saw in the introduction of the category "meaningless" in addition to "true" and "false" the great philosophical achievement of the theory of types.

For those adherents of the theory of types who are ready not only to trade meaninglessness for falsity (or truth, as the chips may fall) in order to achieve standardization but even to tamper with the original ontology for the sake of technical simplicity, Quine shows that a considerable increase in simplicity arises if the null-sets of all types are identified. A different but equally considerable simplification results from the identification of the individuals with each other and with the null-set of the lowest type, as in ***Z***. Still another simple system arises from the identification of each individual with its unit set — counting now '$x \in x$' as true in case x is an individual — bringing this system into line with ***NF*** and ***ML*** (see §§ 3 and 4). This last identification would result in breaking down the exclusiveness of the types; types would now be cumulative, as in the system Σ treated above (§ 6).

These changes do more than just simplify the theory of types. Some of the simplified theories get very close to some of the systems originating with Zermelo. In fact a cumulative theory of types with general

variables and just one memberless object will simply coincide with a Zermelian set theory in which individuals are repudiated, i.e. in effect with system ***Z***; a cumulative standardized type theory with self-membered individuals will come out the same as that variant of Zermelo's original theory in which the individuals are taken as being self-membered and an axiom guaranteeing the existence of the set of all individuals is added.

Half a century after Zermelo and Russell published their theories, independently of each other and starting from seemingly totally different and even contrary approaches, an almost complete reunion of these theories is now in full view. True, the deviations from the original attitudes through which this rapprochement is brought about are quite considerable and some of them might well be regarded as militating against the spirit of both Zermelo and Russell themselves. Many a Russellian will find it hard to swallow the notion of cumulative types; Russellians and Zermelians alike will not easily acquiesce in the notion of self-membered individuals. Those, however, for whom a satisfactory foundation of set theory means not so much a construction that corresponds closely to their intuitions but rather a system from which classical analysis can be effectively reconstructed and which is either demonstrably consistent or at least within which the customary arguments leading up to the standard antinomies cannot be reproduced, will welcome these recent developments. [1])

§ 10. Impredicative Concept Formation

It is now time to deal at some detail with another most interesting, though also highly obscure and controversial ingredient of Russell's original attempt at overcoming the antinomies: the recognition that *impredicative* concept formation is the root of all evil [2]) to be eradicated by strict adherence to the *vicious-circle principle.* (The connection of this conception with the constructivist approach to the foundations of mathematics will be discussed on p. 178.)

Part of the obscurity is caused by the fact that Russell uses in the various formulations he gives the vicious-circle principle at least three different terms which he seems to regard as synonymous: first,

[1]) Another related expression of this development is to be found in Klaua 57.

[2]) Skolem 52 continues to regard this recognition as a natural one and accordingly recommends further study of the ramified type theory.

'*definable*' in "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total" [1]); second, '*presupposes*' in "... it is possible to incur a vicious-circle fallacy at the very outset, by admitting as possible arguments to a propositional function terms which presuppose the function" [2]); third, '*involves*' in "A function is what ambiguously denotes some one of a certain totality, namely the values of this function; hence this totality cannot contain any members which involve the function since, if it did, it would contain members involving the totality, which, by the vicious-circle principle, no totality can do" [3]). Gödel shows [4]) that we have before us three different principles rather than three formulations of the same principle, that the second and third of these principles are much more plausible than the first, and that it is only the first which obviates the derivation of mathematics from logic and militates against procedures in constant use in classical mathematics.

To the presently existing confusions around the concept of impredicativity contributes also the fact that one uses now the adjective 'impredicative' as a modifier for nouns denoting linguistic entities such as 'definition', 'statement' or 'axiom', non-linguistic entities such as 'concept', 'property' or 'set', and even 'procedure' or 'concept-formation', without taking the necessary care in establishing the interconnections between these various uses.

Of late, it has been shown that there exists an important difference in the conceptions of what forms an impredicativity between the two first and foremost proponents of the prohibition of impredicative concept formation in (logic and) mathematics: Poincaré and Russell [5]). This difference is of more than historical interest since it might well throw light on the currently much discussed question where the borderline between the harmful and innocent impredicativities lies.

We shall gain in clarity, though we shall lose somewhat in generality, if we restrict the discussion of impredicativity to systems of the kind treated so far. This enables us to focus our attention to the axiom

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[1]) *PM*$_{\text{I}}$, p. 37.

[2]) *Ibid.*, pp. 37–38.

[3]) *Ibid.*, p. 39.

[4]) Gödel 44, p. 235.

[5]) For the early history of the objections against impredicative procedures, see Fraenkel 28, pp. 247 ff. Cf. also Church 56, p. 347, footnotes 573 and 574.

(-schema) of comprehension (in its various formulations and variants, including the axiom of subsets). We shall say that a certain version of this schema, whose core is — as we recall —

$$(\ ) \qquad (\mathrm{E}y)\,(\mathrm{A}x)\,[x \in y \equiv F(x)]$$

(where, according to the variant, '$F(x)$' is required to fulfil various conditions, and where the superscripts occurring in some variants are omitted as irrelevant in our context), is *impredicative* if '$F(x)$' contains a bound variable of the same kind (type, layer, level etc.) as 'y'; in other words, and in a somewhat less formal and rigid formulation, if the class whose existence is guaranteed by this axiom-schema belongs to the range of a bound variable occurring in the determining condition. Derivatively, we shall call *impredicative* also those conditions themselves, which contain then free variables of lower level than that of some of the bound variables occurring in them, the corresponding classes and the process of forming them, i.e. of proving their existence.

Usually, one formulates the problem of impredicativity in terms of 'impredicative definitions' rather than in terms of 'impredicative concept formation'. This is partly due to the fact that ***PM*** — and other systems — have no axiom of comprehension but instead an (explicit or implicit) rule admitting '$\hat{x}F(x)$' — i.e. 'the class of all x, for which $F(x)$' — as an abstract-expression substitutable for a variable and an axiom(-schema) of conversion: $y \in \hat{x}F(x) \equiv F(y)$. In such systems, the principle of comprehension becomes a derivable theorem. (Indeed, from the tautology $F(x) \equiv F(x)$ we get $x \in \hat{x}F(x) \equiv x \in \hat{x}F(x)$ by conversion, $(\mathrm{A}x)[x \in \hat{x}F(x) \equiv x \in \hat{x}F(x)]$ by universal generalization, $(\mathrm{E}y)\,(\mathrm{A}x)$ $[x \in y \equiv x \in \hat{x}F(x)]$ by the above-mentioned rule and existential generalization, finally $(\mathrm{E}y)\,(\mathrm{A}x)[x \in y \equiv F(x)]$ again by conversion [1].) If now a single symbol is introduced by definition as an abbreviation of such an abstract-expression and '$F(x)$' is an impredicative condition in the sense clarified above, this symbol is introduced by an "impredicative definition". Since this definitory abbreviation is, however, only a minor technical point, we shall continue to see the gist of the problem of impredicativity in the impredicative concept formation, whether through an application of the axiom-schema of comprehension or through the introduction of abstract-expressions.

The vicious-circle principle requires that measures be taken to disqualify impredicative concept formations. As for its justification, we must clearly distinguish between four quite different kinds of argumentation. The first is an argument from pragmatic effectiveness:

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[1]) Cf. Hilbert-Ackermann 28/49, 2nd ed. p. 125, 3rd ed. p. 137.

the application of the principle overcomes the semantic antinomies. Its critics will then point out that other means are known — such as the theory of language hierarchies [1]) — which are equally effective in this respect but less destructive in their effect on the reconstruction of mathematics. The second argument refers to the self-referential character of impredicatively defined terms. After half a century of detailed discussion it should, however, be clear that self-reference as such without some additional qualification, though certainly circular in a sense, is by no means always viciously circular [2]).

The third argument stresses that the applicability of an impredicatively formed concept in a given concrete case is not decidable and that any attempt in this direction must result in an infinite regress. To use an example of Carnap's [3]), modified to apply to system ***P*** (mentioned in § 5): Consider the condition on x^1

$$(Az^2)\,[7 \in z^2 \;\&\; (Ay^1)\,(y^1 \in z^2 \supset Sy^1 \in z^2)] \supset x^1 \in z^2$$

(where 'Sy^1' denotes the successor of y^1), which may be read as 'x belongs to all those sets that contain 7 and together with any natural number also its successor'. The axiom of comprehension guarantees the existence of a set containing all those natural numbers that satisfy this condition. Trying to find out whether 5, for instance, belong to this set, one has to check whether 5 belongs to *all* sets of natural numbers that contain 7 etc., but in order to do this, one has, among other things, also to check whether 5 belongs to that set of natural numbers whose existence is guaranteed by the axiom of comprehension — and we are very clearly running around in a vicious circle. Notice that the argument does not rely on the fact that the checking procedure requires the testing of infinitely many sets. There are indeed authors who object to such procedures in general, but then they reject not only impredicative conditions but all *indefinite* statements, i.e. statements containing essentially unlimited quantifiers of any type whatsoever; we shall deal with this radical attitude in Chapter V. Here we are concerned with those who, though accepting indefinite conditions, reject impredicative ones. But if indefinite statements are not rejected, probably because a general statement is established not by a case-after-case test — which could indeed

[1]) Cf. Page 54.

[2]) See, e.g., the forceful and witty defense of the meaningfulness of self-referential expressions in Popper 54.

[3]) Carnap 37, pp. 163–164.

not be performed as it would consist of infinitely many operations — but rather by the construction of a proof, which is a finite operation, then the third argument is not very convincing either, since the applicability of an impredicative concept can often be established by a proof procedure. In the above-mentioned example, it is indeed very easy to prove that 5 does not belong to the set determined by the displayed condition.

The fourth argument, and certainly the strongest one, refers to the non-constructive character of impredicatively introduced objects. We can hardly be said to have a clear idea of a totality if the membership of a certain object in this totality is determinable only by reference to the totality itself. Not only is this blurred view unsatisfactory from a philosophical standpoint [1]), it might occasionally give rise to antinomies and will in any case cause great difficulties in the construction of models that would prove the consistency of the system. The last point seems to be the decisive one and the one that has in the last years focussed again the attention of the logicians on impredicative concept formations. Let us recall that the consistency of the ramified type theory (with an axiom of infinity but without the axiom of reducibility) has been proved long ago [2]), whereas similar attempts for the simple type theory have failed, though simple type theory without an axiom of infinity is demonstrably consistent [3]).

The systems of Wang and Lorenzen embody a vicious-circle principle in the sense that an object determined by a condition referring to a totality of objects of layer α is not itself of layer α but rather of layer $\alpha + 1$. Since, however, mixing of layers is no longer forbidden, this does not exclude the possibility of there being a layer to which all objects of layer α as well as the object of layer $\alpha + 1$ determined by a condition on these objects belong. In these systems, this holds indeed for layer $\alpha + 1$ itself and for the infinitely many layers of a higher ordinal. The mixing of layers, however repulsive to thinkers of certain strong ontological convictions, does not reduce the con-

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[1]) Intuitionism (see Chapter IV) rejects, of course, impredicative procedures. Cavaillès 47, p. 17, sees in this rejection the distinctive characteristic of the intuitionistic program rather than in the rejection of the principle of excluded middle. This view is explicitly reaffirmed by Gilmore in his review of van Dantzig 51 in *J.S.L.* **21**, p. 323 (1956).

[2]) *Fitch **4**.

[3]) Tarski ***17** (56, IX), Gentzen 36a, *Beth **4**.

structiveness of the system, nor does the use of transfinite ordinals if one restricts oneself here to the "constructive" ordinals of the second number-class.

In all of mathematics, and especially in analysis, seemingly impredicative conditions abound. For those who are disinclined to look for a justification of this use in spite of its militating against the vicious-circle principle, a different approach has been recently outlined. The Finnish logician Hintikka has shown [1]) that there exists a much weaker way of avoiding universal variables than any theory of orders or layers. Utilizing a suggestion of Wittgenstein [2]), later elaborated by Kolmogorov and Zich [3]), he proposes giving bound variables an *exclusive* interpretation, in contradistinction to the customary *inclusive* interpretation. According to the customary interpretation, the statement, 'There are at least two different individuals having the property P' is symbolized by

$$(\mathrm{E}x)\ (\mathrm{E}y)\ [x \neq y\ \&\ P(x)\ \&\ P(y)],$$

whereas in the exclusive interpretation of the bound variables this statement would be symbolized by

$$(\mathrm{E}x)\ (\mathrm{E}y)\ [P(x)\ \&\ P(y)],$$

with the word 'different' being taken care of by the distinctiveness of the variables. Notice that in the second symbolization, under the exclusive interpretation, the variables are not universal since their ranges are not allowed to coincide. The restriction of the universality is, however, not incorporated once for all in a rigid system of type and order indexes but is induced rather by their interplay within any given specific context.

If we change the logical basis of the ideal calculus (of § 1), i.e. the first-order functional calculus, such as to accord with the exclusive interpretation of the bound variables, then the standard derivation of the antinomies will not come through. The derivation of Russell's antinomy, for instance, will fail because step (2) [4]), which is in accordance with the standard (inclusive) first-order functional calculus,

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[1]) Hintikka 56.

[2]) Wittgenstein 22, 5.53–5.5352.

[3]) Zich 48 which contains also, in a footnote, a reference to Kolmogorov's relevant work.

[4]) On p. 138.

would no longer be legitimate in the revised (exclusive) calculus.

Hintikka's innovation is by no means counter-intuitive. On the contrary, he shows that it accords quite well with certain customary ways of expression. Should the set theory constructed on this basis indeed prove to be consistent, the statement often made that the antinomies prove the self-contradictoriness of common-sense logic would have to be revised. Whereas types, orders, layers, language hierarchies, and the various restrictions in the axiom of comprehension may all be regarded as more or less radical deviations from common sense, this is not the case with the exclusive interpretation of the bound variables and of what corresponds to them in ordinary language.

We noticed (on p. 7) that the "village barber" would lose its paradocxial appearance if the word 'other' were inserted between 'all' and 'inhabitants'. This remark was made in accord with the inclusive interpretation of 'all'. According to the exclusive interpretation, 'all' would, in this context, be interpreted from the beginning as 'all other', and not even the appearance of a paradox would have arisen in the original formulation.

However, it is not necessary to revise the customary first-order logic in order to take account of the exclusive interpretation. The same end can be achieved without tampering with the usual formation rules of this logic. The desired exclusiveness can also be taken care of by explicit diversity conditions on the variables. The axiom schema of comprehension of the ideal calculus, revised on a somewhat different line, would now be

$$C_{ex} \qquad ( ) \qquad (Ey)\,(Ax)\,\{(x \in y) \equiv [x \neq y \supset F(x)]\},$$

where each quantifier '(Ez)' in the original formulation of '$F(x)$' is replaced by '$(Ez)\,(z \neq y\ \&\ \ldots)$' and each quantifier '$(Aw)$' by '$(Aw)\,(w \neq y \supset \ldots)$'. Once more, the standard derivation of Russell's antinomy would not work, though this time for a different reason. All we would arrive at, as the reader might check, would be the true formula

$$(Ey)\,[y \in y \equiv (y \neq y \supset \sim y \in y)].$$

Of course, we may expect to have to pay a price for the interpretation of the axiom of comprehension along the exclusive line, an interpretation that seems to be so efficient in overcoming the antinomies. Indeed, we can avoid here Cantor's antinomy, for instance, only by being unable to derive Cantor's theorem (which should not surprise us any longer).

It may be claimed that C_{ex} is the simplest modification of the axiom of comprehension that accords with Russell's vicious-circle principle, since the entity whose existence is asserted by C_{ex} is the only one which is excluded from the ranges of the bound variables that occur in it. This might well be the cheapest materialization of Gödel's hunch [1]) according to which it should be possible to carry out the idea of limited ranges of significance without having to divide up the objects into mutually exclusive ranges. The distinctness clauses would then correspond to the "singular points" which are mentioned by Gödel in his metaphor. Hintikka's system — if indeed it is elaborated into a full-fledged system of which it will be shown that it is consistent at least relative to some of the better-known systems — would then be even more satisfactory than envisaged by Gödel: not only does it validate our logical intuitions up to minor corrections; these minor corrections themselves are not real deviations from our logical intuitions but embodiments of the exclusive interpretation which is the appropriate one in the case of the axiom of comprehension, though our intuitions are certainly not sharp enough to make this clear without the "help" of the antinomies.

The problem of distinguishing, within mathematical reasonings in their pre-formalization stage, between the occurrence of essential, vicious impredicativities and inessential, innocent ones has always been a rather pressing one, from the first discussions between Poincaré and Zermelo [2]) on this point. Nobody ever seriously questioned the legitimacy of such a concept as the maximum of a function in a given interval, though its standard definition as a greatest of the function values in this interval is clearly impredicative. Poincaré's own way of showing that this impredicativity is not essential is by redefining the maximum of a function as the greatest function value for all *rational* arguments rather than for *all* arguments; even if this function value itself turns out to be rational (and within the interval of the argument values), this is not entailed by the definition as such. Hintikka's way is clearly much simpler and relies on the obvious fact that we can redefine the maximum of a function (within a given interval) as any one of these function values which are not smaller than any of the *other* function values (in this interval), a condition to be formalized with the help of a distinctness clause.

[1]) Gödel 44, p. 150.

[2]) See, e.g., Fraenkel 28, pp. 250–251.

However, it seems that Poincaré went his own way not because he simply overlooked a solution along the line later adopted by Hintikka but because he regarded this solution as not sufficient. The borderline he drew between innocent and obnoxious impredicativities was *purposely* different from that proposed by Hintikka, and Poincaré's vicious-circle principle is stronger than Russell's. It is the more remarkable that Russell's theory of orders accords with Poincaré's version and avoids impredicativities beyond what is required by his own version. Should an ideal calculus, revised along Hintikka's exclusive interpretation, turn out to be either inconsistent or else not strong enough to reconstruct all of classical mathematics — we already saw that parts of Cantor's theory cannot be reproduced, but that these parts are not regarded as indispensable by many mathematicians —, we shall the more admire the intuition of such great thinkers as Poincaré and Russell [1]).

§ 11. Set Theories Based upon Non-standard Logics

The reader will by now have come to appreciate the grave obstacles that stand in the way of safeguarding a set theory that will be both strong enough to allow for the derivation of classical analysis as well as weak enough not to allow for the derivation of antinomies. All the systems of set theory discussed so far have in common that they are based either on classical first-order functional logic or coincide with certain natural extensions of this logic. Their adherents believe that it is either too liberal an interpretation of the term 'well-formed formula' or too liberal a use of the axiom of comprehension which is to blame for the occurrence of the (logical) antinomies.

It is, however, understandable that some authors should have come to blame the shortcomings of existing set theories rather on the underlying logic and should, in consequence, have tried to arrive at a more satisfactory set theory by changing this basic logic. The whole next chapter will be dedicated to the most important of these heresies

[1]) For recent discussions of impredicative concept formation, see Kleene 52, pp. 42 ff, Wang 52, 54, 55, pp. 80 ff, Wang-McNaughton 53, *passim*, Church 56, pp. 346 ff, Beth 56, pp. 44 ff.

Added in proof: In Hintikka 57, it is shown that C_{ex} still leads to contradiction. A somwhat stronger revision of the axiom of comprehension is suggested from which this specific contradiction is not derivable. The resulting set theory, however, even if consistent, would now have same strange features.

— the intuitionistic conception; let it, however, be added immediately that this conception, together with a denial of classical logic, rejects the whole view as if set theory, or mathematics in general, were based upon some underlying logic. In this section we shall discuss very briefly some of the less spectacular attempts.

1. Hilbert's ε-operator. In pursuit of his program of proving the consistency of arithmetic and analysis by finitary means (see Chapter V), Hilbert [1]) developed a calculus which differs from ordinary first-order functional calculus by not containing quantifiers among the primitive notation (nor primitive axioms and rules governing them) but instead a new operator (and a new axiom governing it). The importance of having to deal with a quantifier-free basic logic will be discussed later (in Chapter V), here we shall present only an outline of the new features of Hilbert's ε-calculus itself. (It is somewhat unfortunate that the letter 'ε' is being used in a function which has nothing to do with its standard use as a symbol for class-membership. The danger of confusion will, however, be very slight since within the calculus 'ε' in its new function will always carry with it a right subscript. In addition, a different font will be used.)

In brief, the ε-calculus contains, corresponding to each wff '$F(x)$' free in x, the "ε-term" '$\varepsilon_x F(x)$', governed by the "ε-formula"

$$F(x) \supset F(\varepsilon_x F(x))$$

as new axiom(-schema). From the approximate reading of this formula, "if anything has the property F, then $\varepsilon_x F(x)$ has this property", 'ε_x' is determined to be a "selection-operator" that assigns to each condition on x a "representant" from out of the class of entities satisfying this condition (and some arbitrary entity, if this class is empty).

The advantages of this calculus are many. First, it contains the first-order functional calculus. Indeed, through the definitions

$$(\varepsilon_1) \qquad (\mathrm{E}x)F(x) = {}_{Df}\, F(\varepsilon_x F(x))$$

and $$(\varepsilon_2) \qquad (\mathrm{A}x)F(x) = {}_{Df}\, F(\varepsilon_x \sim F(x)),$$

the two quantifiers can be reintroduced and the standard rules governing them easily derived [2]). Second, we get at no extra cost an

[1]) Cf. Hilbert 23, 28, Ackermann 24, Hilbert-Bernays 34-39$_{II}$, § 1.

[2]) Cf. Hilbert-Bernays 34-39$_{II}$, pp. 15 ff.

equivalent of the definite article (for which otherwise the ι-operator would have to be introduced): if there exists just one entity satisfying '$F(x)$' then clearly '$\varepsilon_x F(x)$' is to be read as '*the* entity having the property F'. Third, ε-terms can replace the so-called "Skolem functions" in those metamathematical investigations in which they play a decisive role.

Recently, Wang [1]) has observed that certain technical drawbacks of some modern developments of the first-order logic, in which the techniques of "natural deduction" and subordinate proofs stemming from Gentzen and Jaśkowski [2]) is used — during the last years the use of this technique has become very popular and is described, for instance, in the elementary textbooks of Quine, Fitch, Copi, and Leblanc [3]), — can be obviated by the use of ε-terms. In view of this observation, the early introduction of ε-terms in the teaching of elementary logic should be seriously considered, in spite of the obvious objections.

There clearly exists a close connection between the ε-formula and the axiom of choice. This connection should however not be overstated, as it is occasionally done, in the form that the ε-formula is a kind of logical (or generalized) axiom of choice. Indeed, the ε-formula allows for a single selection only, while the axiom of choice allows for a simultaneous selection from each member of an (infinite) set of sets and guarantees the existence of the set comprising the selected entities. There is, then, no reason to suppose that in a set theory constructed on the basis of an ε-calculus the principle of choice would become generally derivable, unless the specific axioms of that set theory contain ε-terms themselves.

But is not the strengthening of elementary logic provided by the ε-calculus a rather dangerous undertaking? Whatever the philosophical objections against this procedure — and they are in general similar to those that have been launched against the axiom of choice [4]) — the Hilbert school has been able to prove two theorems, the so-called *ε-theorems*, that relieve to a degree the fear of running into troubles [5]). The second ε-theorem states in essence that if in a theory

[1]) Wang 55, p. 67; cf. Hailperin 57, pp. 124-126.

[2]) Gentzen 34, *Jaśkowski 1.

[3]) Quine 50, Fitch 52, Copi 54, Leblanc 55; cf. Gumin-Hermes 56.

[4]) Chwistek, e.g., tells the story (in Chwistek 48, p. 162) how he himself, during his student days in Göttingen, hit upon a similar idea, wrote to Russell about it but was shown the cold shoulder. Later, the ε-formula served for Chwistek as an indication of the idealistic character of Hilbert's system.

[5]) For a not very technical exposition of the two ε-theorems and their comparison with Gentzen's "main theorem", see Ladrière 51 and 57.

based on an ε-calculus, whose specific axioms do not contain ε-terms, a formula is derivable that does not contain ε-terms either, then there exists a derivation of this formula in which ε-terms do not occur at all. This theorem, then, by guaranteeing the eliminability of ε-terms in derivations fulfilling the stated conditions, assures the consistency of a theory based on an ε-calculus relative to that of the same theory based on standard first-order logic. This relative consistency need of course hold no longer if the axioms of the theory do also contain ε-terms; indeed, the consistency of every set theory in which the axiom-schema of comprehension (in any of its variants) is to hold also for conditions containing ε-terms relative to that in which this is not assumed, has not yet been proved.

Altogether, in spite of the large amount of work spent by the Hilbert school in the investigation of ε-calculi, their exact status is not yet completely clear. It is to be hoped that the revival of interest in them by Wang and others [1]) will lead to further clarification.

2. Leśniewski's Ontology. There is probably no country which has contributed, relative to the size of its population, so much to mathematical logic and set theory as Poland. Leaving the explanation of this curious fact to sociology, we shall dedicate the next two subsections to a brief outline of the contributions of two of the most original Polish thinkers to the foundations of set theory: Stefan Leśniewski and Leon Chwistek. Both of them perished during World War II. The fact that many of the most important writings of these authors were published in Polish, that Leśniewski wrote his German papers in a style that made no concessions to the reader — some of them are printed in an esoteric symbolism with hardly a word of explanation in ordinary language —, and that Chwistek often combined an equally eccentric symbolism with obscure asides on metaphysics and philosophy of mathematics and science in general, did not encourage Western logicians to make special efforts to master their writings, and their untimely death — in the case of Chwistek, together with his most gifted pupils — still increased the difficulties. There can be no doubts as to the high originality of the contributions of these thinkers to the foundations of mathematics, and now, after the recent appearance of various authoritative articles in Western

[1]) A rigorous and detailed treatment of the ε-operator is given in Asser 57. See also Maehara 55, 57 and Helena Rasiowa 56.

languages expounding Leśniewski's teaching [1]), and after the appearance of an English translation of Chwistek's principal work [2]) and of an extremely illuminating review of it, also in English [3]), we seem to stand at the verge of a real revival of interest in the work of these two logicians that has already fertilized the thought of many a worker in the foundations of mathematics.

In this subsection, we shall give the barest outline of that part of Leśniewski's system which is most relevant to the theory of sets; the next subsection will do the same for Chwistek's system.

Leśniewski [4]) called that part of his system which dealt with the logic of the particle 'is', or rather of the Polish particle *'jest'*, *ontology*. Though in a sense this theory deals indeed with "the general properties of being", the term 'ontology' carries no metaphysical connotations. Other authors, in order to avoid explicitly the arisal of such connotations, prefer the term *calculus of names* [5]). Within Leśniewski's total system, ontology occupies an intermediate position: it is based on *protothetics* [6]), a generalized propositional calculus, and serves, in its turn, as a basis for *mereology* [7]) which deals with the part-whole relationship. (Leśniewski has no need for functional logic, as the predicational ingredient of this part of standard symbolic logic is taken care of in ontology while its quantificational ingredient belongs already to protothetics.)

Ontology has just one primitive term in addition to those of protothetics, just one additional axiom, and some additional rules of inference. This primitive term is — 'ϵ', which is however not meant to symbolize the class-membership relation but rather to serve as the symbolic counterpart of the ordinary-language *'jest'* (in Polish) or *'est'* (in Latin), and to a lesser degree of 'is' (in English). Modern logicians have often been at great pains to point out that the *copula* of traditional logic is highly ambiguous and that the English term 'is',

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[1]) Sobociński 49-50, Słupecki 53, 55, Grzegorczyk 55.

[2]) Chwistek 48. The introduction by Miss Brodie should be perused with some reservation.

[3]) Myhill 49.

[4]) See Leśniewski 27-31$_V$, 30.

[5]) This is done, e.g., in Kotarbiński 29.

[6]) For the best exposition of this system, see Słupecki 53. Cf. also Tarski 56, Chapter IV (the English translation of *Łukasiewicz-Tarski I), § 5; Grzegorczyk 55, pp. 80 ff; Church 56, p. 28 (who prefers the term 'prototheticˈ — without 's').

[7]) See, e.g., Grzegorczyk 55, pp. 91 ff, Sobociński 55, Lejewski 55.

for instance, stands for class-membership, class-inclusion, and identity, respectively (and fulfils perhaps still other functions on occasion) in such statements as 'Socrates is wise', 'The dog is an animal', and 'Socrates is the husband of Xantippe'. Leśniewski took the contrary view that the Polish *'jest'* and the Latin *'est'* should be regarded as non-ambiguous particles, whatever the situation in English. (The difference is caused by the fact that English — in contradistinction to Polish and Latin — has articles, definite and indefinite, which are obligatory in the contexts 'Socrates is ... man', 'The dog is ... animal', 'Socrates is ... husband of Xantippe' — with statements like 'Socrates is white' and 'Tully is Cicero' still more complicating the situation — whereas the corresponding Latin statements would be *'Socrates est homo'*, *'Canis est animal'*, *'Socrates est coniunx Xantippae'* — and *'Socrates est albus'*, *'Tullius est Cicero'*. We shall spare the reader the corresponding Polish statements.)

Leśniewski is completely successful in providing through his axiom his own term 'ϵ' with a meaning which is a kind of conglomerate of the three mentioned meanings, without falling into contradiction. On the contrary, the consistency of ontology relative to protothetics can easily be proved. Leśniewski's original single axiom of ontology [1]), after some notational adaptations (which would however not have been approved by Leśniewski himself) is:

$$x \epsilon w \equiv (\mathrm{E}y)\,(y \epsilon x)\ \&\ (\mathrm{A}y)\,(\mathrm{A}z)\,[(y \epsilon x\ \&\ z \epsilon x) \supset y \epsilon z]\ \&$$
$$(\mathrm{A}y)\,(y \epsilon x \supset y \epsilon w).$$

Since all symbols in this axiom, with the exception of 'ϵ', are supposed to have their standard interpretation, we see that a Leśniewskian ϵ-statement is true only if to the left of the 'ϵ' stands a non-empty, singular name; this follows from the first and second conjunctive components of the right side of the equivalence, respectively. Accordingly, such statements as *'Hamlet est albus'* or *'Canis est animal'* are treated as false, the first since *'Hamlet'* is empty (i.e. not denoting), the second since *'canis'* is a general name.

That Leśniewski's theory of semantical categories should by no means be regarded simply as a kind of generalized linguistic counterpart of a simple theory of types [2]) is obvious from the fact that the symbols flanking the Leś-

[1]) Cf. Słupecki 55, p. 20.

[2]) This is the impression one could get from a not too careful reading of Tarski 56, pp. 213 ff, or Sobociński 49-50 who often uses the expression 'catégories sémantiques (types logiques)' — and Tarski and Sobociński are two of

niewskian 'ϵ' — and similarly with their ordinary-language counterparts — belong always to the same semantical category, even when the right-hand symbol is a class-symbol: *'Socrates'* and *'homo'*, in the statement *'Socrates est homo'* belong to the same semantical category, i.e. to the category of names, whereas for Russell the denotata of these two words, i.e. Socrates and the class of men, belong to different types.

Since Leśniewski's 'ϵ' is not meant to be a symbol for class-membership, it is preferable to regard his ontology not as a variant of set theory but rather as a rival of set theory for the foundations of mathematics. This view does not exclude that counterparts of many set-theoretical axioms turn out — under a certain notational transformation — to be ontological theorems or that the ontological axiom should be transformed into a type-theoretical axiom. The latter possibility is easily materialized by interpreting '$x \epsilon w$' as 'x is a unit-class of individuals, w is a class of individuals, and $x \subseteq w$'.

How important a rival of set theory ontology is, or could be made to be, is a question which it is still very difficult to decide. But Leśniewski has convincingly shown that the standard arguments leading to the logical antinomies cannot be reproduced, in any of the various plausible rephrasings, in his system; some of the counterparts of these arguments fail to comply with the theory of semantical categories, others require certain steps which are not viable in ontology [1]).

Let us finish this sketch by noting that what might be regarded as the ontological counterparts of the axiom of comprehension are treated at length by Leśniewski under the name of *pseudo-definitions* [2]).

3. The Systems of Chwistek and Myhill. After continuous struggles and many changes of mind, Chwistek — the highly original Polish logician whom we recall as the first influential critic of Russell's ramified type theory [3]) — finally constructed a system which, had it been presented in a less obscure language and a somewhat less forbidding terminology and symbolism, might well have exerted a decisive influence on the reconstruction of mathematics on a consistent "constructive" basis. This final system of Chwistek's was

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the most outstanding pupils of Leśniewski. Cf., however, Tarski 56, p. 213 n, from where it should be clear that Tarski is fully aware of the real situation.

[1]) For an exhaustive discussion of this issue, see Sobociński 49-50.

[2]) For the significance of Leśniewski's use of pseudo-definitions for the formulation of the second-order functional calculus, see Henkin 53.

[3]) See footnote 1 on p. 153.

published posthumously in *The Limits of Science* in 1948. This book is so different from the Polish original, *Granice Nauki*, published in 1935, that it should by no means be regarded as a mere translation of it. Fortunately, Chwistek has now found an equally competent interpreter in the American logician Myhill who was able to bring Chwistek's ideas closer to the main present currents of thought. Even so it is still impossible in the available space to describe Chwistek's approach in any serious detail. Let us therefore content ourselves with stating that Chwistek's set theory is a kind of *"inverted" ramified type theory*, in the sense that classes always have members of "higher" types than themselves. The types are, in addition, cumulative — as, e.g., in Wang's system Σ (§ 6) or in some of the systems discussed by Quine — but again with the direction inverted, each entity belonging to all types lower than any type to which it belongs. By adding to this ramified type theory a certain variant of the axiom of reducibility, Chwistek obtains what he calls a *pure theory of types* which can easily be proved to be consistent; by adding another variant he gets a *simplified theory of types* which, however, combined with a certain powerful metasystem is inconsistent. Chwistek's pure theory of types, in its last stages, looks sufficiently similar to ***PM*** to make it plausible that classical analysis can be constructed on its basis. A completion of this program would therefore mean a proof of the consistency of classical analysis from "nominalistic" assumptions whose soundness can hardly be doudted in any serious sense.

Myhill himself, in an unfinished series of papers [1]), tried to complete the program. His own system is an inverted cumulative type theory with an all-inclusive type 0 but with no highest type. Vicious-circle paradoxes are avoided by arranging that no expression containing a quantified variable be a value of that variable. (In Myhill's — as in Chwistek's — system, expressions, not what the expressions denote, are values of variables; this curious feature is consistently carried through without involving him — or Chwistek, contrary to the general opinion — in fallacious thinking due to the "confusion between use and mention of signs".) He uses a *non-finitary* consequence relation according to which certain formulae are regarded as consequences of appropriate classes of formulae. This means, of course, that the system, though nominalistic in a sense, is anything but

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[1]) See Myhill 49, 51, 51a.

constructivistic. Its consistency can be shown, along the patter n of a proof by Fitch (p. 152, footnote 7), by transfinite induction (up to ε_0, at least). A certain analogue of the axiom of reducibility can be shown to hold in the system. He is finally able to derive in the system (type-restricted) analogues of Bourbaki's axiom system for set theory [1]) (plus an axiom of substitution) including the axioms of choice and infinity, but excluding the axiom of extensionality. Interestingly enough, Myhill believes it to be very unlikely that analysis needs a principle of extensionality, probably being influenced in this by Chwistek who regarded it as a product of metaphysical idealism. It seems, however, that so far Myhill has not been able to accomplish the proof that a non-extensional Bourbaki set theory contains a model of extensional Bourbaki set theory [2]).

4. Fitch's System. After years of constantly refining earlier attempts, F. B. Fitch published a textbook on *Symbolic Logic* [3]) in which he developed a new, and in many respects heterodox, approach to this topic, promising to publish a sequel dealing with the detailed derivation of the more important theorems of mathematical analysis from his system [4]). There is no need to deal here at length with its many notational and pedagogical innovations. Suffices it to state that Fitch's system contains no variables, either free or bound — e.g., the class of entities greater than 2, customarily symbolized by '$\hat{x}(x > 2)$' or '$\lambda x(x > 2)$', is rendered by Fitch as $(3/3 > 2)$ or even (Caesar/Caesar > 2) —, no type restrictions but, on the other hand, no general rule of excluded middle or law of extensionality, the last feature backed up by an extremely narrow treatment of identity, according to which an identity sentence is valid if and only if its two sides, in their disabbreviated form, are notationally the same, in other words, by avoiding the occurrence of notationally different but logically synonymous sequences of primitive terms. (This requires, among other things, to treat one of the just mentioned expressions '$(3/3 > 2)$' and '(Caesar/Caesar > 2)' as an "abbre-

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[1]) Viz., the system presented in Bourbaki 49.

[2]) This was one of the aims he set himself in Myhill 51a, p. 135.

[3]) Fitch 52. For the earlier attemps, see *Fitch 8, 9, 10.

[4]) Among Fitch's later papers that contain further developments of his system of Basic Logic — without yet amounting to a fulfilment of the promise —, we mention only Fitch 56, footnote 1 of which contains further references.

viation" of the other, or both of them as "abbreviations" of a certain arbitrarily chosen expression of the form '$(\ldots/\ldots > 2)$'.)

Antinomies are avoided either through the fact that the relevant derivations fail to satisfy the mentioned restrictions on the proof procedures — this is the way in which Curry's paradox [1]), a variant of Russell's paradox of particular interest insofar as it does not use negation, is overcome — or through the simple device of regarding each such antinomy as a proof of the fact that the proposition involved fails to satisfy the excluded middle. Russell's paradox, e.g., in its original version but in Fitch's notation, deals with the class (or attribute) $(x/\sim[x \in x])$ (where 'x' is *not* a variable) — to be denoted by 'Z' — and winds up by proving that $[Z \in Z] \equiv \sim[Z \in Z]$. But this equivalence is not paradoxical by itself, it becomes so only under the assumption that the proposition $Z \in Z$ satisfies the excluded middle, i.e. that $[Z \in Z] \vee \sim[Z \in Z]$ holds; in that case the really paradoxical proposition $[Z \in Z] \,\&\, \sim[Z \in Z]$ can be easily derived from the equivalence. Since Fitch claims to have shown the consistency of his system, he is able to conclude that $Z \in Z$ is one of those propositions for which the principle of the excluded middle does not hold and which he terms *indefinite* — which should be clearly distinguished from Russell's term *meaningless* [2]). The Liar antinomy is overcome analogously. What this antinomy shows is that the proposition expressed by the sentence, "This proposition itself is false" is indefinite. No structural criterium by which sentences expressing definite propositions would be distinguished from sentences expressing indefinite propositions is given. This is, of course, a drawback, though not so much with regard to ordinary language, for which no more effective criterium should be expected [3]), as with regard to language systems; still it is not fatal, especially if such a system is demonstrably consistent, since in this case at least a sufficient condition for the indefiniteness of a proposition is provided, i.e. that from the assumption

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[1]) This paradox was created in Curry 42. The role of Russell's class of all classes that do not contain themselves as members is taken over there by the class of all classes such that if they contain themselves as members, an arbitrary statement is true. For a full formal derivation of the paradox, see Fitch 52, pp. 107–108.

[2]) As well as, of course, from Carnap's term 'indefinite' mentioned above, p. 177.

[3]) Cf. Bar-Hillel 57a.

of its definiteness a contradiction can be derived. Incidentally, Fitch has a *rule of excluded middle for identity*, according to which every proposition of the form $[a = b]$ is definite.

Fitch's main reasons for rejecting Type Theory are that, on the one hand, certain sorts of self-reference are required for philosophical logic as well as for the development of the theory of real numbers and that, on the other hand, this theory cannot be stated without violation of its own requirements [1]).

Fitch claims that his system is demonstrably consistent. This is achieved through certain restrictions on the proof procedures. These restrictions, however, whatever their intrinsic plausibility, have the effect that one cannot always conclude from the validity of an implication and of its antecedent to the validity of its consequent or from the validity of two statements to the validity of their conjunction but must, in general, take into account the way in which the premisses of these inferences have been obtained. Part of the meaning of a formula would then reside in the specific way of its derivation — which is a rather esoteric idea, to say the least [2]).

In addition, the consistency proof provided by Fitch for his system seems not to be fully constructive, hence in a sense circular. In one decisive place, he uses, metalogically, an argument amounting to the transition from '$(\mathrm{A}x)\ (p \vee F(x))$' to '$p \vee (\mathrm{A}x)F(x)$', which is not constructively valid (and is rejected, for instance, in intuitionistic logic). This metalogical argument is, incidentally, used by Fitch in order to show that an analogous type of argument, *within* the system, is eliminable [3]).

Fitch's system is one of the many erected during the last decades to fight the antinomies through a departure from the classical first-order functional calculus. In Fitch's case, the departure takes the form of constructing the propositional calculus in such a way that the principle of the excluded middle is not generally vaild in it. This, of course, recalls at once intuitionistic logic, to be treated

[1]) This objection has often been made, e.g. by Weiss 28, and as often refuted. Though it certainly is not sound from a purely formal standpoint, it retains a certain heuristic value and is probably one of the strongest reasons why so many thinkers do not like type-theoretical systems. Cf. Gödel 44, p. 149.

[2]) A related idea, however, might prove to be very promising in linguistics; cf. Chomsky 57.

[3]) All these criticisms were launched by Ackermann in his review of Fitch 52 in *J.S.L.* **17**, pp. 266-268 (1952).

at length in the next chapter. Fitch's logic, however, is by no means identical with intuitionistic logic as formalized by Heyting [1].)

Neither Fitch's logic nor the intuitionistic one are, of course, two-valued. Nor are they three-valued, indefiniteness — in Fitch's sense — not being a third truth-value that could possibly be treated on a par with truth and falsity. Three-valued and, more generally, many-valued logics will be treated in the next subsection.

5. Many-valued Logics. It seems rather natural to look for the culprit in the arisal of antinomies in a still different direction: in most, if not all, antinomies the crucial contradiction has, or can be given, the form of an equivalence between a certain statement and its negation. The final line of a derivation of Russell's antinomy, for instance, is usually some variant or abbreviation of

$$[\hat{x}(x \notin x) \in \hat{x}(x \notin x)] \equiv \sim [\hat{x}(x \notin x) \in \hat{x}(x \notin x)],$$

the final line of a derivation of Grelling's antinomy is something like 'heterological' is heterological $\equiv \sim$ ('heterological' is heterological), etc. Now, such statements embody contradiction if the logic of the language in which they are formulated is the classical two-valued logic, in which any statement of the form

$$p \equiv \sim p$$

is indeed self-contradictory. But in some many-valued logics — i.e. logics in which the meta-principle that every (closed) statement has exactly one of two truth-values does not hold — and under some natural interpretations of 'negation' and 'equivalence', statements of the mentioned form are no longer self-contradictory. In certain three-valued logics, such as the famous system $\boldsymbol{L}_3$ of Łukasiewicz [2]), the negation of a statement with the "intermediate" truth-value has itself the intermediate truth-value so that in this case a statement is equivalent (in truth-value) to its negation.

Are then many-valued logics antinomy-free? This would be a rash conclusion. Though the derivation of such paradoxes as Russell's and Curry's would indeed not come through [3]) it can be shown that in a set theory allowing for unrestricted comprehension and based

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[1]) There are many resemblances between Fitch's system and that developed by Ackermann 50; cf. also Ackermann 52-53 and Grize 55.

[2]) See Tarski 56, Chapter IV, 3.

[3]) See, e.g., Bočvar 39,43.

upon certain of these logics, e.g. on the mentioned $\boldsymbol{L}_3$, a Curry-type paradox is derivable [1]).

The shift from the term 'antinomy' to 'paradox' that already occurred above, p. 191, in connection with the first mentioning of Curry's paradox, is not accidental. The term 'antinomy', as defined on p. 1, does not apply as such to negation-free or many-valued logics. Curry's paradox consists not in proving that two contradictory statements are equivalent — notions which obviously involve (two-valued) negation — but in proving any statement whatsoever from seemingly innocuous procedures, thereby proving the system to be (formally) inconsistent (see Chapter V, §4).

However, the avoidance of antinomies was by no means the only incentive that made logicians occupy themselves with many-valued logics. Many logicians treated such logics as pure calculi without insisting from the beginning on interpretation and application [2]). Others believed that such logics might be of use in the analysis of certain epistemologically puzzling situations such as the assignment of truth-values to future contingent events [3]). Still others thought that some of these logics were better suited to deal with certain phenomena in quantum physics than classical two-valued logic [4]).

No serious attempt has been made so far to construct a set theory or a theory of numbers on a many-valued logic. There seems to exist no conclusive apriori reason why certain many-valued logics should not prove to be fruitful and provide a secure foundation for set theory and mathematics. So long, however, as the proponents of these logics have not come forward with a full-fledged set theory or arithmetic, the *onus probandi* doubtless rests upon them, in view of the fact that these theories will certainly become much more complex than they are today. Until then, it is best to refrain from any final evaluation [5]).

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[1]) See Shaw-Kwei 54, Prior 55.

[2]) Cf., e.g., Rosser-Turquette 52, Introduction.

[3]) See *tukasiewicz **3** and, among recent publications, e.g. Prior 53.

[4]) See Reichenbach 44 and, out of the many writings by Destouches and Paulette Destouches-Février dealing with this topic, e.g. Destouches-Février 51a (with the sharply critical review by McKinsey and Suppes in *J.S.L.* **19**, pp. 52–55 [1954]). For a recent attempt to avoid the antinomies on the basis of a "logic of attitudes" showing some resemblances to that of Mme. Destouches-Février, see Moch 56, 56a, 56b.

[5]) After this chapter was sent to the printer, Skolem 57 came to the attention of the authors. There it is shown that a set theory containing an axiom of comprehension, in which no quantifiers occur but otherwise unrestricted, can be erected without contradiction on the basis of an infinite-valued logic.

6. Combinatory Logic. We shall finally just mention that there exists a method of developing logic which is radically different from the „standard" ones. The so-called Combinatory Logic was founded by Schönfinkel [1]) and Curry [2]) and is a highly interesting variant of modern symbolic logic. Since, however, its impact on the foundations of set theory does not seem to be very great, we shall here only call attention to a recent publication by Cogan [3]) in which a set theory is formalized from the point of view of combinatory logic, after an illuminating presentation of the essentials of this logic itself.

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[1]) *Schönfinkel 1.

[2]) *Curry 2.

[3]) Cogan 55. Curry-Feys 58 was not yet available when this chapter was sent to the printer.

CHAPTER IV

INTUITIONISTIC CONCEPTIONS OF MATHEMATICS

§ 1. Historical Introduction. The Abyss Between Discreteness and Continuity

The axiomatization of set theory (Chapter II) undertakes to avoid the antinomies arising from the classical theory by *restricting the concept of set* or its mathematical use to such an extent as is suitable for that aim, but without fundamentally modifying the structure of the theory. Logicism (Chapter III) conceives the antinomies as a danger signal which not only touches set theory but shows that something in the mathematical methods is out of order; therefore logicists attribute the defect to logic and its use in mathematics rather than to mathematics itself and propose a penetrating *reform of logic* which, as one of its incidental consequences, entails the elimination of antinomies.

The attitude that is described in the present chapter is much more radical in its conception as well as in the consequences following from it. The mathematical schools which take this attitude maintain that the *concept of infinity* and its implications have not been sufficiently pondered in mathematics, especially in modern mathematics — though they admit that for mathematics, in contrast with practically all other sciences, infinity is the very lifeblood to such an extent that most mathematical facts not referring to infinity are almost trivial. Yet analysis and geometry as developed since the 17th century and especially since the beginning of the 19th century have — so they argue — utterly disregarded the peculiar traits of infinity and their consequences for mathematics, whereas the supposedly strict methods introduced into real number theory and calculus during the 19th century [1]) from Cauchy to Weierstrass and Cantor, far from reaching the desired goal, have rather raised to an elaborate system the erroneous tendency of treating infinity with methods created for finite domains.

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[1]) For the interrelationship between infinity and the demand for strictness in mathematical proof during various periods, cf. *Pierpont **2.**

Thus a completely wrong conception of mathematics as a whole would have arisen.

According to this view the antinomies appearing at the turn of the century are but a secondary symptom, evolving at a rather accidental spot; actually they are caused by the brittleness of much more "important" branches than set theory or logic. That contradictions emerged just in set theory is due to the fact that in no other branch of mathematics has been made so abundant and unlimited use of infinity. All the same, the contradictions are due to traditional dealing with infinity *in general* and not to its application in the degree involved by set theory. Therefore the significance of the antinomies as warning signals can be met only by a reform of mathematics as a whole; this will automatically exclude not only the antinomies actually found hitherto but any conceivable antinomy.

In addition, the concept of infinity in set theory assumes specific significance in connection with one of the oldest and most intricate notions of science in general, viz. *continuity*. True, the continuum is the very domain to which also analysis and most of geometry refer. However, in these domains the continuum is presumed from the first as a *basis* while set theory ventures to *construct* the continuum in a way which appears as a special case of a more general method (power-set, diagonal method, Cantor's theorem; cf. *T*, pp. 63, 94–97, 151), a method which belongs to the strongest and at the same time to the most daring procedures of set theory. As the starting-point of this procedure serves a *discrete* (discontinuous) aggregate, e.g. the denumerable set of all integers or of isolated points of a line. (Here the order-ingredient of the continuum is disregarded; its inclusion, as in *T*, p. 228, does not change the picture.)

To be sure, the starting-point mentioned is itself an infinite set. But this type of infinity — in one or the other form of conceiving it; for instance, through mathematical induction — is the very foundation of mathematics. Since it is the *conditio sine qua non* of mathematical reasoning the problem is not *whether*, but *how*, we should accept it, and various opinions on this question are displayed in the present book: axiomatic, Platonistic, logicistic, intuitionistic, metamathematical opinions or methods.

Bridging the abyss between the domains of discreteness and of continuity, or between arithmetic and geometry, is certainly a central, presumably even *the* central problem of the foundation of mathe-

matics [1]). Cantor claimed to have bridged the abyss, as in some way had already claimed "classical" analysis; the sharpest criticism of these claims has been expressed by the intuitionistic schools.

To understand the nature of the problem one should stress the fundamental difference [2]) between the *discrete, qualitative, individual* nature of *number* in the "combinatorial" domain of *counting* (arithmetic) and the *continuous, quantitative, homogeneous* nature of the points of *space* [3]) in the "analytical" domain of *measuring* (geometry). Every integer differs from every other in characteristic individual properties comparable to the differences between human beings, while the continuum appears as an amorphous pulp of points each of which is interchangeable with each other. (Therefore it is not astonishing that the axiomatic method has primarily turned to space and not to number whose character rather challenges construction.)

Bridging the abyss between these two heterogeneous domains is not only the central but also the oldest problem in foundation of mathematics and in the related philosophical fields. The existence of incommensurable magnitudes discovered in the fifth century B.C. and for a while kept secret by the experts lest it might offend the public, had opened the first crisis in foundations of mathematics, especially in the Pythagorean school [4]). This crisis found an apparently satisfactory settlement through the Greeks themselves, while the second crisis, originating from the treatment of continuity in calculus and modern analysis in general, seemed to have been overcome in the sixties and seventies of the 19th century. Kronecker's fanatic challenge (p. 202) to recognize integers only and to wipe out of mathematics all other numbers remained then a voice in the wilderness.

As mentioned before, one is in the habit of dating back *the third crisis in the foundations of mathematics* (which is not yet over) to the

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[1]) See *Chevalier 1 where the problem is conceived in its generality, not only in its mathematical and physical aspects. Cf. also *Barry 1, *H. T. Davis 1, *Fraenkel 16 and 20, *Jasinowski 1, *Jørgensen 3, *Rebec 1.

[2]) Cf. *Freudenthal 1, *Weyl 9 and 11; in the first paper also a musical illustration is given.

[3]) Of course, one need not conceive just the *point* as the element of space. Moreover, the same character seems to apply to *time*. ('Point' can evidently be replaced by 'real number'.)

[4]) Aristotle points out the contrast between the discrete notions of reason (thought, counting) and the notions of the external world which seem to be continuous.

antinomies of set theory, i.e. to the turn of the century. However, the real shake began a few years later, with the reactions to Zermelo's first proof of the well-ordering theorem (1904; cf. p. 75), and from 1907 on with the neo-intuitionistic frontal attack against classical mathematics which stands in the center of the present chapter. Now not only set-theoretical antinomies but contradictions on the whole are shrinking to mere *symptoms* of the inadequacy of classical mathematics, which naturally is first visible at the outskirts of the mathematical field; yet without the emergence of contradictions the situation would have been no less disastrous.

In the course of the discussions carried on during the last decades it turned out more and more how closely the present difficulties are related to those that twice seemed to have been overcome, viz. the riddles of the Pythagorean and Eleatic schools and the difficulties which arose in the French and German centers of the theory of functions. Though the arguments have changed the abyss between discrete and continuous is again the weak spot — an eternal point of least resistance and at the same time of overwhelming scientific importance in mathematics, philosophy, and even physics. Exactly those arithmetical theories of real number which a generation before seemed to be the basis of analysis proved now to involve the intuitionistic objections.

Incidentally, it is not obvious from the first which of these two regions, so heterogeneous in their structures and in the appropriate methods of exploring, should be taken as the starting-point. Certainly the discrete admits an easier access to logical analysis, and the tendency of arithmetization [1]) already underlying Zenon's paradoxes has been impressing its mark upon modern mathematics and may be perceived in axiomatics of set theory (Chapter II) as well as in metamathematics (Chapter V). However, the converse direction is also conceivable, for intuition seems at once to comprehend the continuum; principally for this reason Greek mathematics and philosophy were inclined to consider continuity to be the simpler concept and to conceive combinatorial concepts and facts in an analytic view. Whereas some traces of the latter attitude are still visible in some intuitionistic opinions and not even missing in Brouwer's neo-intuitionism (cf.

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[1]) *Lichtenstein 1 (in particular p. 191) incorporates this tendency in the basic ideas of E. Meyerson's philosophy of science.

below § 5) intuitionism in general considers arithmetic, say in the shape of mathematical induction, not only as the primary concept but as the origin of mathematics on the whole (§ 5); it therefore tends to restrict analysis by imposing upon it "combinatorial" methods.

Such intuitionistic restriction of the concept of continuum and of its handling in analysis and geometry, though carried out in quite different ways by various intuitionistic schools, always goes as far as to exclude vital parts of those two domains. (This is not altered by Brouwer's peculiar way of admitting the continuum *per se* as a "medium of free growth".) On the other hand, intuitionists maintain that the restricted system covers all legitimate mathematical processes.

To establish the *necessity* of their restrictions, intuitionists critically analyze the methods of classical mathematics. In particular they claim that the dangers inherent in infinity, as emerging in 18th century's mathematics and further aggravated by Dirichlet's arbitrary function concept, have but seemingly been checked by the "classical" theories of real number, limit, continuity, integral, etc.; these theories are charged with logical circles and exaggerated generality which necessarily leads to contradictions. According to them no actual progress towards guaranteeing the solidity of classical mathematics has been achieved in the 19th century.

On the other hand, it is maintained that the restricted system is *sufficient* for the needs of what may justly be called 'mathematics' — a notion taken, to be sure, in a sense quite different from the traditional one. This claim shall be justified — yet has so far been justified to a very limited extent only — through basing arithmetic and the "meaningful" parts of analysis (including geometry) and of set theory upon the mathematical constructions intuitionistically admitted (cf. below, §§ 5 and 6).

Thus, in striking contrast with the "conservative" steps taken in Chapters II and III to avoid the sources of danger revealed by the set-theoretical antinomies, the present chapter deals with a revolutionary direction meant to alter to the bottom the essence of mathematics and its methods, at least in comparison with the tradition of the last three hundred years. Moreover, if the intuitionistic attitude should oust the classical view it might take generations to save, and to firmly base with intuitionistic methods, those parts of mathematics which do not become meaningless or false according to the new conception.

The group of mathematicians who call themselves *intuitionists* or *neo-intuitionists* [1]) is composed of various and widely diverging trends which, save for few exceptions dating back to the 19th century, arose with the beginning of the present century. The name does not derive from the intuitively creative nature of mathematical discovery or invention [2]) which is accepted universally but from the "primordial intuition" explained in § 5. One may trace back hints of this attitude to earlier periods of mathematical research, even to Greek antiquity [3]).

The number of scholars who adhere to intuitionistic principles, let alone those who actually adjust their methods of mathematical research to intuitionistic restrictions, has been but small so far and shows little tendency of increasing. But it is highly remarkable that they include some of the outstanding mathematicians of the last generation from various countries, as is shown by the names of Kronecker,

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[1]) The latter name is used by Brouwer and his followers. In French literature one often finds the name 'pragmatist' (so Poincaré) or 'realist' while the opponents are called 'idealists'; this terminology may lead to confusion in view of the Platonistic use of the term 'realist'. Some authors who use the term 'empirism' (e.g., *Rivier **1**) seem to misunderstand the intuitionistic credo.

The name *formalists* for the opponents of intuitionism, particulary employed for Hilbert's school of axiomatics and metamathematics, chiefly originates from polemic intentions and does not appropriately characterize that school. In fact "cantorians" or logicists are far more contrary to intuitionism than those "formalists", who in some regards are more finitistic than intuitionists.

[2]) A mathematician as remote from intuitionism as Hadamard beautifully describes (in ***4**) the intuitive character of mathematical creation.

[3]) Cf. *P. Boutroux **3** (especially Chapter IV), *O. Becker **2**, *Ant. Reymond **1**; also *Lamouche **1** and Hadamard's preface to Gonseth 26. According to the analysis of Aristotle's theory of science as given in Scholz 30 (cf. O. Becker 36), the attitude of this classic of ancient logic was rather neo-intuitionistic: he would have regarded intuitionistic mathematics as an *ἐπιστήμη*, mathematical fields beyond this sphere only as a *δόξα*. Also other Greek scientists, including Euclid, display an intuitionistic touch and stricly distinguish between constructibility and abstract existence; see, for instance, O. Becker 33. For the distinction in antique *geometry* cf. *O. Haupt **2**.

This attitude regarding the antique philosophy seems to contradict Brouwer's thesis (see § 3) that classical logic obtained its principles by abstraction from the properties of finite aggregates.

Among the forerunners of modern intuitionism one may also mention I. Barrow, the teacher of Newton, for his criticism of Euclid's theory of proportions.

Poincaré, Borel, Lebesgue, Brouwer, Weyl, Skolem [1]). To a large extent these and other mathematicians independently arrived at their daring ideas, forced as it were by an inability to adapt themselves to the traditional way of mathematical thinking; it is still more surprising that in spite of their relative isolation they showed themselves quite convinced of the final victory of their attitude [2]). It is as if those ideas were at a certain time in the air where, however, only people with a proper scent could track them. The arguments are sometimes just intuitive and dogmatic, at other times based on philosophical motives, and again using strictly mathematical reasons. Accordingly, the various intuitionistic trends are connected by a loose affinity of their basic ideas only while considerably diverging in the further pursuance of these ideas.

For the first time in modern mathematics intuitionistic ideas appeared in Berlin in the seventies and eighties of the 19th century with Kronecker and a few of his pupils [3]). Their fight against the modern methods in the theories of real numbers and functions, as developed especially by Weierstrass and his school, proved on the whole unsuccessful, in spite of Kronecker's great authority; its principal and tragic effect was a considerable suppression of Cantor's ideas during two decades. The theories of real numbers, of functions, and of sets emerged victorious and, owing to their triumphal progress all over analysis, was not even the alarm brought on by the antinomies of set theory at the beginning of the present century felt initially, except at the very outskirts of mathematics.

However, with the first proof of the well-ordering theorem (1904) there began a general attack principally on the part of French scholars, including a number of analysts who had themselves taken an active part in the application of set theory to the theory of functions, such

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[1]) Also the highly gifted British mathematician F. P. Ramsey (who chiefly dealt with foundations problems and died in 1930 at the age of 27) at his final stage took an inclination to intuitionism (according to *B. Russell **10**) after having opposed it in his earlier publications. Cf. Herbrand's attitude.

[2]) This already applies to Kronecker; cf. *Gutzmer **1**, p. 592. Cantor however retorts: This is, so to speak, a question of strength ... it has to turn out *whose* ideas are more powerful, more comprehensive, more productive, Kronecker's or mine; the success only will decide our conflict in due time. (*Schoenflies **12**, p. 12.)

[3]) See, in particular, Kronecker 1887. — Hölder's criticism of the arithmetical theories of the continuum dates from 1892; cf. *Hölder **3**, p. 194.

as Baire, Borel, Lebesgue [1]; distinguished foreign mathematicians, e.g. Lusin, later joined this *Paris School* of intuitionists, while others, for instance Pasch, independently took a related line. Poincaré even ventured to reject set theory in general and to charge the methods of many classical branches of mathematics with the illicit use of impredicative procedures (pp. 181 f) [2]; despite the weakness of many arguments in his attacks on Zermelo as well as on B. Russell, the authority of the foremost mathematician of his generation contributed to creating an atmosphere of alarm and to preparing the ground for the assaults of the Dutch school. In particular the objections against the axiom of choice raised by almost all intuitionists — except for Poincaré whose view in this respect and in the question of mathematical existence in general is *sui generis* [3] — attracted the attention of wider circles, even when many other scruples began losing ground in view of the successful axiomatization of set theory.

In 1907 the Thesis of Brouwer marks the first step in a distinct intuitionistic direction for which he later introduced the term *neo-intuitionism*. Besides several of his pupils in Amsterdam — hence the name *"Dutch School"* — a few others joined his circle, in particular (during the twenties) Weyl, who before had taken an attitude of his own [4] which in some respects resembles the Paris school. Especially in the decade beginning with 1918 Brouwer unfolds his banner to an impetuous attack on traditional mathematics and to a thoroughly new foundation of analysis and set theory, which for some time deeply alarms the mathematical world and provokes no less vehement

[1] Yet most scholars of this trend, in contrast with neo-intuitionists, were not eager to apply their principles in the practice of their own analytical researches. — F. Kaufmann's approach is related to that of the Paris School.

[2] However, the attitude taken by Poincaré (who died in 1913) during the last years of his life was far less radical. One should remember that in 1910 (see *4) he gave a new proof for one of the strongest theorems of set theory, the non-denumerability of the continuum (*T*, p. 65). (It is misleading that the posthumous collection of essays (13), some of which had appeared long before, bears the title *Denières Pensées*.) — Here we may disregard the conventionalistic ingredient of Poincaré's mathematical philosophy, which is essentially restricted to geometry (and physics); cf., for instance, *von Freytag 1, *Rougier 2, *Uta 1 (pp. 97–145).

[3] Cf. Poincaré 08, Chapter V, part III.

[4] See Weyl 18 and 19 and the first part of 21. Later, however, he swerved considerably from neo-intuitionism, in particular from its intolerance of other opinions; cf. Weyl 26 (§§ 10/11) and his remarks at the end of Hilbert 28.

reactions, occasionally producing something like Homeric talks; quite a few of his opponents prove themselves considerably influenced by the new ideas. (While this is usually pointed out for Hilbert and his school from the 20's on, one is in the habit of forgetting that Hilbert's earliest research in logic [1]), which preceded Brouwer's Thesis and then remained almost unnoticed, might be regarded as an intuitionistic program more radical than neo-intuitionism.) Brouwer's principles are far more revolutionary than those of earlier intuitionists, though in the theory of continuum he is less rigid. This is the reason why his attitude found a good deal of both echo and opposition in philosophic circles.

Describing separately the various trends of intuitionism and their implications with regard to the extent of "legitimate" mathematics would require a lengthy exposition. We shall therefore essentially exhibit the *principles of neo-intuitionism* and their consequences while including other trends in the literature and describing some of their chief ideas. The features in which *neo-intuitionism in principle differs from other intuitionistic trends* are chiefly the following:

A) The conception of the *essence of mathematics* and of mathematical existence, involving Brouwer's attitude towards the relation of mathematics to *language and logic* (§§ 2–4).

B) Brouwer's *theory of the continuum*, in particular the *free-choice sequences* (§ 5), based on his peculiar definition of set and species.

Literature dealing with modern mathematical intuitionism in general, both from the mathematical and the philosophical points of view, is listed here with an endeavour at comprehensiveness [2]).

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[1]) *Hilbert 5 (1904).

[2]) Papers cited elsewhere in the present chapter are quoted in this note only if they are particularly significant. Here we use alphabetical order since the chronological moment has little weight. The most important writings or authors are marked by **bold type.**

Literature on neo-intuitionism based chiefly on *mathematical* arguments: *Baldus 1, *Barrau 1, **Beth** 47 (cf. **Kleene** 45, **D. Nelson** 47 and 49), *Bouligand 6, *Bridgman 2 (cf. *Fraenkel 18), **Brouwer** 07, 08, 12, *8, *9, *11, *13, 28, 29, 49, 52/3, *Chandrasekharan 1, *Dienes 2, *Dresden 1–4, *Droste 1, *Enriques 2, Fraenkel 24, 27, *13, Goodstein 48, *Haalmeijer-Schogt 1, *Hedrick 1, **Heyting** *8, 34, *15, 48, 53a, 55, 56, **Kleene** 52 (pp. 150 ff), *Lietzmann 2, **Menger** 28 and 30, *Orloff 1, **Rasiowa** 54, **Rosser** 36, **A. Schmidt** 50 and 50a, *Wavre 5, **Weyl** 21, 26 (and 49, particularly No. 9), *12, 46, **Wilder** 52 (Chapter X), *Wolff 1, Brouwer 54a.

However, literature to particular fields and problems, such as the *principle of the excluded middle* and *intuitionistic logic*, even when imbedded in more general considerations, is given in the following sections. (For literature to the axiom of choice, including its existential character, see Chapter II, § 4.)

To be sure, Brouwer himself refused to acknowledge most interpretations of his attitude, including some of the Dutch school itself.

The most important critics of intuitionism lean either on logicistic or on formalistic attitudes; cf. Chapters III and V. Some critical or polemic papers of a more general (partly philosophical) tendency are given here [1]).

While the Paris school and similar trends will be mentioned in connection with the intuitionistic restrictions in particular branches (§§ 5 and 6), the historical exposition is a suitable opportunity for presenting a somewhat related attitude which, nevertheless, cannot be subsumed under intuitionism and which in many respects is even more radical than neo-intuitionism. It dates back to about 1910 and has, not quite accidentally, the same geographical origin (Amsterdam) as neo-intuitionism; its Dutch name is *Significa* (significs in English) and its character is largely relativistic and pragmatic. In its mathe-

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Mathematical literature dealing chiefly with other *(semi-)intuitionistic* trends: Bockstaele 49, **Borel** 14 (particularly the notes), *4, *5, *6, *Denjoy 1 and 2, Gonseth 26, *6, 39, *Hölder 3 (pp. 193 ff), **Lebesgue** *2, *4, 41, **Lusin** 25, 27, 29, *11, **Pasch** *1 (p. VII of the 2nd ed.), 24-26, *3, *4, *5, **Poincaré** *2, *3, 08, 13, Severi 51, **Skolem** 22/23, 29, *9, 41, 52-53, Stenius 52, Suetuna 51-53, **Weyl** 18, 19, 21, 24, 25, *8, *9, *Wilder 1, Kondô 56. Also **Wang** 54 and 55 may be classified as semi-intuitionistic, and in a different sense Lorenzen's "operative mathematics", see **Lorenzen** 50 and 55 (for its relation to neo-intuitionism cf., for instance, 55, p. 80).

Philosophical literature on (mathematical) intuitionism: *Albergamo 1, ***O. Becker** 1 (pp. 403–419), 2 (cf. *Heinemann 1), 4, ***Beth** 5, 9, 11 (cf. 12), *Betsch 1, **Black** 33 and *3, Bocheński 56 (pp. 342 ff), ***Cassirer** 4 (part 3, Ch. IV; refers especially to *Becker 4), *Cavaillès 2, *Chaslin 1, M. R. Cohen 31 (pp. 186 ff), *Dassen 1 and 2, *Dubislav 1 and 12, *Farber 3, García *2 and 49, *Jørgensen 6, Kaufmann 30, *2, *4, *Korzybski 1, *Larguier 1, *H. Löwy 1, ***Quine** 12, *Sheldon 1, *Stadie 1. — *Nagel 6 does not refer to intuitionism in the present meaning.

[1]) *Ambrose 1–3, *Bentley 2 (Chapter IX), *F. Bernstein 6, *Burkamp 5, ***Cassirer** 4 (pp. 431 ff), *Hölder 4, *Kaulbach 1, *B. Levi 8, *P. Lévy 4, **Ramsey** 26 and *3. **Curry** 51 and Dieudonné 51 derive their arguments against "traditional" intuitionism just from *finitistic* attitudes.

matical direction this attitude has been developed by Mannoury [1]) while it was also applied to other sciences, including sociology and law. Some Dutch mathematical intuitionists, notably Brouwer and van Dantzig, show a close connection to signific ideas. However, these ideas have practically obtained no recognition outside Holland, partly because of the somewhat obscure form of their exposition.

As shown in detail in §§ 2 and 3, the neo-intuitionistic criticism of traditional mathematics is directed against its treatment of *infinite* and *indefinite* aggregates — which, to be sure, form the chief part and the characteristic feature of mathematics on the whole; with respect to *closed finite* aggregates, even if their extent surpasses human imagination, the classical attitude is not altered. Therefore the conception of mathematical statements as objective truth does not become meaningless in neo-intuitionism.

Yet among the *philosophical* interpreters and adherents of Brouwer's doctrine one may perceive "anthropologic-existential" influences [2]) which stress the subjective significance in comparison with the objective one. More radical steps in this direction, long before the rise of existentialism, were made by the signific school.

Significs starts from a critical examination of language and of methods of expression and denotation in general and is thus somewhat related to modern semiotics, especially to its pragmatic branch as conceived by certain Polish and American schools and by the Vienna Circle [3]). Significs, however, has in mind not the ordinary analysis of "meaning" but rather a theory of "psychic associations lying at the bottom of human acts of language". The stress is laid upon the perceptions and emotions connected with the terms or symbols. A characteristic feature of this rather psychological attitude is its conceiving language as an activity by which man tries to influence

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[1]) Mannoury *1, 25, *3, 34, *5. Cf. *Vollenhoven 1 (pp. 232 ff), *Morris 3, *Ducasse 1; the latter address, however, is not connected with significs in its original meaning.

The comprehensive expositions van Dantzig *4 (cf. *1) and 49a and Vuysje 53 — the latter containing an extensive bibliography — give very clear surveys of significs in its original (Dutch) sense. The paper de Iongh 49 contains a criticism of intuitionism from the signific viewpoint and stresses the distinction between mental construction and its linguistic expression (cf. pp. 212 f).

[2]) So especially in *O. Becker 2. Also in Borel's and Lusin's mathematical writings such influence is felt, quite independently of phenomenology.

[3]) Cf., for instance, *Hahn-Neurath-Carnap 1, or *Morris 3.

the behavior and the vital power of others [1]). This applies not only to "acts of volition which demand obedience" but as well to mere "indicative" acts of communication, including the symbolic language of mathematics; according to Mannoury not even in the latter, the social significance of which should not be overlooked, is the subjective, persuasive, emotional character of language altogether lacking [2]). A mathematical formula would, then, not have a meaning *per se* but according to the purpose for which it is used. Thus Mannoury's juxtaposition of mathematics and mystics is not accidental. The emotional and psychological moment, and this alone, is apt to explain the choice of principles (axioms) underlying mathematics and is still more perceptible in mathematical models of the external world as constructed in theoretical physics. Yet even Mannoury admits that the assertions of mathematics are independent of sentiments of like and dislike to a higher degree than those of any other science. In this light one might interpret Kant's well-known maxim, measuring the scientific character of a doctrine by the applicability of mathematics in its field.

§ 2. The Constructive Character of Mathematics. Mathematics and Language

The fundamental thesis of intuitionism in almost all its aspects says that *existence in mathematics coincides with constructibility*. While in neo-intuitionism stress is laid on the very *character of mathematics* as involved by this thesis, in other trends at least the restrictions imposed on the field of admissible *mathematical procedures* derive from the thesis.

Mathematics is, according to Brouwer, not a *theory*, say a system

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1) Here both a congeniality and a contrast with Brouwer's attitude (especially in 29), as described in § 2, become manifest. According to significs, not only the exposition of mathematics but even mathematics itself (mathematical thought) has a partly linguistic character.

2) It is mainly this attitude, related to a psychological foundation of logic (as in the schools of Heymans and Mach), which repels ordinary mathematicians. Cf. the ironical remarks in *Jourdain **6** (p. 88) about evaluating the product $6 \cdot 9$ by working out, from the answers obtained in an examination among schoolboys, the average to six places of decimals; or about evolutionary ethics expecting to discover *what is good* by inquiring what cannibals have thought good.

of rules and statements, but a certain fundamental part of *human activity*, a method of dealing with human experience, consisting primarily in concentrating attention to a single one among our perceptions and distinguishing this one from all others. While acts of distinguishing a single perception lead to the primordial intuition of integer as explained below in § 5, Brouwer associates the totality of perceptions (from which a single one is picked out) with the notions of continuity and of time; for this reason a primordial character is attributed to these two notions as well. Thus somehow the abyss between discreteness and continuity is denied from the first and it is maintained that "the primordial intuition of mathematics and of every intellectual activity is the substratum of all observations of change when stripped of all qualitative properties; a unity of continuity and discreteness" [1]). In § 5, when analyzing the neo-intuitionistic conception and "construction" of the continuum, it will appear what is meant by Brouwer's additional assertion that the unity just mentioned constitutes "the possibility of mentally joining several units, connected by a «between» which is never exhausted by the intercalation of new units".

As the primitive act of intellectual construction in general, neo-intuitionism conceives "the splitting-up of moments of life into qualitatively different parts which, separated only by time, can be reunited". In parallelism to this general conception, the primitive act of *mathematical construction* is maintained, in a remarkable similarity to a well-known idea of Plato's, to be "the process of stripping this splitting-up of any emotional content until the intuition of abstract bi-unity remains" [2]). Accordingly mathematics in its entirety means the universe of mental processes that can be built up by an unlimited sequence of steps repeating such primitive mathematical acts indefinitely — a definition which is more comprehensive than any other and includes logical processes as well as many procedures of natural sciences. No other science — certainly not logic or philosophy — can, then, serve a basis for mathematics. To be sure, this conception does not even consider it an exclusive privilege of *science* to use mathematical procedures but notices them in ordinary life's thinking as done spontaneously and often subconsciously.

[1]) Brouwer 07, p. 8. Cf. the clarification of many obscure formulations of Brouwer (particularly in the programmatic writings 07 and 29) by van Dantzig 47.

[2]) Brouwer 12.

Obviously this neo-intuitionistic "definition" of mathematics does not supply us with an exact determination of what procedures, among those found in traditional mathematics, are reputed to be "constructive" and therefore legitimate within Brouwerian mathematics. Yet the apparently dogmatic character of the above definition is mitigated by certain remarks (above, p. 200) about the necessity and the sufficiency of intuitionistic restrictions.

The emphasis laid hereby on the construction of mathematical entities and even the *identification between existence and constructibility in mathematics* are by no means a novelty. Definitions in mathematics usually raise the question whether an object satisfying the definition can be constructed (effective examples, see p. 57). Also the classical problems of Greek geometry — notably duplication of the cube, trisection of the angle, quadrature of the circle — are problems of construction; in this case, of construction with ruler and compasses. Hilbert, who was being fought by neo-intuitionists as the protagonist of their opponents, after having first solved a fundamental problem of invariants by existential methods [1]) found it worth while to look also for constructive solutions of related problems which in principle were included in the existential procedure.

In § 4 of Chapter II some procedures of a purely existential character were exhibited. Such procedures are found chiefly in analysis, geometry, set theory, but they are not altogether missing even in arithmetic and algebra. Their common feature is that certain mathematical objects (numbers, correspondences, functions, sets etc.) are shown to exist not on account of their derivation from simpler objects by a construction step by step but by means of an argumentation which appeals to logical compulsoriness. Mostly this is done in the way of an alternative between mutually exclusive possibilities in view of the principle of excluded middle (see below § 3), or by showing that the non-existence of an object with the desired quality would involve a contradiction (whereas the nature of the contradiction does not yield a method of constructing a suitable object). A well-known instance,

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[1]) This first proof of the existence of a finite set of invariants (1890) was called by Gordan "theological" because it essentially used an existential reasoning ("there must be an invariant of the lowest possible degree") from which an actual invariant of the kind in question cannot be derived constructively. Cf. the description of the method and of the first reactions to it in *Toeplitz 1.

appearing at the very beginnings of analysis, is the statement (p. 65) that every infinite bounded set of points has at least one accumulation point, which can be proven by continued bisection. During the 19th century demonstrations of this type were not only generally recognized but especially appreciated for their acute and bold character. On the other hand, they have the disadvantage of not granting an insight into the nature of the object secured; if, for instance, the object is a number, a proof of the existence of a number with the desired property need not enable us to estimate, for example, the magnitude of the number.

Neo-intuitionism and most other intuitionistic trends, admitting construction alone as a legitimation for existence in mathematics, deny such procedures any binding power and do not accept the existence of the objects concerned as long as they are not secured in a constructive way. In particular, intuitionists maintain that identifying mathematical existence with non-contradiction would mean degrading mathematics to a mere game [1]). Only Poincaré, though in other directions taking a most radical attitude, regarded freedom from contradiction as a sufficient legitimation for existence (as does the formalistic school) and even accepted the principle of choice.

This principle and the well-ordering theorem resting on it presumably constitute the most characteristic instance of a purely existential mathematical statement [2]). In fact the latter theorem, which asserts the existence of procedures (or sets) that well-order a given set without showing how to obtain such a procedure, is the prototype of an intuitionistically meaningless statement. The uselessness of the well-ordering theorem for ascertaining, say, the place of the power of the continuum in the series of Alephs is interpreted as a mere symptom, to be expected in advance, of the theorem's voidness of meaning.

Far beyond this extreme instance *all* existential propositions appear meaningless to intuitionism; they constitute not proper statements but "statement-abstracts". Still the proof of such an abstract may

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[1]) Among philosophers, in particular the phenomenologic-existential trend (cf. *O. Becker **2**, for instance p. 442) considers intuitionistic mathematics alone to be a science that "discovers actual phenomena which are comprehensible to original and adequate intuition and can be interpreted existentially".

[2]) Cf., however, Suetuna 51-53. (The concept of 'set' used there is somewhat obscure.)

serve as a stimulus for constructively demonstrating a statement which is a substratum of the abstract in question. Normally, however, the abstract has been *derived* from a statement proper; for instance, from the statement '2 is an even prime' we derive the abstract 'there exists an even prime'.

One has compared the situation to that of coming across a document which says that a treasure is hidden in the ground. This is not a statement proper as long as the spot where the treasure is hidden has not been specified; yet the document may cause a treasure-seeker to obtain a statement of the form '*here* a treasure is hidden' by stimulating him to dig and find the treasure. From this statement one may pass on to the abstract which, however, has its only justification in the statement.

In these and similar cases the negative signification of 'constructiveness' is obvious. But one can hardly maintain that its positive meaning is sufficiently clear with respect to procedures employed up to now or to be employed in future in what has always been understood as "mathematics" — save for mathematical induction (§ 5) [1]. As a matter of fact, several attempts and suggestions of defining 'construction' were made, chiefly from outside neo-intuitionism [2]. These suggestions only seem to show that the concept of construction has a *relative* and not an absolute character; there are higher (stronger) and weaker degrees of the concept [3] and this even applies to the domain of arithmetic [4]. Thus the question loses its dogmatic character and,

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[1] For the meaning of 'construction' within the Paris school of intuitionism, cf. § 5.

[2] Cf. Menger 28 (p. 225), 30, 31, Rosser 36, Hermes 37; particularly Kleene 43 (No. 16), 45, 52, Post 44, D. Nelson 47 and 49; also Turing's methods (see Chapter V). Rasiowa 54 gives a topological explanation.

[3] Cf. Z. P. Dienes 52 who distinguishes between different degrees of rigor, the first of which coincides with (neo-)intuitionism.

[4] For instance, with respect to a sequence of integers there is a difference in constructiveness between the existence, for every n, of a rule enabling us to calculate the n^{th} term of the sequence (or the first n terms) by means of a finite number of steps, and the existence of a rule which enables us to calculate, by means of a finite number of steps, the n^{th} term (or the first n terms) for every n. The former means a weaker degree of constructiveness than the latter, a fact sometimes overlooked by neo-intuitionists; cf. Menger 31.

Thus every single digit of the decimal expansion of π or of ρ as defined on p. 256 can be calculated by a finite rule, while decimals which are periodic from

analogically to axiomatic procedure, assumes the form: what parts of mathematics or of a certain mathematical branch can be obtained from a given starting-point by means of such-and-such "constructive" methods? In fact, an opposite (dogmatic) attitude should have induced the Greeks to infer from their conception of geometrical construction that in general there exists no angle equal to the third of a given angle.

Brouwer impetuously opposes this opinion and maintains that there is an *absolute* concept of constructiveness and that this concept settles what is comprehended in mathematics and what parts should be excluded as belonging to pseudoscience. Certainly quite a number of constructive procedures have been described and, which is asserted to be still more important, a certain "mathematical attitude of mind" may be suggested by general hints. But any list of mathematical principles of construction is not only incomplete but would *necessarily remain incomplete*; for, within the limits of the general notion given above for "mathematical" procedures, one must not limit the liberty of creative mind to further extend its constructive faculties. Hence one can never predict what special ways of construction might be needed for reaching a particular goal. The case is compared to a mountaineering expedition for which you may fairly well give a list of "admitted" alpinistic procedures and prohibit others, for instance driving nails into rocks, while you never can foresee all devices that may become life-saving, and therefore admitted, through any possible passage of an ascension.

The conception according to which mathematics is the *mental activity* described above rather than the oral or written *expression* of such activity has a decisive influence on *the relation between mathematics and language* [1]). Apparently the process of thinking, i.e. of mental creation, is not intrinsically connected with a linguistic expression; only for the exchange of thought (the communication of

a certain digit on are defined constructively in the stronger sense. For two sequences of integers which are constructive only in the weaker sense it cannot always be decided by means of finitely many steps whether they are equal or not (cf. the example given in § 6). Menger rightly points out that choosing the one or the other (or a third) concept of constructiveness is altogether arbitrary.

Not constructive in either sense is a sequence whose n^{th} term depends on the validity of Fermat's Last Theorem for the exponent n.

[1]) See, in particular, Brouwer 29, 49, 54a.

ideas) are we in need of language or of its written equivalent. In this context the weakness of mathematical language in comparison to mental construction is stressed, for any language is maintained to be vague and exposed to misunderstanding, even symbolic language (since mathematical and logical symbols rest on ordinary language for their interpretation). Hence mathematical language is ambiguous and defective; mathematical thought, while strict and uniform in itself, is subject to obscurity and error when transferred from one person to another by means of speaking or writing. It would therefore be a fundamental mistake to analyze mathematical language instead of mathematical thought. This pointed distinction between construction in mind and its expression in language is contrary not only to logicism (Chapter III) and metamathematical method (Chapter V) but also to the attitude, not influenced by mathematical aims, of leading philosophers from Plato over Leibniz to W. von Humboldt who assert that all abstract thinking is dependent on language [1]).

This attitude not only applies to mathematical theorems (and definitions [2])) but still more to mathematical *proof*. *The construction itself constitutes the proof* and one should drop the usual idea as though the demonstration were intended to convince the reader of the soundness of an argumentation by basing it, step after step, on recognized principles. Since construction is an activity it can not be communicated adequately; instead one uses the fragile crutch of words or symbols. There is neither exactness nor safety, says Brouwer [3]), in the "transfer of will" by means of language, and this also applies to the transfer of will expressed by the construction of a mathematical system. Hence *for mathematics there exists no safe language* which excludes misunderstanding in talk and prevents mistakes of memory; this defect is regarded as a support of the contention directed against formalism that the exactness of mathematics should be found not "on paper" but in the "mind of man". (To be sure, the intuitionistic

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[1]) Cf. *Schlick **2**. On the other hand several philosophers (for instance *Greenwood **2** and *Lipps **4**), sometimes with a psychological argumentation, accept the distinction between mathematics and its exposition and even extend it.

[2]) The existential character of the usual definitions of equivalence, denumerability etc. has also induced non-intuitionistic mathematicians to specialize in the direction of 'effectiveness'; cf. p. 57.

[3]) Brouwer 29, p. 157. Cf. the epistemologic-psychological analysis in *Beth **6**.

thesis about mathematics and language is inacceptable to the majority of mathematicians for the very reason that it makes mathematics a private affair rather than an organized inter-subjective phenomenon.)

Many outstanding mathematicians from Sylvester to Poincaré have compared mathematics to music. Neo-intuitionism adds a further and more penetrating analogy: as a composer may suggest to an adept how to compose a symphony — not by merely teaching harmonics but by trying to describe how *he* has done it — so a mathematician would initiate a student in the constructive mystery of mathematical production, while the demonstration has a secondary value only.

The neo-intuitionistic conception of language directly influences the problem of the *antinomies*, which are considered mere combinations of words, void of meaning and without any constructive ingredient. Hence it is unnecessary to give them a serious thought, let alone to "solve" them.

It is difficult duly to appreciate the neo-intuitionistic attitude towards mathematical language without plunging into Brouwer's general opinions about civilization of mankind [1]). According to him mathematics, science, and language form the chief functions of human activity through which man rules nature. He finds the origin of these functions in three operational forms, viz. mathematical construction, mathematical abstraction, "imposition of will by sounds". In particular, mathematical construction would be a volitive act serving the human instinct of self-preservation and coming about in the two phases of "temporal engagement" and of "causal engagement". Only at a higher stage of civilization does one proceed to mathematical abstraction which through scientific theories controls intricate causal sequences. Finally language, operating with elementary signals of speech, facilitates the transfer of will in civilized communities, such transfer appearing as "suggestion" or "training to reason" — thus utilizing mathematical construction for the service of parallel volition of human groups or of humanity altogether.

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[1]) These opinions, rather difficult to comprehend, are chiefly explained in Brouwer 29. Cf. Mannoury 34 (see above p. 206).

§ 3. The Principle of the Excluded Middle

In the following sections we shall survey the consequences, both for logic and mathematics, of the fundamental thesis described in § 2. §§ 3 and 4 deal with the consequences for logic, §§ 5 and 6 with mathematics. While in the present section an informal exposition is given, including references to the large literature on the subject, § 4 deals with the intuitionistic *logical calculus*.

From the first [1]) Brouwer raised the question which among the principles of Aristotelian logic could be retained from the neo-intuitionistic point of view. His answer is, the principle of contradiction but not the principle of the excluded middle (*tertium non datur*) in its general sense. This fanfaronade about the *invalidity of the tertium non datur in mathematics* and in infinite domains in general has been made the center of dispute on neo-intuitionism by opponents as well as by adherents. (The Paris school and other intuitionistic trends have not accepted the attitude of the Dutch school to the excluded middle.) To perceive the great importance and the wide application of the principle of the excluded middle in mathematics we may remember that it is the fundament of *indirect proof* (see, for example, *T*, pp. 98, 258, 267 f).

However, it seems that by stressing the *tertium non datur* the Dutch school has produced a pointed slogan rather than described the actual situation. As a matter of fact, the use of the principle of the excluded middle for *finite* domains has never been questioned [2]) and more appropriately one should center the discussion at the subject of *negating general statements*.

To elucidate this subject in the light of simple examples let us consider the following conditions (properties) whose free variable n is restricted to positive integral values.

1) $2^n + 1$ is a prime number;

2) the n^{th} digit after the point in the decimal expansion of $\pi = 3.14159\ldots$ is 7;

[1]) Brouwer 08. Cf. 54a, where he says "the long belief in the universal validity of the principle of the excluded third" ... is considered "as a phenomenon of the history of civilization of the same kind as the old-time belief in the rationality of π ... ".

[2]) Take even an instance as the following. From an urn containing white and black balls a blind man draws a ball and replaces it, nobody else being present. While a constructive ascertainment of the color of the ball drawn seems impossible, most or all neo-intuitionists would admit that the ball drawn was either white or black.

3) every geographical map containing n regions (countries) can be colored [1]) by means of at most four colors;

4) $2^n + 1$ being a prime number, there exists a greater prime number of the form $2^m + 1$ $(m > n)$ [2]);

5) n and $n + 2$ forming a pair of twin primes (i.e. a pair of prime numbers the difference between which is 2), there exists a greater pair of twin primes;

6) there exists a triplet (x, y, z) of natural numbers (a "Fermat triplet") such that

$$x^{n+2} + y^{n+2} = z^{n+2}.$$ [3])

We add the following statement which is related to 2) yet contains n as a bound variable:

7) There exists an integer n such that the n^{th} digit in the decimal expansion of π and its six successors (the $(n + 1)^{\text{st}}$, $(n + 2)^{\text{nd}}$, ..., $(n + 6)^{\text{th}}$ digit) equal 7.

By universal quantification over n we proceed from 3) to the statement:

8) Every geographical map can be colored by means of at most four colors.

By giving n a definite value we may turn 1)–6) also into statements and then raise for each of 1) to 8) the question whether the statement is true or false.

This formulation presupposes that *every meaningful statement p is either true or false*, i.e. $p \vee \sim p$. Let us examine whether this presupposition can be justified in our examples without an *a priori* reference to the principle of the excluded middle.

In the cases 1) and 2), it may at the present stage of research prove difficult for great values of n to reach the actual decision whether the statement is true or false; the same applies to 3) when $n > 37$ (for $n \leq 37$ it is true). Yet *in principle* the decision can be reached; for a given n, a finite number only of tests is necessary to ascertain

[1]) On condition that different colors are given to any two adjacent regions, i.e. to regions that have in common a portion of their boundaries (and not only isolated points). It is assumed that the map is drawn on a plane or on the surface of a sphere.

[2]) Mersenne's numbers, i.e. the primes of the form $2^n - 1$, yield instances similar to 1) and 4).

[3]) The exponent is written in the form $n + 2$ because there exist (infinitely many) solutions (x, y, z) of $x^2 + y^2 = z^2$.

whether n has, or has not, the property in question. For this very reason we may *anticipate* the result: n either has or has not the property, i.e. the statement is either true or false. Therefore a theorem proved both on the assumption that $2^{8192} + 1$ is prime and that it is composite would be true also in the eyes of Brouwer, though we do not know which assumption is true.

The same argument, relying on the possibility of testing by finitely many steps, allows us to state with respect, for example, to the population of London at a certain moment that either each Londoner has an age of at most 99 years or there exists at least one whose age is 100 or more. Thus, according to the Dutch school, *the principle of the excluded middle*, far from being a logical axiom or an *a priori* statement, *only constitutes the anticipation of a result* that may actually be reached by an investigation of a finite character.

On the other hand, there is not the slightest argument in support of such a conclusion when an infinite aggregate is concerned. One should beware, says Weyl, of the idea that, an infinite aggregate being defined, one might — beyond knowing a property characteristic of its members — contemplate the members as if they were spread before our eyes; that, accordingly, one only had to peruse them as a police official peruses his records in order to ascertain whether in the aggregate there is a member of a certain kind. Such procedure is senseless [1]) with respect to an infinite aggregate [2]).

Obviously such is the situation in the instances 4)–8) given above. The instances 4) and 5) resemble each other. It is well known that, m being divisible by an odd integer > 1, $2^m + 1$ is certainly a composite number; among the other values of m, i.e. among the powers of 2, $m = 2^4$ is the greatest one for which $2^m + 1$ is known to be prime (as is the case for $m = 2^0, 2^1, 2^2, 2^3$), $m = 2^{73}$ the greatest one for which $2^m + 1$ is at present known to be composite, while between these two values of m a few only have been successfully

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[1]) To be sure, the term 'senseless' may appear too strong. One might think about perusing the sequence of all positive integers by settling 1 in one minute, 2 in half a minute, n in $\frac{1}{2^{n-1}}$ minute, thus obtaining a convergent series. To this argument cf. *Ambrose **2** and **3** and *B. Russell **12.**

[2]) Weyl 21, p. 41. Cf. the distinction made by philosophers (e.g., Husserl) between "individual" and "specific" generality; see Carnap 37, § 15.

tested (among which are 2^α with $\alpha = 5$ to $\alpha = 12$ but not [1]) $2^{13} = 8192$). At present, then, it is not only unknown whether the set of primes of the form $2^n + 1$ is finite or infinite but whether it contains just six [2]) or at least seven members; that is to say, whether $2^{16} + 1$ is the greatest prime of this form or not. 5) is also unsolved for the time being: we do not know whether there is a *last* pair of twin primes (which seems improbable) or there are infinitely many pairs.

In the case of 6) — contrary to 1)–3) — not even for a given value of n need a decision be obtainable by a finite number of steps, for to achieve it one would have to test infinitely many triplets (x, y, z). In fact, no Fermat triplet is known and it seems probable that none exists; but while for infinitely many values of n the non-existence of a triplet has been proved there remains another infinite set of integers n for which we do not know the answer.

Finally, 7) is somewhat similar to 4) and 5) and is related to 2) though not quite in the same way as 4) is related to 1). For a given n one can in principle decide not only whether the n^{th} decimal of π, a_n, equals 7 but also whether $a_n = a_{n+1} = \ldots = a_{n+6} = 7$ or not. Yet 7) requires an existential quantification over n.

We discussed these instances at length in order to make conspicuous Brouwer's attitude to the would-be principle of excluded middle. Take, e.g., 4) with $n = 16$; the question is whether a prime of the form $2^m + 1$ with $m > 16$ exists. A first possibility is discovering a definite $m > 16$ for which $2^m + 1$ is prime (which, if true, can be stated by a finite number of steps). A second chance is reaching a general proof which shows that $2^m + 1$ is a composite number for every [3]) $m > 16$. But naturally these cases are not exhaustive; it may happen that neither has a suitable m been discovered nor a general proof of the non-existence of such m been obtained — as actually is the case for the time being. This purely negative possibility is called *a third one* by the Dutch school [4]). It has a temporary character, for to-morrow we may discover a suitable m or succeed in finding a general proof; this does not trouble neo-intuitionists

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[1]) According to what is known in 1956.

[2]) For $2 = 2^0 + 1$ has to be added to the values noted above.

[3]) Of course, only powers of 2 need be considered.

[4]) Cf. Moch 51. — Among the philosophical critics of this attitude Carnap 37 (§ 43) should be mentioned for the distinction made between knowing *a priori* how to solve a problem and knowing the structure of a possible solution.

since after the solution of such a problem always another, not yet solved, can be presented [1]). (Neo-intuitionistic literature to the present subject in general utilizes instances of the kind mentioned, undecided for the time being. Preferable to such special problems is a general instance of undecidability in the mathematical branch in question [2]); then the "third case" takes for granted only the assumption that there is no general method of deciding.)

The above example shows, first, that the third case has another character than the two others and that there is no resemblance to a three-valued logic [3]) (p. 193). In fact one can not coordinate the three cases; in the first and the second we have an objective state of matters following from the premisses (here from the axioms of arithmetic) by the rules of inference, while the third case depends on subjective and provisional circumstances (and, as Platonists might add, would vanish for mathematically accomplished beings). Secondly we perceive that the neo-intuitionistic attack does not concern the principle of the excluded middle in its traditional form which in the present case, if p is the statement 'for every $m > 16$, $2^m + 1$ is a composite number', reads: p is either true or false. This alternative contains nothing about a proof or a counter-instance.

The very essence of Brouwer's thesis becomes manifest by analyzing what according to traditional logic is meant by 'p is false'. Obviously, that *there exists* an $m > 16$ such that $2^m + 1$ is a prime. This is a particular case of the assumption of traditional logic that *the negation of a general statement is an existential statement.* Yet according to neo-intuitionists who altogether reject existential statements as autonomous ones, a general statement can be negated

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[1]) See *Brouwer **13**, p. 210; cf. Witt 51a.

[2]) See van der Waerden 30. Here a general criterion of irreducibility for polynomials is dealt with and its non-existence reduced to the undecidability of the question "is there an integer n with an arbitrarily given property $P(x)$?". For branches including the general notion of real number one may use the fact that the trichotomy regarding the order relation between real numbers it not valid intuitionistically (see § 6).

[3]) This fact has been overlooked by the criticism of neo-intuitionistic logic expressed in Barzin-Errera 27 and ***2** (cf. *Errera **2** and **3**, *Dassen **2**, *Toms **1**). Objections to that criticism are raised in P. Lévy 27, Avsitidysky 27, *Dresden **3**, and in particular in Khintchine 28, Glivenko 28, *Heyting **10**. Also the characterization, in *Kattsoff **4**, of Brouwer as taking the line of *modality* can hardly be accepted by the Dutch school.

only by naming a counter-instance; otherwise its negation is meaningless [1]). Hence *the actual fundament of the so-called rejection of the principle of the excluded middle is the impossibility of plainly negating general statements* in neo-intuitionistic mathematics and logic. By their refusal to accept this formulation both adherents and opponents of the Dutch school have distorted the rôle of the principle. In fact, in view of the previous description the "middle" (or the "third case", as it is called in Latin and in French) becomes quite natural; it is the interval between proving a general statement and constructing a counter-instance (from which an existential "statement-abstract" may be derived).

For a *particular statement* p which is neo-intuitionistically meaningful the validity of the *tertium non datur* is not denied by Brouwer [2]). His attitude may therefore be described as adherence to Aristotle's principle $p \vee \sim p$, yet with the restriction that certain types of formulae p, namely general statements p, are not admitted in the principle [3]).

Likewise, whenever the range of the free variable entering the general statement is *finite* — more precisely, when it contains a finite number only of objects each of which can be exhibited individually — the Dutch school admits that by negating the statement one produces an existential statement. For instance, given a finite group, negating its commutativity means that there is at least one pair of members x_0, y_0 such that $x_0 y_0 \neq y_0 x_0$. But then the existential statement obtained by negation has a constructive character since the general statement is but the *conjunction of a finite number of particular statements*. In the instance given above "all Londoners have an age of 99 at most" the meaning is: a_1 is at most 99 and so is a_2 ... and so is a_n, where the set $\{a_1, a_2, \ldots, a_n\}$ represents the population of London. Hence negating the statement means: either a_1 is at least 100 or

[1]) In particular one cannot draw any conclusions from such denial; cf. *Foster 1.

[2]) Cf., e.g., Brouwer 28.

[3]) Cf. *Ambrose 2. — To be sure, in classical logic (so already in Aristotle's *Organon*) the principle is frequently just formulated in the pointed form: either all objects x have the property P, or one x at least has the property *non-P*. One tacitly here restricts oneself to properties that are meaningful for the objects concerned; hence properties as those used in some antinomies are excluded. For the attitude of Aristotle and Leibniz to the *tertium non datur* cf., for instance, des Bancels 26, Schrecker 36, Bocheński 56 (p. 73), Anscombe 56.

a_2 ... or a_n (in the non-exclusive meaning of 'or'). By this procedure at least one counter-instance can be constructed and the supposedly existential statement becomes intuitionistically legitimate.

If, however, the variable 'x' of the general statement '$(Ax)f(x)$' ranges over an infinite aggregate (set) the statement is not equipollent to a conjunction of finitely many particular statements, and an "infinite conjunction" of statements can hardly be considered admissible; nor is then the existential statement '$(Ex)f(x)$' equipollent to a disjunction of particular statements. An infinite aggregate corresponds to a property [1]) which is characteristic of the aggregate and defines it [2]).

Hence the use of the principle of the excluded middle in natural science would depend on the conditions of the finiteness and the atomistic structure of the universe [3]).

To avoid misunderstandings of the present point the importance of which has been pointed out time and again by the Dutch school, one should stress that the legitimacy of negating general statements and of affirming existential ones, hence of applying the *tertium non datur*, does not just depend on the *finiteness* of the domain but beyond this on a *definite limitation* of the domain's extension. This means the exclusion not only of free-choice sequences (§ 5) no matter whether they prove finite or infinite, but also of certain domains which appear in ordinary life and in processes of nature; say, the domain of all plants.

An example of a similar kind is obtained as follows. Certainly the statement 'for all x, x has eight different greatgrandparents' is false when 'x' ranges over all men, neither can it be called true when 'x' ranges over the descendants of a certain person who still has procreative descendants. Yet the existence among those descendants of a person that has less than eight greatgrandparents cannot be stated either. This example is related to the question, broached allegedly

[1]) An aggregate can of course be defined by different properties which have the same extension.

[2]) This conception in many cases also applies to finite aggregates. Think of extreme examples as "Plato has not visited any American town" or "No inhabitant of Mars has participated in the first World-War".

[3]) *Brouwer **18.** Here the remark is added that, as far as mathematics is applied in natural science, the fulfilment of the above conditions referring to nature does not release us from the intuitionistic purification of the mathematical procedures inasmuch as these refer to infinite and continuous domains.

by Aristotle, about the validity of the principle of the excluded middle for future events [1]; in particular for events subject to the freedom of will (or to indeterminism in nature).

For Brouwer, domains which, though finite, lack a definite boundary constitute a species (§ 5) and not a finite set in the ordinary sense.

It is true that the "ordinary" mathematician or philosopher as well as "ordinary common sense" are inclined to oppose the neo-intuitionistic attitude towards the principle of the excluded middle. No matter whether a decision may be reached for the time being or at any time, anyway — so they would maintain — the "objective" state of affairs necessarily implies the statement to be either true or false. Applied to the instance 4) of p. 216, either there exist actually only six primes of the form $2^n + 1$ or there exist at least seven. Such argumentation, answers Brouwer, relies on a double misunderstanding. First on the idea as if mathematics dealt with external facts or with Platonic ideas existing independently of the mathematician's activity; yet just this activity and it alone creates mathematics, as evolves from what was said in § 2. Even if one believes in some "objective truth" its character is metaphysical and mathematical proofs cannot be based upon metaphysical arguments. Secondly, the assertion of the *tertium non datur* is, consciously or not, influenced by an unjustified generalization to infinite domains of a procedure which is legitimate for finite domains only; the process of perusing an infinite domain can never be finished, hence its "result" must not be anticipated, not even in principle.

Take the case of a theorem that is proved both under the assumptions that the statement p is true (six primes only) and that p is false (at least seven primes) [2]; not even in this case may you accept, according to Brouwer, the theorem as settled, as long as the statement is not either demonstrated by a general method — that is to say, by comprehending the infinite domain through a finite or intuitionistically legitimate procedure (as characteristic property, mathematical induction, etc.) — or refuted by a counter-instance. Hence the indirect method of proving is inadmissible in general;

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[1] For these questions, in particular that regarding future events, see A. Becker *1 and 36, *Broad **2** (p. 73), *Burkamp **3**, **Schlick** 31 (p. 158), *Toms 1; cf. *Black **4** and (for the historical side) *Vollenhoven **2**.

[2] Especially in set theory such cases are found; for instance, a theorem proved by assuming first $2^{\aleph_0} = \aleph_1$, secondly $2^{\aleph_0} > \aleph_1$.

that is to say, save for cases where it allows a constructive turn.

A theorem proven in the above way, to be sure, is still admitted by intuitionists to be *non-contradictory*, even in the sense that its contrary cannot be true (cf. § 4). [1]) But then this quality, while having heuristic value as a stimulus inciting us to prove the theorem, does not constitute a genuine confirmation. As mentioned above (footnote 3 on p. 219) some authors tried to refute the rejection of the principle of the excluded middle by tracing out a would-be contradiction. Such attempts cannot but fail; for by *omitting* one of the principles or rules of a logical system one obtains a *restriction* of its operational field and of its consequences and not an extension which might involve contradiction. Against certain misunderstandings it should be stressed that Brouwer *does not intend to replace the principle of the excluded middle by its negation*, i.e. to introduce a third case proper, opening the way for a principle of *quartum non datur* [2]). (This would be rejected by neo-intuitionists also for its methodical implication of inclusion in an abstract logical system.) Actually the "third" case is regarded as one nothing positive can be stated for, which excludes its coordination with the two other cases.

Accordingly, as is confirmed by the symbolic exposition of intuitionistic logic given in § 4, the logic of Brouwer is not in *opposition* to classical logic but constitutes a *part* of it. This torso, in consequence of its renouncing the use of the *tertium non datur*, loses a good deal of the simplicity and lucidity of traditional logic. On the other hand, even for those opposing the Dutch school it is interesting to investigate the remaining part of classical logic and to examine how far one can

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[1]) Therefore a proof performed by means of the *tertium non datur* (called "fictive" by Lorenzen 55) may be useful inasmuch as it shows that the negation of the respective statement cannot be proven, not even in a fictive way. In this respect the situation resembles that of a proof performed by means of the axiom of choice.

[2]) Whereas Brouwer's attitude has to some extent revived the discussions on three-valued logic, it should be stressed that the admission of many-valued logics does not imply Aristotelian logic to be contradictory — as the consistency of non-Euclidean geometries does not mean that Euclidean geometry contains a contradiction.

(While many-valued logics are alternatives to classical logic, H. B. Smith's "non-Aristotelian logic" (*H. B. Smith **2** and **3**; cf. *Kattsoff **1**, *Henle **3**) virtually constitutes an extension of classical logic which contains the latter as a proper part.)

proceed by its means; thus one obtains a domain apt to show the independence of the principle of the excluded middle [1]).

Regarding the principle of contradiction, Brouwer points out that according to his attitude this principle is not actually used when one proves an impossibility by indicating a contradiction. Virtually this only means the failure of a mathematical construction which should satisfy certain conditions [2]).

Throughout this section Brouwer's attitude towards the principle of the excluded middle [3]) has been illustrated by examples referring to integers. Of course it applies all the more to *sequences* or *sets of integers*. Instead of the question "Does there exist in a given set of integers an integer with a certain property?" we then ask "Does there exist in a given set of sequences of integers (for instance, in a given set of decimals) a sequence with a certain property (meaningful for sequences of integers; for instance, periodicity)?". The criticism of neo-intuitionists regarding a question of the latter kind does go further; for

[1]) Korzybski's (*1) "non-Aristotelian systems" renounce, besides the principle of the excluded middle, also the principle of identity. As to the principle of contradiction, cf. *Avey 3; the mutual independence of the principles of contradiction and of the excluded middle is discussed in McGill 39, cf. *Nagel 6.

[2]) Brouwer 07 (p. 127) and 08.

[3]) In addition to literature given in § 1 on neo-intuitionism in general or in the present section in connection with special questions, we list here writings which in particular deal with the principle of the excluded middle, most of them in connection with Brouwer's attitude. Essays whose main subject is many-valued logic are not included; those dealing with intuitionistic logical calculus are found in § 4. The succession in each part of the list is chronological (beginning with 1923).

The few papers with a principally *mathematical* tendency are Wavre 24, *2, *3, *4 (rejoined by *P. Lévy 1 and 2; cf. Dugué 50), **Kolmogoroff** 25 (proves that conclusions of a finite character can also be reached by avoiding the *tertium non datur*), **Church** 28, **Menger** 28 (pp. 218–223), *Härlen 2, *Langford 9, *E. Schmidt 1, **Goodstein** 45, 48, 51, 52, **Ackermann** 50, **Kreisel** 51 (polemic), Lorenzen 54.

There are a great many publications with a *philosophical* tendency, most of them polemic, and of very heterogeneous standards; with few exceptions only those published up to 1936 are mentioned since the later ones may easily be traced through the *Journal of Symbolic Logic*. *Rivier 1, *Vasiliev 1, *Pichler 1, *Burkamp 2, *Carnap 3, *Hartmann 1, *J. Meyer 1, *Bogoslovsky 1, *Lipps 4 (§§ 9 and 10), *Reymond 2, 4, 5, *Kaila 2 (pp. 78 ff), **Nagel** 29, *Farber 1 and 3, Dewey 29 and *2, **O. Becker** 30, Hofmann 31, *Dürr 3, **Beth** 35, 36, 55 (§ 89), *Reiser 1 and 2, *Baylis 3, *Nink 1, *van Os 1, *Mannoury 6, **Feys** 49, Hoenen 49.

here not only is the use of the *tertium non datur* rejected, but the very *notion* of an arbitrary sequence of integers, as meaning something finished and definite, is declared illegitimate. Such a sequence, as long as it is not defined by a given law, may be considered a "growing" object only and not a "finished" one (see § 5). Therefore it would in general be senseless to raise the question of its having such-and-such properties, except for cases specified in § 5.

The abandonment of the principle of the excluded middle, besides its significance for logic (see § 4), has a psychological effect extending across mathematics as a whole, namely on *the conviction that every mathematical problem can in principle be solved.* To be sure, in another sense this conviction has been shaken by Gödel's incompletability theorem (Chapter V). But then the indecidability asserted in this theorem refers to a definite basis from which the attempt to solve the problem is being made. Intuitionism, however, rejects that conviction as a quite groundless belief for any basis and emphasizes this rejection by pointing out that one cannot make a distinction between constructively solving a problem and the "intrinsical" or "objective" truth about the problem, as it were, independently of its solution — the latter being meaningless, a metaphysical speculation drawn from Platonism.

It is quite surprising — and scholars of the present generation are likely to forget it — how thoroughly mathematical outlook and trust have been revolutionized within a few decades. In 1900, at the second International Congress of Mathematicians in Paris, Hilbert opened his historical lecture on (unsolved) Mathematical Problems [1]) with the proud words which then expressed a common belief of mathematicians: all of you certainly share the conviction that each definite mathematical problem necessarily admits of a strict settlement, and you are aware of the continual call addressing you «Behold the problem, seek its solution; you can find it by pure thought». (The term 'solution' of course includes a negative solution as in the classical construction problems of geometry (unsolved by the Greeks), and independence problems as the axiom of parallels.) [2])

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[1]) *Hilbert 4.

[2]) Needless to point out that from any conception of mathematical method the process of thinking that leads to solving a problem is always *finite.* This just is the meaning of 'solution' and 'proof' in mathematics: to reduce a question which at first displays an infinite aspect (e.g., the transcendency of π) to a

The certainty of a solution seems to distinguish mathematics from other (inductive) sciences where the ghost of everlasting failure and of a final *ignorabimus* troubles the mind of the scholar and impairs his efforts. It was this certainty which induced mathematicians to continue, through many centuries, their attempts to solve problems like that of Fermat's last theorem.

The conviction of the solvability of all mathematical problems was not based upon properly logico-mathematical arguments but principally upon actual scientific experience and the reflection that the concepts of mathematics, hence also its problems, originate from the sphere of human thought and (internal) intuition only — in contrast with other, especially natural, sciences where external experience is essential. Accordingly human reason ought to be sufficient to solve the problems put by itself (and it should also be its duty of honor to do so). Yet in addition to such emotional impulses the belief in the logical principle of the excluded middle has been, and still is, a powerful motive for the conviction of solvability; both were straight off identified by Brouwer (cf. p. 236). If, for instance, there is "intrinsically" an answer, positive or negative, to the question whether just six primes of the form $2^n + 1$ exist or more than six, then the question cannot be unsolvable; for its "being unsolvable" precisely means that there exist six only [1]).

Less than the first quarter of the 20th century had elapsed when the belief expressed and glorified by Hilbert, which had been a strong stimulus giving scholars confidence in a final success, was breaking down and even explicitly declared an unsubstantiated prejudice by prominent mathematicians. The rejection of the *tertium non datur*

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finite sequence of inferences. Controversy only arises, first with regard to the mathematical basis, in which certain infinite ingredients (e.g., mathematical induction) are necessarily contained while others (for instance, the axiom of power-set [p. 37] or the axiom of choice) may be accepted or rejected; secondly with regard to the logical basis, particularly to the rules of inference admitted for proving.

[1]) For this complex of questions on a first (lower) level cf. Hessenberg 06 (Chapter XXII), P. Lévy *1, *2, 27, *Wavre 3 and 4, *Kempner 2, *Reymond 5. (Among the philosophical contributions, Goodstein 39 is remarkable; let us quote his saying "in mathematics it is the solution which creates the problem".) On a higher level the questions reappear in connection with the problems of completeness, incompletability, non-contradiction within deductive systems; see Chapter V.

was not the only impulse but an influential one in this surprising development; intuitionism regards as an almost regular feature the trifurcation into (1) positive solution of a problem by a general proof, (2) its negative solution by a constructive counter-instance, (3) lack of solution when neither of these cases takes place. (2) is called by Brouwer *absurdity* of (1) and vice versa (instead of the classical term *negation* which does not apply to the relation between (1) and (2)). The case (3) may hold when a solution, maintained to be given by classical mathematics, has a purely existential character because the *tertium non datur* or the axiom of choice etc. is used. There is no doubt that, far beyond intuitionistic circles, the conviction of universal solvability has been shaken emotionally [1]).

Recently the close connection between the problem of solvability and the theory of certain algorithms (in particular of recursive functions, see Chapter V) on the one hand, the intuitionistic logic (§ 4) on the other, has been stressed in some important papers [2]).

§ 4. Mathematics and Logic. Logical Calculus

From §§ 2 and 3 the neo-intuitionistic thesis about the relation between mathematics and logic becomes evident. Mathematics means mathematical *constructions* in contrast with their oral or written exposition. Logic, on the other hand, is the theory of forms expressing thought, hence the theory of mathematical *exposition* which arises subsequent to mathematical construction and constitutes an abstraction from mathematics. Thus logic is chiefly degraded to a "phenomenon of language"; Brouwer's attitude towards the relation between mathematics and language, as explained on p. 212 ff, automatically operates upon the valuation of logic. Logical laws become the laws of symbolization of thinking, hence they may be applied only to an extent compatible with the intuitive basis and the

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[1]) It is characteristic (though methodically conforming to the line of metamathematics) that Hilbert himself later (in 25; cf. also 18, pp. 412 ff) formulated the problem in the sense that general solvability should be *consistent*, i.e. non-contradictory. It is true that in later essays (especially in 31) he proceeds from 'non-contradictory' to 'true'. Yet these researches, in spite of the title of the paper 31, belong to metamathematics (cf. Chapter V).

[2]) The pioneer work is principally found in Church 36 and 36a, Post 36, Turing 36; cf., also for literature, Kleene 45, 49, 52 (Parts III and IV), 52a.

constructive development of mathematics. In particular, the logical principle of the excluded middle is applicable in finite domains with a definite boundary and is in them both a consequence and an anticipation of a certain property of constructions within such domains (p. 217).

This order of precedence, conceiving logic as succeeding mathematics, is well suited to the character of arithmetic (theory of number) — which is significant since arithmetic is the only basis of mathematics according to intuitionism. In fact arithmetical statements are proved more naturally by construction than by logical deduction from more general laws.

Yet is has remained difficult to grasp from Brouwer's explanation what precisely is meant by *intuitionistic logic* as abstracted from mathematics and still more difficult to comprehend the principles leading to his conception of the relation between mathematics on the one side, language and logic on the other. A good deal of these difficulties derive from the informal and rather rhetorical character of Brouwer's explanations which he justifies, and even insists upon, in view of the informal, unsymbolical, dynamic character of construction itself.

Therefore it was a decisive step towards clearing up the nature of the controversy — the most decisive step since the establishment of neo-intuitionism in 1907 — when in 1930 Brouwer's pupil Heyting undertook *to present the main contents of neo-intuitionistic logic and mathematics in a symbolic form* of essentially the usual kind, subject to a few modifications and additions evolving from certain peculiarities of the new logic [1]).

[1]) Heyting 30 and 30a; cf. 30b, *9, *10, 34, *14, *17, 48, 55, 56. Furthermore, **Freudenthal** *2 and 36 a, *Jørgensen 5, Moisil 38, *Frink 3, Vaidyanathaswamy 43, **McKinsey-Tarski** 48, Rosenbloom 50, Lorenzen 50, Skolem 52-53. (Also *Barzin-Errera 3 and 4, Errera 33; cf. above p. 219.) McKinsey 39 proves the independence of Heyting's primitive symbols (for the calculus of propositions).

Whether the consistency proof of the calculus given in Novikov 43 is intuitionistically valid, as claimed there, seems doubtful.

Fitch 49 extends Heyting's calculus to *modal* concepts such as necessity and possibility. Topological interpretations of Heyting's calculus of propositions are given (independently) in the comprehensive essays Stone 37 (cf. *Tsao-Chen 3) and Tarski 38a; cf. Ogasavara 39, *Pankajam 3, *Stone 8.

A useful modification of Heyting's calculus of propositions (and of Johansson's minimal calculus, p. 241), fit for a splitting-up into subsystems of axioms (cf. Wajsberg 38), is given in Schröter 57. Cf. also Schröter 56a. (Addition in the proof.)

To be sure, the Dutch school does not regard this exposition as an orthodox codification [1]). Such a sceptical attitude results from the alleged impossibility of ever exhausting the totality of processes that may be considered legitimate (see p. 212) and is connected with a more fundamental argument: an exposition in the language of symbolic logic with its static character is, in principle, inadequate to describe the dynamic and never closed domain of mathematical activity and can, therefore, give but a hint at the operational field of admitted constructions. Hence it would be an illusion to think that a system of formulae and inference rules might ever completely describe neo-intuitionistic mathematics.

With such restrictions, however, Heyting's system has been recognized by the Dutch school. Hereby an enormous progress has been obtained for the purpose of comparing traditional (Aristotelian) logic and classical mathematics with their neo-intuitionistic counterparts.

Even before the publication of Heyting's papers Glivenko had ingeniously proved [2]) two important results, viz. (1) whenever a proposition p is provable classically, the absurdity of the absurdity [3]) (hence the non-contradiction) of p is provable intuitionistically, and conversely; (2) whenever the *absurdity of* p is provable classically it is also provable intuitionistically.

We begin with informally exhibiting the characteristic features of Heyting's neo-intuitionistic calculus and its divergencies from classical procedures. After this introduction it will be easy to grasp the symbolic exposition (pp. 235 ff).

Heyting's first paper (30) deals with the intuitionistic *calculus of*

Birkhoff 48 (pp. 128 ff) deals with a lattice-theoretic formulation of *Brouwerian algebras*, i.e. algebraic structures which satisfy Heyting's axioms. Cf. the important extension given in McKinsey-Tarski 46 regarding metamathematics in Brouwerian algebras; also Médvédév 55 and Monteiro 56.

Further literature is given later in this section.

For the connection between Brouwerian algebras and the algebra of closed sets in general topological spaces (and in Lewis' system of strict implication) we refer the reader to the literature, for instance McKinsey-Tarski 46, Mostowski 48, Rasiowa-Sikorski 53.

[1]) Cf. Heyting's own criticism, for instance in 54 and 56.

[2]) Glivenko 29. Cf. Curry 52 where corresponding results are reached. See the exposition in Kleene 52, pp. 492 f.

[3]) The term 'absurd' used by Brouwer for neo-intuitionistic negation signifies, in accordance with its meaning in Dutch, 'contrary to reason' in view of a proof (and not 'nonsensical' or 'ridiculous'). Cf. p. 227.

propositions (statements). Its chief result may be expressed as follows: If $\mathfrak{P}$ denotes a suitable system of axioms and of rules of inference (the *modus ponens* and the rule of substitution) as used in ordinary propositional calculus (cf. below p. 235), one obtains a corresponding system, in particular Heyting's axiom system $\mathfrak{P}'$, for the neo-intuitionistic propositional calculus by replacing [1]) the axiom of the excluded middle $\mathfrak{T}$ (i.e. $p \vee \sim p$) [2]) by a weaker axiom $\mathfrak{T}'$ expressing that a statement which implies both p and $\neg p$ is absurd. Thus neo-intuitionistic propositional logic may be conceived as obtained from ordinary logic by dropping the ingredient of $\mathfrak{T}$ that exceeds $\mathfrak{T}'$.

Now by means of the usual methods one can investigate the consequences of replacing the system $\mathfrak{P}$ by $\mathfrak{P}'$. Whereas in ordinary logic a proposition p is *equipollent* [3]) to its double negation, in neo-intuitionistic logic we only have that p *implies* its double negation and not conversely: by means of an independence proof of the usual kind it is shown that the formula '$\neg\neg p \supset p$' cannot be proved in $\mathfrak{P}'$. Adding this formula to $\mathfrak{P}'$ produces a system which is equipollent to $\mathfrak{P}$.

Hence the intuitionistic propositional calculus is not complete in the sense in which the classical calculus is, i.e. it is not true that the addition of an underivable formula to the axioms makes the resulting system inconsistent [4]).

Negative statements, i.e. statements whose main connective is the sign of absurdity '$\neg$', have a special status in neo-intuitionistic logic [5]): for them the principle of the excluded middle holds true, i.e. $\neg p \vee \neg\neg p$ is valid, and they are equipollent to their double negation, i.e. $\neg p \equiv \neg\neg\neg p$ is valid, as also Glivenko has shown.

[1]) In Johansson 36 the consequences of dropping a further axiom (of denial) are discussed, cf. (also for negationless logic) below p. 241.

[2]) For the meaning of negation in neo-intuitionism and the symbol used for it ($\neg$, instead of the symbol $\sim$ for classical negation), see below p. 235. Cf. Popper 48.

[3]) This still applies to certain calculi developed for the formulation of quantum mechanics; see Birkhoff-von Neumann 36 and *Strauss 1, also *Février 3. (Cf., however, von Weizsäcker 55.)

[4]) Cf. Gödel 32, *Heyting 9. — As to the decision problem (Chapter V) in $\mathfrak{P}'$, special criteria had already been obtained by Glivenko and Heyting; general solutions were then given by Gentzen 34 and in the comprehensive investigation of $\mathfrak{P}'$, Wajsberg 38.

[5]) For instance, Brouwer 29, *Heyting 8; cf. O. Becker 30. This distinction becomes manifest in Glivenko's second result.

The axioms of Heyting's system $\mathfrak{P}'$ are *independent.*

From a formal-axiomatic point of view the incompleteness of $\mathfrak{P}'$ permits the following interpretation: the formula of the *tertium non datur* may either be dropped (as in $\mathfrak{P}'$) or be retained (as in $\mathfrak{P}$) — according to fancy or to the purpose intended. One thus obtains two different kinds of logic one of which comprehends the other, analogically as dropping the axiom of parallels means replacing Euclidean geometry by absolute geometry. The question which logic (or geometry) is "true" is, then, not appropriate. Such an attitude, however, is not in accordance with neo-intuitionism which abandons the axiomatic contrast between 'deducible' and 'non-deducible' (or ('incompatible') in favor of the contrast between 'true' and 'false' ('absurd'); therefore the choice of a system of axioms is not arbitrary but largely compulsory.

The second of Heyting's basic papers [1]) gives, in its first part, an analogous exposition of the neo-intuitionistic *functional calculus* (calculus of predicates) [2]), including Peano's theory of natural numbers. Heyting's original formalization of the functional calculus is rather complicated; it was later simplified by himself and others. See below p. 237.

Besides Gentzen's paper cited below which deals with the intuitionistic calculus in the frame of classical logic, Heyting's treatment of intuitionistic logic was also expanded by the profound investigations of D. Nelson [3]) which, like the preceding work of Kleene, largely depend on a formalization of an extensive part of the theory of recursive functions [4]) (Chapter V). The term 'realizable' in these papers means 'intuitionistically true' [5]).

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[1]) Heyting 30a; cf. Gentzen 34, *Heyting 17, Peremans 49. Gentzen's paper (III. *Abschnitt*) has particular importance (not only regarding intuitionistic logic and mathematics); it illustrates the peculiar rôle of the formula expressing the *tertium non datur* from a formal point of view (cf. Pil'čak 52) and also contains a new proof of the non-deducibility of that formula within neo-intuitionistic logic. A very clear exposition of Gentzen's calculi is given in Kleene 52, ch. XV. Cf. also Kuroda 51, Ohnishi 53, Maehara 54.

[2]) As its classical counterpart, the functional calculus of Hilbert-Ackermann 28 is taken.

[3]) D. Nelson 47 and 49. 47 proves important theorems which were conjectured in Kleene 45; cf. Kleene 49 and the general method developed in Mostowski 48 (cf. Henkin 51). The comprehensive exposition of intuitionistic propositional and functional calculus in Kleene 52, § 80, gives a lucid survey. Cf. Harrop 56.

[4]) See, in particular, Péter 34 and Kleene 36.

[5]) For a more detailed treatment see Kleene 52, pp. 503 ff.

Finally, the second part of Heyting's paper 30a deals symbolically with Brouwer's definition of set (§ 5) and in particular with the "free-choice sequences"; this part is most significant from the neo-intuitionistic point of view.

While intuitionists find little interest in formal expositions, the formalization has for the first time given outsiders the long desired opportunity to scrutinize the new logic in the frame of classical logic and to analyze the meaning and the significance of this particular partial structure. The first and highly interesting result was obtained by Kolmogoroff [1]), namely that Heyting's calculus can be interpreted in classical terms as a *calculus of problems* — 'problem' and its 'solution' left undefined (to be understood "in the ordinary sense") [2]). For instance, neo-intuitionistic conjunction corresponds to solving two problems, disjunction to solving at least one of two problems, implication to reducing the solution of one (the second) problem to the solution of another (the first). The latter interpretation is especially important since the ordinary implication $p \supset q$ is intuitionistically meaningless as long as the truth-values of p and q are unknown. Analogically the rules of inference are interpreted. Regarding the functional calculus, let F be a problem which depends on a variable x; then $(Ax)F$ corresponds to the problem 'indicate a general method of solving F for every x', and $(Ex)F$ to the problem 'name a particular x for which F can be solved'. This attitude confirms again, and from a quite different view, our thesis of p. 220 that Brouwer's restrictions refer to the negation of general propositions rather than to the principle of the excluded middle proper.

We remarked repeatedly that neo-intuitionistic logic so far as the calculus of propositions is concerned forms a *part of classical logic*. Yet this statement should be qualified inasmuch as it depends on the *correspondence* chosen between the primitive concepts of both calculi. It is true only on condition that the intuitionistic connectives are given

[1]) Kolmogoroff 32. Cf. Heyting 34 (pp. 14 f), Jaśkowski *1 and 36, *Février 3, Pil'čak 52. Already in Heyting 30b and *8 one finds an endeavor at interpreting the calculus in the sense of mathematical constructions. Freudenthal 36a criticizes the fundamental attitude underlying the interpretations of Kolmogoroff and Heyting. — Another interpretation of Heyting's calculus (by means of the ordinary propositional calculus with the addition of the predicate 'p is provable') is given in Gödel 33c.

[2]) In Médvédév 55, however, the notion of 'problem' is given a precise meaning.

their "normal" interpretation within the classical calculus. However, there exist other interpretations under which *classical logic appears as a part of intuitionistic logic*, i.e. under which the counterparts of all classical theorems are theorems in the intuitionistic calculus but not vice versa [1]). To be sure, this relative consistency proof of the classical propositional calculus has a restricted value. For adherents of classical logic it is superfluous since the classical calculus can be shown to be even absolutely consistent; for intuitionists it is hardly revealing since it is the "normal" interpretation that interests them. Moreover, the interpretation requires the unanalyzed notion of 'consequence of an assumption' [2]).

Furthermore, starting from Heyting's exposition of mathematics (30a) one analogically obtains even a proof of non-contradiction for arithmetic [3]). In fact Gödel gives an interpretation of the terms of classical arithmetic and logic by intuitionistic terms such that to every formula which can be proved by the formal methods of traditional mathematics, there corresponds a formula which is provable intuitionistically. Hence intuitionistic arithmetic may be conceived in such a way that it includes classical arithmetic, though only subject to a "non-normal" interpretation of the terms used for the primitive concepts. Thus emerges a consistency proof of classical arithmetic [4]) appearing *relative* to a classicist, would-be-*absolute* to an intuitionist whose arithmetic is *eo ipso* consistent.

While intuitionistic arithmetic is generally considered part of classical arithmetic, the methods of Gödel and his successors seemingly invert this picture — in a sense which is not acceptable to the Dutch school. In fact, if we accept (recursive) realizability [5]) as a necessary condition for intuitionistic truth, the formula $(Ax)\,[f(x) \vee \neg f(x)]$ turns out to be intuitionistically untrue since — though classically [6])

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[1]) Gödel 33b and c; cf. the generalization in Rasiowa-Sikorski 54. A similar result was independently obtained by Gentzen 36. Cf. Bernays 35a, Beth 35 and 36, Kleene 52 (pp. 495 ff); also Lorenzen 50, Ridder 50-51, Łukasiewicz 52, Skolem 52-53.

[2]) Cf. Freudenthal 36a. This criticism also applies, in some way, to Heyting's calculus in general.

[3]) Arithmetic is here conceived, say, as in Herbrand 31.

[4]) This proof is not finite in the sense of Hilbert and Herbrand (cf. Chapter V; in particular Herbrand *5 and 30). See the discussion, with exhaustive literature quotations, in Kleene 52, pp. 497 ff.

[5]) Cf. p. 231; see Kleene 52, § 82, and Rose 53.

[6]) Of course, with the classical negation $\sim$.

provable — it is not realizable. On the other hand, the formula $\neg(Ax)\,[f(x) \vee \neg f(x)]$, though classically disprovable, is realizable [1]), hence not intuitionistically untrue. One can therefore add the latter formula to the axioms of the intuitionistic functional calculus, hereby obtaining a calculus *diverging* from the classical one. That such a situation, well-known to mathematicians from geometry, might also arise in arithmetic is rather unexpected.

Still, "classical" mathematicians will interpret Gödel's results to mean that intuitionistic arithmetic is essentially not narrower than classical arithmetic and that an actual divergence only begins with analysis and set theory where intuitionists of all trends, far beyond neo-intuitionism, take offence either at the use of impredicative definitions (p. 174) or else at the palliatives used for avoiding such definitions. The far-reaching correspondence between intuitionistic and classical arithmetic relies partly upon a correspondence between intuitionistic 'absurdity' and classical 'negation'.

We conclude this informal survey of neo-intuitionistic logic with the following remark. The classical propositional claculus with its two truth-values 'true' and 'false' has recently been generalized to many-valued calculi where three and more values are admitted (Chapter III); such systems may, like Heyting's calculus, be considered as parts of classic calculus. As pointed out in § 3, neo-intuitionism does not propose a three-valued system, for its "third" alternative to 'true' and 'false' is not coordinate with these. In fact it has been proved [2]) that Heyting's propositional calculus does not admit of an interpretation by means of any logical system with a finite number of truth-values and that "between" the ordinary and the neo-intuition-

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[1]) See Kleene 52, p. 513, and the subsequent discussion there. Cf. already Kleene 45.

[2]) Gödel 32. In another way such "between"-systems appear in Umezawa 55. — Cf. the use made of certain distributive lattices for the investigation of intuitionistic logic in the (independently written) papers McKinsey-Tarski 48 and Rieger 49. In particular it is shown that if '$p \vee q$' is provable in Heyting's calculus, p or q is provable; also a new proof of Glivenko's results (p. 229) is thus obtained.

Stronger results are reached by means of 'Brouwerian lattices' (cf. p. 229) in Mostowski 48 (cf. Henkin 51) and particularly in Rasiowa 52; here certain results contained in Gödel's completeness theorem (see Chapter II, § 6) are transferred from the classical to Heyting's calculus. Cf. the literature given above at the end of § 3.

istic systems there is an infinite sequence of "decreasing" logical systems. On the other hand, there exists an interpretation which uses denumerably many truth-values [1]).

Regarding Heyting's *formalization* (1930) of the principal parts of neo-intuitionistic logic, we shall deal in detail only with the *propositional calculus* (calculus of statements). For the functional calculus see p. 237; for the mathematical part which mainly concerns the concept of set (species), cf. § 5.

Using here the symbols '$\vee$', '$\wedge$', '$\sim$', '$\supset$' for disjunction, conjunction, negation, and implication respectively (cf. p. 25) we start from the following well-known system of axioms for classical propositional logic, where a, b, c denote propositions (statements), and the formulae should be considered to express general validity: [2])

1) $(a \supset b) \wedge (b \supset c) \supset (a \supset c)$
2) $[a \supset (b \supset c)] \supset (a \wedge b \supset c)$
3) $(a \wedge b \supset c) \supset [a \supset (b \supset c)]$
4) $a \wedge b \supset a$
5) $a \wedge b \supset b$
6) $a \supset a \vee b$
7) $b \supset a \vee b$
8) $(a \supset b) \wedge (a \supset c) \supset (a \supset b \wedge c)$
9) $(a \supset c) \wedge (b \supset c) \supset (a \vee b \supset c)$
10) $a \wedge \sim a \supset b$
11) $a \vee \sim a$.

The usual rules of inference are added to these axioms.

For neo-intuitionistic disjunction, conjunction, and implication we shall continue — for the sake of simplicity, though not quite correctly — to use the same symbols, yet neo-intuitionistic negation (absurdity) will be denoted by Heyting's symbol '$\neg$' instead of '$\sim$'. Then a system of axioms for neo-intuitionistic propositional calculus, closely related to Heyting's, is obtained from the above system by taking the four symbols in their neo-intuitionistic sense and by replacing 11) by

11*) $(a \supset b) \wedge (a \supset \neg b) \supset \neg a$.

While 11) constitutes the axiom of the excluded middle, 11*) is the far weaker axiom mentioned above on p. 230.

It is not difficult to show that '$\neg\neg a \supset a$' cannot be proved by

[1]) Jaśkowski 36; cf. Pil'čak 52 and Rose 53.

[2]) For reasons of simplicity we do not introduce a special symbol of validity.

means of the new axioms, i.e. that the absurdity of the absurdity of a statement a is not generally equipollent to a. Hence the axiom of the excluded middle cannot be proved in the system composed of 1)–10) and 11*).

As to 11*), let us first remark that it would not do just to eliminate 11) and to use the system 1)–10) only; for 11*), which cannot be deduced from 1)–10), serves the indispensable purpose of excluding contradiction (a purpose also answered by the axiom 11) which contains an additional ingredient). Secondly we note that 11) is not replaced by something *negating* it, for instance by a principle of excluded fourth; the ingredient of 11) that transcends 11*) has just been dropped.

From the neo-intuitionistic point of view it would be preferable to interpret the logical axioms according to Kolmogoroff's calculus of problems (p. 232). In fact, while the usual way of explaining implication ('$a \supset b$' is false if and only if 'a' is true and 'b' is false) is intuitionistically meaningless as long as the truth-values of 'a' and 'b' are unknown, in the light of the calculus of problems '$a \supset b$' has the legitimate meaning of reducing the task of solving b to that of solving a (p. 232). Hence the rule of inference 'if a and $a \supset b$ then b' leads us from solved problems again to solved ones. (The meaning of 'reduction' includes, in accordance with axiom 10), that the proof that a cannot be solved reduces the solution of *any* problem to the solution of a.)

What has been said here of implication applies similarly to disjunction. In particular, according to this interpretation the axiom of the excluded middle corresponds to the statement: a being an arbitrary problem, either a can be solved or the solution of a can be shown to be contradictory (absurd). This conception induced Brouwer to identify the axiom with the assertion that every mathematical problem can be solved. On the other hand, corresponding to the true statement of the absurdity of the absurdity of the *tertium non datur*, i.e. to $\neg\neg(a \vee \neg a)$, there is no problem whose unsolvability can be proved ('problem' and 'unsolvability' being understood appropriately).

As to the *functional calculus*, we start again from its treatment in classical symbolic logic [1]) without considering the different meaning of (disjunction and) existence. To the axioms of the propositional calculus

[1]) E.g., Hilbert-Ackermann 28.

we add the axioms:

$$(\mathrm{A}x)f(x) \supset f(y)$$

$$f(y) \supset (\mathrm{E}x)f(x)$$

and the usual rules of inference.

In § 3 it was pointed out informally that, in accordance with the constructivistic conception of existence, the negation of a general statement does not constitute an existential statement. Hence, if f denotes an expression that contains the free variable x, the valid formulae

$$(\mathrm{E}x)f(x) \supset \neg(\mathrm{A}x)\neg f(x)$$

$$(\mathrm{A}x)f(x) \supset \neg(\mathrm{E}x)\neg f(x)$$

$$(\mathrm{E}x)\neg f(x) \supset \neg(\mathrm{A}x)f(x)$$

must not be inverted, while the valid formula

$$(\mathrm{A}x)\ \neg f(x) \supset \neg(\mathrm{E}x)f(x)$$

can be inverted.

Regarding double negation, the valid formulae

$$(\mathrm{E}x)\ \neg\neg f(x) \supset \neg\neg(\mathrm{E}x)f(x)$$

$$\neg\neg(\mathrm{A}x)f(x) \supset (\mathrm{A}x)\neg\neg f(x)$$

must not be inverted.

In the light of the calculus of problems $(\mathrm{A}x)F(x)$ means the task of giving a general method which enables us to solve the problem F for all x, and $(\mathrm{E}x)F(x)$ the task of submitting an x and solving the problem for this x. Neither of these tasks can be reduced to the other. Thus a statement which contains expressions whose truth-values are unknown and is, therefore, neo-intuitionistically meaningless obtains a legitimate meaning when considered as a problem. To be sure, after the problem has been solved, the truth-values are known and the statement becomes trivial.

He who is familiar with the lucidity and simplicity of the classical logical calculus which is due to the excluded middle formula '$a \vee \sim a$', will not readily submit to the complications involved by the restrictions mentioned. It is therefore remarkable that recently a method — more precisely, two methods with different ranges — has been proposed [1]) which seems quite appropriate, not to bridge

[1]) van Dantzig 47 (written in 1942). The "weak interpretation" of intuitionistic calculus was already given in Gödel 33c.

the gap between traditional and intuitionistic attitudes themselves but to facilitate the comprehension of the latter by classical-minded logicians and mathematicians.

The first method, called by van Dantzig *the weak interpretation*, aims at enabling ordinary mathematicians and neo-intuitionists to meet half-way. Frequently and not without reason are intuitionists charged with interpreting mathematical theorems, particularly existential theorems, in a far stronger sense than they are properly meant, viz. in a constructive sense; in other words, p denoting a neo-intuitionistic statement, the ordinary mathematician is not actually interested in proving p but in proving $\sim\sim p$ only. Such a "translation" of intuitionistic statements is not as complicated as it may appear since in accordance with Glivenko's result mentioned on p. 230 we never need consider more than two negations. Accordingly, if we agree to call a statement that is equipollent to (or implied by) its double negation a *stable statement*, every negative statement is stable.

While the negation of any stable statement is clearly stable and the same is easily shown to hold for the conjunction of stable statements, the same does not hold for disjunction and implication. However, what we shall call (in this section only) *quasi-disjunction* defined as '$\neg(\neg p \wedge \neg q)$', and *quasi-implication* defined as '$\neg(p \wedge \neg q)$' [1]), are obviously again stable formulae.

The "transfinite analogue" of conjunction, namely a general statement obtained by universal quantification, retains the stability property. Though existential statements proper do not retain it, *quasi-existential* statements of the form $\neg(Ax)\neg f(x)$ are again stable.

The classical propositional calculus is, therefore, intuitionistically valid if the range of the statement variables is restricted to stable statements and if the classical signs for negation and conjunction and for the universal quantifier are given the normal interpretation while the classical signs for disjunction and implication and for the existential quantifier are given the "quasi"-interpretation.

In view of this interpretation the domain of stable statements constitutes a part of intuitionistic logic which is closed with respect to the elementary operations and fulfils the axioms of traditional logic.

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[1]) Classically but not intuitionistically these formulae are respectively equipollent to '$p \vee q$' and to '$p \supset q$', i.e. '$\sim p \vee q$'. Likewise '$\neg(Ax)\neg f(x)$' is classically equipollent to '$(Ex)f(x)$'.

In particular, the axiom of the excluded middle '$p \vee \sim p$' remains valid in this domain.

Regarding mathematics, the theory of integers and rationals causes no difficulty since equality and inequality between integers are stable relations. Real numbers, however, do cause trouble even when one replaces Brouwer's definition by Cantor's definition which conceives a real number as a "fundamental sequence" of rationals. Nevertheless, by an appropriate modification of the basic definitions and by a rather complicated alteration of some proofs one seems to succeed in giving a stable interpretation to this theory as well, though the use of impredicative definitions may not be avoided throughout. (Incidentally, the new proofs show that certain traditional proofs lack sufficient uniformity.)

On account of such modifications we conclude that no contradiction can occur between classical logic and mathematics and its neo-intuitionistic counterparts, provided the stable interpretation is adopted; the classical theories thus even appear as parts of the intuitionistic ones (cf. p. 233). The contradictions stressed so vehemently and repeatedly by Brouwer are due to his taking a different interpretation.

In fact, without a different "*stronger*" interpretation the neo-intuitionistic calculus introduces nothing new in principle; stable mathematics certainly does not reach further than the system outlined in Weyl's *Kontinuum* [1]). Moreover, for the *applications of mathematics* the attitude taken hitherto is useless. If, for instance, we have in mind an application of the fundamental theorem of algebra (p. 66), it will not be helpful to assert the absurdity of the assertion that a polynomial with a positive degree has no zero; in general, we want an effective method of calculating the zeros of a given polynomial. In what follows we shall hint at the presumably "strongest" interpretation: at *affirmative mathematics*. In other words, the use of negation shall altogether be avoided [2]). A strong argument for this attitude is found in the intuitionistic thesis that by just negating a

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[1]) Weyl 18.

[2]) This direction was taken by van Dantzig independently of Johansson 36 (cf. p. 241). It is true that in the latter paper, as already in Gentzen 34, negation is subsequently introduced, not as a primitive concept but by definition; yet this procedure is not satisfactory for various reasons. Later Griss 44-51 developed similar ideas but without avoiding existential statements of a general (unrestricted) type (see p. 242).

statement p one does not, in general, obtain any statement; hence why should the negation of p be mentioned at all? The idea is, then, restriction to those statements which we affirm without mentioning others. This, of course, includes renouncing negation as a primitive concept [1]).

Along these lines van Dantzig develops the essential parts of mathematics, using only procedures that are intuitionistically legitimate but apart from any intuitionistic dogmatism; no knowledge of the theses of the Dutch school is required. (Cf. below §§ 5 and 6.)

Analysis certainly must be based upon arithmetics; hence we begin with *natural numbers*. This subject may on the whole be introduced along the lines of Peano with the improvements that have meanwhile been added; however, mathematical induction is introduced by a deduction schema, i.e. as a rule of inference. Yet one of Peano's axioms (usually counted as the third), having a purely negative character, cannot be formulated in affirmative mathematics, viz. the statement that 0 is not the successor of any number; after having introduced addition one may replace this axiom by postulating that $x + y = 0$ implies $y = 0$.

The characteristic features of intuitionism turn up with the introduction of *real number*. This may be done by means of approximating finite dual fractions together with an appropriate inequality (as in Cantor's theory of irrational number), which procedure essentially conforms to Brouwer's. Equality as well as the relation $\leq$ between real numbers is easily defined; however, as in Brouwer's system (cf. § 6), inequality and the order relation $<$ cannot be defined. In fact, $x \neq y$ would mean the existence of an integer n such that the digits at the n^{th} place of the dual fractions x and y are different, and this is a purely existential statement, hence not admitted. Therefore a statement of the type $x \neq y$ does not yield any affirmative consequence, in accordance with the fact that the physical measurement of a quantity x will never give us the result that $x \neq 1$, unless either

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[1]) One might recommend the same attitude with respect to disjunction and to existential quantification. In fact, $p \vee q$ does not affirm *per se* anything intuitionistically meaningful. However, excluding disjunction would involve most serious complications; it suffices to think of what may be inferred from the vanishing of a product of integers or rationals. As to existence, the system proposed here uses it in a certain restricted sense only which is reduced to disjunction by means of mathematical induction. Cf. also *Skolem **4**.

'$x \leq 1 - \varepsilon$' or '$x \geq 1 + \varepsilon$' with a definite positive ε can be established. Only from such statements may the "statement abstract" $x \neq 1$ be inferred (cf. above p. 211).

True, a consistent development of these ideas involves considerable inconvenience, for the concepts of convergence (already of a sequence of numbers) and of limit require complicated specializations in the direction of a "fine-structure", involving the notion of speed of convergence. Thus continuity obtains a character rather different from the usual one.

While a definite meaning can be attached to concepts such as mentioned here — a meaning narrower (stronger) than the usual one — there are other classical concepts without any affirmative meaning whatsoever [1]).

Such developments have a definite significance and even a considerable scientific interest also within classical mathematics; yet there one would not consider them as *the* theories of inequality, convergence, etc. but as certain specializations of these theories, particularly useful for the applications of mathematics. On the other hand, while these developments are meaningful from the viewpoint of neo-intuitionism they are far from *exhausting* neo-intuitionistic mathematics.

Whereas van Dantzig's paper may be considered an instructive and profound *interpretation* of neo-intuitionistic logic and mathematics, two other directions have been taken with the intention to *modify* the system of Brouwer-Heyting.

The first is Johansson's *minimal calculus* [2]). It may be described as originating from Heyting's propositional calculus by dropping the axiom 4.1 in Heyting 30, namely $\neg a \supset (a \supset b)$ (cf. 10) on p. 235). An alternative way, of dropping Heyting's second axiom of negation and instead introducing a new primitive element, is also given [3]). In the new (weaker) system, which may be conceived as an intuitionistic calculus with a different interpretation of implication, all theorems of Heyting's system which do not involve negation retain their validity, and even the majority of those that involve negation;

[1]) Cf. van Dantzig 42.

[2]) Johansson 36. Cf. Lorenzen 50 and Ridder 50-51.

[3]) A further alternative is given in Curry 52 (cf. 50), together with an alternative to Heyting's calculus. Cf. also Kanger 55.

$\neg p$ may then be defined as $\sim p$ in the footnote 2 here below. Furthermore, Johansson shows that his system can be interpreted as a weakened form of Gentzen's intuitionistic calculi of inference [1]).

A second direction was first taken by Griss; however, he contents himself with indicating general principles and various examples rather than giving an elaborate system [2]). He endeavors to construct intuitionistic mathematics and logic *without using negation*, while criticizing the neo-intuitionistic attitude which, in his opinion, is not radical enough. Renouncing negation induces him to stress the relation of distinguishability ($\#$) and to start, as does neo-intuitionism (see below, § 5), with the positive integers which are intuitively distinguishable from each other. (That the real numbers a and b are distinguishable may mean, for instance, that a rational number can be named which lies between a and b.) As Bernays has remarked, van Dantzig's attitude should better be called 'negationless' and Griss' 'affirmative'; Griss at first sight seems to go farther than van Dantzig, beyond the mere rejection of negation with its derivatives like disjunction, excluding what he calls 'nullity' (see below). Brouwer has not altogether accepted the attitude of Griss. To be sure, according to

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[1]) The calculi NJ and LJ of Gentzen 34. Gentzen's *Hauptsatz* or "normal form theorem" (cf. Kleene 52, p. 453) for LJ is fulfilled by the minimal calculus. — In his paper 53 Johansson introduces *descriptions*, i.e. certain 'the'-phrases, into the minimal calculus, as well as into Heyting's system.

In connection with Gentzen, Beth's work on the semantic construction of intuitionistic logic should be mentioned: Beth 55 (Livre III, Chapters III–V), 56a, 57 (Chapter IV).

[2]) Griss 44-51, 48, 48a, 49; see also Dequoy 49, 50, 55; Destouches-Février 45, 48, 51; Vredenduin 54; Valpola 55. The notes of Mrs. Février partly take an intermediate position between Brouwer and Griss, in admitting some notion of negation defined in a positive manner, namely $\sim p$ defined by $p \supset \varphi$, where φ stands for a (positively defined) definite false statement of the respective theory, e.g. $0 = 1$. Moreover, Février stresses the connection between negationless logic and a calculus of problems in the sense of Kolmogoroff.

Miss Dequoy applies negationless logic to basing plane projective geometry under the (somewhat problematic) assumption that it will be possible to formulate a theory of real numbers and an analytic geometry satisfying the principles of Griss; in particular, with regard to the intuitionistic treatment of projective geometry in *Heyting 3, Dequoy proposes to replace the form of *reductio ad absurdum* used there by a set-theoretical argument which depends on negationless axioms for distinguishability.

Skolem [1]) it is impossible to build an intuitive set theory in such a negationless system.

A satisfactory understanding and a criticism of the program of Griss was only made possible through Gilmore's profound analysis [2]), who regarding Griss performs a somewhat similar task as Heyting regarding Brouwer and goes beyond this task by his constructive criticism.

Introducing a certain deductive theory H formalized within intuitionistic logic, a subtheory of which contains the theory of the relation $\#$, Gilmore was able to express the consequences of Griss' criticism which would discard many of the predicates and statements of H because they do not satisfy the condition of 'positive non-nullity'. (This condition rejects the empty set [null-class] and any statement which is not true. More precisely, a predicate or statement p is admitted only after the proof of $(Ex)p$, where (Ex) means a row of existential quantifiers [empty when p is a statement], namely one for each free variable of p.) Yet, in addition to the predicates and statements admitted by Griss, Gilmore exhibits a further class of predicates and statements whose introduction does not contradict Griss' principles, namely those which are equipollent to contradiction within H; e.g., $x_1 \# x_1$.

Now Griss' decisive criticism, viz. the rejection of negation, can be overcome since the intuitionistic negation can be defined as the implication of a predicate which plays the rôle of the null-predicate (provided such a predicate exists in the system). Also the limitation in the use of disjunction, conjunction, implication, and quantification can be mastered. The effect of Griss' criticism of the intuitionistic predicate calculus can be expressed, according to Gilmore, by adding a certain axiom for every "atomic" predicate and statement of the logic.

To be sure, Griss would not recognize the deductive theory H on which Gilmore's results depend — analogically to the orthodox neo-intuitionistic attitude towards Heyting's system (p. 229).

Gilmore's achievement is, on the one hand, a step forward within Griss' program of constructing a system of negationless mathematics, but on the other hand a *criticism* of Griss' criticism inasmuch as his 'nullity', the rejection of which is Griss' cornerstone, proves to be a

1) Skolem 52a.
2) Gilmore 53.

relative and not an absolute concept. While seemingly van Dantzig's "affirmative mathematics" and Griss' "negationless mathematics" are in agreement, since both reject negation and demand that theorems expressed by means of negation either be translated into a positive form or, when this proves impossible, be dropped — there is the difference that in Griss' system of mathematics negation can be *defined* and the system can thus be reconverted to a system like Heyting's. This is not applicable to van Dantzig's system which rejects disjunction, the essential tool of that reconversion. (Cf., however, p. 242.)

§ 5. The Primordial Intuition of Integer. Free-Choice Sequences and Brouwer's Concept of Set [1])

The mathematician, tired of dogmatic principles underlying the neo-intuitionistic program and of the painful restrictions imposed on current procedures of definition and proof, will want to learn how, then, mathematics can retain its *infinite character* without which it would essentially be reduced to trivialities. In fact just H. Weyl, who had first created a peculiar semi-intuitionistic system and then accepted the attitude of Brouwer providing it with additional arguments, seems to have created the sentence: mathematics is the science of infinity [2]).

The nucleus of positive intuitionistic principles, common to all intuitionistic trends from Kronecker to our days, is the "primordial intuition" (*Urintuition*) of *positive integer* or of the construction by *mathematical induction*. This induction is also the prototype of those constructions with which mathematical activity is identified (pp. 207 ff). This, of course, does not mean that Brouwer would accept the infinite

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[1]) The manuscript of Chapter IV of this book was finished when Heyting 56 appeared whose subject overlaps §§ 5 and 6 of this Chapter. Thus the alternative arose, either to enlarge these two sections so as to cover the new subject-matters treated by Heyting, or to shorten them and to refer the reader interested in details to Heyting. The second way was taken, considering that Heyting's book is easily accessible and that in these delicate questions the inside information of the Dutch school is more reliable than that of an outsider. For this reason, in cases of a slight discrepancy between Heyting's and our exposition (in §§ 5 and 6) the former should be regarded as more authentic. — Cf. also the lucid review of Heyting 56 by S. Kuroda (*Journal of Symbolic Logic* **21**, 367–371).

[2]) *Weyl **6**, p. 1 (1925).

sequence of positive integers as a mathematical object or as a legitimate idea at all — as Weyl's semi-intuitionism still did; it is the *law* of constructing integers and not their would-be aggregate which is accepted. As Poincaré says, "*Quand je parle de tous les nombres entiers, je veux dire: tous les nombres entiers qu'on a inventés, et tous ceux que l'on pourra inventer un jour .... et c'est ce «que l'on pourra» qui est l'infini.*" It therefore makes no sense to ask whether *there exists* an integer with a given property — except for the cases where an integer with the property can be named, or it can be proved that no such integer exists.

We shall not go into a philosophical analysis of "primordial intuition". Becker (of Heidegger's existentialist school) who devoted two books chiefly to an analysis of neo-intuitionism maintains [1]) that one should not conceive it as a "sensorial" or "empirical" intuition but as the kind of immediate certainty with which we are given the fundamental facts of logic, arithmetic, and combinatorics. This intuition is peculiar to mathematics and can be reduced to the indefinitely repeated bisection of the unit, a process whose fundamental significance was also pointed out by Plato. As will be shown presently, Brouwer utilizes the primordial intuition for defining free-choice sequences and sets, thus extending the realm of intuition from the discrete and countable domain of arithmetic (number theory) to the continuous (and somehow non-denumerable) domain of analysis. From this primordial intuition of number the name *intuitionism*, which is easily misleading, has been derived.

The first modern intuitionist, Kronecker, coined the slogan of the integers made by God, everything else (*scil.* in mathematics) by man [2]). Poincaré, who next adopted a somewhat similar attitude, exhibited in full detail [3]) the idea that mathematics, seemingly constituting an immense tautology consisting of *analytic* statements which are obtained by syllogistic inferences, derives its *synthetic* and creative character chiefly from a synthetic principle *a priori* in

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[1]) *Becker **2**, pp. 446 ff (cf. p. 463) and *Lautman **4**. To the following cf. also *Stenzel **1**. Becker even ventured to base transfinite ordinals upon such intuition.

[2]) *Weber **1**, especially p. 19. Cf. *Scholz **12**. See above, p. 162 footnote 1.

[3]) Particularly in *Poincare **2**, Chapter I.

the sense of Kant [1]), viz. from mathematical induction [2]). Weyl stresses [3]) that mathematics altogether — including "the logical forms of its expansion" — is dependent on natural number. Of course intuitionists do not accept the view, suggested by the axiomatic attitude, that mathematical induction might be considered an ingredient of a *definition* of integers, for then one ought to show the existence of an object satisfying the definition which again would require induction [4]).

The weight attributed to induction by intuitionists of all trends is explained by the fact that constructing through finite procedures can only yield statements of a finite character whose contents can, in principle, be verified by finitely many tests. Yet the main statements of arithmetic do not have this character but are transfinite; for instance, '$x + 1 = 1 + x$ for each positive integer x' or 'for each prime number p there exists a greater prime number in an interval dependent on p'. To be sure, there is a fundamental difference between such statements which can be proved by mathematical induction and between the statements of analysis and set theory which are transfinite in a higher sense; for instance, the theorem that every bounded aggregate of real numbers has a least upper bound, or the well-ordering theorem. The arithmetical statements can be verified finitely in each particular case (each value of x, or of p) though the general statement, just because of its transfinite nature, would require an infinite sequence of verifications which cannot be accomplished. Poincaré believes that this verifiability (of the particular cases) of the intuitionistically legitimate transfinite statements is the very reason that enables mathe-

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[1]) Modern science, to be sure, requires a relativization of the concepts 'analytic' and 'synthetic'; cf., for instance, *Dubislav **3**, *Behmann **7**.

[2]) In later years Poincaré (in particular in 08) modified his earlier attitude of attributing to induction alone a creative character and considered it only the simplest among various creative principles. It seems that he even regarded the axiom of choice as one of them. This shows the fundamental difference between him and the Dutch school. On the other hand, scholars akin to this school (cf. *Mannoury **1** and *van Dantzig **1**) pointed out that also the creation of new formalisms, on account of analogy etc., belonged to the intuitive activity of mathematicians.

[3]) In his comprehensive summary 21 of the development of intuitionism, p. 70.

[4]) Cf. *Hölder **7**; however, the criticism expressed there of Hilbert's proof of the consistency of induction is hardly justified.

maticians — in contrast, for instance, with philosophers — to understand each other in spite of the defects of language; whenever the situation becomes doubtful it can be put to the test by a verification which serves as the supreme arbiter. On the other hand, the transfinite statements of analysis and set theory which cannot be tested in a similar way — the classical instance being the well-ordering theorem — remain obscure and eternally exposed to misunderstanding. It is remarkable that since the 20's and 30's of the present century leading representatives of the so-called school of formalists, beginning with Hilbert, von Neumann, Gentzen, etc., have adopted this intuitionistic attitude and nevertheless obtained remarkable results, notwithstanding the threat of Gödel's incompletability theorem (Chapter V).

While the semi-intuitionists, especially of the French school (including Weyl in his first period), were at least theoretically ready to put up with the blockade of analysis and geometry involved by the restriction to induction as *the* transfinite process in mathematics, just the seemingly more radical trend of Brouwer endeavored to create an outlet through the concepts of *choice sequence* and of *set*.

After the concept of positive integer, the next or equally important mathematical (and scientific) concept is the *continuum* (cf. § 1). One may classify various attitudes in the foundations of mathematics according to the kind of "continuum" they admit. The classical continuum concept of the nineteenth century corresponds, on the one hand, to the Greek conception which accepts the continuum as "given" by nature or as its mathematical reflection, on the other hand to Cantor's conception of set (*Theory*, §§ 1 and 9) according to which a set is given if, for any object, it is "internally settled" whether it belongs to the set or not. The deficiency of these conceptions compels us either to take an axiomatic attitude which renounces a definition of set altogether (Chapter II) or to put up otherwise with more restricted conceptions. In this respect semi-intuitionists have taken different and not always consistent attitudes [1]) which were sometimes interpreted in the sense of "extensional definiteness" [2]);

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[1]) The presumably last expression of Lebesgue's attitude (of 1938) is found in 41; cf. Sierpiński 41, p. 137.

[2]) So, in particular, Weyl's "atomistic" continuum (Weyl 18, cf. 19 and 21; cf. the philosophical analysis of these and the neo-intuitionistic attitudes in *O. Becker 1, pp. 403 ff, also Skolem 29, § 3). In Becker's work the alleged dependence of mathematics on the notion of *time* and in general the anthropo-

at any rate, the set ot all properties of integers, which essentially coincides with the classical continuum, is not legitimate in this sense. (Cf. the *impredicative definitions*, Chapter III.) Hence the continua of Weyl, Lebesgue, Lusin, etc. are denumerable, though they cannot be counted within the system.

Brouwer has introduced a novel and important moment into this discussion. His constructivistic attitude certainly conforms to the views mentioned, as far as a single real (irrational, transcendental) number is conceived as a law defining it and as the classical continuum cannot be attained by arithmetical operations [1]). Nevertheless, he maintains that a veritable continuum which is not denumerable in any sense can be obtained as a *medium of free development*; that is to say, besides the points which exist (are ready) on account of their definition by laws, such as e, π, etc., the other points of the continuum are not ready but *develop* as so-called *free-choice sequences*. As it has been said, the free-choice sequences "free the infinite from the concept of law".

Thus both the rejection of geometry altogether (Weyl) and the distinction between a discrete-atomic analysis and a continuous geometry (Hölder, Lusin) become unnecessary for Brouwer.

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logic-existentialistic arguments are closely connected with particular intuitionistic arguments (especially in Becker ***2**, ***4**, 30); cf. the well-based refutation in *Cassirer **4**.

Weyl does not pretend that his continuum reflects the classical notion; on the contrary, he despairs of the possibility of such a reflection for logical reasons and is by force satisfied with extracting discrete "atomic drops" from the continuous "pulp" by rather arbitrary principles of construction. A similar attitude is established by Borel's restriction to *nombres calculables*, i.e. real numbers which can be approximated by rational numbers; however, Borel's notion is rather involved (cf. Sierpiński's [21, p. 114] example of an effectively defined *integer* which is not "calculable").

[1]) It is remarkable that as early as 1892, when such ideas were rather unusual, Hölder (cf. ***3**, pp. 193 f, 349 ff, etc.) denied this possibility. However, not being ready to abandon the mathematical branches which depend on the classical continuum, he accepted it, if not as an arithmetical-analytical construction, at least — analogically to the Greek attitude — as a kind of aprioristic schema which cannot be constructed by thinking but may serve as an object of thinking. Similar ideas appear among the French semi-intuitionists; cf. Borel 14, note IV, and Lusin 27, p. 33. The systematic danger inherent in this attitude is illuminated by certain philosophical doctrines which even nowadays assert the compulsoriness of Euclidean geometry because of its aprioristic origin.

To explain the concept of free-choice sequence [1]) we may, remembering the expansion of real numbers into dual fractions, start with intervals λ_k between $\frac{a_k}{2^k}$ and $\frac{a_k + 1}{2^k}$ where a_k is an integer. A constructively given real number then determines a sequence (λ_k) of nested intervals such that λ_{k+1} is a proper part of λ_k. Conversely, one may successively choose such a sequence of intervals λ_k in an *arbitrary* way — a process which at first appears to depend on temporal and subjective ingredients; these may subsequently be eliminated by a rule which restricts statements about a free-choice sequence to such as refer *to any continuation* of the sequence. There also remains the possibility of "restricting the freedom to choose successive intervals" by a rule [2]) given in advance, called *set-rule*, which rule may be further limited from a certain step on by a restrictive supplement. For instance, the statement that a certain free-choice sequence is greater than $\frac{1}{2}$ is true if this is guaranteed by a set-rule fixing, for example, λ_2 as the interval $(\frac{3}{4}, 1)$.

The concept of free-choice sequence is not restricted to sequences of intervals (with the intention to real numbers): sequences of integers, and in general of any mathematical objects, are admitted as well. One might fancy fixing the successive digits of a hexadic fraction by throwing dice, but then nothing can be affirmed about the resulting real number before an appropriate set-rule is added. Less trivial examples are given in the following section. One may consider this attitude to be a return to certain ideas of Galileo and of the first inventors of calculus, ideas which fundamentally can be traced back to Aristotle and even Anaxagoras [3]).

Naturally the free-choice sequence does not endeavor to yield a single object (say, a real number) but to produce an aggregate which itself is continuous. The conception of the continuum as an aggregate of existing *points* (members), which is at the bottom of nineteenth century analysis and of Cantor's set theory, is replaced by an aggreg-

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[1]) For the following cf., in particular, Brouwer 18-19, *13, 25-27, 30, *24, *25, 50; also Heyting 30a and 34, p. 19. Brouwer's term is *Wahlfolge*. — Some points which may appear obscure in this explanation will be elucidated below when Brouwer's notion of *set* is explained (pp. 250 ff).

[2]) Brouwer's German term is *Gesetz*, whereby an effective process or algorithm (see Brouwer 23 and 27) is intended.

[3]) Cf. *Morris 1, *Dehn 1.

ate of *parts* which are partially overlapping and each of which is continuous itself. Instead of the membership relation the part-whole relation becomes fundamental (cf. Chapter II, p. 22).

We conclude with explaining Brouwer's notion of set (or spread) [1]) which in its original formulations [2]) is all but incomprehensible. Following Brouwer rather closely, we may say that a set arises in the following way (cf. Kleene, *l.c.*). Let an object be chosen arbitrarily from a given effectively denumerable class C_0 of objects, then another, etc., in the sense of free-choice sequences; C_0 may be conceived as the class of all positive integers. After the n^{th} choice of an object from C_0, one of the following three cases applies, depending on the set-rule regarding the first n choices. Either a certain object from another effectively denumerable class C_1 (possibly also the class of positive integers) is attached to the chosen object of C_0, and this may happen *without* terminating the process which is afterwards continued by another choice (first case), or *with* terminating the process (second case). The third case applies when the process of the first n choices is *checked*, meaning that no object of C_1 is attached, the result of the previous attachments is discarded, and no further choice admitted. The set, then, is the rule which after each choice from C_0 determines which of the three cases applies, and in the first two cases, what object of C_1 is attached. The "set-elements" are the finite or infinite sequences of objects from C_1 which arise from the free-choice sequences as far as they are not checked.

However, set-elements which arise from infinite free-choice sequences are not considered to be given in their entirety but only by stages consisting of the objects chosen in the first k steps, k meaning any integer. The process may be continued either through new choices made freely or under restriction by some rule [3]).

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[1]) In most of his papers Brouwer used the term *Menge* (set) or *Mengenspezies* (see § 6), which he later (see, e.g., 54a) replaced by *spread* (*spreiding* in Dutch) so that the idea of a "totality of members" should be avoided.

[2]) Especially in Brouwer 18-19 and 25-27. Among papers which try to elucidate the definition and to ponder its consequences, Menger 28 and 30, Beth 47 and 56 (Chapter V), and Kleene 52 (partly originating from 1941) are most important. Cf. Kleene 45 and D. Nelson 47 (see above p. 231) where a connection is established between intuitionistic logic and arithmetic and the theory of recursive functions. Cf. also *Euwe 1; here Brouwer's definition is used to form the set of all games of chess.

[3]) In particular, a set is called *finite* if the unchecked free-choice sequences

The essential contents of this definition may be stated more technically as follows. If (a_k) is an arbitrary sequence of positive integers, e.g. $a_k = k$, or generally $a_k = f(k)$, a set is a rule by which an object $\varphi_n = \phi_n\,(a_1, a_2, \ldots, a_n)$ corresponds to the initial segments of the free-choice sequence (a_k). For each n there are the three possibilities called above "cases". In particular, the φ_n can also be assumed to be positive integers.

Now Beth, and in a more explicit way Kleene, suggest that this definition should be made precise by conceiving the rule as a pair of recursive functions, by the first of which one of the values 0, 1, 2 corresponds to each initial $(a_1, a_2, \ldots, a_n)$, according to the three cases mentioned, while the second is defined in the first two cases only, and then coincides with ϕ_n.

This means the first step in *reconstructing neo-intuitionistic set theory* (see the following section) *in the light of the theory of recursive functions*. The next step is the introduction of functions of set-elements, namely functions $\zeta(\alpha)$ recursive with respect to functions α which define the free-choice sequences $\alpha(a_k)$. Besides the technical difficulties involved, there is the difficulty of a reinterpretation of the logical notions, in particular of the existential quantificator. In fact, an important result of Brouwer's set theory [1]) would fail if the range of the function-variables were restricted to recursive functions only — which confirms the remark of p. 204 that the admission of arbitrary free-choice sequences constitutes a decisive difference between neo-intuitionists and the other intuitionistic trends whose "continuum" is denumerable in an absolute sense.

Of these "semi-intuitionistic" continua, the continuum introduced by Weyl in 1918 (and later abandoned by him in favor of Brouwer's continuum) has been examined most thoroughly, notably by Grzegorczyk [2]). The latter's formalization of Weyl's informal restriction to certain "e(lementary) d(efinable)" analytical methods starts with the arithmetic of integers; the class of e.d. relations shall be closed with regard to the logical operations of the propo-

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contain only numbers $\leq$ some fixed bound B; in other words, if the third case applies whenever a number $> B$ is chosen. (Preferable is Brouwer's later term "bounded spread"; in fact, the above condition does not imply that there are only finitely many elements.)

[1]) Viz. the statement that if to each element ξ of a "finite set" a positive integer ζ_ξ corresponds, there is an integer m such that ζ_ξ is already determined by the first m of the choices generating ξ. See Brouwer 27, Theorem 2.

[2]) Weyl 18 and 21, Grzegorczyk 54.

sitional calculus and to universal quantification over an integral-variable. The notion of an e.d. function is defined by means of the minimum (least integer) of an e.d. number-theoretical relation, provided a minimum exists. An e.d. functional is analogically defined over a finite number of functions and such that it assumes integral values.

By means of these notions, e.d. analogues of the classical concepts of real number and of sequences and sets of real numbers are defined. It is confirmed that the "semi-continuum" (field) of e.d. real numbers is denumerable, and an essential part of the e.d. analysis of continuous functions is developed. Differentiation and integration on families of continuous functions prove e.d., and an e.d. continuous function defined on an interval whose ends are e.d., assumes its maximum at an e.d. value.

Thus far we have examined the connection between Brouwer's definition of set and certain fundamental concepts of the foundations of mathematics in general, notably recursive functions. But no less important is the "practical" question, *what category of sets in classical mathematics is represented by Brouwer's sets.* Menger 28 shows that there is a close relation [1]) between the sets in Brouwer's sense and the abstract analytic sets [2]); however, the *tertium non datur* is required to show this relation. To clarify this connection we start with the concept of *ramified set* [3]).

Let D be a class of objects in which an operation is defined such that, to any (finite or denumerable) sequence of objects from D, an object corresponds which need not belong to D. If $d_1, d_2, \ldots, d_k, \ldots$ are objects of D given in this succession, we denote the result of the operation by $\psi(d_1, d_2, \ldots, d_k, \ldots)$.

Furthermore, given any *finite* sequence of positive integers $(n_1, n_2, \ldots, n_k) = N$, let to N correspond an object of D which shall be denoted by $d_{n_1 n_2 \ldots n_k}$.

For a given infinite sequence of integers $\nu = (n_1, n_2, \ldots, n_k, \ldots)$ we contemplate the sequence $(d_{n_1}, d_{n_1 n_2}, \ldots, d_{n_1 n_2 \ldots n_k}, \ldots)$ and denote $\psi(d_{n_1}, d_{n_1 n_2}, \ldots, d_{n_1 n_2 \ldots n_k}, \ldots)$ by d_ν. Each of the objects $d_{n_1}, d_{n_1 n_2}, \ldots, d_{n_1 n_2 \ldots n_k}$ is called a *constituent* of d_ν. Then the set of all objects d_ν, when ν runs over all infinite sequences of positive integers, is called a *ramified set*; more precisely, the ramified set obtained from the class of objects $\{d_{n_1 n_2 \ldots n_k}\}$ by the operation ψ, where $k, n_1, n_2, \ldots$ are any positive integers. $\{d_{n_1 n_2 \ldots n_k}\}$ shall be called the class *producing* the ramified set.

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[1]) Heyting criticizes Menger for not appropriately considering free-choice sequences.

[2]) The concept of analytic set was first introduced by M. Souslin in 1917; therefore some authors, for instance Hausdorff, use the term 'Souslin's sets'. The first expositions of the theory of analytic point-sets and also of abstract analytic sets are found in Hausdorff 27, §§ 19 and 32, and in Lusin 27 (cf. *5).

[3]) Menger 28, p. 213. (This concept is not connected with the *ensembles ramifiés* in the sense of *Kurepa **2**.)

If the objects of D are *sets* and ψ is the operation of intersection $\cap$, the ramified set is the set of all intersections d_ν, ν denoting all infinite sequences of positive integers. The union of all sets d_ν, i.e. the sum-set of the ramified set, is called the *analytic set* produced by the class of sets $\{d_{n_1 n_2 \ldots n_k}\}$. (The same analytic set may be produced by different classes of sets.)

One easily sees that there is a close connection between this concept and Brouwer's concept of set. In particular, the third case in Brouwer's definition (p. 250), meaning that a finite sequence of choices is checked, can be transferred to the language of ramified sets by introducing a "null-object" O of D such that if $d_{n_1 n_2 \ldots n_k} = O$, a sequence of integers beginning with $n_1, n_2, \ldots, n_k$ does not contribute at all to the ramified set. (Whether such an O exists depends on the nature of D.) Then, by essentially using the principle of the excluded middle, Menger shows that *the sets in the sense of Brouwer's definition coincide with the ramified sets having the property that every object $\neq O$ of the producing class is a constituent of at least one d_ν*. Thus, by means of the *tertium non datur*, every ramified set proves to be a set in Brouwer's sense.

The same connection applies between ramified point-sets and point-sets in Brouwer's sense.

The main importance of these results lies in the fact that analytic sets have been studied extensively and are, incidentally, considered by many (especially French) mathematicians, who are remote from neo-intuitionism, to be the only "definable" sets. On the other hand, the results have not much significance from the neo-intuitionistic point of view because they depend on the principle of the excluded middle.

§ 6. Mathematics as Trimmed according to the Intuitionistic Attitude [1])

After the preceding sections it is clear that the historical field of mathematics has to undergo serious restrictions in order to conform to the principles imposed by intuitionists. In this final section we shall give a survey of the effect produced by these restrictions and of what can be saved; mostly we shall refer for details (in a positive or negative direction) to intuitionistic literature, elaborating the arguments in

[1]) Cf. the note at the beginning of § 5, p. 244.

a few characteristic cases only. In general we limit ourselves to neo-intuitionistic attitudes, while restrictions eliminating negation (pp. 241 ff) shall be disregarded altogether.

Arithmetic and algebra in the strict sense are less affected than other branches. In those parts of algebra where only discrete aggregates appear such that for any two members it can be decided whether they are equal or not, one has just to keep to constructive methods which bring on a result by a finite number of steps, as already Kronecker did in his treatment of certain arithmetical problems, rejecting the "classical" (simpler) methods. The finite extensions of the field of rationals (or of a field with p^n members) belong to this part of algebra. But far-reaching modifications prove necessary wherever the distinction between $\neq 0$ and $\# 0$ (see below) is unavoidable; for instance in the field of real numbers. Hence the very definition of a field becomes complicated, for not to every member which is $\neq 0$ need an inverse exist. In the field of all real numbers, for instance, a^{-1} only exists if $a \# 0$, i.e. if a rational number can be named which lies between 0 and a [1]).

In general a *distance relation* $\#$ is defined with the following properties: 1) $a = b$ and $a \# b$ are incompatible, 2) the absurdity of $a \# b$ implies $a = b$, 3) if $a \# b$ we have, for every c, $a \# c$ or $b \# c$ (or both). If in 3) we replace c by a we obtain that the distance relation is symmetric. Furthermore, from $a = b$ and $b \# c$ follows $a \# c$, and the absurdity of the absurdity of $a = b$ is equipollent to the truth of $a = b$. This is essentially the consequence of 2); for, denoting the absurdity of $a = b$ by $a \neq b$, we obtain that $a \# b$ implies $a \neq b$, hence the absurdity of $a \neq b$ implies the absurdity of $a \# b$, i.e. the truth of $a = b$. Thus the absurdity of $a \neq b$ proves equipollent to $a = b$, while in general (without a distance relation at our disposal) the three cases $a = b$, $a \neq b$, absurdity of $a \neq b$ have to be distinguished (see above p. 230).

While the theory of *groups* with a distance relation is not greatly different from the ordinary theory, the properties of *rings* and *fields* become rather involved. In a ring without divisors of zero or in a field one cannot, from $ab = 0$ and $a \neq 0$, conclude $b = 0$; if this conclusion is generally valid the ring is called 'regular', and if it moreover contains for every $a \# 0$ an inverse a^{-1}, a 'field'. Accor-

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[1]) See to this, and to the following, Heyting 34, pp. 20 ff, and 41, pp. 4 ff.

ding to this definition, the rationals as well as the real or the complex numbers constitute fields.

Also for polynomials a corresponding distance relation can be defined.

The theory of *linear equations* [1]) may be developed in two essentially different ways, without determinants or with using them. The distance relation involves the need of additional differentiations regarding linear dependence; by means of determinants certain criteria of dependence can be released from their transfinite-existential character.

For the theory of polynomials a relation $\not\equiv$ of distance-congruence is introduced, defined by '$f \not\equiv g$ (mod. h) if, for every polynomial k, $f - g \# kh$' [2]). Regarding the decomposition of polynomials, only *negative* results can be reached, because the theorem of the (unique) decomposition of a polynomial in a given field becomes invalid; one can only prove that a polynomial cannot be decomposed in two essentially different ways into irreducible factors.

These results of Heyting, referring to general fields, may be conceived as generalizations of results of Kronecker [3]) obtained sixty years before, results with regard to a constructive decomposition of polynomials in the field of rational numbers.

Of course, those parts of algebra (infinite extensions of fields and rings) which rest upon the axiom of choice or equipollent principles become intuitionistically meaningless [4]).

In *analysis* the difficulties are much more fundamental, and the restrictions involved prove rather catastrophic. With the introduction of irrational numbers or free-choice sequences defining real numbers, the comparison of numbers according to magnitude becomes impossible in general [5]); in other words, given real numbers a and b, the trichotomy $a \gtrless b$ ceases being valid. A characteristic example

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[1]) See Heyting 28 and 41, § 2. The former paper considers also non-commutative fields.

[2]) The relation '#' between polynomials in x expresses that the coefficients of at least one equal power of x are distant.

[3]) Kronecker 1882 and 1883. Cf. van der Waerden 30, Vandiver 34-35 and 36, and the generalization in Fröhlich-Shepherdson 56.

[4]) Nevertheless, in a modified sense many fundamental results of algebra may be saved by "semi-intuitionistic" methods, as Lorenzen 56 has shown. (This important lecture, which shows in a new light the use of the axiom of choice for transfinite induction and the Skolem paradox, appeared after Chapter II had been sent to the printer.)

[5]) See also Brouwer 50, in addition to his earlier papers.

is the following. It is unknown (see p. 218) whether in the expansion of π into a decimal a sequence of (at least) seven 7's will occur; if so, let h denote the place after the decimal point (i.e. the index of the respective digit in $\pi = 3.a_1a_2 \ldots a_h \ldots$) where for the first time such a sequence starts. Now we define a decimal ρ, which begins with $0.777\ldots$, as follows: if there exists h as defined, replace the digit 7 at the h^{th} place after the point in ρ by 6 if h is odd, and by 8 if h is even; save for this, the digits of ρ are 7, hence *all* are 7 "if no h exists". The real number ρ is effectively defined since, simultaneously with the expansion of π, the successive digits of ρ can be calculated. But without the *tertium non datur* it cannot be asserted that one of the cases $\rho = \frac{7}{9}$, $\rho < \frac{7}{9}$, $\rho > \frac{7}{9}$ holds true, or even that either $\rho = \frac{7}{9}$ or $\rho \# \frac{7}{9}$. This does not contradict the fact that the rational numbers are comparable; for ρ cannot, according to our present knowledge, be considered rational — though after a possible decision regarding the integer h (non-existent, odd, even) ρ proves to be rational in each case.

Recently Brouwer [1]) even constructed a real number which is $\neq 0$ (in the sense of absurdity of equality) while it cannot be proven to be either positive or negative. In a critical analysis of Brouwer's paper, van Dantzig improved the logical frame, confirmed the result, and also made Brouwer's recent publications more comprehensible to logicians by explaining the essence of its philosophical ingredients and by analyzing certain peculiarities of Brouwer's terminology.

The incomparability of real numbers fatally affects many classical proofs in analysis. While in a few cases one has succeeded in replacing certain intuitionistically void proofs by constructive ones, for the majority this has not been achieved nor is there a prospect of achieving it; hence the respective theorems have to be abandoned.

An elementary example may illustrate the situation. For the so-called *fundamental theorem of algebra* (see p. 66), which contains a statement that is properly analytic and not only algebraic, dozens of proofs have been given since the end of the 18^{th} century. Many of the proofs use various theorems from the theory of (real or complex) functions, sometimes also topological facts, while others (as the second proof of Gauss, simplified by Gordan) are altogether arithmetical but

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[1]) Brouwer 48; cf. 49 and van Dantzig 49.

for the use of the elementary analytic theorem stating that every algebraic equation with real coefficients and an *odd* degree has a real root. Proofs of the first kind had already been rejected before the turn of the 19th century by intuitionists such as Kronecker and Mertens.

Yet the "arithmetical" proofs, too, contain ingredients which are not acceptable to intuitionists; in fact, in 1924 new proofs of the fundamental theorem have been given independently by scholars of three different countries [1]) who maintained that no sound proof had been given before. An essential shortcoming of most "arithmetical" proofs can be described as follows. In order to get rid of multiple roots of the given equation one has to ascertain whether its discriminant equals 0. For instance, the equation $x^3 - \frac{4}{27}x + 2\rho^3 = 0$ has the discriminant $D = -108((\frac{7}{9})^6 - \rho^6)$ where ρ is the real number defined above. In this case D is intuitionistically neither equal 0 nor distant from 0. Generally speaking, the impossibility of basing the proof on the alternative that the discriminant is either 0 or a positive or a negative number, makes the proof illusory. Therefore a proof is required (and can be given) which enables us to calculate the roots of the equation numerically with any desired precision, starting with an approximation of the coefficients by rational numbers; for the discriminant of an algebraic equation with *rational* coefficients is also intuitionistically either equal to, or distant from, 0.

In the foundations of analysis, the theorem of Bolzano-Weierstrass [2]), the convergence of a bounded monotone sequence of real numbers, the existence of a zero for a continuous real function $f(x)$ with $f(a) < 0$ and $f(b) > 0$ [3]), the theory of Dedekind cuts (T, p. 215), the existence of the least upper bound for a bounded set (species) of real numbers and of a maximum for a continuous real function in a closed interval are among the victims of the neo-intuitionistic attitude; these theorems prove either false or meaningless. In the case of Bolzano-Weierstrass, for instance, the obstacle is the alter-

[1]) Brouwer-de Loor 24, *Skolem 5, Weyl 24. In *de Loor 1 an analysis of earlier proofs from the intuitionistic point of view is given. Cf. *van der Corput 1, Rosenbloom 45, Rice 54. Fundamentally this argumentation, too, goes back to Kronecker (1882; cf. Fine 14, Vandiver 34–35 [*Annals of Math.*]).

[2]) For the use of the *tertium non datur* in this theorem, cf. Billing 49 and Rice 54.

[3]) The assumption about f may be conceived in the stronger sense of distance, i.e. that a rational number between 0 and $f(a)$ or $f(b)$ can be named.

native between mutually exclusive cases which uses the *tertium non datur*. It is also used for the comparison of two Dedekind cuts $\kappa = (K_1|K_2)$ and $\lambda = (L_1|L_2)$ in the form: either all rationals of L_1 belong to K_1, or else there exists in L_1 a rational which is not contained in K_1.

Regarding convergence and the theory of infinite series, Brouwer claimed [1]) that the notion of convergence ought to be split up into different notions, and Belinfante in a series of ingenious papers [2]) showed that Brouwer's distinction between "positive" and "negative" convergence leads to two quite different theories of which only the first is similar to the classical theory. Absolute convergence and summability are also dealt with by Belinfante.

The classical foundations of *calculus*, all the more the modern theory of *real functions* including the Lebesgue integral, clearly become meaningless in this light [3]). Only such theorems on uniformly continuous functions as can be checked by calculating are retained. On the other hand, Brouwer showed that *any* real function which is defined in a closed interval is there uniformly continuous — owing to his concept of function [4]). The proof is far from being trivial. To make things plausible, let $f(x)$ have a jump at $x = a$; then in its neighborhood f cannot be generally defined, for the value of f "should become calculable with increasing precision when the argument is specified with sufficient precision, which is not fulfilled in the neighborhood of a" [5]).

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[1]) Brouwer 25.

[2]) Belinfante 29, ***2**, ***4**, ***6**, ***7**. See also Dijkman ***1** and 52 and van Rootselaar 52. In particular, Dijkman modifies the concept of "negative convergence" so as to preserve the classical theorems about the convergence of the sum and the product of convergent series.

[3]) For the so-called intersection theorem of Cantor for compact sets and a related theorem of Peano's on differential equations, see van Dantzig 42. For Brouwer's concept of measure and integral (and also derivative), see Brouwer 23; cf. Heyting 51 (and 56), Brouwer 54, and van Rootselaar 54 (particularly for the theory of derivation).

[4]) Brouwer 23 and 27; for intuitionistic analysis cf. also ***13**, 25, 25-27. — Among theorems which are problematic to intuitionists because of their existential character, the Riesz-Fischer theorem should be pointed out; in fact the theorem proven in Heyting 51 differs from the classical form of the theorem.

[5]) Cf. what is said below (p. 260) about splitting up a continuum.

In the theory of *complex analytic functions* the situation is easier; it looks as though large parts of this theory, in the sense of Weierstrass and possibly also of Cauchy, might be developed along intuitionistically legitimate lines. This is not astonishing since the continuum of this theory can be ruled by denumerable sets of points. The steps taken in this direction by Belinfante [1]) have, however, so far hardly been continued by others. Generally, in arithmetic and still more in analysis even those results which remain meaningful and true in the eyes of intuitionists mostly require new proofs which are very complicated; one of the reasons is the *impossibility of using indirect proofs* (see above p. 215) [2]). Accordingly, for pedagogical reasons even intuitionists would be compelled first to teach the "illusive" classical proofs before criticizing and replacing them with their own demonstrations.

There exists no autonomous *geometry* for neo-intuitionism, no more than for Weyl's semi-intuitionism — except for the very concept of continuum which is saved by the free-choice sequences. To be sure, one may start with certain axiom systems and derive theorems from the axioms [3]). But this frame remains empty until a model is given by means of analytic geometry, which reduces geometry to analysis. In a somewhat paradoxical way the crisis of the *tertium non datur* influences geometry; Desargues's theorem, for instance, after having been proven on the one hand for triangles situated in the same plane and on the other for such whose planes are distant, should not be considered to be proven altogether because "in general" a decision between both cases cannot be reached [4]).

[1]) Belinfante 31 proves some of the fundamental theorems, including the integration of the logarithmic derivative (which yields an intuitionistic proof of the fundamental theorem of algebra) and Picard's theorems. Cf. Goodstein 51 (Chapter VI) where, besides functions of a complex variable, also measure theory and constructive topics of topology are treated from a viewpoint not much different from Brouwer's. (Belinfante died during the German occupation.)

[2]) Cf. the historical exposition Goodstein 48.

[3]) As done for projective geometry by *Heyting 1 and 3 (cf. 13), and by Dequoy 49, 50, 55 even in negationless intuitionism; yet Dequoy did hardly accomplish her program along the lines of Griss and rather remained within Heyting's logic without using negation. — Hilbert spaces are investigated in Heyting 53.

[4]) *Heyting 3, p. 511.

In *topology*, especially in combinatorial topology and Euclidean spaces, no decisive difficulties seem to arise except for the foundations, namely for the concept of space and the determination of those spaces which are intuitionistically meaningful. After more special results of Brouwer [1]), Freudenthal thoroughly examined the intuitionistic space concept and proved the fundamental theorems required for intuitionistic topology [2]). In particular — contrary to previous conjectures of the Dutch school — he succeeded in developing topology independently of metric notions.

The present section and most of §§ 2–5 refer to *neo*-intuitionism. The various trends of semi-intuitionism (p. 247) also involve serious modifications and restrictions of classical mathematics, but in general do not go as far as the Dutch school since they do not reject the *tertium non datur* in principle. We content ourselves with giving some references to literature [3]).

The most revolutionary changes, naturally, are found in the *theory of sets* (abstract sets and sets of points). Brouwer's concept of set and of free-choice sequences were explained in the preceding section. The linear continuum, then, is a set (spread), and by suitably restricting the freedom of choice one may define certain sets of points included in the continuum. On the other hand, the continuum cannot be split up into continuous parts; for instance, the union of the open intervals (0, 1) and (1, 2), including the point 1, does not constitute a continuum since the points whose order relations to 1 are unknown are not contained in the union [4]).

Two set-elements (p. 250) are *equal* if, for every n, the n^{th} choice assigns the same object, and the property of being equal to an element

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[1]) Brouwer 26 (intuitionistic formulation of Brouwer's classical researches on dimension theory; see *T*, p. 140), in a particular point anticipated in 18-19$_{\text{II}}$. In Brouwer 25a Jordan's curve theorem is proven intuitionistically.

[2]) Freudenthal 36.

[3]) The most important material is found in Weyl 18, 21, 24; *Skolem **4** and **5**; Lusin 27; Heyting 34 (Chapter I, §§ 2–4) and ***12**. Cf. *J. König **5** and *Menger **1** (especially p. 115). The treatment of the second number-class in the light of the Paris school, as given in *Lusin **4**, is remarkable also from the classical viewpoint. In Lorenzen 56 new light is thrown on the possibility of developing analysis with semi-intuitionistic methods. Cf. also p. 205.

[4]) Cf. Shiraishi 54 where the relation between intuitionistic indistinguishability and Zeno's argument (*T*, p. 11) is elaborated.

of the set is called a set-species [1]). (Brouwer calls set-elements and set-species "mathematical entities".) Now a hierarchy of species (cf. the theory of types, Chapter III) is constructed. A species of the first order corresponds to, or is, a property which only a mathematical entity can possess; for instance, a property of integers. A species of the second order, then, is a property of entities and of first-order species, etc. Thus 'species' means approximately what is meant by 'set' in the constructive stage of classical set theory; Brouwer avoids the term 'set' to exclude the idea of comprehension in the sense of the axiom of subsets (p. 38) or of (*B*) on p. 112. While in Brouwer's definition of set the origin of the members of the set is included, the members of a species are defined independently (previously) and on account of this definition it is established whether they belong to the species. Impredicative definitions are automatically excluded, yet only "intuition" informs us of what a mathematical property is. 'Subspecies' is understood as by Definition I in Chapter II, § 2. In contrast with what was said in the preceding paragraph about splitting-up the continuum, for a species S it may occur that S' and S'' are disjoint subspecies of S such that their union equals S; Brouwer then calls S 'separated' into S' and S'', and S' or S'' a 'separable subspecies' of S [2]). This is the case, for instance, if S contains all integers > 1, S' all primes, S'' all composite numbers.

The theories of cardinals, ordered sets, and ordinals (well-ordered sets), as derived from the definitions of set and species, are not only limited to an infinitesimal part of the classical theories but are also much more complicated within the restricted extent. We have to distinguish between 'equivalence' and 'cardinal equality' (*Gleichmächtigkeit*) of two species S and T. The latter relation means that a one-to-one mapping can be established between S and T, while S and T are called equivalent if there is a mapping of S onto a subspecies

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[1]) To this notion, see Menger 28, p. 221, and van Dantzig 47, p. 927, footnote 17. Brouwer (18-19, p. 4) conceives a species as a "property which mathematical entities can be assumed to possess". Cf. the somewhat related distinction made in Moch 51 between *ensemble* and *espèce*. The latter is introduced without a type restriction, as the extension of a predicate.

[2]) This notion yields a distinction regarding infinite species. While a species A is called 'infinite' when it has a subspecies cardinally equal (see above) to A, it is in particular called 'reducibly infinite' when the subspecies is separable. (This notion of 'infinite' was called 'reflexive' on p. 62.)

of T and a mapping of T onto a subspecies of S. Since the equivalence theorem (T, pp. 98 f) is not available, equivalent species need not have equal cardinals. Brouwer adds further relations between species, which classically are equipollent to equivalence; the ensuing complication may be measured by the fact that the classical concept of denumerability is split up into eight different properties which are largely independent [1]).

Brouwer assigns to every species which cardinally equals the set (species) of positive integers, a "cardinal" $\boldsymbol{a}$, and to every species cardinally equal to the continuum (i.e., to the set of unrestricted sequences of positive integers) a "cardinal" $\boldsymbol{c}$. A relation $<$ is defined, and the very concept of free-choice sequence yields $\boldsymbol{a} < \boldsymbol{c}$. On the other hand, nothing seems to be known about higher cardinals [2]); at any rate, neither the diagonal procedure nor the notion of power-set (Cantor's theorem) is available. A simple instance shows why the restrictions introduced prevent us from attaining higher cardinals in the usual way. The "insertion-species" of the linear continuum C into itself, which classically has the cardinal $\aleph^{\aleph} > \aleph$ (T, p. 159), or in Brouwers notation $\boldsymbol{c}^{\boldsymbol{c}}$, contains just the uniformly continuous functions with values in C (see p. 258), which prevents the application of the diagonal procedure.

Regarding the theory of order [3]), Brouwer calls a species P *virtually ordered* if for the members of a certain subspecies of the species of the pairs of members of P there exists a relation $<$ with the following "properties of order".

1) $a = b$, $a < b$, $b < a$ exclude each other.
2) $a < b$, $a = a'$, $b = b'$ imply $a' < b'$.
3) The absurdity of both $a < b$ and $a = b$ implies $b < a$.
4) The absurdity of both $a < b$ and $b < a$ implies $a = b$.
5) $a < b$ and $b < c$ imply $a < c$.

This does not correspond to a relation based on the notion of distance; in fact, if $a \ll b$ for real numbers a, b means that a rational number between a and b can be named, property 3) is not satisfied by $\ll$.

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[1]) See *Heyting 4.

[2]) See, in particular, *Brouwer 3 (1909). Cf., also for the preceding paragraphs, Brouwer 25-27, 30, ***24**, ***25** where additional particulars are given; Wilder 52, pp. 235–242, closely follows Brouwer's exposition.

[3]) Brouwer $25\text{-}27_{II}$; cf. Heyting 34, pp. 26 f. In Brouwer 50 also partial order is introduced.

A virtual order proves not-extensible; that is to say, to the existing statements of the form $a < b$ no other can be added. Cantor's determination of the order-type η (T, pp. 221 ff) can be retained, and for the order-type of the linear continuum (T, p. 228) a determination is given which replaces the method of Dedekind cuts.

For *well-ordered* species one cannot develop a general theory on account of a definition (T, pp. 242-246) but has to content oneself with constructing well-ordered species by means of two "principles of construction", namely a) joining two well-ordered species or a finite number of such, b) joining a sequence of well-ordered species. Hereby certain early approaches of Cantor, starting with the concept of derivative (T, p. 232), are revived. Beginning with finite species and the ordered species of positive integers, one only reaches denumerably infinite species [1]). While a well-ordered species, just as a classical well-ordered set, includes no subspecies having the type $*\omega$, the property that every non-vanishing subspecies has a first element is not retained, neither the comparability of two well-ordered species. On the other hand, the well-ordered species play an original and intuitive rôle in intuitionistic mathematics in so far as each mathematical construction consists of a well-ordered species of single steps.

A survey of Brouwer's set theory shows that many essential points still remain obscure. If this theory shall be maintained a fundamental recast seems indispensable. On the other hand, the Dutch school regards Cantor's theory of sets as "an interesting pathological case in the history of mathematics" likely to give the creeps to coming generations.

In view of the mutilated shape which mathematics assumes according to intuitionistic principles it is not astonishing that a small minority only of mathematicians has been ready to accept the intuitionistic attitude; this situation will probably not change very much when the adherents of the Dutch or the Paris school will succeed in formulating their principles and inferences in a less dogmatic and more comprehensible and consistent form than done up to now (even considering the progress due to Heyting, Freudenthal, and van Dantzig). True, the school of Hilbert and some other trends show, in spite of themselves, full understanding for the basic attitude

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[1]) Brouwer 25-27$_{III}$ and 54a. Brouwer extends the principles by admitting "nullelements".

stressed by intuitionists, and one should emphasize their significance in this respect more than usually done. However, even after the partial failure of the Hilbert school regarding consistency proofs one must not forget that the very existence of mathematics and the wide range of its applications during many centuries seem to show that modern analysis is not just nonsensical or meaningless. A comparison with speculative metaphysics is instructive: there dubious and mutually contrasting premisses and methods yield contradicting results, while in analysis and set theory the variance of principles has led to *more, or less, far-reaching* results but never to such as *contradict* each other. Hence we may trust that, sooner or later, methods will be found by which those methodical doubts which have not been answered so far can be dissolved [1]). In short, the prevailing conviction is that one should not cut off the leg to heal the toe.

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[1]) Kreisel 51 is characteristic in this respect. He declares the intuitionistic attitude straight off to be wrong since it involves the rejection of large parts of mathematics. While his hopes regarding the success of Hilbert's school may be too optimistic, he uses the instance of the theorem that a monotonically increasing bounded sequence of rationals converges, to make it plausible that this and similar more general theorems can be defended against intuitionistic objections.

CHAPTER V

METAMATHEMATICAL AND SEMANTICAL APPROACHES

§ 1. The Hilbert Program

So far, we have discussed three main approaches to the problem of rebuilding the foundations of set theory, the "naive" Cantorian conception of which was so badly shaken by the antinomies. The *Brouwerians* believed that this conception was wholly wrong from the beginning. They accused it of misunderstanding the nature of mathematics and of unjustifiedly transferring to the realm of infinity methods of reasoning that are valid only in the realm of the finite. By regaining the right perspective, mathematics could be constructed on a basis whose intuitive soundness could not be doubted. The antinomies were only the symptoms of a disease by which mathematics was infected. Once this disease was cured, one need worry no longer about the symptoms. All *Russellians* thought that our naiveness consisted in taking for granted that every grammatically correct indicative sentence expresses something which either is or is not the case, and some — among them Russell himself — believed, in addition, that through some carelessness a certain type of viciously circular concept formation had been allowed to enter logico-mathematical thinking. By refining the rules of formation — and proscribing the dangerous types of concept formation — the known antinomies could be made to disappear. Their faith in the consistency of the resulting, somewhat mutilated, systems was less strong than that of the Brouwerians, since certain intuitively not too well founded devices had to be used in order to restore at least part of the lost strength and maneuverability. *Zermelians*, finally, thought that our blunder consisted in naively assuming that to every condition there must correspond a certain entity, namely the set of all those objects that satisfy this condition. By suitable restriction of the axiom of comprehension, in which this assumption is formulated, they tried to construct systems that were free of the known antinomies yet strong enough to allow for the reconstruction of a sufficient part of classical mathematics. Their

faith in the consistency of the resulting systems was based on nothing more — but also on nothing less — than the fact that the usual ways of deriving the known antinomies could not be reproduced.

The adherents of the axiomatic approach to the foundations of set theory, and to a somewhat lesser degree the adherents of the type-theoretical approach, were badly in need of a *proof* of the consistency of their systems. The classical method of providing such a proof, viz. the exhibition of a *model* taken from a theory whose consistency was not in doubt — in the tradition of Beltrami who in 1868 proved that certain non-Euclidean geometries were consistent (relative to Euclidean geometry, as we would say now) by constructing a model for them within Euclidean geometry, or of Hilbert who in 1899 constructed a model for Euclidean geometry within real number theory, thereby proving its consistency (relative to real number theory) — could not be applied: finite models were obviously not suitable, and no conceptual framework within which an infinite model could be constructed might be regarded as safe, in view of the antinomies. A different method was needed. It was Hilbert who supplied it, dimly in 1904, but with increasing precision and pregnancy from 1917 [1]: it must be shown that the standard mathematical proof procedures are strong enough to derive *all* of classical mathematics — including *all* of Cantor's set theory — from suitable axioms but not so strong as to derive a contradiction. That classical mathematics was basically sound was an act of faith with Hilbert which was never shaken by the antinomies.

Hilbert intended to carry his program through in two steps: first, all of mathematics — as a matter of fact, he was thinking mainly of arithmetic, analysis and set theory, — had to be *formalized* [2], i.e. a *formal system*, or *formalism*, had to be constructed from whose axioms should be derived at least the beginnings of mathematics to a degree corresponding to that achieved, say, in ***PM***, with the help of a certain definite set of rules of inference. The system was to be formal in the sense that only the kind and order of symbols, upon whose sequences the rules of inference were to operate, was to be taken into account but not, for instance, their "meaning". Such a system could be sufficiently mastered by a minimum of intuition, the so-called "global intuition"

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[1] Cf., e.g., Hilbert 22, p. 160.

[2] *Ibid.*, p. 174.

required to decide whether two symbol tokens are the same or not, more precisely, whether they do or do not belong to the same symbol type, a kind of intuition that requires no intellectual powers at all and can be built into suitably constructed machines. Whether a series of symbol sequences is or is not a proof of its last sequence can, in such a system, be mechanically checked by using essentially nothing more than a purely mechanical operation of matching. This complete formalization goes far beyond the kind of formalization provided in *formal axiomatics*. There one relies on an "understood" logic and is satisfied with a complete listing of all specific primitive terms and an enumeration of all assumptions formulated in these terms necessary for the derivation of a certain body of theorems and takes pain that the intended meaning of the specific, extra-logical terms of the theory under treatment should never enter into the derivations unless explicitly expressed by the axioms. (Incidentally, it was Hilbert himself who brought the *formal axiomatic method* to its perfection.) Here, the intended meaning of *all* terms is to be disregarded, including that of the logical terms. If one wants to infer the fact that ϕ is true from the fact that ϕ and ψ are true, then this can no longer be done by implicit, or even explicit, reliance on the meaning of 'and' but this inference has to be made on the basis of suitable axioms and rules exclusively. Hilbert's kind of formalization is based on what is sometimes called the *logistic method*. Hilbert was, of course, in a position to rely on existing axiomatizations of certain parts of mathematics, such as Peano's axiomatization of number theory and Zermelo's axiomatization of set theory, as well as on existing formalizations of logic, provided by Frege and Russell-Whitehead, though especially the system of ***PM*** was not quite up to the standards necessary for his purposes. The various systems of formalized arithmetic, developed by Hilbert and his school [1]) over many years of hard work, started indeed from an almost mechanical superposition of Peano's axiom system on the ***PM*** first-order functional calculus, though later developments went far beyond this stage and exhibited, especially in the introduction of Hilbert's ε-operator [2]), great originality.

In the second step, Hilbert planned to show that the application of the rules of inference to the axioms could never lead to contradiction,

[1]) For a careful and very detailed description of these systems, see Hilbert-Bernays 34–39.

[2]) See Chapter III, pp. 183 ff.

or rather to *formal inconsistency* in one of the senses of this term (with which we shall deal later, in § 4, at length), e.g. that there could exist no valid formal proof the endformula of which would be '1 = 2'. The argumentation by which this impossibility metatheorem was to be established had to be of such an elementary character that its soundness could not possibly be doubted. All those kinds of argumentation which the intuitionists found objectionable *within* mathematics, such as the use of the *tertium non datur* for infinite sets or the inference from the falseness of an universal statement to the truth of a certain existential statement, not to mention impredicative concept formations and the use of the axiom of choice, were to be banned from that metatheory in which the proof procedures of mathematics were to be investigated and which Hilbert called therefore *metamathematics* or *proof theory*. As a matter of fact, he even went beyond the intuitionistic strictures when he insisted that only *finitary* arguments were to be allowed in proof theory. But just as the intuitionists never committed themselves to a complete specification of what proof procedures were admissible — such a commitment would have militated against their basic stand — so Hilbert never produced a univocal statement as to what procedures were regarded by him as finitary. The nearest to such a specification can perhaps be found in the following quotation from Herbrand, that highly gifted French Hilbertian whose early death put an end to great hopes: [1])

> We understand by an intuitionistic [i.e. finitary] argument an argument that fulfils the following conditions: one always deals with a finite and determined number of objects and functions only; these are well defined, their definition allowing the univocal calculation of their values; one never affirms the existence of an object without indicating how to construct it; one never deals with the set of all the objects x of an infinite totality; and when one says that an argument (or a theorem) holds for all these x, this means that for every particular x it is possible to repeat the general argument in question which should then be treated as only a prototype of these particular arguments.

It should be stressed that the task of metamathematics was not only to show the consistency of mathematics proper by safeguarding it

[1]) Herbrand 32, p. 3, footnote 3.

against the antinomies; among other things, it was meant to protect mathematics from the restrictions that other schools, especially those of intuitionistic provenience, were trying to impose upon it [1]).

Had Hilbert succeeded in carrying out his original program, this might have been the end of foundational research for most mathematicians as it would have been for Hilbert: this kind of research for him was a not too pleasant duty that he felt obliged to perform but which distracted him from other more attractive occupations. True, a minority would still have claimed then, as they do now, that consistency of a formalism as such is by no means a sufficient condition for the material truth of any of its interpretations. Even if a formal system in which, say, an axiom of choice is contained, is demonstrably consistent, *selection functions* — so this minority would argue — just *don't exist.* To which Hilbert — in this respect being, strangely enough, in the good company of Poincaré — would have replied that mathematical existence is nothing but the consistency of the system [2]). But we shall leave the discussion of mathematical existence to the last section.

Fortunately — if we may be allowed to be for a moment somewhat light-hearted on such a serious matter — neither Hilbert nor any of his brilliant followers and associates did succeed in accomplishing this program, not because of any lack in ingenuity but — just because it could not be done. However, during the pursuit of this, as we now know by hindsight, Utopian aim — as it so often happened in the history of mathematics — an enormous wealth of new theories, concepts, and techniques was developed which have already proved to be extremely interesting and fruitful and promise to become even more so in the future. The Gödel theorems of 1931, from which the inaccomplishability of the original Hilbert program can be deduced (see § 6), did shatter certain illusions, to be sure, but they have also been hailed — rightly, we believe — as belonging to the greatest achievements of abstract human thinking in recent times. Hilbert was wrong when he attempted to belittle the crisis into which mathematics had be thrown by the antinomies, and his belief in the essential decidability of all mathematical problems turned out to be unjustified, at least *if* decidability is understood as decidability in one specific

[1]) See Hilbert 22, p. 174.

[2]) For some recent discussions of the relationship between consistency and mathematical existence, see Bernays 50 and Beth 56.

formal system. No unique and universally accepted way of reconstructing mathematics exists or is in view, and in this sense the foundational crisis is still in force. But many a scientist wishes that his field were in as "critical" a state as mathematics, and few are the mathematicians who are really depressed by the existing uncertainties in the foundations. *Dealing with these foundations has, surprisingly enough, turned out to be not only a job that had to be undertaken for reasons of intellectual sincerity or philosophical meticulousness but something that was infinitely rewarding, exciting, and fruitful.*

We shall not try to present the rise and decline of the Hilbert program in all its historical details, interesting as such a presentation would doubtlessly be [1]). We shall rather introduce first the concepts in which its present status can be understood and then describe in outline the main results of the theories that were developed during its pursuit. At the end of the chapter, and thereby of the book, we shall deal with some of the philosophical problems connected with the foundations of set theory.

§ 2. Formal Systems, Logistic Systems and Formalized Theories

Let us start with a general description of the structure of a certain important class of formal systems and then illustrate it by a partial description of a certain number-theoretic formalism as well as by a complete description of a set-theoretical formalism such as might arise from a complete formalization of the set theory **Z** presented in the first sections of Chapter II.

We have no intention of describing here the structure of all possible formal systems. This is not even quite done by Carnap who dedicated more than 120 pages of his masterwork, *The Logical Syntax of Language*, to General Syntax, i.e. to the theory of formal systems in general [2]).

The following remark as to terminology is appropriate: for any of the technical terms to be used in this and the following sections, there exist many, occasionally very many, synonyms and near-synonyms. On the other hand, many of these terms are homographs and carry different meanings, sometimes with the same author in different or even in the same publications. The reader would only have become bewildered had we tried to list all the synonyms in every case. Therefore, we shall use in general only one or two terms for each concept. When comparing our statements with those found in the literature, great care

[1]) For one such presentation, see Heyting 55, pp. 36 ff; for another, older one, see Bernays 35a.

[2]) Carnap 37, pp. 153–275; see especially p. 167.

will therefore have to be taken in determining what corresponds to what. As an illustration only let us notice that 'formal system' has the following synonyms: *formalism, calculus, formal calculus, uninterpreted calculus, abstract calculus, syntactical system, formal language, formal logic, codificate,* and many others.

A formal system [1]) is determined by the following five sets:

(1) A set of *primitive symbols*, the (*primitive*) *vocabulary*, divided into various kinds, such as *variables, constants, and auxiliary symbols* [2]). If the vocabulary is finite, it can be given simply in form of one or more lists. If it is infinite, and in practice also if it is very large, its membership is established in the metalanguage by inductive means, using syntactical variables. In any case, being a primitive symbol has to be an effective notion, enabling us to determine in a finite number of steps whether or not a given symbol is primitive. The primitive symbols are to be regarded as indivisible whatever their external look and whatever the way of their specification.

In a certain number-theoretical formalism, the number of variables belonging to the primitive vocabulary is infinite. The following inductive specification would still make the notion of being a variable an effective one:

(a) 'x' is a variable.

(b) If ξ is a variable, then ξ_1 is a variable.

(c) The only variables are those provided by (a) and (b).

Any finite string of symbols is called an *expression* [3]). The number of expressions is therefore at least denumerably infinite even if the vocabulary is only finite.

(2) A set of *terms* as a subset of the set of expressions, determined by effective rules.

Continuing our illustration, we might have the following rules:

(d) Each variable is a term.

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[1]) Curry's views on the nature of formal systems, expressed in Curry 50, 51, 54 (and other publications), though idiosyncratic and occasionally somewhat obscure, should be consulted by any serious student of this problem.

[2]) Or technical symbols; these form only a subset of what Church 56, pp. 32 ff, calls 'improper symbols'. Church, for good — though not generally accepted — reasons, includes among the improper symbols the so-called logical constants such as the sentential connectives, the quantifiers and other operators.

[3]) In spite of our general decision not to mention divergent terminologies, a comparison with the terminology used in Church 56 is often indicated, in view of the extraordinary meticulousness with which this terminology has been prepared and explained. Church uses 'formula' instead of our 'expression' and 'well-formed formula' instead of both our 'term' and 'formula'.

(e) If α [1]) is a term, then $S(\alpha)$ [2]) is a term.
(f) If α and β are terms, then $(\alpha + \beta)$ and $(\alpha \cdot \beta)$ are terms.
(g) The only terms are those provided by (d), (e), and (f).

It can again easily be checked that being a term is indeed an effective notion in this formalism.

(3) A set of *formulae* as a subset of the set of expressions, determined by effective rules with the help of the notion of term, whenever this notion has been determined (which is not always the case).

We might have, for instance, the following rules:
(h) If α and β are terms, $\alpha = \beta$ is a formula.
(i) If ϕ is a formula, then $\sim(\phi)$ is a formula.
(j) If ϕ and ψ are formulae, then $(\phi) \supset (\psi)$ is a formula.
(k) The only formulae are those provided by (h), (i), and (j).

Once more the effectiveness of the notion of formulahood is evident.

One might occasionally want to introduce terms and formulae by a simultaneous induction. We could have, e.g., such a rule as
(ff) If ϕ is a formula and ξ is a variable, then $(\lambda\xi)\phi$ is a term (where 'λ' is a certain primitive constant).

(4) A set of *axioms* as a subset of the set of formulae. If this set is finite, the axioms can, at least in principle, be listed. If not, they may be given through *axiom-schemata* formulated in the meta-language with the help of syntactical variables. The rules have to be such as to guarantee the effectiveness of the notion of axiomhood.

Still continuing our illustration, we might have, among others, the following axiom-schema:
(l) All formulae of the form $(\alpha = \beta) \supset ((\phi) \supset (\psi))$ are axioms, if α and β are terms, ϕ and ψ are formulae, and ψ differs from ϕ only in containing β at a place where ϕ contains α.

It is again obvious that checking whether a formula is or is not an axiom in accordance with (m) is an effective procedure.

(5) A finite set of *rules of inference* according to which a formula is *immediately derivable* as *conclusion* from an appropriate finite set of formulae as *premisses*.

Most formalisms contain a rule of inference either identical with or equivalent to the *rule of detachment* (or *modus (ponendo) ponens*):

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[1]) Where α is a *syntactical* variable ranging over the expressions of the formalism.

[2]) Where 'S' is a constant of the formalism; 'S' is used autonymously — see Chapter II, p. 26 — in '$S(\alpha)$'. Had there been some point in avoiding such a usage, we could have said instead: "... then 'S' $\frown \alpha$ (the *concatenation* of 'S' and α, i.e. the sequence consisting of 'S' followed by α) is a term".

(n) From a class of formulae consisting of two formulae of the form ϕ and $(\phi) \supset (\psi)$, ψ is immediately derivable.

A finite sequence of one or more formulae is called a *derivation from the class Γ of premisses* if each formula in the sequence is either an axiom or a member of Γ or immediately derivable from a class of formulae preceding it in the sequence. The last formula of the sequence is called *derivable from Γ*. A derivation from the empty class of premisses is called a *proof* of its last formula, hence of each of its formulae. A formula is called *provable* or a *formal theorem* if there exists a proof of which it is the last formula. (The relativization of these definitions to a given formal system is always tacitly to be understood.)

It follows that the notion of being a proof is effective. But it does not follow that the notion of theoremhood is effective. There need not exist, from all we know so far, a general method that would, for a given formula, either tell us how to construct its proof or else show that no proof is possible. This does not exclude, of course, that theoremhood might still be a demonstrably effective notion for certain formal systems. Once upon a time, some mathematicians had hoped that all of mathematics could be formalized in a formal system of this kind.

The rules determining the membership of the first three sets are called *rules of formation*, those determining the membership of the last two sets *rules of transformation*.

A formal system may be schematically regarded as an ordered quintuple of sets fulfilling certain requirements [1]).

Terms and formulae will be called *open* [2]) if they contain at least one

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[1]) For an especially thorough, and somewhat divergent, treatment of the question of what constitutes a calculus, see Schröter 41.

[2]) Church 56 has 'propositional form' instead of our 'open formula'. The term 'propositional function' should better be avoided in this connection; 'function' should be reserved, in concordance with best mathematical standards, to denote certain types of relations and not be used to denote certain notations — 'propositional function' should therefore be reserved for a function whose range of values are truth-values; cf. Church 56, p. 28.

An interesting light is thrown on the intricacies of the terminological situation by the fact that Carnap, who until 1950 was employing the term 'open sentence' for our 'open formula', started using the term '(sentential) matrix' in a book — Carnap 50 — published that year, thereby adopting a term advocated by

free occurrence of a variable, *closed* if they contain no free variables. Instead of 'closed formula' we shall usually say 'sentence'.

We are now ready to present one of the many formal systems into which the set theory **Z** can be formalized.

A. *Rules of formation*

1. α is a *primitive symbol* (of Z) if and only if it is one of the following symbols:

a. *Individual variables:* 'x', 'x_1', 'x_{11}', ... (ad infinitum). For the 10 first variables following 'x' we shall use as abbreviations the following symbols, in this order: 'y', 'z', 'w', 'u', 'v', 'r', 's', 't', 'p', 'q'.

b. *Primitive predicates:* the binary predicate 'ϵ'.

c. *Logical constants*

α. *Connectives:* '$\sim$', '$\vee$', '$\&$', '$\supset$', '$\equiv$'.

β. *Quantifiers:* the *universal quantifier* (A), the *existential quantifier* (E).

d. *Auxiliary symbols:* '(', ')'.

3. ϕ is a *formula* if and only if it has one of the following forms:

a. $\xi \epsilon \eta$, where ξ and η are variables; a formula of this form is called *atomic*.

b. $\sim(\psi)$, $(\psi) \vee (\chi)$, $(\psi) \,\&\, (\chi)$, $(\psi) \supset (\chi)$, $(\psi) \equiv (\chi)$, where ψ and χ are formulae.

c. $(A\xi)(\psi)$, $(E\xi)(\psi)$, where ξ is a variable and ψ is a formula.

B. *Rules of transformation*

4. ϕ is an *axiom* if and only if it has one of the following forms:

a. *Axiom-schemata of the propositional calculus*

$(\psi) \supset ((\chi) \supset (\psi))$, $(\psi \supset (\psi \supset \chi)) \supset (\psi \supset \chi)$ [1], $(\psi \supset \chi) \supset ((\chi \supset \varrho) \supset (\psi \supset \varrho))$,
$(\psi \supset \sim \chi) \supset (\chi \supset \sim\psi)$, $\sim\psi \supset (\psi \supset \chi)$, $((\psi \supset \sim\psi) \supset \chi) \supset ((\psi \supset \chi) \supset \chi)$,
$(\psi \,\&\, \chi) \supset \psi$, $(\psi \,\&\, \chi) \supset \chi$, $(\psi \supset \chi) \supset ((\chi \supset \varrho) \supset (\psi \supset (\chi \,\&\, \varrho)))$,
$\psi \supset (\psi \vee \chi)$, $\chi \supset (\psi \vee \chi)$, $(\psi \supset \varrho) \supset ((\chi \supset \varrho) \supset ((\psi \vee \chi) \supset \varrho))$,
$(\psi \equiv \chi) \supset (\psi \supset \chi)$, $(\psi \equiv \chi) \supset (\chi \supset \psi)$, $(\psi \supset \chi) \supset ((\chi \supset \psi) \supset (\psi \equiv \chi))$,

where ψ, χ, and ϱ are formulae [2].

b. *Specific axioms* (and *axiom schemata*)

I (*Axiom of Extensionality*)

$$(Ax)\,(Ay)\,[(Az)\,(z \epsilon x \equiv z \epsilon y) \supset (Aw)\,(x \epsilon w \equiv y \epsilon w)].$$

Quine in publications prior to 1950. Quine himself, however, in a book published in that very year — Quine 50 — decided to use 'open sentence', admittedly following Carnap's usage; see Quine 50, p. 90n.

We ourselves shall continue to use the term 'statement' instead of 'sentence' as an informal term applying to ordinary language and unformalized theories.

[1] In this and the following formulae, some parentheses are omitted, under certain obvious tacit conventions.

[2] This is the axiom system adopted by Hermes-Scholz 52 and is a slight variation of the well-known system of Hilbert-Bernays 34–39$_1$. About its main advantages see Hermes-Scholz 52, pp. 35 ff.

II (*Axiom of Pairing*)

$$(Ax)\ (Ay)\ ((Ar)\ (x \in r \equiv y \in r) \vee (Ez)\ (Aw)\ \{w \in z \equiv [(Au)\ (w \in u \equiv x \in u) \vee \vee (Av)\ (w \in v \equiv y \in v)]\})\ ^{1}).$$

III (*Axiom of Sum-Set*)

$$(Ax)\ \{(Ey)y \in x \supset (Ez)\ (Aw)\ [w \in z \equiv (Eu)\ (w \in u \ \&\ u \in x)]\}.$$

IV (*Axiom of Power-Set*)

$$(Ax)\ (Ey)\ (Az)\ [z \in y \equiv (Aw)\ (w \in z \supset w \in x)].$$

V (*Axiom-Schema of Subsets*)

$$(\ )\qquad (Ax)\ (Ey)\ (Az)\ [z \in y \equiv (z \in x \ \&\ \psi)],$$

where 'y' does not occur free in the formula ψ and where '( )' stands for a string of universal quantifiers binding all the free variables of ψ (except 'z').

VI (*Axiom of Choice*)

$$(Ax)\ [((Ay)\ (Az)\ \{[y \in x \ \&\ z \in x \ \&\ \sim(Aw)\ (y \in w \equiv z \in w)] \supset [(Eu)\ (u \in y)\ \& \ \& \sim(Ev)\ (v \in y \ \&\ v \in z)]) \supset ((Er)\ (As)\ \{s \in x \supset (Et)\ (Aq)\ [(Ap)\ (q \bullet p \equiv t \in p) \equiv \equiv (q \in r \ \&\ q \in s)]\})].$$

VII (*Axiom of Infinity*)

$$(Ex)\ [(Ay)\ \{[\sim(Ez)z \in y] \supset y \in x\}\ \&\ (Aw)\ (w \in x \supset (Au)\ \{(Av)\ [v \in u \equiv \equiv (Ar)\ (v \in r \equiv w \in r)] \supset u \in x\})].$$

VIII (*Axiom-Schema of Substitution*)

$$(Ax)\ ((Ay)\ (Az)\ (Aw)\ \{[y \in x\ \&\ \psi(y, z)\ \&\ \psi(y, w)] \supset (Av)\ (v \in z \equiv v \in w)\}) \supset \supset (Er)\ (As)\ \{s \in r \equiv (Et)\ [t \in x\ \&\ \psi(t, s)],$$

where ψ is a formula containing two free variables none of which is 'x' or 'r'.

IX (*Axiom of Foundation*)

$$(Ax)\ \{(Ey)y \in x \supset (Ez)\ [z \in x\ \&\ \sim(Ew)\ (w \in x\ \&\ w \in z)]\}.$$

The reader will have noticed that the symbolic versions of the specific axioms are in general more cumbersome than those given in Chapter II. This is due to the fact that in the formulation presented here we have completely avoided any use of defined symbols such as '0' or '{ }' [2]. This was done first in order to show that it is possible to

[1]) All parentheses, brackets, and braces are to be regarded as variants of the same two auxiliary symbols.

[2]) Where the braces are this time used not as a variant of parentheses but as a single defined symbol denoting the-set-containing-...-as-its-only-member(s).

comply with the stipulation of some axiomaticians that all axioms should always be formulated in primitive notation, and secondly because we wished to dodge the troublesome problem of the status of definitions in formal systems. Let us only remark that this problem is graver than is generally assumed. If definitions are to be allowed as part of a formalism (and not as metatheoretical devices), certain rules of definition have to be laid down with which the definitions will have to comply.

5. ϕ is *immediately derivable* from (the class of) the formulae ψ and χ if

a. (*Rule of detachment, modus ponens*) χ has the form $\psi \supset \phi$

or else if ψ has the form $\varrho \supset \sigma$ and ϕ has the form

b. (*Rule of antecedent universalization*) $(A\xi)\varrho \supset \sigma$

c. (*Rule of consequent universalization*) $\varrho \supset (A\xi)\sigma$, where ξ is not free in ϱ

d. (*Rule of antecedent existentialization*) $(E\xi)\varrho \supset \sigma$, where ξ is not free in σ

e. (*Rule of consequent existentialization*) $\varrho \supset (E\xi)\sigma$.

f. (*Rule of relettering*) [We shall not formulate this rule here. Its effect is to allow for replacing, under certain conditions, a variable by some other variable. Its exact statement would be too tiresome for our purposes[1]).]

Rules 4a and 5a form a complete set of transformation rules for the propositional calculus. Together with the remaining rules of inference, they form a complete set of transformation rules for the functional calculus of first order without equality. We can now see more exactly in what sense system **Z** has this calculus as its basic logical discipline. Containing one primitive predicate, the binary predicate 'ϵ', but no propositional or predicate variables whatsoever, it is a *simple applied functional calculus of first order*.

The notion of formal system, as developed so far, is a rather narrow one, with strong requirements of effectiveness, and is probably very close to, if not identical with, the one Hilbert had originally in mind. For certain purposes, however, it is useful, and for other purposes even necessary, to investigate systems with weaker requirements of effectiveness. Whether we call such systems 'formal systems', too, perhaps with some qualifying phrase, or artificially confine one or more of the synonyms mentioned above (p. 271) for this wider conception, or coin some new terms, is of course a purely terminological matter. We shall from now on use *'logistic system'* for 'formal system with strong requirements of effectiveness' and employ qualifying phrases, where necessary, to distinguish between the different shades of formal systems.

Until recently, only formal systems with an at most denumerably

[1]) For a very exact and detailed treatment of the rules of inference for functional calculi, see Schröter 55, especially pp. 65 ff.

infinite vocabulary were investigated. Lately, however, the revival of the algebraic approach to logic has also caused an increase of interest in systems with a non-denumerable vocabulary [1]).

Expressions are usually regarded as *finite linear* symbol strings, but occasionally expressions of infinite length are taken into consideration [2]) — something that is especially easy when syntax is arithme-

[1]) This approach dates back to Boole but was later thrown into temporary oblivion by Frege, Peano and Russell. Its recent revival, though of no great direct impact on the foundations of set theory proper — so far, at least —, is such a decisive factor in the evaluation of the present situation in the foundations of mathematics in general that it must be mentioned, even if only in the form of a footnote.

Originating, with a topological flavor, in such works as Tarski 35 and Stone 37, and with a combinatorial flavor, in Post 43, this approach flourishes now in such centers as Berkeley (Tarski, Henkin, Myhill, Vaught and a large number of pupils of Tarski) and Warsaw (Mostowski, Łoś, Sikorski, Rasiowa and a similarly large number of other pupils of Mostowski) and is practiced by many logicians and mathematicians all over the world. Tarski, Henkin, A. Robinson and others also used this approach in reverse, i.e. applied metamathematical methods to algebraic problems. The straight algebraic and topological results achieved were impressive to a degree sufficient to raise the interest of many mathematicians who were formerly left cold by mathematical logic. Together with recursive function theory — see below, pp. 300 ff — modern algebra of logic (and logic of algebra) are the two great methodological achievements of the second quarter of our century in the field of foundations of mathematics, and they are both only in the beginning of their career. It needs no great foresight or special powers of prophecy to predict that the coming years will bring these two methods to fruition, with results which in their specificity can hardly be foretold.

Out of the vast literature, we should like to mention only the following works that have appeared during the last ten years (in alphabetical order):

Beth 51a, 55b, 56a, Birkhoff 48, Fraïssé 55, Fröhlich-Shepherdson 56, Halmos 56, 56a, 56b, 56c, 56–57, Hasenjaeger 52, Henkin 49, 51a, 53b, 55, 56, 56a, Hintikka 54, Kleene-Post 54, Kreisel 56/57, Łoś 55, Łoś-Mostowski-Rasiowa 56, Łoś-Suszko 55, Lorenzen 51b, McKinsey-Tarski 48, Markwald 56, Mostowski 49, 51, 53, 55, 56a, Mycielski 57, Rasiowa 52, 53, 54, 55, Rasiowa-Sikorski 51, 52, 53, 54, Rieger 49, 51, A. Robinson 51, 51a, 52, 54, 55, 55a, 56, 56a, 56b, 57, Rosenbloom 50, Scholz 51, Suszko 51, Tarski 49, 52, 54–55, Tarski-Vaught 57, Vaught 54.

For the specific issue of the employment of systems with a super-denumerable vocabulary, see Henkin 49, A. Robinson 51, 56, Tarski 52, 54–55, Vaught 54, Schröter 55 (especially p. 42), Tarski-Vaught 57.

[2]) See Carnap 37, p. 169. During the Summer Institute of Symbolic Logic at Cornell University in 1957, two talks were given on calculi with infinitely long expressions, one by Tarski, the other by Tarski and Dana Scott.

tized (see below, § 5) — and often a second dimension is used in our practical notational systems. No theoretical problem is however raised thereby as it is always possible to treat, through some artificial device, a two-dimensional array of symbols as linear [1]).

Then there are formal systems in which the notion of formulahood is not effective. To mention just one example, Hilbert and Bernays introduce the iota-operator of definite description in such a way that an expression of the form $(\imath\xi)\phi$, where ξ is a variable occurring free in ϕ, is regarded as a term only when the "uniqueness condition" $(\mathrm{E}\eta)\ (\mathrm{A}\xi)\ [\phi \equiv (\xi = \eta)]$ is provable [2]). Since provability is, in general, not an effective notion, as we saw above, there exists no generally effective method of determining whether $(\imath\xi)\phi$ is a term, hence no effective method of determining whether certain expressions containing $(\imath\xi)\phi$ as a part are formulae [3]).

With regard to axioms, we had already allowed their number to be infinite, even for a logistic system, so long as their specification within the metalanguage is such as to provide for an effective procedure of testing whether a given formula is or is not an axiom. Nevertheless, since systems with a finite number of axioms have definite theoretical advantages, the problem of the *finitizability* of a given infinite axiom system, i.e. the establishing of a finite axiom system whose set of theorems coincides with the set of theorems of the former system, is an important one that has often been discussed with respect to the various set theories (cf. below, p. 320).

Sometimes, however, one has occasion to deal with formal systems whose notion of axiomhood is only *semi-effective* in the sense that though there exists a mechanical procedure which, for any given

[1]) Cf. Carnap 42, p. 5.

[2]) Hilbert-Bernays $34\text{–}39_1$, pp. 383 ff. Other authors, following Frege and Russell, take care that iota-expressions should always be terms; cf. Carnap 47, pp. 33 ff and Schröter 56.

[3]) The arguments brought forward in Church 56, pp. 52–53, to the effect that systems, whose rules of formation and transformation are non-effective, are not suitable for purposes of communication do not sound too convincing. Communication may be impaired by this non-effectiveness but is not destroyed. Understanding a language is not an all-or-none affair. Our quite efficient use of ordinary language shows that a sufficient degree of understanding can be obtained in spite of the fact that "meaningfulness", relative to ordinary language, is certainly not effective.

For a discussion of calculi for which some or all of the notions of formula, axiom, and rule of inference are non-effective, see especially Carnap 37, § 45.

formula, would determine after a finite number of steps that it is an axiom if it is one, there exists no mechanical procedure that would determine, after any number of steps, that a formula is not an axiom if it is not one [1]).

With regard to the rules of inference, whereas for logistic systems the number of premisses in any immediate derivation is by definition always finite, there are formal systems that contain rules of inference in which a conclusion is immediately derivable from a denumerably infinite number of premisses. The best-known of such rules is the so-called *rule of infinite induction* which states, for the specific case in which the system in question contains symbols for all natural numbers, that the formula $(A\xi)\phi$, where ξ is the only free variable of ϕ, is immediately derivable from the infinitely many formulae resulting from replacing each free occurrence of ξ in ϕ by the symbols for the natural numbers, in order. The extreme intuitive plausibility of such a rule does not alter the fact that it transcends the framework allowed in a logistic system. We shall come back to this rule later.

Finally, occasions arise to investigate systems whose set of sentences, meant to be the counterparts of the true sentences of the intuitive theories which these systems are designed to systematize, can no longer be assumed altogether, or at least *ab initio*, to be identical with the set of sentences formally derivable from some set of axioms. Let us not stretch the term 'formal system' any further to make it cover even such systems but let us rather use the term '*formalized theory*' for that purpose (each formal system being a formalized theory but not vice versa, just as we decided before to use the term 'logistic system' such that each logistic system is a formal system but not vice versa). Whereas a formal system is determined by its set of *provable* sentences, a formalized theory is determined by its set of what we, following Tarski [2]), shall call *valid* sentences. The exact extension of this term

[1]) It has, however, been proved that for any formal systems with a semi-effective set of axioms there exists an equivalent formal system with an effective set of axioms. This is the content of *Craig's theorem:* see Kleene 52b and Craig 53. The theorem as such is formulated not in the intuitive terms 'effective' and 'semi-effective' but rather in their strict counterparts '(general) recursive' and 'recursively enumerable' to be discussed below, pp. 301 f.

[2]) Tarski-Mostowski-Robinson 53, p. 10. Having decided to use the term 'valid' in this sense, we shall have to use different terms for the other, more frequently used senses of this word. The reader will be well advised to keep this in mind when consulting the literature.

has to be defined from case to case, the only general condition which such a definition will have to fulfil being that the set of valid sentences should be *closed with respect to derivability*, i.e. that every sentence derivable from valid sentences by the rules of inference should be valid itself; this again entails that all *logical axioms* as well as the sentences derivable from them, i.e. all *logically valid* sentences, will be among the valid sentences. The schematical representation of a formalized theory will then consist of an ordered sixtuple of sets: a set of symbols, a set of terms, a set of formulae, a set of logical axioms, a set of rules of inference, and a set of valid sentences, with various relations obtaining between, and various conditions fulfilled by, these sets.

Whether a given theory, presented initially as a formalized theory, can also be equivalently represented as a formal system, perhaps even as a logistic system, in other words, whether the set of valid formulae of this theory can be exhaustively described as the set of sentences derivable from some initial set of axioms by some rules of inference, these axioms and rules fulfilling more or less rigid requirements of effectivity, is a problem, often a very difficult one, always a decisive one. In still other words, the problem is whether a given formalized theory is *axiomatizable*, i.e. equivalent to an *axiomatically built* formal system. As a matter of fact, Hilbert and many other mathematicians and logicians had assumed that all of mathematics is axiomatizable. Anticipating results to be mentioned later, let us state, however, that many mathematical theories, including such "simple" ones as the arithmetic of natural numbers, have turned out to be nonaxiomatizable, to the amazement of them all.

§ 3. Interpretations and Models

A formalized theory is usually set up in order to formalize some intuitively given theory. Whether, and to what degree, this aim is achieved can only be determined after the formalized theory is provided with an *interpretation*, with the help of suitable *rules of interpretation*, turning thereby into an *interpreted calculus*. These rules can take many forms, but their common function is to provide each sentence of the formalized theory with a meaning such that it turns into something that is either true or false, i.e. into a statement, though no effective method need of course be provided for deciding whether it is the one or the other.

We shall sketch here one way of providing an interpretation for

theories with standard formalization [1]) (which coincide, in the case of formal systems, with what we called above *simple applied first-order functional calculi with equality*), both because they are the most important and best studied sorts of formalized theories and for the sake of simplicity.

A full-fledged theory with standard formalization T contains some standard set of logical constants (including a symbol of equality) and auxiliary symbols, a denumerable set of variables ranging over the same set of entities, and an at most denumerable set of extra-logical constants comprising individual constants, singulary, binary, ..., n-ary predicates and operation symbols; this last set is assumed to be ordered in some sequence without repetitions. (The auxiliary symbols and the operation symbols are theoretically superfluous but are in practice often very convenient.) Such a theory is interpreted by providing the logical symbols with their usual signification, by fixing the *universe (of discourse)* U over which the variables range, and by assigning, through *rules of designation*, to each individual constant some member of U, to each singulary predicate a certain subset of U and in general to each n-ary predicate a certain n-ary relation whose field is a subset of U, finally to each n-ary operation symbol an n-ary function from ordered n-tuples of U to members of U. Calling the sequence consisting of U and these individuals, sets, relations, and functions ordered in a way similar to that of the constants to which they are assigned, a *structure, semi-model,* or *(possible) realization* [2]) of T, *rules of truth* finally determine under what conditions a formula of T is *true in a given structure,* relative to a given *value-assignment* to its free variables if it contains such. (These rules determine, of course, also truth conditions for all the *sentences.*)

An interpretation of a given theory T is called *sound* if under it all

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[1]) Following most of the time Tarski's exposition in Tarski-Mostowski-Robinson 53.

[2]) These three terms form only a sample out of the many terms in use. 'Structure' is used, e.g., in A. Robinson 56 and Beth 57 (in French), 'semi-model' in Kemeny 56 (p. 5) and Beth 56a (in English), '(possible) realization' in Tarski-Mostowski-Robinson 53. Kleene 52 (p. 24) has 'system of objects', Tarski 54–55 has 'relational systems', Church 56 uses 'interpretation'. Many authors have 'model', e.g. Hasenjaeger 55 and Henkin 57; these authors have then to use something like 'admissible model' or 'satisfying model' for our 'model'. Still others are using 'model' ambiguously with occasionally rather confusing results.

the valid sentences of T become true in the structure determined by the interpretation; this structure itself — and often also the assigning function — is then called a *model* of the theory.

In the special case, when the given formalized theory is a formal system, the axioms of this system become true under any sound interpretation, and its rules of inference must be *truth-preserving*, i.e. leading always from true premisses to true conclusions, in order that a sound interpretation be possible, in other words, in order that there exist a model of the system.

Let us illustrate by giving a complete set of rules of interpretation for $\bar{\boldsymbol{Z}}$, i.e. $\boldsymbol{Z}$ minus the axiom VIII of substitution, hence of a set theory more similar to the original theory of Zermelo. ($\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ are, strictly speaking, not theories with standard formalization, as the symbol of equality is not treated in them as a logical constant, but this does not matter for our purposes here.)

a. If ϕ and ψ are sentences, then $\sim(\phi)$ is true if and only if ϕ is not true, $(\phi) \vee (\psi)$ is true if and only if either ϕ or ψ (or both) are true, etc.

b. The universe U contains the null-set O, its power-set $\boldsymbol{C}O$, the power-set of $\boldsymbol{C}O$, i.e. $\boldsymbol{C}\boldsymbol{C}O$ or $\boldsymbol{C}^2O$, ..., $\boldsymbol{C}^nO$ (for all finite n), the union of all these sets, say E, and $\boldsymbol{C}^nE$ (for all finite n).

c. 'ϵ' designates the relation being-a-member-of (with its field restricted to U), i.e. the set of all the ordered couples whose first element is a member of the second one, where both belong to U.

d. An atomic formula of the form $\xi \epsilon \eta$ (where ξ and η are variables) is true, relative to a given value-assignment to ξ and η, if and only if the set assigned as value to ξ is a member of the set assigned as value to η.

e. A formula of the form $(A\xi)\phi$ (where ϕ is a formula and ξ a variable) is true, relative to a given value-assignment to all the free variables of ϕ, if and only if ϕ is true, relative to any value-assignment that differs from the given one at most by the value assigned to ξ.

f. A formula of the form $(E\xi)\phi$ (where ϕ is a formula and ξ a variable) is true, relative to a given value-assignment to all the free variables of ϕ, if and only if ϕ is true, relative to at least one value-assignment that differs from the given one at most by the value assigned to ξ [1]).

Let us now introduce a few semantical terms which we shall need in the sequel. An open formula will be called *universal* in a given structure [2]) if it is true in this structure relative to every value-assignment to its free variables, *satisfiable* if true relative to at least one such value-

[1]) In order to see that the interpretation provided by these rules is sound, we have to verify that all axioms of $\boldsymbol{Z}$ are true in the structure $\langle U$, is-a-member-of$\rangle$. This is easy, in general. For the verification of the axiom of choice, however, we assume a corresponding principle in the metatheory.

[2]) This requires, of course, that it should have been previously established what are the structures corresponding to the given formalized theory.

assignment, *empty* if true relative to no such value-assignment [1]). A sentence true in every structure is *logically true* [2]), and an open formula universal in every structure — *logically universal.* A sentence true in every structure in which all the sentences of a given set of sentences are true is called a *logical consequence* [3]) of this set.

When a sentence ϕ is derivable from a set of sentences Γ, it is also a logical consequence of Γ, but the converse does not generally hold. It does hold, however, with regard to theories with standard formalization, according to results obtained by Gödel in 1930 [4]). For such theories we have then also that a formula is logically true if and only if it is logically provable, in other words, that the proof procedures of the underlying logic are complete [5]).

Whereas *sound interpretability* (*having a model*) is an absolute semantic property of a formalized theory T, *interpretability in* T' (*having a model in* T') is a relational syntactic property of T, to the definition of which we turn now [6]). Let then T and T' be two theories with standard formalization and with disjoint extra-logical vocabularies. We say that T *is interpretable* (*has a model*) *in* T' if there exists a theory T'' which is a common extension of T and T' (i.e. whose set of valid sentences comprises those of T and T') and a set Γ (with effectively decidable membership) of sentences valid in T'' containing, for each extra-logical constant of T, just one possible definition of this constant in T', such that each valid sentence of T'' is derivable in T'' from the union of the valid sentences of T' and Γ. (To illustrate the notion of possible definition: If ϕ is a formula of T' free in just ξ and η and ρ is a new binary predicate, the sentence

$$(\mathrm{A}\xi)\ (\mathrm{A}\eta)\ [\rho(\xi, \eta) \equiv \phi]$$

is a *possible definition* of ρ. ρ does not belong to T', of course, but rather

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[1]) These terms are Carnap's who has, however, 'non-empty' instead of 'satisfiable'; see Carnap 42, p. 48. It is mandatory to relieve the pressure from the overburdened term 'valid'. 'Universal' is a fitting term for one of its senses, 'logically true' and 'logically universal' for two more.

[2]) It is to be deplored that Carnap's notational method of prefixes — e.g. 'L-true' for 'logically true' — has not found favor in the eyes of most mathematicians and logicians. One should perhaps try again.

[3]) See Tarski 36 (56, XVI).

[4]) Mentioned in Chapter II, p. 106; see also below, p. 288.

[5]) Cf. the remarks on p. 172 in regard to elementary (or standard) theories.

[6]) We follow again Tarski-Mostowski-Robinson 53, pp. 20 ff.

to any extension of T' that contains the predicate ρ.) One says that T is *weakly interpretable* in T' if T is interpretable in some consistent extension of T' with the same vocabulary as T'. The proof that T has a model in T', if not given in some intuitive theory (such as recursive arithmetic — cf. below, p. 303), has to be carried out in some formalized theory T^* which will in general be different from either. The value of a relative consistency proof for T carried out through showing in T^* that T is interpretable in T' is therefore not only dependent on the relative safeness of T' but also on that of T^*.

Interpretability in T' was defined for T just in case both theories are with standard formalization. A related notion, however, is applicable even in case the two theories do not contain the same logical apparatus, if only they contain notations for negation. This notion was called *translatability* by Wang [1]). Its definition, as modified by Kreisel [2]), is essentially this: T is said to be T^*-*translatable* into T' if there exists in T^* a function that effectively maps the formulae of T into those of T' such that the image of the negation of a sentence of T is the negation of the image of this sentence in T' and such that the image of any valid sentence of T is a valid sentence of T'. If T is T^*-translatable into T', then it can be proven in T^* that T is consistent if T' is. The method of proving relative consistency by translatability is clearly more powerful than that by interpretability. It is only through translatability that the consistency of the functional calculus relative to the propositional calculus — hence the absolute consistency of the functional calculus — can be proved; and likewise for the consistency of classical number theory relative to the intuitionistic one [3]).

A still different relationship between two theories has recently been investigated by Kreisel [4]). He reserves for it the term 'interpretability', using the 'model'-terminology exclusively for the first relationship discussed above (whereas Tarski uses exclusively the 'interpretability'-terminology for it). Again he gets the result that from the existence of

[1]) Wang 51, pp. 284–285.

[2]) Kreisel 55, pp. 30–31. We are using, however, Tarski's terminology. Kreisel's own use of 'axioms' (instead of 'valid sentences') has the somewhat disconcerting consequence that not all systems of axioms are axiomatizable.

[3]) Of these two results, the first appears in every textbook on symbolic logic; see, e.g., Church 56, pp. 180 ff. For the second, see Chapter IV, p. 233.

[4]) Kreisel 55, pp. 33 ff.

a T^*-interpretation (in his sense) of T in T' the consistency of T relative to T' can be proved in T^*. This third relationship is of special interest in the framework of Kreisel's highly interesting reformulation of the finitary approach but is too involved to be discussed here [1]).

§ 4. Consistency, Completeness, Categoricalness, and Independence

A formalized theory is interesting only if it is free from contradiction. It gains in interest if it answers all pertinent questions. And one might wish that it should uniquely characterize its subject matter.

These formulations are of course very vague. They form nevertheless the starting-point for the rigorous definitions of the many metatheoretical concepts to which we shall turn now. Whereas at the beginning it was thought that the rigorous concepts answering the three above-mentioned desirable properties of theories would turn out to be relatively simple and unambiguous, at least in the case of axiomatically built theories, we have come to realize that there exists a whole battery of terms [2]) explicating the various aspects of these properties. We shall not try to be exhaustive, especially since some of these terms refer to exceedingly complex situations whose description would require many pages; these terms are then better left for more specific monographic treatment.

A formalized theory [formal system] T is called formally *consistent* if not every sentence of T is valid [provable], otherwise formally *inconsistent* [3]). If T is a theory with standard formalization (but not only then), its formal consistency implies that there is no sentence ϕ of T such that both ϕ and $\sim\phi$ are valid [provable] in T. A class of formulae of a given formal system is called *consistent as to derivability* [*as to consequences*] [4]) if not every formula of this system is derivable

[1]) See Kreisel 51, 51–52, 52a, 52/53, 53/54.

[2]) Many of these terms were used before in this book informally or semiformally, sometimes with forward references. We are now paying our debts.

[3]) It is again deplorable that the term '*comprehensive*' proposed in Carnap 42 for what is generally called 'inconsistent' has not been accepted. 'Inconsistent' could then have been reserved to denote a formal system that contained a sentence simultaneously provable and refutable.

[4]) These are Church's terms; see Church 56, p. 327. Hermes-Scholz 52, pp. 31–32, use 'syntactically consistent' and 'semantically consistent' in approximately the same sense.

[is a logical consequence] from this class. It follows that the class of extra-logical axioms of a consistent formal system is consistent as to derivability.

One can easily deduce from these definitions, together with those of the preceding section, that a formal system is soundly interpretable (has a model) if and only if its class of extra-logical axioms is consistent as to consequences. On the other hand, it is clear that if a class of formulae is consistent as to consequences, it is consistent as to derivability. It results that if a formal system has a model, it is formally consistent. For theories with standard formalization, the converse of this last statement holds too, since in these theories the concepts of logical consequence and derivability are entirely equivalent.

One way, then, of proving the formal consistency of a theory is to show in its metatheory that it possesses a model. Such a consistency proof, however, can be regarded as *absolute* only if the metatheory is unimpeachable. Otherwise, it carries conviction only *relative* to the degree in which we are convinced of the consistency of the metatheory. For certain theories, such as number theory, analysis, and set theory, it looks hopeless to find a suitable metatheory that would not be at least as suspect as these theories themselves, in fact, that would not contain counterparts of these very theories themselves [1]). It was this hopelessness that made Hilbert invent a different, namely the metamathematical, method of providing consistency proofs.

Another way of proving the formal consistency of a theory T is by establishing a normal truth definition for it in some other theory T'. One says, following Tarski [2]), that a formal theory T' possesses a *truth-definition* for another formal theory T if there exists in T' a predicate, say 'Tr', such that all sentences (of T') obtainable from the expression

$$Tr(x) \text{ if and only if } p$$

by substituting for 'x' a name (or some other designation) of any sentence of T and for 'p' a translation of this sentence into T' are provable in T'. The truth-definition is called *normal*, following Wang [3]),

[1]) Cf. Kleene 52, p. 131.

[2]) Tarski 35/36 (56, VIII, pp. 187 ff). For other definitions, see McKinsey 49 and Martin 51.

[3]) Wang 52b; this important paper contains a very thorough investigation of the whole problem of the relationship between truth-definitions and consistency proofs.

if in addition the formal counterpart of '$(\mathrm{A}x)$ [x is a theorem of $T \supset$ $\supset Tr(x)$]' is provable in T'.

The possibility of establishing a truth definition in T' for T does not necessarily mean that T' is stronger than T [1]) — though T' must not be identical with T, or a subtheory of T, in view of Tarski's truth theorem (cf. below, p. 306) — but this feature does not by itself guarantee the possibility of establishing the consistency of T in T', contrary to what was generally believed for a time [2]). Only the possession of a *normal* truth definition does guarantee the consistency, but then T' must be definitely stronger than T, which once again destroys the epistemological value of the consistency proof. In addition, the model whose existence is guaranteed by a consistency proof via normal truth definition need not be a standard one [3]).

Formal systems with the natural numbers as individuals and containing a notation, primitive or defined, for each such number — we may then as well assume that they contain the ordinary numerals '0', '1', etc. — are called *ω-consistent* [4]) if there exists no formula ϕ free in ξ such that $\phi(0)$ (i.e. the formula resulting from ϕ by substituting everywhere '0' for ξ), $\phi(1)$, $\phi(2)$, ..., and $\sim(\mathrm{A}\xi)\phi$ are theorems, otherwise *ω-inconsistent*. ω-consistency clearly implies formal consistency but the converse does not hold.

A formally consistent but ω-inconsistent system would not be intuitively satisfactory. A mathematical system might well be de-

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[1]) *Ibid.*, pp. 269 and 272.

[2]) This belief originates with Tarski 35/36 (56, VIII, p. 236) who created the method of proving consistency through the establishment of a truth definition. Wang 50 and Mostowski 51a showed that this belief was not quite justified, as the metatheory T' might not be powerful enough to contain a certain strong form of the principle of mathematical induction. This could happen, for instance, if the formation of impredicative sets was not guaranteed by the rules of the metatheory. Such sets are indeed dispensable for defining truth but are seemingly indispensable for consistency proofs, as they are for the definition of the stronger notion of normal truth, introduced for this purpose in Wang 52b. Cf. also the daggered footnote on p. 237 of Tarski 56, added for the English edition of Tarski 35/36.

[3]) In the sense of Henkin 49 and Rosser-Wang 50.

[4]) The term was introduced in Gödel 31 and studied in Tarski 33 (56, IX). It was later generalized in many different directions; see, e.g., Henkin 54. Cf. also the review by Kreisel, *Math. Reviews* **16**, p. 103, 1955, an abstract by the same author in *J.S.L.* **22**, pp. 108–109, 1957, and Orey 56. For a so far informal notion of *degree of ω-consistency*, see Feferman 57, pp. 166–167.

monstrably consistent but, being ω-inconsistent, might still contain certain theorems that would be regarded as intuitively false. A similar remark holds for the related notion of so-called *external inconsistency* [1]).

A formalized theory T is called *formally complete* if there is no proper consistent extension of T with the same vocabulary, otherwise *formally incomplete*. If T is a theory with standard formalization (but not only then), its completeness implies that, for every sentence ϕ of T, either ϕ or $\sim\phi$ is valid in T. A class of formulae of a given formal system T is called *complete as to derivability* [*as to consequences*] if from it, for every sentence ϕ of this system, either ϕ or $\sim\phi$ is derivable from [is a logical consequence of] this class. It follows that the class of extra-logical axioms of a formally complete theory with standard formalization is complete as to derivability, hence also complete as to consequences.

In addition to the absolute notions of formal consistency and completeness, there are a host of relative notions of consistency and completeness applying to formal systems, where the relativization concerns either some property of the provable formulae or some intended interpretation. If in a formal system all sentences that are true under all possible interpretations, or under all intended interpretations, or under all interpretations of a certain kind, or under some specific interpretation, are provable, then this system is said to be *complete with respect to truth under all interpretations*, etc. [2]). Of special interest are the two concepts of completeness with respect to truth under *all* interpretations and under *all intended* interpretations, either of which — not properly distinguished until the investigations of Henkin in 1947 [3]) — might have been referred to when one was speaking about (*semantic*) *completeness of a formal system* (without qualification).

The best known completeness theorem is the so-called *Gödel completeness theorem* to the effect that certain standard first-order functional calculi are complete with respect to the interpretation of these calculi in the domain of the natural numbers, i.e. every sentence of such a calculus which is true in every structure whose uni-

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[1]) Cf. Finsler 26; Hilbert-Bernays 34–39$_{II}$, p. 282; Kleene 52, p. 212; Quine 53b.

[2]) Cf. Kleene 52, p. 131; Wang 53a, p. 440.

[3]) Cf. Henkin 47, 50; the idea as such probably originates with Skolem.

verse is the set of natural numbers is provable in this calculus [1]).

The situation is different with regard to higher-order functional calculi. The pure second-order functional calculus, for instance, differs from the pure first-order calculus in that its predicate variables are quantifiable. Its rules of interpretation must provide the predicate variables with suitable ranges of values. In the intended (principal) interpretations, the range of each n-ary predicate variable is the set of *all* n-ary relations over the universe U, i.e. the set of *all* sets of ordered n-tuples of members of U. If the ranges are arbitrary subsets of these sets, we get a larger class of interpretations among which those that are non-intended and sound are called the *secondary* interpretations. It follows from Gödel's incompletability theorem (below, p. 303) that the pure second-order functional calculus is not complete with respect to truth under all principal interpretations, but Henkin [2]) succeeded in proving that it is complete with respect to truth under all sound interpretations, principal and secondary [3]), as is the calculus of order ω (the simple theory of types).

Models of second and higher order calculi, which correspond to the principal interpretations, are called *standard* by Henkin, those corresponding to any sound interpretation are called *general*, general models corresponding to secondary interpretations are called *non-standard*. We have then, in these terms, that all sentences true in every general model of the second-order calculus are provable in it but that this is not the case for all the sentences which are true in every standard model (of which there are of course more).

Rosser and Wang [4]) use the notion of standard model in a generalized sense applicable also to other systems besides the pure higher functional calculi. They do not present, however, any rigorous definition but work rather with some necessary conditions for standard-

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[1]) There exist already many different proofs of this theorem, in addition to Gödel's own proof in Gödel 30, with many variants and generalizations. Some of these proofs use algebraic and/or topological methods, indicating that Gödel's completeness theorem has also a straight mathematical (and not only metamathematical) content. In addition to the proofs mentioned in Chapter II, p. 105, footnote 1, let us mention here: Malcev 36; Kleene 52, pp. 383 ff; Hermes-Scholz 52, pp. 42 ff; A. Robinson 51; Rieger 51; Dreben 52; Łoś 54, 55; Copi 54; Wang 55; Quine 55a; Henkin 56 (Appendix); Kanger 57.

[2]) Henkin 47, 50.

[3]) See Church 56, § 54.

[4]) Rosser-Wang 50.

ness; these are good enough for them to arrive at their important results regarding the impossibility of the existence of standard models for such theories as ***NF*** [1]).

Among the many relative consistency notions, let us mention only *consistency with respect to satisfiability under some interpretation with a non-empty universe*, known simply as (*semantical*) *consistency*. A formal system is then *semantically consistent* if its set of provable formulae is satisfiable under some interpretation with a non-empty universe, in short, if it has a model. It follows from Gödel's completeness theorem that if a theory with standard formalization is formally consistent it has a model. Since the converse of this follows from the definition of the terms — as mentioned above, p. 286 —, we have the important result that for such theories semantical and formal consistency coincide, hence that the in general non-finitary notion of semantical consistency is replaceable, for such theories, by the finitary notion of formal consistency. The proof Gödel himself gave of his theorem is, however, non-constructive. But Hilbert and Bernays were able to prove by finitary means a somewhat weaker completeness theorem [2]).

Formal systems fulfilling the conditions mentioned above in connection with the definition of ω-consistency are called *ω-complete* if, for every formula ϕ free in ξ, $(A\xi)\phi$ is a theorem whenever $\phi(0)$, $\phi(1)$, $\phi(2)$, ... are theorems, otherwise *ω-incomplete* [3]).

[1]) This result has already been mentioned in Chapter III, p. 145; cf. Chapter II, p. 107, especially footnote 5.

For the whole issue of intended interpretations and standard models, see the interesting paper Kemeny 56, but the reader should beware of divergencies in the terminology.

Let it finally be mentioned that in another sense of completeness even the first-order functional calculus is incomplete; see Hasenjaeger 50.

[2]) See Hilbert-Bernays $34–39_{II}$, pp. 252–253; Kleene 52, pp. 395–397.

[3]) The concept of ω-completeness — for a larger class of formal systems — was introduced in Tarski 33 (56, X). An important generalization, under the name of Γ-*completeness*, is due to Henkin 57. A still other variant, called *model-completeness*, was introduced in A. Robinson 55a, developed in 56 and 56a, and further generalized in 57. It is closely related to Γ-completeness but incomparable with formal completeness, in the sense of neither including it nor being included in it.

A good semi-formal discussion of various notions of completeness is to be found in Copi 54. But the reader should notice that Copi uses 'model' as a synonym for 'non-empty universe'. Cf. also De Sua 56.

The modern treatment of the methodology of the deductive sciences may be dated with Tarski 30 (56, V) and 35–36 (56, XII).

A formal system is called *categorical* (or *monomorphic*) if it has a model and all its models are isomorphic with each other, i.e. if there exists a one-to-one correspondence between the universes of any two models such that the relationships are preserved [1]).

It can easily be proved that every categorical set of axioms is complete as to consequences. In fact, were it not so, there would exist a sentence ϕ in the theory under discussion that would be true in some models of the axiom set and false in others; therefore not all models would be isomorphic [2]).

On the other hand, there exist sets of axioms which are complete as to consequences, or even as to derivability, and non-categorical. Of this kind is, for instance, the elementary theory of succession in the natural numbers, i.e. the simple applied first-order functional calculus with the Peano axioms but without the recursive "definitions" of sum and product as its extra-logical axioms [3]).

For some time it was thought that the notion of (absolute) categoricalness would provide the basis for a significant dichotomy of *all* kinds of theories, separating *one-model theories* (see below, p. 345) such as ordinary arithmetic from *many-model theories* such as group theory. However, when it was proved that no consistent theory with standard formalization that possesses an infinite model is categorical, if only for the simple reason that each such theory possesses models of arbitrary infinite power [4]), absolute categoricalness lost its signific-

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[1]) For a rigorous definition, see, e.g., Church 56, pp. 327–330 or Tarski 56, X, p. 390. The first formulations of the idea of categoricalness are due to Huntington and Veblen (the latter of which was the first to use the *term*, following a suggestion by the philosopher John Dewey), though it was known in essence already to Dedekind and Cantor; cf. Fraenkel 28, pp. 341 ff. A comparison of the statements made there — thirty years ago — with those contained in the present book, with regard to categoricalness of arithmetic and other aspects of axiomatics, should be quite revealing. Cf. also Chapter II, p. 89, footnote 4.

[2]) The formulation of theorem I in Wang 53a, p. 417 — *Tout système catégorique est complet* — is somewhat misleading: by *'complet'* is meant there formal completeness — see Df. 5 on p. 416 —, and it is not true that *every* categorical formal system is formally complete. It is clear, however, that Wang intended to refer to elementary systems only, for which his theorem indeed holds.

[3]) See, e.g., Wang 53a, p. 422.

[4]) For theories with an at most denumerable vocabulary, this result was obtained by Tarski; see Skolem 34, p. 161. It was later proved in A. Robinson

ance with regard to such theories. Weaker notions of *relative categoricalness* were therefore introduced and, in the last years, extensively studied.

A somewhat lengthy terminological and historical digression is requisite at this stage [1]). Though we have no intention of going here into the foundations of arithmetic in any detail, arithmetic is the central mathematical discipline, multiply connected with set theory, so that terminological confusions relating to arithmetic are apt to induce parallel confusions in set theory.

Following Church, we shall call *elementary arithmetic* any logistic system obtained by adding appropriate axioms to a simple applied first-order functional calculus. One such system, called $\boldsymbol{A}^0$ by Church, has two ternary predicates — for sum and product — as its only extra-logical constants, twelve axioms governing them, and an axiom-schema for the infinitely many axioms of mathematical induction. (No stronger formulation of mathematical induction is possible with such a weak logic.)

By *first-order arithmetic* we understand any logistic system obtained by adding appropriate axioms to a non-simple applied first-order functional calculus. The system called $\boldsymbol{A}^1$ by Church, for instance, has the same vocabulary as $\boldsymbol{A}^0$, the same twelve axioms, but instead of the infinitely many additional axioms of mathematical induction it has just one; since the underlying logic contains predicate variables, not being any more a *simple* calculus, this formulation is now possible.

In general, we call *n-th order arithmetic* (n being any ordinal number greater than or equal to 1, finite or infinite) any logistic system obtained by adding appropriate axioms to an n-th order applied functional calculus. It is well-known that in second-order arithmetic the ternary predicates for sum and product are already definable in terms of the binary predicate for the relation of *having a successor* [2]).

Peano's well-known axiom system, first formulated in 1889 [3]), is

51 and Henkin 53b, independently, under a less restrictive assumption; cf. Vaught 54, p. 467, footnote 3. A still stronger result is given in Tarski-Vaught 57, pp. 92 ff.

[1]) This digression is based upon Church pp. 318 ff which pages contain the best presentation of postulate theory we know, and — for the historical part — upon Wang 57.

[2]) See, e.g. Church 56, p. 322.

[3]) This axiom system is, roughly and informally, presented in Chapter II, p. 82, footnote 4.

symbolized within an applied functional calculus containing predicate variables, hence certainly not within a *simple* calculus. At that time, no distinction was explicitly drawn between functional calculi of the various orders. In Peano's system, the successor relation (or operation) is regarded as primitive, whereas the relations (or operations) of sum and product are regarded as being introduced by definition, through recursive equations. We now know that this way of introducing sum and product is legitimate only within a functional calculus of at least second order; in the framework of a first-order calculus, these equations must be considered as being additional axioms. If the term '*Peano('s) arithmetic*' is then to be used at all, it should be reserved as a common name for arithmetics of at least second order, including the systems $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, ..., $\boldsymbol{A}^n$, ... of Church [1]).

Peano himself acknowledges his debt [2]) to Dedekind [3]) whose characterization of the natural numbers, however, is not quite axiomatic in the modern sense. Dedekind, in his turn, recognizes a kinship to Frege's work [4]), though Frege himself tended rather to stress the differences. A definition of sum and product through informally stated recursive equations was already given by Peirce in 1881 but was not known to either Frege, Dedekind or Peano.

On the other hand, since all the axiomatic systems of arithmetic that were treated by Hilbert and Bernays in the *Grundlagen der Mathematik* [5]) were either elementary or of first order, the term '*Hilbert('s) arithmetic*' is appropriate as a common name for such systems, including those denoted $\boldsymbol{A}^0$ and $\boldsymbol{A}^1$ by Church.

Finally, since Skolem was the first to point out the limitations of the axiomatic method in characterizing within a first-order calculus the set of all true arithmetical statements [6]), the term '*Skolem('s) arithmetic*' would serve as a useful abbreviation for those systems of non-axiomatic arithmetic with standard formalization whose notion of validity is defined as truth under the ordinary interpretation.

Elementary arithmetic is not categorical; it has been shown that

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[1]) Tarski-Mostowski-Robinson 53, pp. 30–31, introduce this term in the sense of what we propose further on to call Skolem's arithmetic.

[2]) See, e.g., *Peano **5**.

[3]) Dedekind 1888 had appeared just the year before.

[4]) He refers especially to Frege 1884.

[5]) Hilbert-Bernays 34–39.

[6]) Skolem 34.

this system possesses also models whose universe contains additional entities [1]), besides the "regular" natural numbers, i.e. the number 0 and the numbers obtained from it by applying a finite number of times the successor operation. The customary arguments for the categoricalness of the arithmetic of natural numbers do not prove the absolute categoricalness of *elementary* arithmetic. What they do prove is that this system is categorical *relative* to the regular natural numbers (which is rather trivial). All models of elementary arithmetic whose universe contains only these numbers are indeed isomorphic [2]).

One could try to impose this limitation to regular numbers by adding an *axiom of restriction* (limitation; cf. Chapter II, p. 89) to some standard axiom system of elementary arithmetic, say of the form

$$(\mathrm{A}x)\,[x = 0 \vee x = S(0) \vee x = S(S(0)) \vee \ldots].$$

But this expression is clearly, and unfortunately, of infinite length, hence not legitimate within the framework of a theory with standard formalization. And there is no way of reformulating it legitimately. It can be replaced by an additional rule of inference,

If $\phi(0)$, $\phi(S(0))$, $\phi(S(S(0)))$, ... are theorems, so is

$(\mathrm{A}\xi)\phi$, for any formula ϕ free in ξ,

but this rule — which we recognize as the rule of infinite induction discussed above (p. 279) — transcends again the given frame [3]). (This rule, incidentally, would have ensured the ω-completeness of the resulting system.)

The absolute non-categoricalness of the usual elementary set theories (in which number theory can be developed) follows immediately from their incompleteness (see Gödel's incompletability theorem, below, p. 303). It is also a direct consequence of the Löwenheim-Skolem theorem (Chapter II, § 6) which guarantees the existence of models with denumerable universes, whereas the intended models of these set theories have non-denumerable universes. There arises

[1]) See Skolem 33, 34 and 55; the last of these papers contains simplifications of the well-known proofs and constructions presented in the second one as well as outlines for the construction of non-standard models for various fragments of elementary arithmetic. In Henkin 50, p. 91, the result is mentioned that every non-standard denumerable model for the Peano axioms has the order type $\omega + (^*\omega + \omega)\eta$, where $^*\omega + \omega$ is the type of the integers and η the type of the rationals, both in their natural order. Malcev 36 had already proved that elementary arithmetic has models of any given infinite cardinal. Cf. also Hasenjaeger 52a.

[2]) Wang 53a, pp. 422 and 425.

[3]) *Ibid.*, p. 423; cf. Schütte 51.

therefore the highly interesting question whether these theories are categorical in some relativized sense, e.g. whether they are categorical relative to their natural numbers or relative to their ordinal numbers. This problem is still unsolved [1]).

Various notions of relative categoricalness have been developed, of which categoricalness relative to the natural numbers of the theory is just an example. More generally, one says that a formal system S is *categorical relative to its subsystem* S' if and only if any two models of S that are extensions of two isomorphic models of S' are isomorphic, and one says that S is categorical *relative to a set of predicates* definable in it if and only if any two models of S in which the predicates of this set are isomorphically interpreted are isomorphic. (Categoricalness relative to the natural numbers, for instance, is categoricalness relative to the set of three predicates interpreted as denoting the property of being a natural number, of being equal to zero, and the successor relation, respectively.)

A related notion is that of *categoricalness in a given power*, introduced independently by Vaught and Łoś [2]). A formal system is called *categorical in a given power* if any two of its models whose universes are of this power are isomorphic. Systems which are absolutely non-categorical may well be categorical in some power, in many powers, even in all powers [3]).

[1]) *Ibid.*, p. 425 and Wang 55, pp. 69–70; cf. also Wang 53b.

[2]) See Chapter II, p. 105, footnote 6.

[3]) The sense in which we are using here the adverb 'absolutely' is not the technical one in which Tarski 56, X, p. 311n uses it, in distinction from what he calls 'intrinsically categorical'. Section 2 of this paper (pp. 308–319) contains important remarks on various other aspects of the notion of categoricalness and its application to scientific theories in general. Cf. also the closing remarks of XIII, pp. 390 ff, dealing, among other things, with the notion of *non-ramifiability*, another explicatum of the intuitive notion of completeness, and its relation to categoricalness and formal completeness. In this essay, originally published in 1935, the axiomatically built arithmetic of natural numbers is regarded as categorical, but Tarski refers there not to $\boldsymbol{A}^0$ but to $\boldsymbol{A}^\omega$.

The notion of *notational* (or *expressive*) *completeness with respect to a given subject matter* deserves at least to be mentioned. Its meaning should be clear. As an illustration, let us only mention that the ordinary propositional calculus, based upon '$\sim$' and '$\supset$' as the only connectives, is notationally complete with respect to truth-functions, in short: is truth-functionally complete, since it can easily be shown that all truth-functions are expressible on this basis. This result, together with a proof of the formal completeness (and decidability) of the propositional calculus, was first obtained in *Post 1.

One says that an axiom, within a formal system, is *independent as to derivability* [*as to consequences*] if it is not derivable from [not a consequence of] the set of the remaining axioms. It follows immediately that an axiom is independent as to derivability if it is independent as to consequences and that an axiom is independent as to consequences if and only if the set of the remaining axioms possesses a model in which this axiom is false [1]). Establishing the independence of an axiom through exhibition of an *independence example* is, of course, the classical method that was used in effect already by Beltrami and Klein in their investigations on the consistency of hyperbolic geometry relative to Euclidean geometry and later by Peano and Hilbert in their studies in the axiomatics of arithmetic and geometry, respectively. In Chapter II, the independence of the axiom of choice from the remaining axioms of a set theory that differed from **Z** mainly through the admission of individuals, i.e. of different memberless non-sets — or else through the admission of "extraordinary sets" —, was established by this method.

It follows immediately from the definition that an axiom ϕ is independent, within a theory with standard formalization T, if and only if the theory T' resulting from T through replacement of ϕ with $\sim\phi$ is consistent. One says, in this case, that $\sim\phi$ is *compatible* (or *consistent*) with the remaining axioms.

A set of axioms is called *irredundant* if each of its axioms is independent, otherwise *redundant*. A generation ago, there was a great deal of discussion around the importance of irredundancy in the evaluation of a formal system [2]). Though there are even today some axiomaticians who regard irredundancy as a *conditio sine qua non* for a set of sentences to serve as a set of axioms, the consensus of most authors is that irredundancy as such has at most some aesthetic and didactic value, but that independence investigations and proofs are apt to provide important insights into the structure and scope of a given theory.

A rule of inference is called *independent within a theory* T if the set of theorems of T includes the set of theorems of the theory T', resulting from T through omission of this rule, as a proper part.

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[1]) Cf. Church 56, p. 328.

[2]) For the older literature on this and related notions see *E. H. Moore 1, *Sheffer 1, 2, *Huntington 6, 9, 10, 11, *B. A. Bernstein 6, and, for an early critique, e.g. *Ingraham 1.

In view of the far-reaching correspondence existing within axiomatic systems between axioms and (proper) theorems on the one hand, primitive and defined terms on the other hand, investigations in the irredundancy of the set of axioms of such a system are often supplemented by studies in the *irredundancy of the primitive vocabulary*, understood as the indefinability of any primitive term (with respect to the remaining primitive symbols) [1]).

§ 5. Decidability and Recursiveness; Arithmetization of Syntax

The last property of formalized theories to which we shall turn now is decidability. The intuitive meaning of this notion is clear enough: a formalized theory T is *decidable* if there exists an effective, uniform method — a so-called *decision method* — of determining whether a given sentence, formulated in the vocabulary of T, is valid in T, and *undecidable* otherwise. The problem whether there exists a decision method for T, in other words whether T is decidable, is then called the *decision problem* for T. If T is a formal system, we have instead of the general decision problem *for validity* the more specific decision problem *for provability*. One occasionally discusses also decision methods for other notions, such as formulahood, axiomhood, etc.

Decidability is clearly a highly desirable property of theories. In a decidable theory, every problem that can at all be formulated in its vocabulary has an answer that can be obtained by mechanically following a fixed recipe or, to use the terminology of computing machinery, a routine. It was hoped for a time — and it was the task of the Hilbert program to realize this hope — that the formal system into which classical arithmetic and analysis could be systematized and adequately axiomatized would be decidable as to provability as well as (formally and semantically) complete and categorical. The Hilbert school believed, in addition, that all this, together of course with consistency, could be proved by finitary metamathematical methods.

The pursuit of this aim led to a host of important insights, with regard to both the various interesting general connections between the

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[1]) For a recent investigation along this line, see Beth 53a where the older relevant literature such as *Padoa 1 and Tarski 35a (56, X) is taken into account. Cf. also A. Robinson 56b, 57, and Craig 57a.

mentioned metatheoretical notions and the consistency, completeness, categoricalness and decidability of various sub-theories of the envisaged total logico-mathematical theory. Before we proceed to mention some of the results of these insights, let us briefly describe the technique of the *arithmetization of syntax* that has allowed us to deal with all these matters by a unified method whose major tool was the *theory of recursive functions*, and at the same time enabled us to replace the somewhat vague notions of effective and semi-effective procedures by the precisely defined concepts of (*general*) *recursiveness* and *recursive enumerability*.

Let us recall then that at a certain stage in the exposition of the axiomatic approach to the foundations of set theory we claimed that by this approach the antinomies were eliminated (pp. 95–96): the logical ones, because of the limitations which the axiom system imposed upon the existence proofs, the semantical ones, because those semantical terms like 'true', 'definable', 'denotes', etc. that occupied the central place in their derivation could not be reproduced within the framework of these systems. But whereas the first contention was (more or less) proved, though not in detail, the second one was left unjustified, hoping that the reader would agree that the mentioned semantical terms could not well be expressed on the basis of a vocabulary that included, in addition to the logical terms of the first-order functional calculus, just the symbol of set-membership.

As a matter of fact, however, this second contention is far from being obviously correct. Most set theories developed (or outlined) in previous chapters of this book are strong enough, in spite of their poor primitive notation, for having all, or at least certain large parts, of classical arithmetic and analysis developed within them. Could not a method be found of developing within them their own syntax and semantics? And would then not the semantical antinomies be once more reproducible?

This was the question that Tarski and Gödel, independently, put themselves in the late twenties. The answer at which they arrived was highly surprising and led to further unexpected and interesting results. The story has often been told, in all degrees of formality and rigor, and we shall be very brief.

For purely technical reasons of expediency, let us consider a formalization of **Z** which is somewhat different from the one given above, pp. 274 ff. According to the new rules of formation, there are just 9

primitive symbols:

$$\epsilon,\ x,\ |,\ \sim,\ \supset,\ \equiv,\ \mathrm{A},\ (,\ ).$$

(It is very easy to see that this primitive notation is sufficient. There is no need to show this in detail. Nor shall we indicate the changes in the rules of transformation required for the adaptation as they, too, would be purely mechanical.) Let the numbers 1 through 9 be assigned to these symbols, say in the given order, and let to every expression (string of symbols) be assigned as its *Gödel number* the number expressed by the corresponding string of digits. The assignment is clearly one-to-one.

The Gödel number of '$x_{|} \epsilon x_{|||}$' would be, for instance, 2312333, and the Axiom of Power-Set, whose unabbreviated formulation would now be

$$(\mathrm{A}x)\ (\sim(\mathrm{A}x_{|})\ (\sim((\mathrm{A}x_{||})\ ((x_{||} \epsilon x_{|}) \equiv ((\mathrm{A}x_{|||})\ ((x_{|||} \epsilon x_{||}) \supset (x_{|||} \epsilon x)))))))),$$

will have

87298487239848872339882331239688723339882333123395823331299999999

assigned as its Gödel number.

The set of variables is represented through this device by a certain set of integers, viz. the set {2, 23, 233, ...}; similarly for the set of atomic formulae, of formulae in general, of axioms etc. The corresponding sets of integers are easily recursively defined.

The number assigned to a series of formulae, such as a derivation, composed of the formulae $\phi_1, \phi_2, \ldots, \phi_n$ in order, is $\prod_{i=1}^{n} p_i^{g_i}$, where p_i is the i-th prime number and g_i the Gödel number of ϕ_i. Thereby the syntactical notions of derivation, proof, etc. become arithmetically representable. Notice that 'ϕ is a theorem' is represented by the arithmetical sentence 'there exists a number x which is the Gödel number of a proof such that the Gödel number of ϕ is the power of the largest prime number in the decomposition of x into a product of prime number powers' [1]).

We recall that the notion of proof in a logistic system is effective in the sense that there exists a mechanical procedure of testing whether a given series of formulae is or is not a proof, but that the notion of theoremhood is in general only semi-effective in the sense that there

[1]) The most extensive treatments of Gödel representation and the arithmetization of syntax are to be found, in addition to Gödel 31 itself, in Hilbert-Bernays 34–39, Carnap 37, Kleene 52, and Ladrière 57 (cf. already 49). A simple recent account is to be found in Wilder 52.

exists in general no mechanical procedure of testing whether a given formula is a theorem, though there exists a mechanical procedure of generating all theorems, one after the other, such that if the formula under investigation is a theorem, it will, after finitely many steps, appear in this list. (But there does not in general exist a mechanical procedure that would simultaneously grind out all non-theorems one after the other.)

This characterization of effectiveness and semi-effectiveness suffers from the defect that the notion of mechanical procedure which plays a decisive role in it is so far left in a state of unanalyzed vagueness. Many different, partly or wholly independent, attempts were made to explicate it and, interestingly enough, almost all of the strict notions brought forward in this attempt have proved to be equipollent [1]). This being so, we can restrict ourselves here to mentioning the most intuitive explication of the notion of effective procedure provided independently, and with some variation, by Turing and Post in 1936 [2]) in respect of numerical computation: there exists an effective procedure of computing the value of a function from (n-tuples of) natural numbers to natural numbers, if a machine of a certain specific design (of a somewhat idealized type, as its "memory" is assumed to be infinite or at least infinitely extensible) can be programmed to perform this computation for any arguments. For an exact description of these machines — the so-called *Turing machines* — we refer to the abundant literature [3]). We can now define: A given n-ary number-theoretical

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[1]) More specifically, the following notions were proved to be equipollent: (1) *general recursiveness*, introduced in Herbrand 31 and Gödel 34 in generalization of the notion of *primitive recursiveness* used in Gödel 31, pp. 179–180, and given its definite determination in Kleene 36 who also generalized it further to *partial recursiveness*; (2) *λ-definability*, introduced in Church 36, pp. 346 ff; (3) *computability by appropriate machines* in Turing 36 and Post 36; (4) *reckonability* in Gödel 36; (5) *'regelrecht auswertbar'* ("which can be evaluated according to rule") in Hilbert-Bernays 34–39$_{II}$, pp. 392 ff; (6) *binormality* in Post 43; (7) *normal algorithm* in Markov 54 (and earlier publications dating back to 1947). The equipollence of (1) and (2) was proved in Church 36 and Kleene 36e, that of (2) and (3) in Turing 37; see Kleene 52, pp. 320–321.

[2]) Turing 36 and Post 36.

[3]) The most readable descriptions are given in Péter 51 and Kleene 52. Among more recent publications, four papers by Claude E. Shannon, M. D. Davis, John McCarthy and others in *Automata Studies*, ed. by Shannon and McCarthy, *Annals of Mathematics Studies*, No. 34, Princeton 1956 and two

function f is called *Turing-computable* if there exists a Turing machine that computes its value for any n-tuple for which it is defined.

The thesis that Turing-computability (or general recursiveness, or any other of the equipollent notions mentioned above, p. 300, footnote 1) is an adequate explicatum for the pre-systematic notion of effective calculability is known as *Church's thesis* [1]). In view of ever recurring misunderstandings (and misleading formulations in terms of 'is identical with' instead of 'is an adequate explicatum of'), it must be stressed that a thesis of this kind, claiming that a notion rigorously defined with respect to some formalized theory adequately explicates some intuitive notion, is, by definition, not susceptible to exact proof. The evidence for the correctness of Church's thesis is nevertheless overwhelming. In addition to the fact already mentioned that almost all explications proposed so far for effective calculability have proved to be equipollent, every function acknowledged to be effectively calculable, for which general recursiveness has at all been investigated, has indeed been found to be so [2]). Since a number-theoretical property (or relation) is *effectively decidable* if and only if its *characteristic function* — i.e. the function whose value for a given n-tuple of natural numbers as argument is 0 if this property (or relation) holds for this tuple, and 1 otherwise — is effectively calculable, we see that the notion of effective decidability of number-theoretical properties and relations, hence of number-theoretical predicates, is adequately explicated by general recursiveness. Since all syntactical (metamathematical) predicates are mapped by arithmetization into number-theoretical predicates, we realize that all syntactic questions such as whether a given formal system is decidable are mirrored into corresponding number-theoretical questions such as whether a certain number-theoretical property is general recursive.

A set S of natural numbers is called *general recursive* if the property

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papers by Wang in *Journal of the Association for Computing Machinery* 4 and in *Ztschr. f. math. Logik und Grundlagen der Math.* 3, pp. 69–80, both in 1957, should be mentioned.

[1]) This thesis was first formulated in Church 36. Its best exposition and justification is in Kleene 52, pp. 300 ff.

[2]) L. Kalmár, however, in a talk given before the Colloquium on Constructivity in Mathematics held in Amsterdam, August 1957, voiced some doubts which were not completely dispelled in the subsequent discussion; cf. already Kalmár 56.

of being a member of S is general recursive, and *recursively enumerable* if it can be enumerated by a general recursive function f, i.e. if there exists a general recursive f such that the sequence of values $f(0)$, $f(1)$, $f(2)$, ... enumerates (allowing repetitions) the members of S. A set of metamathematical objects (expressions, formulae, axioms, proofs, etc.) is called *general recursive* [*recursively enumerable*] if the set of corresponding Gödel numbers is so. It can be shown that a general recursive non-empty set of natural numbers is recursively enumerable and that, classically, a set of natural numbers is general recursive if and only if it and its complement are recursively enumerable. An example of a set which is recursively enumerable but not general recursive can be effectively constructed. That there exist also sets which are not even recursively enumerable follows already from a simple consideration of cardinalities: the set of all recursively enumerable sets is only denumerable [1]).

[1]) For all these results and in general for an admirable treatment of the theory of recursive number-theoretic functions, we refer once and for all to Kleene 52. During the 35 years of its existence (we are probably entitled to date its origin with Skolem 23), this theory has developed into a full-fledged and constantly expanding new branch of mathematics. The most elementary and best readable introduction into it is still Péter 51, its by now already classical text-book is Kleene 52. Some of the many hundreds of papers that were published in this field during that period have opened new vistas. Several new text-books are due to appear shortly, among them books by Martin Davis and Hartley Rogers, Jr. The related *theory of algorithms* has been mostly developed by A. A. Markov and other Russian authors and has so far received its authoritative presentation in Markov 54. During the last years recursive methods of investigation were extended into analysis, and a good part of the talks given before the Colloquium on Constructivity in Mathematics, Amsterdam 1957, were dedicated to this topic.

In addition to the works mentioned above, in footnote 1 on p. 300, the following publications up to 1950 deserve to be especially mentioned: Skolem 23, Ackermann 28, Péter 34, 50, Curry 41, Kleene 43, 44, Post 44, Goodstein 45, Skolem 46, 47, Mostowski 47, Nelson 47, R. M. Robinson 47, Quine 48/49, Specker 49, Davis 50, Julia Robinson 50, Rosenbloom 50.

The following list of publications on recursive function theory, from 1951 onwards (hence not mentioned in the bibliographies of Péter 51 and Kleene 52), should be rather exhaustive. (Papers by Markov prior to his book Markov 54 are not listed.)

Goodstein 51, 53, Kreisel 51, 53, 53/54, 55, Mostowski 51, 52, 55, 55a, 55b, 56a, 57, Péter 51a, 53, 55, Bereczki 52, Davis 52, 53, Hermes-Scholz 52, Novikov 52, 55, Julia Robinson 52, Skolem 52–53, 54, 55, Dekker 53, 53a, 54, 55, Détlovs 53, Grzegorczyk 53, 54, 55b, 56, Myhill 53b, 55, Rice 53, 54, 56, Trahténbrot 53,

It follows from the definition of 'logistic system' that the expressions of a logistic system admit an effective Gödel numbering and that their properties and relations and the operations upon them are all *recursively representable.* More specifically, the axioms of a logistic system form a general recursive set, and the rules of inference determine general recursive derivability relations. The set of theorems of a logistic system is therefore recursively enumerable.

The original Hilbert program amounts, then, in these terms to the construction of a consistent, complete, decidable, and categorical logistic system whose set of provable sentences would coincide with the set of intuitively true mathematical statements and such that it could be shown, via arithmetization, in *recursive arithmetic,* i.e. in that part of arithmetic which contains general recursive functions, properties, and relations exclusively, that the system possesses all these desirable properties.

§ 6. The Limitative Theorems of Gödel, Tarski, Church and their Generalizations

That the attempt to carry out the Hilbert program, as formulated in the preceding paragraph, would run into troubles was not too surprising: any logistic system strong enough to have all of classical mathematics developed in it would certainly contain all of classical number theory, hence presumably all of its own syntax and semantics. But does this not entail that the semantic antinomies, of the Liar and Richard types, could be reestablished within such a system?

As already mentioned above, Gödel and Tarski asked themselves precisely this question. The partial answer at which Gödel arrived is

Gödel's (incompletability) theorem: [1]) **Every logistic system rich enough to contain a formalization of recursive arithmetic is either ω-inconsistent or else contains an undecidable (though true) formula,** i.e. a formula that can be neither proved nor refuted with the means of this system (though it can be shown to be true with

55, Uspénskij 53, 55, 56, Janiczak 54, 54a, Kleene-Post 54, Markwald 54, 55, Schütte 54, Kleene 55, 55a, 55b, Kuznécov-Trahténbrot 55, Lacombe 55, Médvédév 55, 55a, Myhill-Shepherdson 55, R. M. Robinson 55, 56, Spector 55, 56, Fitch 56, Friedberg 56, 57, Kalmár 56, Klaua 56, Mućnik 56, Riguet 56, Shapiro 56, Church 57, Feferman 57.

[1]) Gödel 31.

the help of additional means outside of the system); in other words, any given ω-consistent system of this kind is (syntactically and semantically) incomplete and even *incompletable.*

Gödel established his theorem *constructively*; for a given logistic system fulfilling the mentioned condition, the Gödelian undecidable sentence can actually, time and patience allowing, be written down in primitive notation. The truth of this undecidable sentence follows from the fact that, suitably interpreted, it states that this very sentence itself (more precisely, the sentence with a certain Gödel number — which then turns out to be the Gödel number of the sentence in which this statement is made) is not provable (in the given system) [1]).

It follows immediately from Gödel's theorem that *Skolem's arithmetic is not axiomatizable.*

This, then, is the revenge of the Liar's ghost: the attempt to arrive at an antinomy through the construction of a sentence within formalized arithmetic that (intuitively) states that it itself is non-provable does not meet with success, but only because — contrary to expectation and hope — a sentence non-provable within such a formal system is not thereby necessarily refutable, there being sentences which are neither. That completeness' price is inconsistency, for logistic systems rich enough to contain recursive arithmetic, including all set theories worth their name formalized as such systems, is a result which was doubly unexpected: first, for its content, second, for the fact that it could be proved according to standards of rigor which were the highest known, higher even than those customarily used in mathematical proofs. With its philosophical implications we shall deal later on.

Since there exist many excellent descriptions of the background of Gödel's theorem, its methods, generalizations and ramifications, ranging from the monographs of Ladrière [2]) — containing, among other things, an almost exhaustive bibliography up to 1955 — and Mostowski [3]) — unifying Gödel's results with those obtained by Tarski to be mentioned shortly —, through Kleene's exceedingly rigorous treatment [4]) and the detailed and painstaking

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[1]) There are many authors who are still unable to swallow the fact that there should be a sentence which is not provable — in some object-language — but can be shown to be true — in its metalanguage. For a recent discussion, see, e.g., Goddard 58; cf. also Wittgenstein 56, pp. 50–54.

[2]) Ladrière 57.

[3]) Mostowski 52.

[4]) Kleene 52; cf. Kalmár 43, 49, 50, Kleene 50.

description given by Hilbert-Bernays [1]), to the semi-formal exposition by Rosser [2]) and the informal, but still responsible and highly readable account by Findlay [3]) — not to mention the many shorter outlines of which the best is probably still Gödel's own informal introduction to his 1931 paper —, we did not deem it appropriate to go here into further details. Let us only mention that though Gödel's undecidable sentences remind us strongly of the Liar, the argument by which he arrived at his discoveries is reminiscent rather of Richard's antinomy and is, like it, decisively based upon a diagonalization process [4]).

Gödel's theorem holds for ω-consistent logistic systems. How much can his result be generalized? Rosser [5]) showed that ω-consistency can be replaced by the weaker property of simple consistency and that the requirement of the general recursiveness of the set of axioms can be weakened to the requirement of recursive enumerability. Other generalizations are due to Tarski and Mostowski [6]). They are based on the notion of (semantic) *definability* introduced by Tarski [7]), later generalized by Mostowski [8]) and finally still more generalized by Tarski himself [9]). We shall present here the latest version.

Let A be any formalized theory whose logical basis includes at least the first-order functional calculus with equality and in which an infinite sequence of terms $\Delta_0, \Delta_1, \ldots, \Delta_n, \ldots$ containing no variables is available. A subset P of the set N of all natural numbers is said to be *definable* in A if there is a formula ϕ of A, free in some fixed variable ξ, such that $\phi(\Delta_n)$ — i.e. the sentence arising from ϕ through replacing ξ everywhere by Δ_n — is valid in A whenever $n \in P$ and $\sim\phi(\Delta_n)$ is valid whenever $n \in N$ and $n \notin P$. In a similar way, definability is defined for relations and functions.

Let now $G(\phi)$ be the Gödel number of ϕ, according to some Gödel numbering which may be left here undetermined, and let E_n be that expression of A whose Gödel number is n. Let D (the *diagonal function*)

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1) Hilbert-Bernays 34–39.

2) Rosser 39.

3) *Findlay 1; cf. Kreisel 52/53.

4) See, e.g., Mostowski 52, pp. 7 ff. Wang 55b shows how each semantical antinomy can be made to generate undecidable sentences.

5) Rosser 36a; cf. Kalmár 50a.

6) See, especially, Tarski 39, Mostowski 52, pp. 97 ff, and Ladrière 57, pp. 335 ff.

7) See Tarski 56, VI (the [enlarged] English translation of Tarski 31).

8) See Mostowski 52, pp. 74–75.

9) Tarski-Mostowski-Robinson 53, pp. 44 ff.

be the function defined by

$$D(n) =_{\mathrm{Df}} G(E_n(\Delta_n))$$

(the value of the diagonal function for the argument n is the Gödel number of the expression obtained from the expression whose Gödel number is n by replacing everywhere the variable ξ by the term Δ_n; if E_n does not contain ξ, $E_n(\Delta_n)$ is, of course, just E_n itself), and let, finally, V be the set of Gödel numbers of all valid sentences of A. The following result can then be established:

TARSKI'S UNDEFINABILITY THEOREM: **If the theory** A (fulfilling the above-mentioned conditions) **is consistent, then the diagonal function** D **and the set** V **of all valid sentences** (of A) **are not both definable in** A.

From this theorem follows immediately that if in a consistent theory the diagonalizing function is definable, its set of valid sentences is not definable. Since D can be shown to be general recursive, no theory strong enough to contain recursive number theory may contain a (semantic) definition for its V. We obtain then, finally, as a special case of Tarski's undefinability theorem — replacing, for historical reasons, 'validity' by 'truth' —

TARSKI'S TRUTH THEOREM: [1]) **The notion of truth** (the set of all true sentences, the truth-set) **of a consistent formalized system containing recursive number theory is not definable in this system** [2]).

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[1]) See Tarski 56, p. 247; the footnote there tells the story. Cf. Quine 52.

[2]) An extremely elegant, simple and direct proof of Tarski's theorem was recently given in R. Smullyan 57. It is based upon replacing the *substitution* procedure which is central in Gödel's *diagonalization* method — to diagonalize the formula ϕ free in ξ means to substitute the quotation of ϕ for all occurrences of ξ in ϕ — by the far easier formalizable (arithmetizable) *concatenation* procedure which is basic in Smullyan's *normalization* method — to normalize an expression ϕ means to produce its *norm*, i.e. to concatenate ϕ with its quotation (or with its Gödel numeral, i.e. the numeral designating its Gödel number, if the syntax is arithmetized).

Smullyan calls a sentence which is true (in a given semantical system S) if and only if it is a member of a given set W, a *Tarski* sentence for W. (If W is, for instance, the set of all expressions not provable in some formal system C, then the Tarski sentence for this W is just the Gödel sentence of C, which is true in S if and only if it is not provable in C.) Let S be strong enough to contain a Tarski sentence for any set W definable in it. Then the sentence ψ, "W

Whereas Gödel's theorem shows that the *deductive power* of any given sufficiently rich system is intrinsically limited, Tarski's theorem discloses the limitations in the *expressive power* of such systems. Paraphrasing a colorful dictum by Quine, we may say that the ontology bitten off in such systems is larger than they are able to chew. This is not as surprising as it might look at the first moment. After all, the set of definable properties and relations is only denumerable — since the set of defining formulae is so, of course — whereas there are nondenumerably many subsets of the set of all natural numbers. What was not quite expected is that the set of the Gödel numbers of all true sentences of a given consistent logistic system should inevitably be among those sets for which no defining expression exists in this system.

Let us mention that Gödel's theorem has been shown to hold also for certain formal systems which are not logistic systems in the sense that they allow transfinite rules of inference such as the rule of infinite induction [1]). In what sense systems which transcend the Gödel-Tarski limitations, so-called *non-Gödelian systems*, can be regarded as satisfactory foundational systems of mathematics is a question that deserves most careful investigation. Wang's system Σ, treated in Chapter III, § 6, is admittedly not a logistic system. Other systems proposed in this connection either contain no proper negation sign, such as certain systems constructed by Fitch and Myhill [2]), or no proper universal quantifier, such as systems by Church and Myhill [3]),

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contains the norm of 'W contains the norm of' ", is a Tarski sentence for W. Indeed, ψ says that the norm of the expression 'W contains the norm of' is contained in W. But this norm is just ψ itself. ψ is therefore true if and only if it is a member of W.

Using a highly ingenious and tricky Gödel numbering — which is however not essential to his method and only serves to highlight its efficiency —, he is able to accomplish the whole arithmetization of syntax (of a certain arithmetical system) by just one simple definition whose effect is to assign as the Gödel number of the norm of an expression whose Gödel number is x — the number $x \cdot 10x$.

We may look forward with expectation to Smullyan's forthcoming paper, *Abstract structure of unsaturated theories*, in which a procedure generalizing both diagonalization and normalization will be investigated.

[1]) See Rosser 37; Tarski 39; Kleene 43, p. 68; Mostowski 49a.

[2]) *Fitch 8, 9, Myhill 50, 50a. Later systems of Fitch contain negation which does not, however, obey the *tertium non datur*; see, e.g., Fitch 52.

[3]) Church 34 (see also *Church 9), which contains an infinite hierarchy of universal quantifiers, and Myhill 50a.

or no proper sign of material implication, such as a system by Church [1]). The exact scope and efficiency of such heterodox systems still remains to be explored [2]).

Just as Gödel's theorem showed that the (semantical) notion of arithmetical, hence mathematical, truth could not be exhaustively mirrored by the (syntactical) notion of provability within one logistic system, thereby destroying the basis of simple-minded formalistic attitudes towards the foundations of mathematics, so did a corollary of this theorem show that the foremost aim of the original Hilbert program could not be achieved. This aim, we recall, consisted in proving the formal consistency of number theory with the help of only a part of the proof procedures employed in number theory itself, viz. the "finitary" ones. But however liberal we are ready to be in interpreting this equivocal term, so long as the finitary proof procedures are taken to transcend the proof procedures admitted in logistic systems only moderately — if at all — (one such moderate transcension would be the admission of a rule of infinite induction), the possibility of obtaining a finitary consistency proof of number theory is excluded by

GÖDEL'S COROLLARY [3]): **No sentence that could adequately be interpreted as representing the consistency of a logistic system containing number theory is provable within this system.**

The enormous efforts made by Hilbert and his school to carry out his program [4]) enabled them to prove by strictly finitary means the consistency of a very extensive subsystem of arithmetic, the remaining gap consisting in that this subsystem contained the inductive principle only in a weakened form which did not allow for its application to quantified sentences. Gödel's corollary shows that their failure to achieve the aim in its totality was not due to lack of ingenuity; on the

[1]) The system of Church 34 (and *9) exhibits also an infinite hierarchy of signs for material implication.

[2]) Some of the "heterodox" systems treated in the last sections of Chapter III are non-Gödelian, too. Cf. also Turing 39.

[3]) For its proof, see, e.g., Kleene 52, pp. 209–210.

[4]) Cf. Hilbert 18, 22, 23, 25, 28, 31; Ackermann 24, 28, 40; von Neumann 27; Herbrand 28, 30, 31, 32; Bernays 30, *10, *11 — to mention only the major publications of the members of Hilbert's "inner circle" and those of its perhaps most outstanding foreign associate. All these contributions — with the exception of Ackermann 40 — are incorporated in the *magnum opus* of this school, Hilbert-Bernays 34–39.

contrary, we now know that they went about as far as they possibly could.

The question then arises whether the consistency of number theory could not be proved by the use of methods which, though in general forming only a proper part of all the methods used in number theory itself, transcend these methods in just one respect, the additional method, no longer finitary in the original sense, still being "finitary" in some extended sense that would conform to the basic philosophical attitude of the formalists. However, since this attitude has not been univocally and authoritatively formulated [1]), no unique and universally acceptable answer to this rather vague question can be expected. As a matter of fact, a few years after the publication of Gödel's corollary, Gentzen [2]) was able to prove the consistency of number theory by using, as the only method transcending those used in number theory proper, *transfinite induction* up to the first ε-number ε_0. This method is finitary in a certain well-determined extended sense, or *constructive*, as we shall prefer to say, in order not to have constantly drag along those bothersome qualifications, "in a certain sense". That Gentzen proved his theorem for a formal system which differs in many essential features from the logistic systems considered so far — whereas the calculi we dealt with embody *theorem logics*, Gentzen's calculi embody another major form of logic, viz. *rule logic* [3]) — is of no importance for our purposes, since his results can be easily transferred to standard logistic systems [4]).

A third type of limitation on formal systems is provided by those theorems in which it is stated that the decision problem for validity [provability] is unsolvable for certain kinds of formal systems, i.e. that these systems are *undecidable* as to validity [provability] [5]).

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[1]) Cf. above, p. 268.

[2]) Gentzen 36, 38a.

[3]) Hermes-Scholz 52 distinguish two main forms of logic: *Satzlogik* and *Regellogik* (*Folgerungslogik, Konsequenzenlogik*). Kleene 52, p. 441, distinguishes between Hilbert-type systems and Gentzen-type systems. The systems of natural deduction mentioned in Chapter III, p. 184, are of course rule logics.

[4]) Cf. Gentzen 36 and Ladrière 57, pp. 208 ff.

[5]) The reader should beware of the trap caused by the equivocal functioning of 'decidable'. A formalized theory that contains an *undecidable sentence*, i.e. a sentence which is neither provable nor refutable within the theory, is thereby *formally incomplete* but not necessarily undecidable itself, and an undecidable theory need not contain an undecidable sentence. Gödel's incompletability

Using once more the direct method of diagonalization, Church was able to prove in 1936 [1]) that first-order arithmetic (as well as certain fragments of it) is undecidable. Since an axiomatized theory resulting from an undecidable axiomatic theory through omission of a finite number of axioms (but keeping the vocabulary) is clearly undecidable itself, Church was able to show [2]), by omitting all the finitely many extra-logical axioms from a suitably modified fragment of elementary arithmetic, that the first-order functional calculus is undecidable, in other words, that the set of its theorems, though of course recursively enumerable, is not general recursive, in still other terms, that though this calculus contains a complete proof procedure it contains no complete disproof procedure. Since Rosser was able to prove in the same year [3]) that every consistent extension of first-order arithmetic is undecidable, in short that this system is *essentially undecidable*, we have altogether the following two basic results:

CHURCH'S UNDECIDABILITY THEOREM: **The first-order functional calculus in undecidable.**

CHURCH-ROSSER'S UNDECIDABILITY THEOREM: **Elementary arithmetic is essentially undecidable** [4]).

Investigations into the decidability and undecidability of the various logical and mathematical theories received an additional impulse through the advent of the fast electronic computers. With their help, it was hoped, many mathematical problems might become *practically* solvable if their *theoretical* solvability could somehow be guaranteed, in the simplest case through a proof of the recursive decidability of

theorem has often been called an undecidability theorem, a usage we have avoided in order not to increase the danger of confusion.

For the distinction between 'undecidable' and 'unsolvable' as qualifying problems, and for the inappropriateness of speaking about "unsolvable problems" (instead of the appropriate "infinite class of problems"), see the revealing discussion in Myhill 52a, pp. 167 ff; however, not all formulations given there are watertight. Cf. also Turing 54 and Rabin 58, p. 175.

[1]) Church 36.

[2]) Church 36a.

[3]) Rosser 36.

[4]) Whereas in Tarski-Mostowski-Robinson 53 essential undecidability is understood as implying that every consistent extension *with the same vocabulary* is undecidable, the Church-Rosser theorem holds also in the stronger sense, obtained by omission of the italicized qualification. Kleene 52, p. 437, uses this term in the stronger sense.

the theory in which these problems were formulated. It was probably under this impact that Tarski, for instance, finally published his *decision method for elementary algebra and geometry* [1]) which he had already obtained, in essence, in 1930. And though this method is still not practical enough for even the fastest existing computers, there are good chances that through simultaneous improvements of the method and of the computers we will shortly reach the stage when open problems in these and other fields will be handled and solved by these machines.

Altogether, the list of logical or mathematical theories which have been proved to be decidable is not large. Let us mention only some of the most important ones, for the sake of illustration: the classical propositional calculus [2]), the singulary first-order functional calculus, with [3]) and without equality [4]), the singulary second-order functional calculus [5]), the elementary theory of linear order [6]), the theory of addition of integers [7]), elementary Boolean algebra, the elementary theory of Abelian groups [8]).

On the other hand, many theories were proved to be undecidable. For some theories this was done directly, mostly through a kind of diagonal argument, as was the case with Church's theorem. For others, use was made of the theorem that an essentially undecidable theory induces this property in every consistent theory which is interpretable in it (or in some inessential extension of it) [9]). Since elementary arithmetic is essentially undecidable and interpretable in many axiomatic set theories, *all these theories* — if consistent — *are essentially undecidable.* The set theories of Bernays and Gödel, being finitely

[1]) Tarski 51. For elaborations and variants of this method, see Seidenberg 54, Meserve 55, Schwabhäuser 56.

[2]) Post 21.

[3]) Löwenheim 15, who deals, however, with universality rather than with provability.

[4]) *Behmann 1.

[5]) *Skolem 1.

[6]) *Langford 3.

[7]) *Presburger 1.

[8]) Wanda Szmielew 50, 55. Two other theories, whose decidability has been proved, are mentioned below, p. 322.

[9]) See Tarski-Mostowski-Robinson 53, p. 22, Theorem 7. T' is called an *inessential extension* of T if the vocabulary of T' differs from that of T at most by containing additional individual constants and if every valid sentence of T' is derivable in T' from a set of valid sentences of T.

axiomatizable, thereby exemplify the existence of essentially undecidable and finitely axiomatizable theories.

The scope of undecidability proofs was greatly increased when it was proved, on the one hand, that an essentially undecidable and finitely axiomatizable theory T induces essential undecidability in every theory T' (as well as in every subtheory of T') which is *weakly interpretable* in T (or in some inessential extension of T) [1]) and, on the other hand, an example of an essentially undecidable and finitely axiomatizable theory was found which, in contradistinction to the ones mentioned above, was extremely simple and therefore had a good chance of being interpretable in other theories [2]). Within a few years, the undecidability of the elementary theories of groups, of rings, of fields, and of lattices as well as that of the theories of more specific structures, such as commutative rings, ordered rings, non-densely ordered rings, modular lattices, complemented modular lattices, distributive lattices, Brouwerian algebras, closure algebras, abstract projective geometries etc., was proved [3]).

Of special interest for us is the fact that by this method the essential undecidability of a small fragment of set theory could be shown [4]).

[1]) *Ibid.*, pp. 24–25, Theorem 8.

[2]) Cf. the abstract by Mostowski and Tarski, *J.S.L.* **14**, p. 76 (1949). This theory was later simplified in R. M. Robinson 52. The story is told in Tarski-Mostowski-Robinson 53, p. 39n. Because of its amazing simplicity, the axiom system of this theory shall be reproduced here. Its extra-logical constants comprise four symbols: the individual constant '0', the unary operation symbol 'S', the two binary operation symbols '$+$' and '$\times$'. The axioms are the universal closures of:

$$\mathrm{S}x = \mathrm{S}y \supset x = y$$
$$0 \neq \mathrm{S}y$$
$$x \neq 0 \supset (\mathrm{E}y)\,(x = \mathrm{S}y)$$
$$x + 0 = x$$
$$x + \mathrm{S}y = \mathrm{S}(x + y)$$
$$x \times 0 = 0$$
$$x \times \mathrm{S}y = (x \times y) + x.$$

[3]) A very detailed proof of the undecidability of the elementary theory of groups can be found in the third chapter of Tarski-Mostowski-Robinson 53, written by Tarski. There, and throughout the book in general, the other undecidability results are mentioned, together with a bibliography of 40 items. We mention only Julia Robinson 49, R. M. Robinson 51, 52, Szmielew-Tarski 52, Grzegorczyk 52, 56, Janiczak 53, Hermes 55a, Henkin 55a, Rogers 56, Myhill 56, Putnam 57, Ehrenfeucht 57.

[4]) Tarski-Mostowski-Robinson 53, p. 34.

This fragment contains 'ϵ' as the only extra-logical constant and just three extra-logical axioms, viz. the axiom of extensionality (in the form Ia of Chapter II, p. 33), an axiom stating the existence of a null-set, and an axiom guaranteeing, for any given sets x and y, the existence of a set z whose members are the members of x or are identical with y (cf. Bernays' II (2), above p. 111). This is a stronger result than the one mentioned two paragraphs before.

For theories whose undecidability has been proved or for which decidability has so far been neither proved nor refuted, *restricted decision problems* arise, i.e. problems as to whether certain proper subsets of the set of all their valid sentences are general recursive. For Skolem's arithmetic, e.g., which is a complete, non-axiomatizable and essentially undecidable formalized theory, the problem as to the general recursiveness of the set of all valid sentences of the form

$$(\mathrm{E}\xi_1)\ (\mathrm{E}\xi_2)\ \ldots\ (\mathrm{E}\xi_n)\ (\alpha = \beta),$$

where α and β are terms (polynoms) all whose free variables are among the $\xi_1, \xi_2, \ldots, \xi_n$, is nothing but the famous tenth problem of Hilbert [1]; it is, incidentally, still unsolved, in spite of certain partial results obtained by R. M. Robinson, Davis and Putnam [2].

For elementary group theory, the problem of whether the set of all valid sentences of the form

$$(\mathrm{A}\xi_1)\ (\mathrm{A}\xi_2)\ \ldots\ (\mathrm{A}\xi_n)\phi,$$

where ϕ is a formula without quantifiers, is general recursive has only very recently been proved to be unsolvable. This problem, better known in a different but equivalent formulation [3], is the famous *word-problem* which was proposed by Thue in 1914 and had defied mathematicians for almost 40 years. Already in 1947, Post and Markov had independently shown its unsolvability for semi-groups [4]. In 1950, Turing [5] showed that this problem is unsolvable for semi-

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[1] Raised in his talk to the International Congress of Mathematicians in Paris, 1900 (Hilbert *4).

[2] Cf. Davis 53 and R. M. Robinson 56; further results were presented by Davis and Putnam in a talk before the Summer Institute in Symbolic Logic held at Cornell University in 1957.

[3] Cf. *McKinsey 13, p. 68.

[4] Post 47, Markov 47; cf. Kleene 52, pp. 382 ff.

[5] Turing 50.

groups with cancellation. But only in 1952 did Novikov [1]) succeed in proving the recursive (or algorithmic — as he, in common with other Russian authors, prefers to call it) unsolvability of the word-problem for groups.

The reader might have already asked himself whether there might not exist a decision method by which it would be effectively determined for every given formalized theory, whether it is decidable or not. However, a rather simple argument shows that, at any rate for finitely axiomatizable theories with standard formalization, no such general method could possibly exist, hence that this second-degree decision problem, so to speak, is unsolvable [2]). Other higher-degree decision problems deal, for instance, with the existence of an effective procedure of deciding, for every given *presentation* of a certain algebraic structure, such as of a semi-group, of a semi-group with cancellation, or of a group, whether the structures defined by these presentations have certain algebraic properties such as cyclicity, finiteness, simplicity, decomposability into a finite product, etc. By combining Markov's methods with Novikov's mentioned result, Rabin has recently been able to prove the recursive unsolvability of a large number of group-theoretic problems [3]).

Many restricted decision problems have turned out to be solvable, many others could be reduced to more special ones. Of paramount importance are the reductions to and in the pure first-order functional calculus. Indeed, it is an immediate consequence of certain proofs of Church's theorem that the problem of whether or not a given sentence is provable in a given logistic system (whether or not a given natural number is in a given recursively enumerable set) can be effectively reduced to the question whether or not a particular formula of the first-order functional calculus is true (in a denumerable universe). Since the decision problem for truth in the first-order functional calculus is, of course, unsolvable, every reduction of this problem to more specific ones will automatically yield the undecidability of further theories. Kalmár's reduction of the general problem to that of the set of formulae containing just one binary predicate shows that

[1]) Novikov 52, 55.
[2]) Tarski-Mostowski-Robinson 53, p. 35.
[3]) Rabin 58.

the elementary theory of a single binary relation is undecidable [1]). Further reductions prove the undecidability of the elementary theories of a single symmetric relation, even of a single symmetric-reflexive relation, or of two equivalence (symmetric-reflexive-transitive) relations. The theory of a single equivalence relations, on the other hand, is decidable. An interesting open problem is the reducibility of the general decision problem of the first-order functional calculus to that of the theory of a single total order relation, hence also the problem whether this theory is decidable [2]).

Other aspects of reducibility problems were treated by Post who introduced [3]) the term *'degree of recursive unsolvability'* which served as the basis for a classification of functions, properties, relations, and sets into equivalence classes created by the reflexive, symmetric, and transitive relation 'A is recursive in B and B is recursive in A' [4]). The least degree is that of the general recursive functions etc., the decision problem of which is recursively solvable, of course [5]). Though this issue is quickly becoming one of the most interesting and fruitful fields of mathematical logic, it is too intricate to be treated here in any detail [6]).

There exist, of course, also many cases for which restricted decision problems have been shown to be solvable. The interested reader is referred to the exhaustive treatment of these problems by Acker-

[1]) *Kalmár 7.

[2]) For these results and problems, see Rogers 56.

[3]) Post 44.

[4]) For the notion of (general) recursiveness of a function in other functions, see Kleene 52, p. 275; for the other notions, see *ibid.*, pp. 276 and 307.

[5]) The word 'unsolvability' is therefore slightly misleading in this context, as is, incidentally, the word 'degree', the "degrees" being *classes* and not, as one might have thought, numerical measures of some sort.

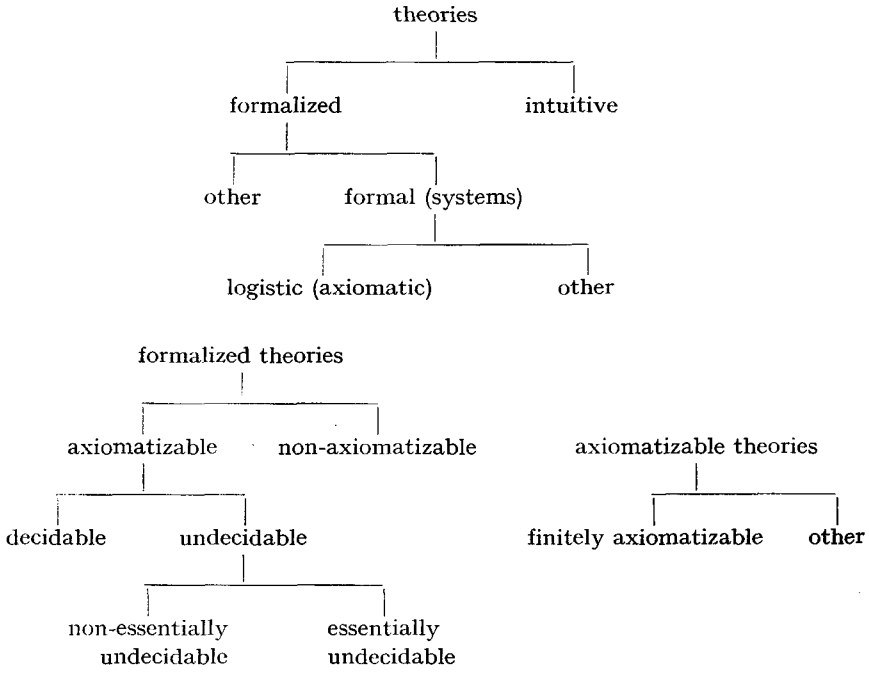
[6]) The most important publications dealing with this issue are Kleene-Post 54, Médvédév 55a, and Spector 56. In 1956, a question raised in Post 44, which has become known as *Post's problem* and which has been considered as one of the most important open foundational problems, was almost simultaneously, but independently, answered in the negative by a 19-year old American student, R. Friedberg (see Friedberg 56, 57), and a 17-year old Russian student, A. A. Mučnik (see Mučnik 56). Post's problem was whether all degrees of unsolvability could be linearly ordered by the relation is-recursive-in. Friedberg and Mučnik were able to construct two recursively enumerable sets whose degrees of unsolvability are incomparable with respect to this relation.

mann [1]), Klaua [2]) and Church [3]); their publications contain also ample references to the older literature.

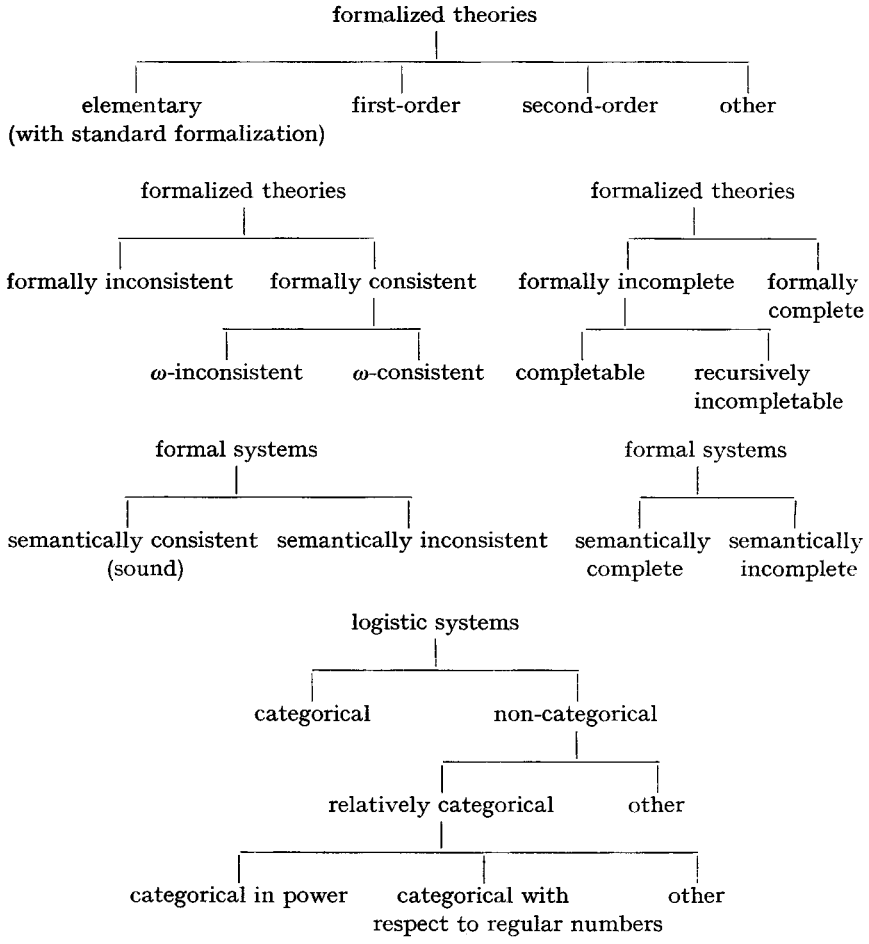
§ 7. The Metamathematics and Semantics of Set Theory

We are now ready to present in a systematic fashion the major results of the metamathematical and semantical investigations into the foundations of the various set theories treated in the preceding chapters. First, however, we shall summarize the generic relationships of some metatheoretical terms introduced in this chapter with the help of a few tree-diagrams. This should be helpful, in view of the fact that the terminological situation in this field is in general still rather confused, that the technical terms preferred in this book are not always the most customary ones, and that the meaning of most terms is by no means self-explanatory.

Among others, we made the following distinctions:



[1]) Ackermann 54.
[2]) Klaua 55.
[3]) Church 56, pp. 246 ff.



The first question that confronts us with respect to intuitive theories is whether these theories can and should be formalized. It is well known that one of the basic tenets of the intuitionistic approach is that no formalized theory can do full justice to intuitive (which is for them intuitionistic) mathematics or any of its subtheories. Heyting, who formalized intuitionistic propositional and functional logic (cf. Chapter IV, § 4), has always been very insistent that his own formalization should by no means been treated as an adequate representation of intuitionistic modes of argumentation, and that in general

> no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambi-

guity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof [1]).

It is therefore the more interesting that Beth has recently succeeded in proving [2]), in spite of this explicit disavowal of its originator, that Heyting's propositional and first-order functional calculi are complete with respect to intuitionistic arguments, in a very pregnant sense, hence that this part of intuitionistic logic at least can be exhaustively formalized and even axiomatized. A notion of (intuitionistic) truth can be satisfactorily defined for intuitionistic elementary logic under which the resulting formalized theory is complete and Heyting's logistic system semantically complete.

There are very many mathematicians, and even more so other scientists, who doubt it very much whether mathematical (and other) theories should be formalized even if they can be so in principle, suspecting that the fruits of formalization are not worth the effort. It is very difficult to discuss this issue *in abstracto*, but the present book will not have attained its aim unless the reader has become convinced that only through formalization can many important problems be given a formulation which makes it worthwhile to attempt their solution. The two outstanding examples discussed in this book are the formalization of intuitionistic logic just mentioned which enabled many mathematicians and logicians, who had no intuitionistic inclinations, to react to intuitionistic mathematics otherwise than by just shrugging their shoulders, and the formalization of Zermelo's concept of "definiteness" (cf. Chapter II, pp. 39 ff) [3]).

Unless a formalized theory is presented *ab initio* as an axiomatic system, so that its set of valid sentences is from the start identified with the set of sentences provable in this system, or — to put it positively — whenever the set of valid sentences of a theory is determined semantically rather than syntactically, the question arises whether this set is recursively enumerable, i.e. whether the theory is axiomatizable.

With regard to Skolem's arithmetic, the answer is of course negative:

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[1]) Heyting 56, p. 102.

[2]) Beth 56a.

[3]) For an interesting, balanced discussion of the merits and demerits of formalization, see Wang 55a; cf. also Dubarle 55.

the (recursive) non-axiomatizability of this theory is an immediate consequence of Gödel's incompletability theorem. Skolem's arithmetic possesses a well-determined semantic notion of validity [1]) and is complete under this notion, its completeness being a simple consequence of a precise truth-definition; but no axiomatic subtheory is semantically complete and all sufficiently rich axiomatic subtheories are even formally incomplete.

With regard to set theory, the situation seems to be different. There exists at present no formalized theory having the same status relative to (intuitive) Set Theory as Skolem's arithmetic has relative to (intuitive) Number Theory. There exists no generally accepted truth definition for a formalized set theory from which its completeness could be deduced, since there exists no universe of "regular" sets that could in any serious way be compared with the universe of regular natural numbers. Though no axiomatic set theory, which is adequate in the sense of allowing for the development of arithmetic on its basis, is completable, it is doubtful whether this result can be interpreted as implying the existence of some definite formalized set theory which is not axiomatizable. This point, seemingly of the greatest importance for the understanding of what set theory is about, has hardly been discussed in the literature, and the few existing discussions are contradictory and inconclusive. We shall later return to it.

Among the axiomatizable theories, those that are from the beginning based upon a finite number of axioms form an important subclass, though the exact significance of this feature is not quite clear, in all its generality. We know, for instance, that the classical propositional calculus is finitely axiomatizable, if a rule of substitution is admitted among its rules of inference, but is no longer so if *modus ponens* is its sole rule of inference. But here the explanation is relatively trivial. It is only with regard to *applied* functional calculi that the issue becomes both important and complex. Finite axiomatizability, for such calculi, means that the number of their *specific* axioms is finite, disregarding

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[1]) A brief characterization of Skolem's arithmetic: Its set of extra-logical constants contains just two ternary predicates (or else two binary operation symbols) for sum and product (and, in some formulations, also the binary predicate '$>$' and either or both of the individual constants '0' and '1'), its rules of formation are obvious, and validity is defined as truth under that interpretation that has the variables range over the regular natural numbers and provides the constants with their ordinary meaning.

the way in which their basic logic is formalized. That the schema of mathematical induction — standing for an infinity of axioms — in elementary arithmetic cannot be replaced by a finite number of axioms (formulated in the original vocabulary) has only recently been proved [2]). The proof utilizes the non-categoricalness of this theory, but there exists no simple general connection between finite axiomatizability and categoricalness. Among the set theories treated in previous chapters some, such as those of von Neumann, Bernays and Gödel, were finitely axiomatized from the start, others that had an infinite number of axioms in their original presentation, such as ***NF***, were later shown to be finitely axiomatizable, of others it could be proved that they were not finitely axiomatizable — this is the case for Zermelo's original theory, for ***T*** and ***Z*** [3]) — and for still other theories, finally, their status in this respect has not been settled so far — ***ML*** being one of the most important systems of this kind. With respect to ***Z*** an even stronger impossibility proof has recently been obtained [4]) when it was shown that not only is this system not finitely axiomatizable as such but that it cannot even be represented as a finite extension of Zermelo's original system, essentially that the axiom-schema of substitution is not replaceable by a finite number of axioms in this framework. For the case of set theories, some general results have been obtained. Wang, for instance, has been able to show [5]) that, in general, impredicative axiom-schemata of set-formation, such as the axiom-schema of comprehension and its various derivatives, cannot be replaced by a finite number of their instances, provided, of course, that the systems in question are consistent. (The whole problem of finite axiomatization makes sense only for such systems.)

We already saw above, p. 312, the import of finite axiomatizability in connection with undecidability proofs. This feature is equally important for other metamathematical investigations [6]). Finitizability of a given infinite axiom system is therefore always an interesting

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[1]) See, e.g., Tarski 56, p. 236.

[2]) Ryll-Nardzewski 53.

[3]) Wang 50d, 52.

[4]) Montague 56.

[5]) Wang 52; the earlier note, Wang 50d, dealing with this topic, contained an erroneous assertion that was corrected in the later paper.

[6]) Apparently also for consistency proofs; see Wang 52, p. 244.

problem and finitizing such a system often a decisive achievement. The question whether finitely axiomatizable systems have any real *epistemological* advantage over other axiomatic systems is more complex. So long as an axiomatic system is axiomatizable with the help of a finite number of axiom-schemata (in addition, perhaps, to a finite number of axioms proper), no serious advantage, from an epistemological point of view, can apparently be claimed. The situation is different with regard to those systems whose set of axioms, though recursive, is not known to be characterizable by a finite number of schemata. The axiomatizations guaranteed, for instance, by Craig's theorem (cf. p. 279, footnote 1) are of this kind, and Craig himself took pain to point out the limited impact of his theorem on the methodology of empirical sciences [1]).

The proof of the existence of non-axiomatizable formalized theories is doubtless one of the most important achievements of recent foundational research. The philosophical and epistemological implications of this result have not yet been exhaustively evaluated.

Among the axiomatizable theories, the distinction between those that are decidable and those that are not (and among them, especially those that are essentially undecidable) is of paramount importance, for reasons that were explained above. Let us add here a couple of theorems that exhibit the connection between formal consistency, formal completeness, axiomatizability and decidability. The first states that for a complete theory, axiomatizability is sufficient for decidability [2]), from which follows as an easy corollary that for a complete theory the conditions of undecidability, essential undecidability, and non-axiomatizability are equipollent. The second states that a theory is essentially undecidable if and only if it is consistent and none of its consistent and complete extensions with the same

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[1]) In "Replacement of auxiliary expressions", *Philos. Rev.* 56, 38–55, 1956; cf. also C. G. Hempel's paper "The theoretician's dilemma", *Concepts, theories, and the mind-body problem* (Minnesota Studies in the Philosophy of Science, vol. II), 37–98, 1958, especially pp. 76 ff, for a thorough discussion of the issue.

Of similar restricted epistemological value, though of considerable logical impact, is Kleene's result that axiomatizable theories with standard formalization can always be finitely axiomatized through enlarging their original vocabulary; see Kleene 52b. And the same holds for the enforcement of finite axiomatizability through adding (artificial and opaque) rules of inference to the underlying logic, shown in Hermes 51 to be always possible.

[2]) Kleene 43, p. 56; Janiczak 53.

vocabulary are axiomatizable [1]). Its proof relies on an argument used by Lindenbaum to prove his well-known theorem, according to which every consistent theory has a consistent and complete (but, of course, not necessarily axiomatizable) extension [2]).

Formally complete, axiomatizable theories being decidable, it is obviously important to determine whether a given axiomatic theory is complete. An interesting sufficient criterium for the formal completeness of a theory with standard formalization has recently been given by Vaught [3]). According to *Vaught's test*, such a theory is complete if it is categorical in a sufficiently high power [4]). As examples for axiomatic (or axiomatizable) theories whose completeness (and therefore decidability) has recently been proved, we mention the classical result of Tarski, according to which the elementary theory of real-closed ordered fields is complete [5]), and the result of A. Robinson, according to which the elementary theory of completely divisible ordered abelian groups, with at least two different elements, is complete [6]).

Let us conclude this section with a survey of results concerning consistency, absolute and relative, of set theories and their underlying logics. Many of these results were already mentioned before, scattered in Chapters II and III.

Starting with logic, we notice that the (formal and semantical) consistency of the classical propositional calculus can be easily established; a proof is given in any elementary textbook of formal logic. Only slightly more difficult is the consistency proof for the classical pure first-order functional calculus; it is based upon a translation of this calculus into the propositional calculus [7]). But

[1]) Tarski-Mostowski-Robinson 53, p. 14; cf. also Tarski 51.

[2]) Tarski-Mostowski-Robinson 53, pp. 15 ff.

[3]) Vaught 54. In the same paper, Vaught also proves that a theory with standard formalization is decidable if it is axiomatizable, categorical in some infinite power, and all its models are infinite. In the abstract, Henkin 55a, it is announced that finite axiomatizability and categoricalness in some infinite power also entail decidability, for theories of this kind.

For an earlier test, of restricted applicability, see Kustaanheimo 49.

[4]) For this notion, see above, p. 295; for a test for model-completeness (see footnote 3 on p. 290), see A. Robinson 56, pp. 16 ff.

[5]) Tarski 51.

[6]) A. Robinson 56, p. 77.

[7]) For these two proofs, see, e.g., Church 56, pp. 108–109 and 180 ff.

whereas the consistency proof of the propositional calculus yields immediately a decision procedure for provability, this is not the case with the consistency proof of the first-order functional calculus indeed; as was mentioned above (p. 310), no decision procedure exists for this calculus as a whole. The consistency of the second and higher order functional calculi can be proved by a similar method of translation, though not into the propositional calculus but rather into the *extended propositional calculus*, which contains also bindable propositional variables and is in its turn demonstrably consistent [1]). A very detailed proof of the consistency of the singulary functional calculus of order ω, for instance, has been given by Gentzen [2]), who likewise proved the consistency of arithmetic — using in the metatheory transfinite induction up to ε_0 as the only ("constructive") principle transcending the deductive means of arithmetic itself [3]).

We already mentioned the consistency proofs for the simple theory of types without an axiom of infinity [4]) and for the ramified theory of types without an axiom of reducibility [5]). These proofs were given both proof-theoretically for formal consistency and model-theoretically for semantic consistency. Since these systems are adequate for recursive arithmetic, the proofs require, in view of Gödel's corollary, a metatheory which is in some sense stronger than these systems themselves. But these type theories are not strong enough for the development of analysis. True, assuming the existence of some infinite universe, U, of its power-set, $\boldsymbol{C}U$, of $\boldsymbol{CC}U$, etc., simple type theory with an axiom of infinity can easily be shown to be semantically consistent, this hierarchy forming an obvious model for it. But one's belief in the existence of this intuitive model cannot be very strong — unless one happens to be an out-and-out Platonist — in view of the fact that the axiom-schema of comprehension allows for the existence of impredicative sets, something that must blur our view of the step-by-step construction of the hierarchy starting from the domain of individuals.

On the other hand, it seems that a reasonably convincing consistency proof is available for those systems which — like the systems of Wang

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[1]) Church 56, pp. 306–307; cf. also p. 290 and footnote 462, there.

[2]) Gentzen 36a.

[3]) Gentzen 36, 38a, 43; cf. Ackermann 40, Stenius 52.

[4]) See Chapter III, p. 178, footnote 3; cf. also Herbrand 30.

[5]) See Chapter III, p. 152, footnote 1.

and Lorenzen — extend the ramified type theory into the constructive transfinite, modifying it at the same time from a homogeneous to a cumulative one.

NF, in spite of the stratification condition in its axiom of comprehension, allows for the existence of self-membered classes, such as the universal class. The intuitive model of the simple type theory is therefore not available for ***NF*** [1]; in fact, no standard model is available for ***NF*** at all [2]. With the consistency of ***ML*** we shall deal below.

In order to deal systematically with the consistency problem of set theories, it might help to start with that fragment of ***Z*** which is composed of the axioms I–VI and IX, to the exclusion of the axioms (VII) of infinity and (VIII) of substitution. This fragment lies between what we called in Chapter II, p. 33, *general set theory*, from which it differs through the inclusion of the axiom (IX) of foundation, and $\bar{\boldsymbol{Z}}$ (roughly, Zermelo's original theory), from which it differs through the lack of the axiom of infinity. Following Wang-McNaughton [3], we denote this fragment by '$\boldsymbol{T}_1$'. Let $\boldsymbol{T}_2$ be defined as follows: it contains two sorts of variables, say lower-case and upper-case ones, and correspondingly an additional kind of atomic formulae of the form '$x \in X$'. Whereas identity for lower-case variables is defined by Definition IIb (Chapter II, p. 32), identity for the upper-case variables is defined by

$$X = Y =_{\mathrm{Df}} (\mathrm{A}x)\ (x \in X \equiv x \in Y).$$

Its axioms are those of $\boldsymbol{T}_1$ together with

$$(\mathrm{A}x)\ (\mathrm{A}y)\ [x = y \supset (\mathrm{A}X)\ (x \in X \equiv y \in X)],$$

in which we recognize the "second-order" version of the axiom of extensionality, and

$$(\ )\qquad (\mathrm{E}X)\ (\mathrm{A}x)\ [x \in X \equiv F(x)]$$

(where '$F(x)$' does not contain 'X' free), which is, of course, just the "second-order" axiom-schema of comprehension.

Let $\boldsymbol{T}_n$, for any finite n, contain n different sorts of variables etc., and result from $\boldsymbol{T}_{n-1}$ just as $\boldsymbol{T}_2$ results from $\boldsymbol{T}_1$. Tarski [4] proved that

[1] Wang-McNaughton 53, p. 24.
[2] Rosser-Wang 50; cf. Chapter III, p. 145.
[3] Wang-McNaughton 53, p. 27; cf. Chapter II, p. 124.
[4] Tarski 39.

$\boldsymbol{T}_n$ contains a normal truth-definition for $\boldsymbol{T}_{n-1}$. The consistency of $\boldsymbol{T}_n$ is therefore provable in $\boldsymbol{T}_{n+1}$ (though, of course, not in any $\boldsymbol{T}_m$ with $m \leqslant n$). The systems $\boldsymbol{T}_n$ form a monotonous sequence of growing strength, with no element of the sequence ever reaching the strength of $\overline{\boldsymbol{Z}}$ or $\boldsymbol{T}^*$ (p. 141). In view of their structure, $\boldsymbol{T}_1$ may be considered to be a *set theory for natural numbers* [1]), $\boldsymbol{T}_2$ a *set theory for real numbers* (i.e. for sets of natural numbers), and $\boldsymbol{T}_3$ a *set theory for real functions*.

Let $\boldsymbol{I}_n$ differ from $\boldsymbol{T}_n$ only in having its $n - 1$ axiom-schemata of comprehension provided with the additional restriction that '$F(x_i)$' ($i < n$) shall contain no variables of a type higher than $i + 1$. This restriction is not strong enough to make its axioms of comprehension predicative — since '$F(x_i)$' may still contain variables of type $i + 1$ itself — but reduces, so to speak, the degree of impredicativeness in comparison with the corresponding axioms of $\boldsymbol{T}_n$. (This is true only for $n > 2$; $\boldsymbol{I}_2$ and $\boldsymbol{T}_2$ are clearly the same system.) McNaughton [2]) showed that the consistency of $\boldsymbol{I}_n$ relative to $\boldsymbol{I}_{n-1}$ can be formalized in $\boldsymbol{T}_3$. Since the absolute consistency of $\boldsymbol{I}_2$ ($= \boldsymbol{T}_2$) is provable in $\boldsymbol{T}_3$ [3]), too, the absolute consistency of every $\boldsymbol{I}_n$ is provable in $\boldsymbol{T}_3$, a rather dramatic exhibition of the strength gained by an increase in the degree of impredicativeness.

Another no less dramatic exhibition of the strength (and thereby vulnerability) added through greater impredicativeness is given through Kemeny's success in constructing a normal truth-definition for $\boldsymbol{T}^*$ in $\overline{\boldsymbol{Z}}$ and thereby proving the consistency of $\boldsymbol{T}^*$ in $\overline{\boldsymbol{Z}}$ [4]), though the intuitive models of these two theories seem to be rather of equal status. We recall the intuitive models for $\overline{\boldsymbol{Z}}$ (above, p. 282) and for $\boldsymbol{T}^*$ (above, p. 323). If we let U in the model of $\boldsymbol{T}^*$ correspond to E in the model of $\overline{\boldsymbol{Z}}$, we clearly obtain a one-to-one correspondence between the models. $\overline{\boldsymbol{Z}}$'s greater strength is then due to the fact that in its axiom-schema of subsets, the variables of '$F(x)$' range over the whole universe, whereas $\boldsymbol{T}^*$'s axiom-schema of comprehension allows each variable of its '$F(x^i)$' to range only over a restricted universe [5]).

We shall not present here the proof for the consistency of each $\boldsymbol{I}_n$

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[1]) For the development of number theory within $\boldsymbol{T}_1$, see p. 118, footnote 2.

[2]) McNaughton 53.

[3]) This follows from the fact mentioned above that $\boldsymbol{T}_n$ contains a normal truth-definition of $\boldsymbol{T}_{n-1}$.

[4]) In his thesis; cf. Kemeny 48.

[5]) Cf. Wang-McNaughton 53, p. 350

relative to $\boldsymbol{I}_{n-1}$. Instead, we shall briefly describe a proof for the consistency of ***ML*** relative to ***NF***. This is of greater intrinsic interest, and in addition, ***ML*** results from ***NF***, as we recall, through the addition of a new sort of variables for classes (or so, at least, the transition may be described for certain purposes), the addition of an axiom of extensionality for classes and a new axiom of comprehension guaranteeing the existence of classes of elements fulfilling a given condition, hence in a way which is analogous to the way in which $\boldsymbol{I}_n$ results from $\boldsymbol{I}_{n-1}$. Indeed, from the assumption that ***NF*** is consistent it follows, by Löwenheim-Skolem, that it has a (non-standard) model in the domain of natural numbers. But then ***ML*** has a model in the domain of sets of natural numbers, i.e. in the domain of real numbers. Since such a model is generally considered to be trustworthy, ***ML*** is consistent. This proof is due to Wang [1]). McNaughton [2]) later showed that it can be formalized within $\boldsymbol{T}_3$.

In ***Z***, the consistency of $\overline{\boldsymbol{Z}}$ can be proved: With the help of the axiom of substitution the domain of the sets which form the intuitive model for $\overline{\boldsymbol{Z}}$ can easily be proved to exist in ***Z***. It follows that the consistency of ***Z*** relative to $\overline{\boldsymbol{Z}}$ cannot be proved in ***Z***, as otherwise we would have a proof of the (absolute) consistency of ***Z*** in ***Z***, contrary to Gödel's corollary.

The relationship between ***B*** and ***Z*** is in many respects similar to that between ***ML*** and ***NF***, but there exists one important difference. Replacing 'η' by 'ϵ' in Chapter II, pp. 112 ff, it can be shown that, under an appropriate formulation, ***B*** differs from ***Z*** only in having a new sort of variables (upper-case italics), an additional set of atomic formulae, the new sort of variables being allowed to flank 'ϵ' at its right (but not at its left), in having the axiom-schemata of subsets and substitution replaced by corresponding single axioms — replacing, for instance '$F(x)$' by '$x \epsilon X$' (and putting '(AX)' before the resulting formula) —, and finally in adding an axiom-schema of comprehension for classes of sets, the schema (B) of p. 112. (There it is stressed that (B) is derivable from a finite number of proper axioms, thereby ensuring the finitizability of ***B***.) The decisive difference between (B) and C_{ML} (on p. 146) does not lie in the seemingly different formulation of the displayed formulae — by changing 'η' to 'ϵ' in (B) and eliminating

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[1]) Wang 50b.

[2]) McNaughton 53.

the '$(Ez)x \in z$' clause through the use of the second kind of variables, these displayed formulae could be made to coincide completely — but in the fact that the '$F(x)$' in (B) is more restricted than the '$F(x)$' in C_{ML}: Bound upper-case variables are not allowed to occur in the condition of (B) but may occur in the corresponding condition of C_{ML}, causing ***B*** to be a *predicative extension* of ***Z*** (cf. p. 122), whereas ***ML*** is an *impredicative extension* of ***NF***. The impredicativity of the extension is, however, not a strong one. Indeed, Shoenfield [1]) has recently been able to prove both the consistency of ***B*** relative to ***Z*** and of ***ML*** relative to ***NF*** is recursive number theory, so that the proof is formalizable already in $\boldsymbol{T}_1$.

As might have been expected, the fact that ***B*** is only a predicative extension of ***Z*** makes for some shortcomings of ***B***. Mostowski [2]) has shown, for instance, that ***B*** runs into certain troubles over mathematical induction which are absent in ***ML***. But we are by now fully acquainted with the dialectics of maneuverability versus safety. Though the consistency proof of ***ML*** relative to ***NF*** is less convincing than one might have wished, it still *is* a consistency proof. It would certainly gain in its power of persuading mathematicians to pay more attention to it, if the consistency of ***NF*** relative to some of the systems with which they are more familiar, such as ***B*** or even ***Z***, could be proved [3]). This, however, has not been done so far, and there are many who cherish grave doubts about the feasibility of such a proof.

The converse problem has been treated and solved by Rosser [4]). $\boldsymbol{T}_1$ can be shown to be consistent relative to ***NF*** through a modelling process, $\overline{\boldsymbol{Z}}$ is consistent relative to ***NF*** plus an axiom postulating the

[1]) Shoenfield 54. Cf. Chapter III, p. 172, footnote 2.

[2]) Mostowski 51a. Wang 53, p. 403, makes the interesting, though not too surprising remark that this weakness of the predicative systems as against the systems which allow for impredicative class formations could perhaps be overcome by the use of modes of reasoning which transcend those customarily admitted in logistic systems but might still be safer than the use of impredicativities. He is able to show that in one particular case, at any rate, a single instance of the rule of infinite induction is able to replace impredicative class formation. As against the fact that this rule is not finitary stands the fact that a system which is ω-consistent to start with remains demonstrably (and trivially) consistent after a single use of this rule; and the ω-consistency of many systems can be shown through the use of an intuitive (though not necessarily constructive) model.

[3]) Cf. Quine 53, p. 100.

[4]) Rosser 56.

existence of a strictly Cantorian set (p. 145), and ***Z*** itself is consistent relative to the such strengthened ***NF*** plus an axiom of substitution for strictly Cantorian sets.

Orey, finally, was able to prove [1]) the consistency of Gödel's set theory relative to slightly strengthened ***ML*** and to show that in such strengthened ***ML*** the proof of a statement expressing the consistency of Gödel's system can be formalized. He also showed that (very much) strengthened versions of ***NF*** or ***T**** — strengthened, for instance, through adding an axiom postulating the existence of inaccessible numbers — contain models of Gödel's system. In a similar fashion, the consistency of ***Z*** relative to strengthened versions of ***NF*** can be shown.

Having dealt at some length with the trees in the forest, not to say jungle, of the various set theories, more specifically — in the last pages — with their consistency and strength (absolute and relative), let us summarize part of what we said, together with additional material adapted from recent publications by Wang, Mostowski, Firestone-Rosser, Rosser, Shepherdson, Shoenfield, Mendelson, Orey, and A. Lévy (and abstracts by Tarski, Vaught and Montague) [2]), in the form of hierarchies of set theories, as much as possible.

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[1]) Orey 56.

[2]) The relevant publications are: Wang 49, 50, 50a, 50d, 51, 51a, 52a, 53a, 53c, 55, Mostowski 49a, 51a, 53, 55, 56, Firestone-Rosser 49, Rosser-Wang 50, Kreisel 50, 53, 55, Shepherdson 51–53, McNaughton 53, Wang-McNaughton 53, Shoenfield 54, Kreisel-Wang 55, Rosser 55, Mendelson 56, Orey 56a, A. Lévy 58 (Tarski 55, 55a, Montague 56, Vaught 56).

The remainder of this section utilizes much of the material presented in A. Lévy 58. Since this thesis was submitted in May 1958, when the first four chapters of the present book were already printed, its many other contributions could no longer be taken into full considerations. Let us therefore only mention the two following important results: (1) It contains a proof of the consistency of the set theory recently developed in Ackermann 56 (which — as remarked in Chapter II, p. 23, footnote 5 — reached the authors after the second chapter was sent to the printer) relative to the system denoted below by '***Z***$\sharp$', a proof which holds even if Ackermann's system is strengthened by suitable axioms of foundation and choice and a suitable version of the continuum hypothesis. Strengthening Ackermann's system by an axiom of substitution analogous to Gödel's C4, one arrives at an extremely strong system. (2) Very strong axiom-schemata of comprehension are formulated, for both ***Z*** and ***B***, whose addition to these theories pushes the universe(s) of the models for the enlarged systems high up in the intuitive model of Cantorian set theory, in a certain sense maximally so, without reintroducing the antinomies.

First, let us distinguish between set theories for which intuitive models are available and those for which they are not. Many mathematicians and logicians would, for philosophical reasons, reject set theories with no intuitive models from the beginning. To this category belong, as we recall, Quine's ***NF*** and ***ML*** (and related theories). Simple type theory without an axiom of infinity and general set theory possess very simple intuitive models, and their consistency may be regarded as definitely established beyond all reasonable doubt. They may still be rejected by thinkers tending to intuitionism, not because they are suspect of inconsistency but because they, or their underlying logic, do not reach up to the intuitionistic standards in other respects. But the intuitive models of the theories described in Chapter II and of some related ones are no longer constructive and therefore open to doubt in proportion to how far we have to go in the intuitive hierarchy of sets in order to exhibit these models.

In preparation for the first hierarchy of systems, let us introduce by transfinite induction a sequence of sets starting with the null-set and in which every other set is the union of the power-sets of all its preceding sets; denoting the ξ-th set in this sequence by $\psi(\xi)$, we define

$$\psi(\xi) =_{\mathrm{Df}} \bigcup_{\eta<\xi} C\psi(\eta).$$

Consider now the following hierarchy:

1. General set theory (with the axiom of foundation), $\boldsymbol{T}_1$ (or $\boldsymbol{I}_1$).
2. Its predicative extension, $\boldsymbol{T}_2$ (or $\boldsymbol{I}_2$).
3. Zermelo's original theory (with a slightly strengthened axiom of infinity '$(\mathrm{E}z)\{O \in z \,\&\, (\mathrm{A}x)\,(\mathrm{A}y)[(x \in z \,\&\, y \in z) \supset x \cup \{y\} \in z]\}$') which, for the sake of simplicity, we shall also denote by '$\bar{\boldsymbol{Z}}$', though it must, for other purposes, be distinguished from the system we usually denote by this symbol.
4. Its predicative extension, to be denoted by '$\bar{\boldsymbol{B}}$'.
5. $\bar{\boldsymbol{Z}}$ plus the axiom of substitution, $\boldsymbol{Z}$.
6. Its predicative extension, $\boldsymbol{B}$.
7. $\boldsymbol{Z}$ plus an axiom postulating the existence of at least one non-trivial (i.e. greater than ω) inaccessible number, to be denoted by '$\boldsymbol{Z}^{\sharp}$'.
8. $\boldsymbol{B}$ plus such an axiom, $\boldsymbol{B}^{\sharp}$.
9. $\boldsymbol{Z}$ plus an axiom postulating the existence of at least two non-trivial inaccessible numbers, $\boldsymbol{Z}^{\sharp\sharp}$.

It can now be shown that the axioms of these 9 systems hold true when their only extra-logical predicate 'ϵ' is interpreted as designating the membership relation and their universe (respectively universes, for the systems containing two sorts of variables, i.e. systems 2, 4, 6, and 8) are taken to be, in order:

1. $\psi(\omega)$.
2. $\psi(\omega)$ and $\boldsymbol{C}\psi(\omega)$.
3. $\psi(\omega + \omega)$.
4. $\psi(\omega + \omega)$ and $\boldsymbol{C}\psi(\omega + \omega)$.
5. $\psi(\xi_1)$, where ξ_1 is the first non-trivial inaccessible number.
6. $\psi(\xi_1)$ and $\boldsymbol{C}\psi(\xi_1)$.
7. $\psi(\xi_2)$, where ξ_2 is the second non-trivial inaccessible number.
8. $\psi(\xi_2)$ and $\boldsymbol{C}\psi(\xi_2)$.
9. $\psi(\xi_3)$, where ξ_3 is the third non-trivial inaccessible number.

In addition, the existence of the set(s) forming the universe(s) of the n-th theory in this hierarchy is in each case provable already in the $n+1$-st theory (and that of the ninth theory in a theory resulting from $\boldsymbol{Z}$ by adding an axiom postulating the existence of at least $\aleph_0$ inaccessible numbers).

In preparation for the next hierarchy, let us remark that each of the theories to be considered contains arithmetic, hence the arithmetization of its own syntax. For any given theory of this kind, T, let $\mathrm{Con}(T)$ be one of the sentences expressing that T is consistent. To be more specific, let $P(x, y)$ be the arithmetical formula of T expressing that x is the Gödel number of a proof whose endformula has the Gödel number y; then $\mathrm{Con}(T)$ is the sentence: $\sim(\mathrm{E}x)P(x, a)$, where a is the Gödel number of '$0 = S(0)$'. According to Gödel's corollary, this sentence is not provable in T, provided T is consistent. Let us also remark that the principle of mathematical induction to its full extension — as already implied above, p. 327 — is independent of $\boldsymbol{B}$ (and a fortiori of any of its subtheories). We are now ready for the second hierarchy:

1. General set theory, $\boldsymbol{T}_1$.
2. $\boldsymbol{T}_1$ plus $\mathrm{Con}(\boldsymbol{T}_1)$, $\boldsymbol{T}_1'$.
3. The predicative extension of $\boldsymbol{T}_1$ plus the axiom of mathematical induction, $\boldsymbol{T}_2^*$.
4. $\bar{\boldsymbol{Z}}$.
5. The predicative extension of $\bar{\boldsymbol{Z}}$ plus mathematical induction, $\bar{\boldsymbol{B}}^*$.

6. $\boldsymbol{Z}$.

7. The predicative extension of $\boldsymbol{Z}$ plus mathematical induction, $\boldsymbol{B}^*$.

8. $\boldsymbol{Z}$ plus an axiom postulating the existence of at least one non-trivial inaccessible number, $\boldsymbol{Z}^\sharp$ [1]).

Each of the last seven of these theories contains arithmetical theorems stating the consistency of the preceding theories in the hierarchy. (This is trivial with regard to $\boldsymbol{T}_1'$ which contains Con($\boldsymbol{T}$) as an axiom but far from being so for the remaining cases.) Each of the last six of these theories contains a normal truth definition for all preceding theories. Therefore, if these theories are consistent, in none of them may its own consistency relative to a preceding theory be provable. Relative consistency proofs by T^*-translation (cf. above, p. 284) are therefore impossible for the theories in this hierarchy. Altogether, if a theory T' is strong enough to contain a consistency proof for some (weaker) theory T, the consistency proof of T' relative to T must be a tough problem. If, on the other hand, the consistency proof of T' relative to T can be given in some weak theory then T' cannot be that much stronger than T as to contain an absolute consistency proof for T.

We recall, for instance, that the consistency of $\boldsymbol{B}$ relative to $\boldsymbol{Z}$ is provable in recursive arithmetic, hence certainly in $\boldsymbol{B}$ itself. Therefore, the additional strength of $\boldsymbol{B}$ over $\boldsymbol{Z}$ is not enough to allow for the establishment of the absolute consistency of $\boldsymbol{Z}$ in $\boldsymbol{B}$ (though all the sets needed for the model of $\boldsymbol{Z}$ can be proved to exist in $\boldsymbol{B}$). $\boldsymbol{B}^*$, on the other hand, is strong enough to allow for a consistency proof of $\boldsymbol{Z}$, but its relative additional strength over that of $\boldsymbol{Z}$ is now too great to allow for the establishment of a relative consistency proof in any theory not stronger than $\boldsymbol{B}^*$ itself. Notice what difference is made by the full-strength principle of mathematical induction.

That impredicative extensions of set theories are so much stronger than their predicative extensions — where the additional impredicativity is obtained through adding a full-strength principle of mathe-

[1]) Between $\boldsymbol{B}^*$ and $\boldsymbol{Z}^\sharp$, the system resulting from $\boldsymbol{B}$ through adding an axiom to the effect that $\boldsymbol{B}$ possesses a *proper complete model* — this axiom is the negation of Shepherdson's axiom G (cf. Shepherdson $51\text{–}53_{III}$, pp. 147 ff) — which fulfils also condition (b) on p. 183 of Shepherdson $51\text{–}53_{I}$, can be inserted. Existence of a proper complete model is equivalent to the formula $\Delta(\epsilon)$ of Mostowski 49a, and equivalent also to the existence of a standard model in the sense of Rosser-Wang 50 (cf. Shepherdson $51\text{–}53_{II}$, pp. 165 ff), and is a stronger condition than mere consistency.

matical induction, an axiom postulating the existence of inaccessible numbers, such axiom-schemata as an axiom-schema of substitution, or a "second-order" axiom-schema of comprehension — can also be exhibited in the following interesting way: According to an important theorem of Mostowski [1]), predicative extensions contain no new theorems in the old vocabulary of the basic theory. Should ϕ be a theorem of $\boldsymbol{B}$, for instance, which is not provable in $\boldsymbol{Z}$, then ϕ is not a formula of $\boldsymbol{Z}$ either. This is not the case with regard to impredicative extensions. These may well contain new theorems in the old vocabulary. Indeed, $\boldsymbol{B}^*$, $\boldsymbol{Z}^\sharp$, and the theory resulting from $\boldsymbol{B}$ through the addition of a second-order axiom-schema of comprehension contain each of them the theorem $\mathrm{Con}(\boldsymbol{Z})$ which is a formula of $\boldsymbol{Z}$ though, of course, not provable in $\boldsymbol{Z}$ (so we hope). Predicative extensions of set theories are of no help in reducing the number of (natural or real) number-theoretical sentences which are formulable in the basic theory and undecidable in it, impredicative extensions do so.

Every consistency proof of a theory T' relative to one of its subtheories T yields immediately a compatibility proof of the additional facilities of T' relative to the facilities of T, and since compatibility of an axiom means independence of its negation, we also obtain various independence proofs. For the specific results obtained in these regards, we refer the reader to the literature mentioned above [2]).

§ 8. Philosophical remarks

On many occasions, when our discussions reached a certain ticklish, "philosophical" stage, they were disrupted by the remark that the issue will be taken up "later on". It is now high time that we pay our accumulated debts. Not that the reader is likely to rise, after the reading of this last section, with the feeling that all his problems have now found their final solution. Very few judgments will be passed here, and the only progress that might possibly be made will consist in formulating some of these problems and the various views on them in a more systematic fashion which could contribute to a better understanding.

Our first problem regards the ontological status of sets-not of this

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[1]) Mostowski 51a, footnote 6; cf. Chapter II, p. 121, footnote 2.

[2]) On p. 328, footnote 2.

or the other set, but of sets in general. Since sets, as ordinarily understood, are what philosophers call *universals*, our present problem is part of the well-known and amply discussed classical problem of the *ontological status of the universals*. The three main traditional answers to the general problem of universals, stemming from medieval discussions, are known as *realism*, *nominalism*, and *conceptualism*. We shall not deal here with these lines of thought in their traditional version [1]) but only with their modern counterparts, known as *Platonism* [2]), *neo-nominalism*, and *neo-conceptualism* (though we shall mostly omit the prefix 'neo-' since we shall have no opportunity to deal with the older versions). In addition, we shall deal with a fourth attitude which regards the whole problem of the ontological status of universals in general and of sets in particular as a metaphysical pseudo-problem.

A *Platonist* is convinced that corresponding to each well-defined (monadic) condition there exists, in general, a set, or class, which comprises all and only those entities that fulfil this condition and which is an entity on its own right of an ontological status similar to that of its members. Were it not for the antinomies, the calculus that would best represent his intuitions would be the ideal calculus ***K*** (p. 137) or something of this kind, whose main feature is an unrestricted axiom-schema of comprehension. Things being as they are, he reluctantly admits that his vision of what constitutes a well-defined condition might be slightly blurred and declares himself ready to accept certain restrictions in the use of the axiom-schema of comprehension, temporarily working with a type theory or a set theory of a Zermelian brand, but hoping that sooner or later someone will be able to show that much less radical interventions will do the trick. Of course, some Platonists may convince themselves, or become convinced by others, that the objects of the world they live in are *really* stratified into types and orders and, as a consequence, accept type theory not as an *ad hoc* advice but as an expression of hard fact.

A *neo-nominalist* declares himself unable to understand what other

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[1]) For an able, modernized description of these classical views, see Stegmüller 56–57; these papers present equally well some of the contemporary views.

[2]) This term, in the present sense, seems to have been used first in Bernays 35. Whether Plato was, or even would have been, a Platonist is a moot question. Cf., e.g., P. Henle's review of Goodman 51 in *J.S.L.* **17**, 130–133, 1952.

people mean when they are talking about sets, unless he is able to interpret their talk as a *façon de parler*. The only language he professes to understand is a calculus of individuals, constructed as an applied first-order functional calculus. With regard to many locutions used in scientific or ordinary discourse, which *prima facie* involve sets, he has little trouble in translating them adequately into his restricted language. This is the case, for instance, for such a common statement as 'the set of the a's is a subset of the set of the b's', which he renders as 'for all x, if x is a, x is b'. With regard to other locutions and devices he has greater trouble. The quite common kind of concept formation by which the ancestral of a given asymmetric and intransitive relation is formed — the resulting relation then being transitive — is easily formulable in set theory. Assuming, e.g., that the relation is-greater-by-one-than in the domain of integers is available (but not yet is-greater-than), one defines: x is-greater-than y if and only if x is-different-from y and x belongs to *all sets* which contain y and all integers greater-by-one-than any of its members. The corresponding concept formation within a calculus of individuals calls, in certain cases, for a considerable amount of ingenuity and seems to be hardly feasible in other cases [1]). It is well known that expressions of the kind "The cardinal number of the set a is 17" (or "... at most 17", or "... at least 17", or "... between 12 and 21" etc.) can be readily rendered in first-order functional calculus with equality. But a sentence like "There are more cats than dogs" causes again grave difficulties, and though these can be overcome in this and any other particular case, no general method is available for a nominalistic rendering of "There are more a's than b's" [2]).

The difficulties in rephrasing *all* of classical mathematics in nominalistic terms seem, and probably are, insurmountable. Inasmuch as Cantorian set theory, the theory of transfinite cardinals, and similar theories are concerned, nominalists are only too happy to get rid of them and will regard the "loss" incurred with equanimity. But they have a healthy respect for those parts of mathematics which are used in the sciences and many would rather renounce their philosophic intuitions than curtail the useful mathematics. The only serious ways out of their predicament are either to go on using all the useful parts of

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[1]) Cf. Goodman-Quine 47, Goodman 51, 56, Quine 53.

[2]) See Goodman 51, pp. 37 ff.

mathematics in the hope — admittedly not too well founded [1]) — that one day someone will produce an adequate rephrasing in nominalistic terms, or else to declare that all higher mathematics is an uninterpreted calculus which remains manageable despite its lack of interpretation through the fact that its syntax is formulated, or formulable, in a well-understood nominalistic metalanguage [2]). How exactly an uninterpreted (and directly uninterpretable) calculus is able to perform its useful function of mediating between interpreted empirical statements is an issue that is still far from being definitely clarified, in spite of the great efforts put into this task by many philosophers of science [3]). We recognize here a relationship with the formalistic (Hilbertian) approach which regards a certain part of mathematics — essentially recursive number theory — as being interpretable and the remainder as an uninterpreted calculus useful as a means of transformation of meaningful statements into other meaningful statements and compares this status of the "ideal" parts of mathematics to the status of the "ideal" points in affine geometry.

It is only one step from here to the adoptation of an "As-if" philosophy, and Henkin [4]) intimates that a finitistic nominalist, i.e. one who believes that the universe, which for him is always just one homogenous domain of individuals — whatever these individuals may be —, comprises only finitely many elements, could very well assume the existence of infinitely many objects as a useful pretense (the older word was 'fiction'). He sees, of course, that as soon as one is ready to pretend one might as well pretend that there are universals and use a full-fledged Platonistic language — while still denying that one thereby accepts the ontological commitments usually connected with such languages — but feels that there is some difference between these two pretenses, a difference which makes it easier for a conscientious nominalist to accept the first than the second pretense; Henkin admits that he knows of no objective criterion for this distinction. He is certainly right that this kind of behavior, using linguistic forms without ac-

[1]) For reasons why it is hopeless to find an interpretation for an axiom of infinity which would be palatable for a finitistic nominalist, see Henkin 53a, p. 27.

[2]) Cf. Goodman-Quine 47.

[3]) For a thorough, recent discussion of a closely related topic, namely the status of theoretical terms in empirical science, see Carnap 56 and Hempel's paper mentioned above, p. 321, footnote 1.

[4]) Henkin 53a, p. 28.

cepting the conjugate ontological commitments, does look somewhat frivolous and is therefore in need of further clarification [1]).

There are authors who are attracted neither by the luscious jungle flora of Platonism nor by the ascetic desert landscape of neo-nominalism. They prefer to live in the well-designed and perspicuous orchards of *neo-conceptualism*. They claim to understand what sets are, though the metaphor they prefer is that of *constructing* (or *inventing*) rather than of *singling out* (or *discovering*), which is the one cherished by the Platonists, these metaphors replacing the older antithesis of *existence in the mind* versus *existence in some outside (real or ideal) world*. They are ready to admit that any well-determined and perspicuous condition indeed determines a corresponding set — since they are able in this case to "construct" this set out of a stock of sets whose existence is either intuitively obvious or which have been constructed previously — but they are not ready to accept axioms or theorems that would force them to admit the existence of sets which are not constructively characterizable [2]). Therefore, they do not accept sets that correspond to impredicative conditions (unless, of course, these conditions are demonstrably equivalent to predicative ones) and deny the validity of Cantor's theorem in its naive, absolute interpretation as endowing the power-set of a given set with a higher cardinal than has the given set itself. Absolute non-denumerability is declared to be void of sense, though an infinite set may not be enumerable with certain given means.

A nominalistic interpreted set theory, with 'ϵ' interpreted as 'is-a-member-of', is, of course, a *contradictio in adiecto*. But we already mentioned that some nominalists are ready to use set theory as an uninterpreted calculus fulfilling transformational functions. Both Platonists and conceptualists insist that set theory (and mathematics in general) must be interpretable and understood as such and have no use for uninterpretable calculi. They differ in their conception of intelligibility.

It goes without saying that each of these broad philosophical views

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[1]) See Carnap 50a, 56, Alston 58, Issman 58. It may be assumed that the forthcoming volume, The philosophy of Rudolf Carnap, *The Library of Living Philosophers*, will make some substantial contribution towards this much-needed clarification.

[2]) For a discussion of this point, as well as of the whole issue treated in this subsection, see Beth 56, pp. 41 ff.

splits into many specific ones, that their borders are blurred, and that it will often be very difficult to pin some author down to one of them. Logicism is usually regarded as one brand of Platonism, but Russell himself, during his 60 years of philosophic activity, expressed many ideas which were conceptualistic and even nominalistic. Ramified type theory has a definite conceptualistic flavor, but the axiom of reducibility is obviously Platonistic. When he professed a *no-class theory*, this was understood by many — most noticeably by Vienna Circle's Hans Hahn [1]) in the early thirties, but probably also by Russell himself, for a time — as a strictly nominalistic continuation of the use of Occam's razor. (This was, however, definitely a misunderstanding, partly created by the ambiguity in Russell's use of the term 'propositional function' for 'open formula' and 'attribute' simultaneously. Russell indeed showed how to eliminate classes in favor of "propositional functions", but these functions were just attributes (properties or relations), hence at least as "universal" as classes; Russell, due to his ambiguous usage, deceived himself in thinking that they were linguistic forms [2]).) Gödel is now usually regarded to be a Platonist, but his first publications were strongly influenced by the Hilbert school and even by Skolem's still more radically conceptualistic thinking. His postulate of constructibility (p. 121) is clearly conceptualistic and has been hailed and accepted as such by conceptualists, but Gödel himself refuses to regard it as a true set-theoretical statement. Hilbert is the father of modern formalism, but his metamathematics is strongly conceptualistic and the talk about the "ideal" nature of most higher mathematical notions is far from being unambiguously classifiable into any of the standard views. Lorenzen's operationism must be dubbed a blend between conceptualism and nominalism of the "As-if" brand, but this characterization is of only little help in revealing the idiosyncrasies of his approach. Quine, starting as a logicist, has for many years tried to uphold a nominalistic position but he feels now that conceptualism is a position into which he can lapse when tired of his quixotic attempts at nominalistic reconstruction, while allaying "his puritanic conscience with the reflection that he has not quite taken to eating lotus with the platonists" [3]). Tarski's first publications exhibited an attitude, derived from Leśniewski, which he character-

[1]) *Hahn 7, 9, 10.

[2]) Cf. Quine 53, pp. 122–123.

[3]) *Ibid.*, p. 129.

ized as *intuitionistic formalism*, but this is no more his present attitude [1]). Whereas he formerly had troubles in justifying operating with infinite sets of sentences, he now operates, apparently with few pangs of conscience, with languages whose set of individual constants is of any cardinality.

It would be easy, far too easy, to continue in this vein. There are very few contemporary logicians and mathematicians who consistently and unflinchingly have adhered all their lifetime to one philosophic view. Among the exceptions we may count Brouwer who has been a whole-hearted and uncompromising conceptualist all the time (though this attitude is occasionally bracketed in some of his "classical" contributions to topology), Church who has always professed a straightforward, though never dogmatic, Platonism, and Goodman who so far has not yielded to the conceptualist temptation and continues to adhere to a stead-fast extreme nominalism, which if anything is growing more radical in time. It must be noted, however, that his nominalism is of a special brand which has very little in common with classical nominalism. It is what we might call purely *syntactical nominalism*, insisting that the only legitimate language form is a first-order functional calculus but putting no restrictions, at least no official ones, on the ontological status of the individuals themselves which, for all he cares, might even be intimations of immortality, numbers, or sets, which would, however, be rather "sets" since such sets could not be said to contain members. To put it in slogan form: Goodman has no objections against sets, he is only unable to understand sets-of [2]).

Most authors who occupied themselves with the foundations of mathematics have exhibited a curious unsteadiness in matters philosophical. It was only natural for them to ascribe these changes of mind to their increasing maturity of thinking and to regard their later positions as better justified than their earlier ones, in whatever direction the shift might have gone.

It is understandable, on the other hand, that some thinkers should have seen in these vagaries a confirmation of their view that the three major ontological conceptions treated above should all of them be

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[1]) Cf., e.g., Tarski 56, p. 62.

[2]) For the clearest description of this brand of nominalism and for a very able defense of its many unusual contentions against various objections, see Goodman 56.

objectively irrelevant to the foundations problem, whatever those who upheld these conceptions thought about the matter and however strong their feelings were in this respect. Set theories, so these authors came to think, should not be judged by their ontologies (in Quine's sense) but by their fruits. Whether there are impredicative sets or not is not a matter to be decided by theoretical arguments nor a matter of (irrational?) belief based upon intuition or conscience. The prevalent opinions to the contrary are caused by a fusion of, and confusion between, two different questions: the one whether certain existential sentences can be proved, or disproved, or shown to be undecidable, *within a given theory*, the other whether this theory as a whole should be accepted. Whether the existence of a set which is the (union sum-set) of three given sets is provable in $\boldsymbol{Z}$ is a serious question though easily answerable in the affirmative, as we know. Whether the non-existence of a non-trivial inaccessible number is provable in $\boldsymbol{Z}$ is an even more serious question which is so difficult that we don't know the answer. The same question with respect to $\boldsymbol{Z}^\sharp$ is trivially answerable in the negative. For still other theories the answer is affirmative, trivially or deeply so. Whether $\boldsymbol{Z}$ or $\boldsymbol{B}$ or $\boldsymbol{Z}^\sharp$ or $\boldsymbol{T}^*$ or $\boldsymbol{ML}$ or Σ or what have you should be accepted is another very serious question but of an entirely different kind. It is a matter of practical decision, based upon such (theoretical) considerations as likelihood of being consistent, ease of maneuverability, effectiveness in deriving classical analysis, teachability, perhaps possession of standard models, etc. It is by fusing these two questions that such pseudo-problems as whether there exist non-denumerable sets (as such, absolutely, not within a given theory) are posed, leading either to futile pseudo-theoretical discussions or to the feeling that such a question is answerable only through an appeal to intuition and philosophical conscience, on the basis of which the Platonist would anwer this question with a clear 'yes', the conceptualist and nominalist with an equally clear 'no', though out of entirely different intuitions.

The foremost exponent of this fourth, anti-ontological view is Carnap. In one of his later formulations [1]), he coined the terms *internal* and *external existence questions* for the two kinds of questions mentioned above, though he did not directly apply this distinction to the foundations of set theory. This application does look to us, however, rather

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[1]) See Carnap 50a.

straightforward and we strongly hope not to have misrepresented Carnap's view on this point.

This view is not without its own difficulties. We shall not discuss them here. Let us only stress that the presentation given here probably exaggerates the degree of disagreement between such authors as Quine and Carnap. Though Quine is used to say that by accepting a certain theory you have taken upon yourself certain absolute ontological commitments and Carnap denies just this, it is not yet clear to what degree this clash is not wholly or mostly verbal [1]).

Among the neo-conceptualists, we already had opportunity (Chapter III, p. 177) to mention those who reject not only impredicative concept-formations but the more extensive class of indefinite (in Carnap's sense) concept-formations, who reject — to formulate it metalinguistically — languages with unlimited quantification. These authors, among whom may be reckoned Poincaré, Brouwer, Wittgenstein, Kaufmann, Skolem and Goodstein, arrive at their rejection of these transfinite operations from the observation that there exists no decision procedure for the truth of quantified statements. Identifying meaningfulness with effective verifiability [2]), they immediately arrive at the conclusion that sentences containing unlimited quantifiers are in general meaningless.

Though this position, *qua* philosophical attitude, is highly questionable — the major counter-argument being that it would cripple mathematics just as the parallel view concerning empirical statements would cripple empirical science —, theories complying with it have, of course, their attractions. An arithmetic, for instance, that starts with relations (or operations) which are effectively decidable in each specific instance and proscribes the use of unlimited quantifiers in further concept-formations, remains intuitive all the way and is one of the safest and least doubt-ridden theories dealing with an infinite universe. It is understandable that Hilbert should have wanted to prove in this highly intuitive *recursive number theory* that mathematics is formally consistent. Skolem was able to develop a great part of classical arithmetic in this theory and Gödel succeeded in showing

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[1]) Cf. the last sections of Carnap 50a and Quine 53, Essay II (p. 46), respectively.

[2]) The corresponding identification in respect of empirical statements stems from Peirce and played, as the *verifiability criterion of meaning*, a central role in the early stages of logical empiricism. For the story, see, e.g., *Carnap 18.

that it suffices for the arithmetization of the elementary syntax of any formal system [1]).

In definite languages, in spite of the fact that classical propositional logic holds in them, those uses of the principle of the excluded middle to which intuitionists object and which are partly responsible for the antinomies are obviated in that they just cannot be formulated. Unrestricted generality is expressible by means of free variables, but unrestricted existentiality is not expressible at all: asserting '$F(x)$' is to assert that all x are F, but asserting '$\sim F(x)$' is not to assert that not all x are F but rather that all x are not-F or that no x are F.

That the ban on ordinary (unlimited) quantifiers does not have those grave restrictive effects which one might have expected can be illustrated by the following rather trivial example. Assuming that the binary predicate 'D' ("divides") has already been defined in some theory of natural numbers, one would normally define the singular predicate 'P' ("is-prime") in something like the following fashion:

$$P(x) =_{\mathrm{Df}} x > 1 \;\&\; (\mathrm{A}y)\,[D(y, x) \equiv (y = 1 \vee y = x)].$$

Simply replacing '$(\mathrm{A}y)$' with '$(\mathrm{A}y)_{<x}$' — read: 'for all y from 0 up to and including x' — will now turn the trick. In general, whenever a decidable attribute is introduced, one only needs to find some upper bound of the numbers involved in order to be able to replace the unlimited quantifiers by limited ones.

It has therefore been proposed [2]) to see in a definite language, such as Language I of Carnap, the realization "in a certain sense" of the more radical among the conceptualistic tendencies, sometimes called 'finitary' or 'constructivist'. While such authors as Skolem or Goodstein would probably agree to such a formulation of their views, the intuitionists would not, though perhaps for no other reason than that no formalization adequately expresses their intuitions.

Lorenzen, on the other hand, while he strongly rejects impredicative concept-formations, most definitely accepts unlimited quantification [3]), refusing to be hampered by the verifiability criterion. His philosophy is not conceptualistic — sets for him are nothing but propositional forms [4]), conditions with free variables, and not, as with a normal

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[1]) In Goodstein 57, this theory has found its authoritative textbook. Nothing is presupposed, not even propositional calculus. Cf. already Goodstein 54; for the philosophy behind it, see Goodstein 52.

[2]) See Carnap 37, p. 46.

[3]) Lorenzen 55, p. 6.

[4]) Or rather, they are obtained by abstraction from equipollent propositional forms, the same set corresponding to all equipollent forms.

conceptualist, the extra-linguistic entities corresponding to these forms. Nor is Lorenzen a syntactical nominalist and least of all a Platonist. But he is not a Carnapian either. Mathematics for him is not an uninterpreted linguistic framework to be judged by its properties of fruitfulness etc., but is an interpreted theory of schematically operating with uninterpreted calculi.

In spite of many divergencies, Curry's philosophy of mathematics [1]) is most closely related to that of Carnap. Like him, he rejects any ontological commitments [2]) and stresses *acceptability* [3]) as the criterion by which mathematical theories should be judged. He calls his view *empirical formalism* to distinguish it from Hilbert's version of formalism, from which it diverges indeed considerably; *pragmatical formalism* would probably be a better label. Curry's insistence that the formalist definition of mathematics (as he gives it) requires no philosophical presuppositions and that philosophical differences should be transferred rather to the level of acceptability squares well with Carnap's present views, and his distinction between discussions around the truth of some given mathematical statement within a given system and that of the acceptability of the system as a whole is probably equivalent to that between internal and external existence questions of Carnap.

Curry goes on to deflate the importance of provable consistency for acceptability, in contradistinction to Hilbertian formalism. This difference in attitude is admittedly only a matter of degree, since Hilbert himself did not see in consistency a sufficient condition for acceptability [4]). And intuitive evidence is, in Curry's view, a luxury which mathematics can easily afford to forego. "So far as acceptability for physics is concerned, analysis has no more need for a consistency proof than it has of intuitive evidence" [5]).

[1]) Curry's views underwent many changes with the years. In addition, many of his publications appeared — due to World War II — years after they were written, sometimes after the appearance of later compositions. This, and frequent changes of terminology, tends to blur the evaluation of Curry's contributions to the foundations of mathematics. The present passage is based mostly on Curry 51 (originally written in 1939), which was later condensed in Curry 54.

[2]) Curry 51, p. 31.

[3]) *Ibid.*, pp. 59 ff.

[4]) Cf. Hilbert 25, p. 163.

[5]) Curry 51, p. 62.

Curry's final plea for tolerance in matters of acceptability [1]) mirrors, probably on purpose, Carnap's famous tolerance principle [2]). Any author who, for reasons of intuitive convictions, insists that only mathematical systems of a certain kind have a *raison d'être* would do well to ponder once more whether his intolerance does not hamper the progress of science rather than channel it into the only promising road. Whereas constructibility could well be a necessary condition for the acceptability of a mathematical theory for certain purposes, say for metamathematics or for electronic computation, so that *theories of the constructible* — to use a confrontation recently made by Heyting [3]) — deserve to be studied by mathematicians of any philosophical conviction and has indeed been studied by authors with strongly divergent convictions as well as by authors with no philosophical convictions, the claim that the only legitimate mathematics is *constructible mathematics* has little chance of convincing anybody who does not share the specific convictions of the intuitionists.

We have had no intention to present here a summary of all current philosophies of mathematics. A few more remarks on this topic are however appropriate.

We had not mentioned at all that conception of mathematics which sees in it an empirical science, distinguished at most in degree from other empirical sciences. We had not done it so far, because we cannot imagine what justification there might be for the belief that "the source and ultimate *raison d'être* of the notion of number, both natural and real, is experience and practical applicability" [4]), though this is the belief of Mostowski, and similar formulations are to be found in many other publications, starting with John Stuart Mill. Unless, of course, by this formulation nothing beyond the trivial view is meant that experience has led humanity to develop mathematics. This trivialization is however very unlikely, since it is hard to see how from this interpretation one can "draw the conclusion that there exists only one arithmetic of natural numbers, one arithmetic of real numbers and one theory of sets" [5]). But in what other sense can infinite sets be

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[1]) *Ibid.*, p. 64.

[2]) Carnap 37, p. 51.

[3]) In the Syposium on Constructibility in Mathematics, Amsterdam 1957.

[4]) Mostowski 55, p. 16.

[5]) *Ibid.*

said to have their source in experience? (We have little quarrel with the view that the ultimate *raison d'être* of the notions of number and set are practical applicability, but we fail again to see how from this view the uniqueness of number theory and set theory can be derived.)

This attempt to abolish the qualitative distinctness of the formal sciences (logic and mathematics) from the real (empirical) sciences, which does not seem to us to have been substantiated, should not be confused with another recent attempt, undertaken by Quine [1]) and others, to abolish the borderline. It differs from the first attempt, to put it in slogan form, in claiming that the empirical sciences are less "empirical" than one usually thinks rather than in claiming that the formal sciences are less "formal". The arguments for this view are rather convincing, but the conclusion is not necessarily forthcoming. One could as well, perhaps even better, draw the conclusion that in an empirical theory a theoretical sub-theory should be distinguished from an observational sub-theory, so that mathematics, or appropriate sections of it, would form together with the specific theoretical sub-theory a calculus which is not directly interpreted at all but receives a partial and indirect interpretation through rules of correspondence which connect the theoretical terms with the observational terms of the observational sub-theory [2]).

Many attempts have recently been made to interpret some metamathematical theorems such as the Löwenheim-Skolem theorem or Gödel's incompletability theorem as discrediting certain ontological views and bolstering others. We do not believe that these attempts were successful. We had already opportunity to express our doubts in this respect with regard to Löwenheim-Skolem (Chapter II, p. 109). With regard to Gödel's theorem, we would like to endorse Myhill's penetrating critique [3]) of the argument from the divergence of 'provable' and 'true' and insist, like him, that this argument does not disprove nominalism (though we would by no means concur with Myhill's psychological interpretation of the limitative theorems of Gödel, Church, etc.). On the contrary, we believe it to be unlikely that any new mathematical or metamathematical results will ever definitely refute any ontological stand-point, though they might conceivably have some influence on the readiness to adopt such a stand-point, for

[1]) The *locus classicus* is Quine 53, I.

[2]) This is the view of Carnap 56.

[3]) Myhill 52a; cf. Turquette 50.

reasons which are extra-rational. Should one want to go on from here and conclude that all ontological views on mathematics, since irrefutable, are thereby also irrelevant for mathematics, though not necessarily for mathematicians, we see no good reasons against such a conclusion.

We are now in a position better to evaluate, though perhaps not to solve, a problem which was raised above (p. 318). There we saw that the incompletability of certain logistic systems can sometimes be interpreted as the non-axiomatizability of certain formalized theories. But whereas this interpretation was rather natural with regard to arithmetical theories, it was quite dubious with regard to set theories. While there exists at least one formalized arithmetic which is complete under a perspicuous and natural notion of validity, viz. Skolem's arithmetic, nothing of this kind seems to hold for set theory.

In what sense then, if in any, does there exist a unique notion of set (natural number) governed by a unique Theory of Sets (Theory of Natural Numbers), of which existing axiomatic set theories (arithmetics) are incomplete approximations? We already saw that some empirical realists, such as Mostowski, would answer this question by claiming that there exist sets and natural numbers in (approximately) the same sense in which there exist animals and stones and that Set Theory and Arithmetic are unique in the same sense in which Zoology [1]) and Mineralogy are unique. It is conceivable that other empirical realists would want to make here a distinction and assert reality and uniqueness only of numbers and their theory but not of sets. We already declared ourselves unable to understand either stand.

All Platonistic realists believe in the uniqueness of numbers, not as empirical entities but as Platonic ideas, and of their theory. (It is unimportant, for our purposes, what terms are used to denote the specific "mode of being" of these entities to distinguish it from the mode of being of animals and stones. Some use qualifying adjectives, others distinguish between 'being', 'existence', 'subsistence', 'reality', etc.) Gödel, for instance, believes "that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption

[1]) Notice that even Russell, for some time, expressed himself in a similar vein. In *Russell 5, p. 169, we read: "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features". The last clause, of course, raises some doubts about the seriousness of this mode of expression, which he abandoned very soon in any case.

of physical bodies and there is quite as much reason to believe in their existence" [1]). But it is not clear whether this view entails the uniqueness of classes and concepts, or whether various, perhaps even mutually incompatible, systems of such entities could fulfil the task of allowing to "obtain a satisfactory system of mathematics". We are not convinced that the distance between the *pragmatic Platonism* of Gödel and the *pragmatic formalism* of Carnap and Curry is as great as the customary formulations would make one think. Believing in the existence of sets because they are necessary for obtaining some satisfactory system and accepting some set theory because it is helpful for obtaining some satisfactory system — is the abyss between these views really so deep?

Conceptualists and nominalists have little reason to believe in the uniqueness of the notion of set, though most conceptualists would believe in the uniqueness of the natural numbers series which serves them as the major basis for their constructions. But the constructions themselves need not proceed in a unique fashion.

For the anti-ontologists, the whole problem does not arise.

It is easy to understand the urge for the belief in the uniqueness of Set Theory. Set-theoretical notions enter everywhere in non-elementary theories, and if set theory itself is treated as an elementary axiomatic theory, every non-elementary theory can as well be regarded as the union of two elementary theories, an elementary set theory and some elementary theory which is specific for the discipline treated. The decisive notion of an "absolute model" of a non-elementary theory, i.e. a model in which all set-theoretical notions receive their standard interpretation, is unique only to the degree that there exists one unique standard interpretation of these notions. It is true, therefore, that the notion of an absolute model "will gain essentially in value only when the difficult problems of the foundations of the theory of sets are solved; this will enable mathematicians to agree to one method of establishing that theory" [2]). But so far we don't see any reason compelling us to believe that there will be a unique solution to the foundational problems of set theory which will induce all mathematicians to accept one such theory as *the* Set Theory. It is doubtful whether such a belief is pragmatically necessary in the sense that

[1]) Gödel 44, p. 137; cf. also 47, from which we already quoted on p. 93 of Chapter II.

[2]) Mostowski 55, p. 12.

otherwise a chaotic situation would arise in which every mathematician would work with his own set theory. The pragmatic criterion of acceptability should suffice to keep the situation under control. The existence of many competing set theories, at least so long as they induce little changes in the day-by-day work of the mathematician and physicist, is hardly harmful enough to justify the imposition of some *credo* or other in this respect. So long as the belief in the objective reality (whatever this may mean) and the resulting uniqueness of the notion of set and its theory is a kind of tranquilizer and does not lead to the dogmatic rejection of proposed set theories — and notice that even Mostowski states in no uncertain terms that "there are no criteria indicating the proper choice among all these numerous [set theories]" [1] — it remains a harmless, and in a certain sense even helpful, metaphysical act of faith. But there is often only one step from the belief in the existence of an objective criterion that would uniquely determine the issue between competing theories and the belief that one has found this criterion and is therefore entitled to disqualify all these theories, except possibly one, in the name of some earthly or heavenly reality. There are many authors who prefer perturbation out of freedom to tranquillity out of external coercion.

The attitudes on how set theory might be given a satisfactory foundation are as yet widely divergent, and a host of problems connected herewith are far from being solved. Nevertheless, the great majority of mathematicians refuse to accept the thesis that Cantor's ideas were but a pathological fancy. Though the foundations of set theory are still somewhat shaky, these mathematicians continue to apply successfully its concepts, methods, and results in most branches of analysis and geometry as well as in some parts of arithmetic and algebra, confident that future foundational research will converge towards a vindication of set theory to an extent that will be identical with, or at least close to, its classical one. This attitude is compatible with a readiness to interpret set theory in a way which might diverge considerably from the customary ones, in line with the apparently existing need for a reinterpretation of logic and mathematics in general.

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[1] *Ibid.*, p. 19.

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Each item contained in the bibliography is quoted in this book at one or several places, for its connection with a certain subject.

As to the selection of the material, up to 1918 only works of particular importance are listed, with the exception of the subject of antinomies in connection with which – for the sake of comprehensiveness – also some minor publications are included that were not mentioned in the bibliography of *Theory*. Of works published between 1919 and 1946 which are contained in the bibliography of *Theory*, only those of special relevance to the topics discussed in the present book have been listed again. For the period 1947–1956, the present bibliography is intended to be rather comprehensive with respect to many subjects, especially those for which no recent easily accessible bibliography exists otherwise. Some publications that appeared in 1957 are also included, as are about half a dozen 1958 publications, but these could only be taken into account sketchily, if at all, for obvious reasons.

For many publications, especially those with whose languages the authors were not familiar, they relied on the (almost always highly authoritative) reviews in the *Journal of Symbolic Logic*, without indicating this in every case.

Regarding the date of the publication, differences of one year may occur, since in general the year of the appearance of the *Volume* is stated, yet in particular cases the year indicated on the fascicle or the reprint.

Reviews and Abstracts are only included in exceptional cases.

An Index of those authors who (also) appear outside the alphabetical order is added at the end of this Bibliography.

ABBREVATIONS USED IN THE BIBLIOGRAPHY

(Self-evident abbreviations such as Acad., Enzykl., Intern., Math., Philos., Ph(ys)., Psychol., Soc., Univ., etc. are not mentioned. "The" is mostly omitted.)

A.M.S. = American Mathematical Society

Abh. Hamburg = Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität

Acad. R. P. Roum., Bull. Sc., Math. Ph. = Acad. de la République Populaire Roumaine, Bulletin Scientifique, Sect. de Sc. Math. et Phys.

Acad. U.S.A. = Proceedings of the National Academy of Sciences (U.S.A.)

Act. Sc. Ind. = Actualités Scientifiques et Industrielles

Acta Szeged = Acta litterarum ac scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio scientiarum mathematicarum

Afd. = Afdeeling

Alg. = Algemeen

Am. = American

Ann. = Annales

Anzeiger Akad. Wien = Akademie der Wissenschaften in Wien, Mathematisch-Naturwissenschaftliche Klasse, Anzeiger

Appl. = Applicazioni

appl. = appliquées

Ber. = Bericht

Boll. = Bolletino

Bull. = Bulletin

C.N.R.S. = Centre National de la Recherche Scientifique

C.R. = Comptes Rendus

C.R. Paris = Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris)

C.R. Varsovie = Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III

Časopis = Časopis pro Peštováni Matematiky a Fysiky

Cl. = Classe

Colloq. = Colloquium, Colloque, etc.

Congr. =Congress(o), Congrès, etc.

Congr. Amsterdam 1954 = Proceedings of the International Congress of Mathematicians, Amsterdam 1954

Congr. Bologna 1928 = Atti del Congresso Internazionale dei Matematici, Bologna 1928

Congr. Cambridge Mass. 1950 = Proceedings of the International Congress of Mathematicians, Cambridge (Massachusetts) 1950
D.M.V. = Deutsche Mathematiker-Vereinigung
Ens. = Enseignement
f. = for, für, etc.
Fac. = Faculty, etc.
Forschungen zur Log(ist)ik etc. = Forschungen zur Log(ist)ik und zur Grundlegung der exakten Wissenschaften
Fund. = Fundamenta
Grundl. = Grundlagen
Helv. = Helvetici
I.M. = Indagationes Mathematicae
Int. Enc. Un. Sc. = International Encyclopedy of Unified Science
Ist. = Istituto
J. = Journal
J.f.M. = Journal für die reine und angewandte Mathematik (Crelle)
J.S.L. = Journal of Symbolic Logic
Jahrb. = Jahrbuch
Jahresb. = Jahresbericht
Koll. = Ergebnisse eines mathematischen Kolloquiums (herausgegeben von K. Menger)
Kon. = Koninklijke
Kongr. Zürich 1932 = Verhandlungen des Internationalen Mathematiker-Kongresses, Zürich 1932
Les Entretiens de Zurich = Les Entretiens de Zurich sur les Fondements et la Méthode des Sciences Mathématiques 1938
Monatsh. = Monatshefte
N.S. = New Series, Neue Folge, Nieuwe Reeks, etc.
Nachr. Göttingen = Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse
Nat. = National, etc.; Natural, etc.
Naturvid. = Naturvidenskab
Naz. = Nazionale
Ned. = Nederlandse
Norsk Vid. Selsk. Forh. = Det Kongelige Norske Videnskabers Selskab, Forhandlinger
p. = pures
phänomen. = phänomenologisch
Phenomen. = Phenomenological
Polon. = Polonaise
Proc. = Proceedings
Proc. Amsterdam, = Kon. Nederlandse Akademie van Wetenschappen te Amsterdam, Proceedings of the section of sciences
Publ. = Publications
R. = Royal(e)
R.M.M. = Revue de Métaphysique et de Morale
Rendic. = Rendiconti

Rendic. Palermo = Rendiconti del Circolo Matematico di Palermo
Rev. = Revue, Review, etc.
Riv. = Rivista
s.a. = sine anno
Sc. = Science(s), etc.
Scand. = Scandinavica
Sem. = Seminar, etc.
Semesterberichte Göttingen = Mathematisch-physikalische Semesterberichte zur Pflege des Zusammenhanges von Schule und Universität (Göttingen)
Semesterberichte Münster = Semesterberichte zur Pflege des Zusammenhangs von Universität und Schule (Math. Seminar, Münster i. W.)
Sitz. Berlin = Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Mathemat. Klasse
Tr. = Transactions
u. =und
v. = van, voor
Verh. = Verhandlungen, Verhandelingen
Vid. = Videnskab
Wetensch. = Wetenschappen
Wiss. = Wissenschaft(en)
Ztschr. = Zeitschrift
(2) = second series (and similarly for other numerals in parentheses)

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